Model Dependence of the Asymptotic Behavior of Form Factors for Relativistic Composite Models

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The large-momentum-transfer behavior of the electromagnetic form factors of composite hadrons is shown to be model-dependent. Integer-spin bound states are treated in the ladder approximation to the two-body Bethe-Salpeter equation. Both spinless and spin-$\frac{1}{2}$ constituents are considered, interacting through regular or singular renormalizable interactions. It is found that the asymptotic behavior of the form factors depends on the kind of singularity at the origin for the regular interactions and on the coupling constant for the singular ones. This dependence is different for spinless and spin-$\frac{1}{2}$ constituents. For spinless constituents, the form factor vanishes more rapidly than $1/q^2$ in all significant cases, but for spin-$\frac{1}{2}$ constituents it does not; in all cases, however, it vanishes at infinity, and the behavior is better than in the corresponding elementary-particle case.

I. INTRODUCTION

THE asymptotic behavior of electromagnetic form factors has been recently investigated by various authors[1-5] by using composite models of nucleons and other hadrons. For a completely bootstrapped particle the form factors seem to be exponentially falling,[1] while if the nucleon is thought of as a bound state of two elementary particles, the $(1/q^2)^2$ behavior is obtained in some simple models,[2-5] in which the binding force is assumed to be given by the exchange of a scalar particle. In a previous paper[4] Menotti and the present author also examined some more singular interactions and pointed out, though not giving the precise asymptotic behavior holding in these cases, that the $(1/q^2)^2$ behavior no longer holds.

The aim of this paper, which is intended to be the continuation of the previous one, is to examine to a greater extent the model dependence of the large-q^2 behavior. The general hope is to obtain results which are independent of the particular interaction, such as an upper asymptotic bound to the form factors of a composite particle, as opposed to an elementary one. However, the only general features we shall find are that the form factors vanish at large q^2 and that the asymptotic behavior seems to be better than the "elementary" one. Another suggested result, e.g., superconvergence,[6] will be shown to be model-dependent and therefore not peculiar to composite particles.[7,8]

The models we shall examine refer only to integer-spin, two-body bound states, in the ladder approximation to the Bethe-Salpeter[9] (BS) equation. Thus, nucleons are not considered here. However, both spinless and spin-$\frac{1}{2}$ constituents will be treated, so that the results apply to realistic models of pseudoscalar (and vector) mesons. The mathematical technique we shall use is not completely rigorous, being based on consistency requirements for some ansatz made for the high-momentum behavior of the wave functions.

For spinless constituents (Sec. II) we consider both regular and singular renormalizable interactions.[10] We assume an attractive and continuous four-dimensional potential which in coordinate space has a singularity R^{-s} at the origin, and we let s vary from 0 to 4, $s=4$ corresponding to the singular interactions.[11,12] The simplest cases, $g\phi^3$ and $g\phi^4$ theories, give $s=2$ and $s=4$, respectively. The intermediate cases (which need a continuous mass distribution) are not by themselves completely academic, because the exchanged particles may be composite and so the singularity of the potential may depend on their propagators and vertex functions. The first result, rather surprising at first sight, is that the large-q^2 behavior depends on s for regular potentials also ($0\leq s<4$), becoming less and less convergent as s approaches 4. The critical behavior is $(q^2)^{-1}$ (which is also essentially the elementary-particle behavior in the

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[1] J. D. Stack, Phys. Rev. **164**, 1904 (1967); J. Harte, *ibid.* **165**, 1557 (1968), and references therein.

[2] D. Amati, R. Jengo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Letters **27B**, 38 (1968).

[3] J. S. Ball and F. Zachariasen, Phys. Rev. **170**, 1541 (1968).

[4] M. Ciafaloni and P. Menotti, Phys. Rev. **173**, 1575 (1968).

[5] D. Amati, L. Caneschi and R. Jengo, CERN Report No. TH900, 1968 (unpublished).

[6] For a suggestion in this direction, see, e.g., S. D. Drell, Comments Nucl. Part. Phys. **2**, 36 (1968).

[7] The fact that a fixed pole of, e.g., photoproduction is removed (see Ref. 8) by the assumption of compositeness of the pion seems to be dependent only on the vanishing of the vertex function of the composite particle. However, the extent to which the fixed pole is shifted to the left of the l plane is probably model-dependent.

[8] H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. **167**, 1441 (1968).

[9] Y. Nambu, Progr. Theoret. Phys. (Kyoto) **5**, 614 (1950); J. Schwinger, Proc. Natl. Acad. Sci. U.S. **37**, 452 (1951); **37**, 455 (1951); M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951); E. E. Salpeter and H. A. Bethe, *ibid.* **84**, 1232 (1951).

[10] The interaction is said to be regular if the leading behavior of the BS wave functions (Ref. 9) does not depend on the strength (or on the singularity) of the interaction; singular otherwise. For classifications of the interactions according to their singularity, see Bastai *et al.* (Ref. 11) and Domokos *et al.* (Ref. 12).

[11] A. Bastai, L. Bertocchi, S. Fubini, G. Furlan, and M. Tonin, Nuovo Cimento **30**, 1512 (1963).

[12] G. Domokos and P. Suranyi, Nucl. Phys. **54**, 529 (1964).

absence of the Born term) and therefore the form factor is always superconvergent for regular interactions. However, the $(1/q^2)^2$ behavior is obtained only for $s=2$.

In Sec. II the $g\phi^4$ model is also treated for $J\geq 1$, and the asymptotic behavior of the electric-charge form factor $F(q^2)$ is explicitly given. This behavior is always worse than that of the regular case, and joins continuously with it for $g\to 0$.

We shall not deal with $J=0$ bound states for singular interactions. In fact, if they exist (this is possible only for vector interactions; see Sec. II), they can hardly be interpreted as composite particles since they lie on a fixed Regge cut. This is perhaps not mere coincidence, because otherwise they would presumably have a form factor asymptotically worse than $1/q^2$, as indicated from the higher-partial-wave analysis mentioned above.

Spin-$\frac{1}{2}$ constituents (e.g., for quark-antiquark bound-state models of mesons) are examined in Sec. III. Only the case of the exchange of a vector elementary particle is treated in detail; this interaction is singular and leads to a λ-dependent asymptotic behavior of $F(q^2)$. The form factor always vanishes at infinity, but, for $J=0$ the behavior varies from $(q^2)^{-1/4}$ to $(q^2)^{-1/2}$ for $\lambda\to 0$. Therefore $F(q^2)$ can be superconvergent only for higher partial waves, or for sufficiently regular interactions. To understand this behavior one must realize that the elementary-particle behavior with bare coupling to spin-$\frac{1}{2}$ particles is presumably bad, being logarithmically divergent with q^2 for some perturbative graphs (Fig. 3).[13]

The main conclusion of this paper is that the asymptotic behavior of form factors strongly depends on the model and especially on the spin of the constituent (elementary) particles. The q^2 behavior of the form factor determines, depending on the spin of the constituents, the behavior of the (attractive) interaction at small distances. Whether these results can be generalized in some way to three-body bound states remains an open question.

II. SPINLESS CONSTITUENTS

Let us consider a bound state of mass m of two spinless particles of equal mass M. We shall calculate the electromagnetic form factors according to a high-energy model[14] which has been previously adopted[15,16] and which is consistent with the ladder approximation for the bound-state wave functions. By looking at Fig.

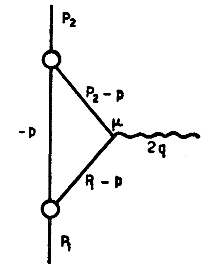

FIG. 1. Ladder approximation to the composite-particle electromagnetic current. The circles are the vertex functions of the composite particles.

1, one obtains the matrix elements of the electromagnetic current

$$\langle 2|J_\mu|1\rangle = 2i\int d^4p\,\bar{\phi}_2(p+\tfrac{1}{2}P_2)(p^2+M^2)\times[\tfrac{1}{2}(P_1+P_2)_\mu+p_\mu]\phi_1(p+\tfrac{1}{2}P_1)\,. \quad (1)$$

Here $\phi_i(p)$ are the Fourier transforms of the BS wave functions[9] of momenta P_i,[17] and $\bar{\phi}_i(p)$ are the conjugate wave functions.[18,19] From Eq. (1) the various form factors are easily obtained.[4] The wave function $\phi(p)$ of total momentum P is[20] assumed to satisfy the BS equation (cf. Fig. 2)

$$\phi(p)=\lambda G(p)(-i)\int d^4k\; V(p-k)\phi(k)\,,$$
$$G(p)\equiv(p_1^2+M^2)^{-1}(p_2^2+M^2)^{-1}, \quad p_{1,2}\equiv\tfrac{1}{2}P\pm p\,, \quad (2)$$

where $V(p)$ is a function of p^2. For regular interactions we shall assume the representation

$$V(p)=\int_0^\infty \frac{d\mu^2\sigma(\mu^2)}{(p^2+\mu^2)^2},\quad \sigma(\mu^2)\underset{\mu^2\to\infty}{\simeq}(\mu^2)^\Delta,\quad |\Delta|<1. \quad (3)$$

It is easily seen from Eq. (3) that $V(p)\simeq(p^2)^{-1+\Delta}$ for large p^2, while its Fourier transform $\tilde{V}(R)\simeq R^{-2(\Delta+1)}$ for $R\to 0$. So when Δ varies from -1 to $+1$ the exponent s of the singularity (cf. Introduction) varies from 0 to 4.

The asymptotic behavior of the form factors is better understood by using the DGSI[21] spectral representation of the wave function. For $J=0$ and $0\leq\Delta<1$,[22]

[13] For electrons see, e.g., M. Cassandro and M. Cini, Nuovo Cimento, **34**, 1719 (1964), and references therein. In this case a class of radiative corrections to the Born approximation shows an oscillating behavior which may sum up to give a convergent behavior for $q^2\to\infty$. We do not know whether similar things happen in our case.

[14] S. Mandelstam, Proc. Roy. Soc. (London) **A233**, 248 (1955); K. Nishijima, Progr. Theoret. Phys. (Kyoto) **13**, 305 (1955).

[15] Cf. Refs. 2–5 and Ciafaloni and Menotti (Ref. 16). In Ref. 3 the approximation of Fig. 1 is called "triangle approximation" but, in the simplest cases, it is the same thing as the ladder approximation of Ref. 14.

[16] M. Ciafaloni and P. Menotti, Nuovo Cimento **46A**, 162 (1966).

[17] We shall use 4-vectors with imaginary fourth component and Hermitian γ-matrices. Initial and final states have covariant normalization.

[18] The relation between ϕ and $\bar{\phi}$ is given through an analytic continuation in the p_0 variable [G. C. Wick, Phys. Rev. **96**, 1125 (1954)] and is written $\bar{\phi}(\mathbf{p},p_0)=-[\phi(\mathbf{p},p_0^*)]^*$ (see Ref. 19) in the spinless case.

[19] M. Ciafaloni and P. Menotti, Phys. Rev. **140**, B929 (1965).

[20] We stress the point that the value of γ [Eqs. (7) and (8)] which determines the asymptotic form factor cannot be determined from the $P=0$ case. Also, the fact that for singular interactions $\gamma\neq\delta$ [Eqs. (22) and (23)] is an indication of the fact that the $P\neq 0$ case gives rise to new features, at least for singular interactions.

[21] S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. **115**, 731 (1959); M. Ida, Progr. Theoret. Phys. (Kyoto) **23**, 1151 (1960) (DGSI).

[22] For $-1<\Delta<0$ the most suitable spectral representation has a fourth power in the denominator, the p^2 asymptotic behavior being better than $(p^2)^{-3}$. The final results, Eqs. (10) and (11), hold unchanged, however.

we take

$$\phi(p)=\int_0^{\infty}dt\int_{-1}^{+1}dz\frac{g(z,t)}{[\frac{1}{2}(1+z)(p_1{}^2+M^2)+\frac{1}{2}(1-z)(p_2{}^2+M^2)+t]^3}=\int\frac{dzdt\,g(z,t)}{(p^2+P\cdot pz+M^2-\frac{1}{4}m^2+t)^3}. \quad (4)$$

By substituting (4) into Eq. (1), it has been shown in Ref. 4 that the leading contribution to $F(q^2)$ for large q^2 $[q=\frac{1}{2}(P_2-P_1)]$ is given by

$$\text{const}\times\int\frac{dzdz'dtdt'd\alpha\,g(z,t)g(z',t')(1-\alpha^2)^2}{[M^2+\frac{1}{2}(1+\alpha)t+\frac{1}{2}(1-\alpha)t'+\frac{1}{4}(1-\alpha^2)(1-z)(1-z')q^2]^3}. \quad (5)$$

Now, if there were a bare coupling constant g_0 (Fig. 3) between our scalar particle and its constituents, we would have, for regular potentials[12,23,24]

$$\lim_{p\to\infty}G^{-1}(p)\phi(p)=g(z,\infty)=g_0\neq 0. \quad (6)$$

and the corresponding behavior of the form factor would be, apart from the Born term, the known[3] "elementary" behavior $(q^2)^{-1}[\ln(q^2/M^2)]^2$. Actually, for composite particles $g(z,\infty)=0$,[12,23] and one expects a better behavior than $1/q^2$. More precisely, if

$$g(z,t)\underset{z\to\pm1}{\simeq}(1\mp z)^{\gamma}g_1(t),\quad g(z,t)\underset{t\to\infty}{\simeq}t^{-\delta}g_2(z) \quad (7)$$

one has that[4]

$$F(q^2)\simeq\text{const}\times(q^2)^{-1-\gamma}\ln(q^2/M^2),\quad(-1<\gamma<2\delta) \quad (8)$$

provided that $-1<\gamma<2\delta$ is satisfied. But this condition will be shown to be always true in our cases, so that we shall not consider other possibilities. The meaning of γ and δ is cleared up by the following results[4]:

$$G^{-1}(p)\phi(p)\simeq(p^2)^{-\delta}\text{ for }p_1{}^2\simeq p_2{}^2\simeq p^2\to\infty$$
$$\simeq(P\cdot p)^{-\gamma}\text{ for fixed }p_2{}^2,\quad p_1{}^2\simeq P\cdot p\to\infty. \quad (9)$$

Then we see that, if $\gamma\neq\delta$, the asymptotic behavior of the vertex function $\Gamma(p_1{}^2,p_2{}^2)\equiv G^{-1}(p)\phi(p)$ in the $p_1{}^2$, $p_2{}^2$ plane is not uniform. This fact is peculiar to singular interactions, as we shall see soon.

First let us consider the regular interaction case of Eq. (3). As is known,[9] the regular wave function in coordinate space goes to a constant, independent of the value of s. The asymptotic behavior in momentum space is, however, s-dependent, because γ and δ are. In fact, one has

$$\gamma=\delta=1-\Delta=\tfrac{1}{2}(4-s). \quad (10)$$

A brief derivation of this result is as follows.[25] Since $V(p-k)\simeq[(p-k)^2]^{-1+\Delta}$, from Eq. (2) one has that $G^{-1}(p)\phi(p)\simeq(p^2)^{-1+\Delta}$ for $p^2\to\infty$, the argument being self-consistent because $\int\phi(k)d(k)<\infty$ with the above-stated behavior of $\phi(p)$. Analogously, $G^{-1}\phi\simeq(P\cdot p)^{-1+\Delta}$ for $p^2\simeq P\cdot p\to\infty$. Equation (10) follows from Eq. (9). By substituting this result in Eq. (8), one finally obtains[22]

$$F(q^2)\simeq\text{const}\times(q^2)^{-1-(1/2)(4-s)}\ln(q^2/M^2). \quad (11)$$

There are some interesting points to be noted. First, this behavior is s-dependent, although the potential is regular. This is because the large-p behavior of the wave function is not determined by the leading term in R (R^0 and possibly R^2), but instead by the next-to-leading one, which is s-dependent.[26] The form factor is always superconvergent, its behavior starting from $(q^2)^{-3}$ for $s=0$ and approaching $(q^2)^{-1}$ for $s\to4$, the latter being essentially the "elementary" behavior discussed above. Only for $s=2$ is the $(1/q^2)^2$ behavior obtained.

Analogous results hold for nonrelativistic potential scattering. It can be shown in this case that $F(q^2)\simeq q^{-3-(2-s)}$ for large q,[27] the $1/q^4$ behavior being reached for a Coulomb-like singularity ($s=1$). The critical behavior, approached for $s\to2$, is q^{-3} instead of q^{-2}, while for singular potentials (varying as $-\lambda/r^2$) one obtains $F(q^2)\simeq(q^2)^{-1-(1/4-\lambda)^{1/2}}$ $(0<\lambda<\frac{1}{4})$.

For higher partial waves the ansatz for the wave

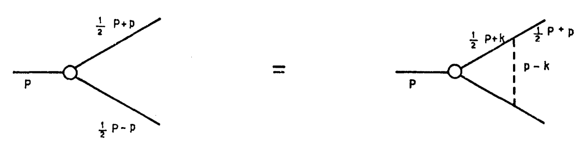

FIG. 2. Ladder approximation to the Bethe-Salpeter equation of the composite particle with elementary constituents.

[23] For singular interactions the limit (6) gives rise, for $g_0\neq0$, to a divergent, or cutoff-dependent behavior. However, the cutoff dependence can be factorized (Ref. 24) for renormalizable interactions and the bound-state condition implies $Z\simeq g_0/g=0$ in this case also.

[24] L. Bertocchi, S. Fubini, and G. Furlan, Nuovo Cimento **32**, 745 (1964); G. Furlan and G. Mahoux, *ibid.* **36**, 215 (1965).

[25] This argument, which is similar to those of Ref. 3, must be taken with care because it works in this case, but not for Eq. (32), even if a regular interaction is assumed. The result (10) can be proved by means of the same techniques adopted for singular interactions (Secs. II and III).

[26] The s-dependent term is R^{4-s}, and for $2<s<4$ it dominates the R^2 term.

[27] The assumption of continuity of the potential is important to obtain this result. In fact, for a square-well potential, $F(q^2)$ goes as q^{-4}, not as q^{-5}. See S. D. Drell, A. C. Finn, and M. H. Goldhaber, Phys. Rev. **157**, 1402 (1967).

function is

$$\phi(p)=Y_{lm}(\mathbf{p})\int\frac{dzdt\,g(z,t)}{(p^2+P\cdot pz+M^2-\frac{1}{4}m^2+t)^{l+3}}$$
$$\equiv Y_{lm}(\mathbf{p})\phi_l(p)\,, \qquad (12)$$

where Y_{lm} are solid harmonics of the 3-vector $\mathbf{p}$. The analogs of Eqs. (9) under assumption (7) are

$$G^{-1}(p)\phi_l(p)\simeq(p^2)^{-\delta-l}\text{ for }p^2\rightarrow\infty\,,$$
$$\simeq(P\cdot p)^{-\gamma}\text{ for }p^2\simeq P\cdot p\rightarrow\infty\,,$$
$$p_\perp^2/P\cdot p\rightarrow 0\,, \qquad (13)$$

where we have specialized P to be parallel to the 3 axis, and $p_\perp^2=p_1^2+p_2^2$. The determination of γ and δ can be carried through as before by using the partial-wave interaction kernel, and the results are[28]

$$\gamma=\delta+l=l+\tfrac{1}{2}(4-s)\,,$$
$$F_l(q^2)\simeq\text{const}\times(q^2)^{-1-l-(1/2)(4-s)}\ln(q^2/M^2)\,. \qquad (14)$$

We come now to singular potentials. Both $g\phi^4$ and vector-exchange theories have been treated by various authors[11,12] in different contexts. We recall that the partial-wave behavior in coordinate space is $R^{\beta_\mp}$ for $R\rightarrow 0$, where, for suitable normalization of the coupling constants, one has[29]

$$\beta_\mp^2=1+(l+1)^2\mp[4(l+1)^2+\lambda]^{1/2},\quad(g\phi^4)$$
$$=1-2\lambda+(l+1)^2$$
$$\mp2[l(l+2)+(1-\lambda)^2]^{1/2}\quad\text{(vector)}. \qquad (15)$$

The requirement of the existence of two regular solutions for $R\rightarrow 0$ implies $\beta_-^2>0$, and therefore, for $\lambda>0$,

$$\sqrt{\lambda}<l(l+2),\quad(g\phi^4)$$
$$\lambda<l(l+2)\ \text{ if } l\neq0, \qquad (16)$$
$$\lambda<1\qquad\text{if } l=0,\ \text{(vector)}.$$

These conditions give rise to the well-known[30] fixed cuts in the l plane. The point $l=0$ always lies on the cut, but, while for the $g\phi^4$ theory the bound-state condition is meaningless, in this case it can be given a meaning for vector-exchange theory. There is, however, another ambiguity[12] for $l=0$ due to the possibility of adding a $\lambda'\delta(x)$ potential arising from renormalization terms. Owing to this ambiguity and to the fact that a possible $l=0$ bound state cannot lie on a Regge trajectory, its interpretation as a composite particle is rather doubtful and its treatment is not attempted here.[31]

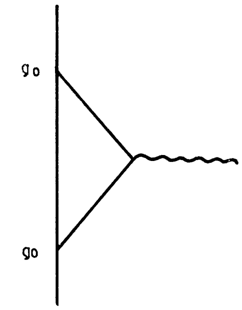

FIG. 3. Diagram belonging to the class represented in Fig. 1 only when the external particle has a bare coupling constant with its constituents.

Let us now look at the $l\neq0$ case for $g\phi^4$ theory. We make an ansatz of the form (12) but with $(l+3)$ replaced by $(l+2)$, because the wave function is expected to be less convergent for large p than in the regular case. The BS equation reads

$$\phi(p)=\lambda G(p)\left(\frac{-i}{4\pi^2}\right)^2\int\frac{d^4k}{(p-k)^2}\frac{d^4q}{(k-q)^2}\phi(q)\,, \qquad (17)$$

where the renormalization terms are not explicitly shown because they give a vanishing contribution for $l\geq1$. Inserting in Eq. (17) the spectral representation of the wave function, we get the integral equation

$$32g(z,t)=\lambda\int K_l(z,t\,;z',t')g(z',t')dz'dt'\,, \qquad (18)$$

where

$$K_l=\theta\left(t'-\frac{t}{R}\right)t^{l-1}\int_{t'}^{\infty}d\tau\,f_l(\tau,z')+\theta\left(\frac{\tau}{R}-t'\right)$$
$$\times t^{l-1}\left[f_{l-1}\left(\frac{t}{R},z'\right)-t'f_l\left(\frac{t}{R},z'\right)\right], \qquad (19)$$

$$f_l(\tau,z')=\int_\tau^\infty dy\,y^{-l}$$
$$\times[M^2+\tfrac{1}{4}m^2(1-z'^2)+y]^{-2}\underset{\tau\rightarrow\infty}{\simeq}(l+1)^{-1}\tau^{-l-1}\,,$$

$$R=R(z,z')\equiv\theta(z-z')\frac{1-z}{1-z'}+\theta(z'-z)\frac{1+z}{1+z'}.$$

We now want to find δ and γ by requiring that the ansatz of Eq. (7) be consistently reproduced in Eq. (18). For large t, or for $z\rightarrow\pm1$, by using the asymptotic

[28] We consider here only the electric-charge form factor, the behavior of the other form factors being determined by kinematical considerations. If l is allowed to take continuous values through an analytic continuation, we notice the amusing result that the $1/q^2$ behavior is reached, for fixed m, only if $\lambda\rightarrow0$, being in this case $l(\lambda)\rightarrow-\frac{1}{2}(4-s)$ for the leading trajectory [A. Bastai, L. Bertocchi, and M. Tonin, Nuovo Cimento **29**, 247 (1963)].

[29] We assume that accidental degeneracy is absent so that a given angular momentum l contains all four-dimensional angular momenta $\alpha\geqslant l$.

[30] R. F. Sawyer, Phys. Rev. **131**, 1384 (1963). See also Ref. 12 and G. Cosenza, L. Sertorio, and M. Toller, Nuovo Cimento **31**, 1086 (1964).

[31] Detailed calculations seem to show that the ansatz (4) cannot be consistently reproduced in this case, the resulting spectral functions $g(z,t)$ being too singular near $z=\pm1$ $(\gamma<-1)$. This is an indication that the spectral representation does not exist in this case.

properties of f_l, we obtain

$$g_2(z)t^{-\delta}\simeq\tfrac{1}{32}\lambda\int dz'g_2(z')\left[\frac{t^{l-1}}{l(l+1)}\int_{t/R}^{\infty}dt't'^{-l-\delta}+\frac{t^{-1}}{l}R^l\times\int_0^{t/R}dt't'^{-\delta}-\frac{t^{-2}R}{l+1}\int_0^{t/R}dt't'^{-1-\delta}\right]\simeq\lambda t^{-\delta}\left[\int dz'g_2(z')\times[R(z,z')]^{l+\delta-1}/32(l+\delta-1)(2-\delta)(1-\delta)\right]. \quad (20)$$

By integrating over z from -1 to $+1$, and using the relation

$$\int_{-1}^{+1}dz[R(z,z')]^{\alpha}=2/(\alpha+1), \quad (21)$$

one obtains the consistency equation

$$16(l+\delta)(l+\delta-1)(2-\delta)(1-\delta)=\lambda \quad (22)$$

which has the solution $\delta=1-\frac{1}{2}(l-\beta_-)$, as is expected in order to obtain the behavior R^{β_-} in coordinate space. More interestingly, Eq. (20) gives at once the value of γ,

$$\gamma=l+\delta-1, \quad (23)$$

from which it follows that

$$F_l(q^2)\simeq\text{const}\times(q^2)^{-l-\delta}\ln(q^2/M^2), \quad (24)$$

which is the desired result.

As $1-\frac{1}{2}l<\delta<1$ for the values of λ allowed by Eq. (16), it is clear that the "regular" behavior of Eq. (14) is approached for $\lambda\to 0$, while for any allowed λ, the function $F(q^2)$ vanishes faster than $(q^2)^{-1-l/2}$. Equation (24) can be given a meaning also for continuous l through an analytic continuation of Eq. (18). It is then interesting to notice that the "elementary"[3] $1/q^2$ behavior is approached only for $l\to 0$. However, the point $l=0$ cannot be reached for $\lambda>0$, as pointed out previously.

III. SPIN-$\frac{1}{2}$ CONSTITUENTS

All realistic composite models of mesons are of fermion-antifermion type, so the analysis of the form factors in this case is needed in order to have reliable predictions. The ladder approximation of the electromagnetic current (Figs. 1 and 2) can be extended in a straightforward manner to spin-$\frac{1}{2}$ particles, giving rise to the expression

$$\langle 2|J_\mu|1\rangle=-\int d^4p\times\text{Tr}[\bar{\phi}_2(p+\tfrac{1}{2}P_2)\gamma_\mu\phi_1(p+\tfrac{1}{2}P_1)(ip\cdot\gamma+M)], \quad (25)$$

where the wave function $\phi(p)$ of total momentum P is a 4×4 matrix function obeying the BS equation

$$(ip_1\cdot\gamma+M)\phi(p)(-ip_2\cdot\gamma+M)=\lambda\left(\frac{-i}{4\pi^2}\right)\int\frac{d^4k\,\Gamma\phi(k)\Gamma}{(p-k)^2+\mu^2}, \quad (26)$$
$$p_{1,2}=\tfrac{1}{2}p\pm p.$$

In this equation the exchanged particle is scalar, pseudoscalar, or vector according to $\Gamma=1$, γ_5,[32] or $i\gamma_\mu$. From now on, we shall concentrate on the vector interaction, which gives opposite forces for the $f-\bar{f}$ and $f-f$ interactions, and also gives rise to a simpler asymptotic problem. We recall,[33] however, that, for any Γ, Eq. (26) gives rise to a singular problem because in coordinate space the interaction function has a singularity R^{-2} which is, roughly speaking, of the same type as that of the second-order differential equation on the left-hand side. The behavior of the wave function at the origin is, in this case, $R^{-1+[(J+1)^2-4\lambda]^{1/2}}$.

Our purpose is to find out the large-q^2 behavior of the form factors according to the composite model of Eqs. (25) and (26). We notice, however, that the corresponding elementary behavior is not as clear as in the spinless case. The graph of Fig. 3 gives in our case, after renormalization, a logarithmically divergent behavior. But it is not certain, the potential being singular, that it dominates the asymptotic behavior or that, by summing a series of diagrams, a better behavior is not obtained.[13] In any case the elementary behavior seems to be worse than in the spinless case.

To give now the composite-particle behavior we need the asymptotic properties of $\phi(p)$. To this end we make the reasonable assumption that M and μ can be neglected in Eq. (25) compared with P and p. The asymptotic equation then reads

$$-i(\tfrac{1}{2}P+p)\cdot\gamma\phi(p)i(\tfrac{1}{2}P-p)\cdot\gamma=\frac{i\lambda}{4\pi^2}\int\frac{d^4k\,\gamma_\mu\phi(k)\gamma_\mu}{(p-k)^2}\equiv-\gamma_\mu V_0\phi\gamma_\mu, \quad (27)$$

where also a vector interaction is explicitly assumed. It is now easily realized that Eq. (27) admits two separate $J^P=0^-$ solutions of the form

$$\phi=\gamma_5(\varphi_0-\varphi_1[p\cdot\gamma,\tfrac{1}{2}P\cdot\gamma]) \quad (28)$$

and of the form

$$\phi=\gamma_5(\varphi_0'ip\cdot\gamma+\varphi_1'i\tfrac{1}{2}P\cdot\gamma), \quad (29)$$

where φ_0, φ_1, etc. are functions of p^2 and $P\cdot p$. This means that only two scalar functions are coupled together instead of four as in the general case.[34] There-

[32] This interaction is attractive for, e.g., the symmetric $N-\bar{N}$ interaction in the isospin-1 channel.

[33] J. S. Goldstein, Phys. Rev. **91**, 1516 (1953).

[34] See, e.g., M. Ciafaloni, Nuovo Cimento **51A**, 1090 (1967), and references therein.

fore, the complete Eq. (26) has two types of solutions, of which one is asymptotically essentially type (28), and the other type (29). Both of these have the right quantum numbers[35] of, e.g., the pion ($J^P=0^-$, $C=+1$) and we shall deal with the first one because, for $P^2\to 0$, it corresponds to the ground-state solution.[34] By substituting Eq. (28) into Eq. (27), one obtains two coupled equations, one of which is simply an algebraic relation between φ_0 and φ_1, because $\gamma_\mu\sigma_{\lambda\rho}\gamma_\mu=0$ (this is peculiar to the vector interaction).[36] The final equations are

$$[(p^2)^2-(P\cdot p)^2]\varphi_0\simeq\lambda p^2V_0\varphi_0,\quad \varphi_1=(p^2)^{-1}\varphi_0. \tag{30}$$

We now insert in Eq. (30) the ansatz

$$\varphi_0=\int\frac{dzdt\, g(z,t)}{(p^2+P\cdot pz+M^2-\frac{1}{4}m^2+t)^2} \tag{31}$$

to get the equation

$$\varphi_0=\frac{p^2}{(p^2)^2-(Pp)^2}\lambda\int_0^1\frac{dx}{x}\int\frac{dzdt\, g(z,t)}{(p^2+P\cdot pz+\rho^2+u)},$$
$$u(x,z,t)=[t+Q(z)]/x-Q(z), \tag{32}$$
$$Q\equiv M^2-\tfrac{1}{4}m^2(1-z^2),\ \rho^2\equiv M^2-\tfrac{1}{4}m^2.$$

We could obtain, as in Sec. II, an integral equation for the spectral function to determine the values of γ and δ in this case. This can be done, but we prefer to try a consistency argument directly on Eq. (32) because it seems clearer. As usual we make the ansatz of Eq. (7) and then we let $p_\perp^2=p_1^2+p_2^2\to\infty$, P being parallel to the 3 axis. By means of the change of variables

$$t=x(p^2+P\cdot pz+\rho^2)\tau \tag{33}$$

and the analogous one on the right-hand side, we obtain

$$(p^2)^{-1}[(p^2)^2-(P\cdot p)^2]\int\frac{dz\, g_2(z)}{(p^2+P\cdot pz+\rho^2)^{1+\delta}}I_1\simeq\frac{\lambda}{1-\delta}\int\frac{dz\, g_2(z)}{(p^2+P\cdot pz+\rho^2)^{\delta}}I_0, \tag{34}$$

where

$$I_n\equiv\int_0^\infty d\tau\,\tau^{-\delta}(1+\tau)^{-1-n},\quad I_1=\delta I_0. \tag{35}$$

We now consider the asymptotic behavior of (34) in the directions of Eq. (9). If $P\cdot p/p_\perp^2\to 0$, then one can extract p^2 in the integrals of both sides and, because of relation (32), one obtains the consistency condition

$$\delta(1-\delta)=\lambda\Rightarrow\delta=\tfrac{1}{2}+(\tfrac{1}{4}-\lambda)^{1/2}. \tag{36}$$

This means that, for $p^2\to\infty$, $\varphi_0(p)\simeq(p^2)^{-1-\delta}$ as for the case $P=0$.[4] If, on the other hand, we let $p^2=\pm P\cdot p+p_\perp^2\to\infty$ and $p_\perp^2/P\cdot p\to 0$, the asymptotic behavior of both sides of Eq. (34) depends on γ. Now we note that, in this case,

$$(p^2)^{-1}|(p^2)^2-(P\cdot p)^2|\simeq 2p_\perp^2. \tag{37}$$

On the other hand,

$$\int\frac{dz\, g_2(z)}{(p^2+P\cdot pz+\rho^2)^{1+\delta}}\simeq\frac{1}{(p_\perp^2)^{1+\delta}}\left(\frac{p_\perp^2}{P\cdot p}\right)^{1+\gamma}\int_0^\infty\frac{dy\, y^\gamma}{(1+y)^{1+\delta}} \tag{38}$$

provided that $\gamma<\delta$. The condition $\gamma<\delta$ must be satisfied because otherwise, if $\delta\le\gamma$, the left-hand side of Eq. (34) would behave essentially as $(P\cdot p)^{-1-\delta}$ and the right-hand side as $(P\cdot p)^{-\delta}$. Then it also follows that $(1+\gamma)<\delta$ because $(P\cdot p)^{-\delta}$ is the most convergent behavior one can obtain on the right-hand side [the value $\gamma=(\delta-1)$ being ruled out by the presence of a logarithmic term on only one side of (34)]. Therefore, the integrals being convergent, one obtains, also on the right-hand side, the behavior

$$I_1(p_\perp^2)^{-\delta+\gamma+1}(P\cdot p)^{-1-\gamma}\int_0^\infty\frac{dy\, y^\gamma}{(1+y)^\delta}, \tag{39}$$

which, because of Eqs. (37) and (38), is consistent with the left-hand side, provided that the coefficients are equal. This leads to the condition (obtained by relating the y integrals to each other)

$$\gamma=-1+\tfrac{1}{2}\delta, \tag{40}$$

from which the asymptotic behavior

$$F(q^2)\simeq\text{const}\times(q^2)^{-\delta/2}\ln(q^2/M^2) \tag{41}$$

follows after substitution of (28) into Eq. (25).[4]

The solution (27) can be generalized to higher partial waves by simply inserting solid harmonics (this is because $\gamma_\mu\sigma_{\lambda\rho}\gamma_\mu=0$) and the analogous result for spin J is, for the electric-charge form factor,

$$F_J(q^2)\simeq\text{const}\times(q^2)^{-(J+\delta_J)/2}\ln(q^2/M^2), \tag{42}$$

where

$$\delta_J\equiv\tfrac{1}{2}(1-J)+[\tfrac{1}{4}(1+J)^2-\lambda]^{1/2}. \tag{43}$$

We note that, according to these equations, the form factors always vanish asymptotically. This is in fact what is expected for any bound state for which the representation (4), or (38), makes sense ($\gamma>-1$). In our case the best behavior one obtains for $J=0$ is $(q^2)^{-1/2}$. Therefore one expects that this behavior can be approached also by regular potentials when $s\lesssim 2$. This is indeed what happens.[37] The J dependence of

[35] We do not consider here internal quantum numbers in detail; but we refer to a particle-antiparticle case, just to make sure that the BS wave function is not abnormal, i.e., that it has the right time parity (Ref. 34) $\tau=-CP=+1$.

[36] W. Kummer, Nuovo Cimento 31, 219 (1963); 34, 1840 (1964).

[37] We do not discuss this case here. The essential difference with the spinless case arises from the behavior of the Green's function in Eq. (37).

Eq. (39) is rather curious and we have been unable to attach any particular significance to it.[38,39]

The differences between spinless and spin-$\frac{1}{2}$ constituents are impressive. In particular, while in the spinless case binding by a regular potential meant superconvergence, it is not so for the spin-$\frac{1}{2}$ case. An indication of superconvergence or of $(1/q^2)^2$ behavior of the pion form factor, would imply that either the interaction is strongly regularized by some mechanism (e.g., bootstrap), or that the high-energy model we used is wrong.

[38] We do not know whether the possible existence of $J=0$ singularities in the J plane (Ref. 39) for more sophisticated models can invalidate the composite-particle interpretation of the bound-state solution studied above.

[39] See, e.g., S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967), and references therein.

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Bootstrap of Meson Trajectories from Superconvergence

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In this paper we study the reactions $\pi\pi\to\pi\omega(1^-)$, $\pi\pi\to\pi A_2(2^+)$, and $\pi\pi\to\pi\omega_3(3^-)$ as a bootstrap system for natural-parity trajectories. We start from the solution of our previous work that gave, among other results, expressions for the trajectory and residue functions as well as mass formulas, in agreement with experiment. Here we study in detail the sum rules as a function of momentum transfer t. We find a set of residue functions $\beta(t)$ that are self-consistent and such that the Regge and resonance sides of the equations are almost equal in a large region of t. We study also a step-by-step approximation that, at each stage, enlarges the region where the equations are valid. We find, however, that the leading Regge trajectories, even if infinitely rising, cannot bootstrap themselves. We outline two possible (not incompatible) ways of implementing the bootstrap. The first way demands an optimized choice of the cutoff parameter and considers the whole family of reactions $\pi\pi\to\pi X_J$ (X_J being a normal-parity state of spin J). Our results for $J\leq 3$ show that this is a definite possibility. The second way is to consider a whole family (parent and daughters) as participating in the bootstrap. We find this possibility also attractive, and as a consequence we find that daughters must be parallel to the parent, for linear trajectories. The properties of our parametrization are also discussed—in particular, the Khuri paradox and the coupling of high-spin resonances to the system. We also compare our results with experiment whenever possible. Our A_2 trajectory, for instance, follows the Gell-Mann mechanism, and the exponential t dependence of our residue functions is perfectly consistent with the one found in recent phenomenological fits to inelastic reactions.

1. INTRODUCTION

It seems that a very promising attempt in elementary-particle theory today can be found in blending the general principles of S-matrix theory, embodied in analyticity, crossing, and unitarity, with the dynamical elements contained in Regge-pole theory. The resulting scheme will, it is hoped, put strong enough restrictions on the scattering amplitudes that the Regge trajectories and their residue functions will be uniquely determined. As a consequence, the spectrum of particles and their couplings will be completely determined and their bootstrap accomplished.

A large number of papers, dealing with the question of analyticity at $t=0$ when the external masses are not equal, have shown that Regge trajectories must appear in families.[1] The Regge functions of the members of the family must obey relations at this point but are undetermined elsewhere. These results have been reached by means of powerful group-theoretical techniques by Toller and collaborators[2] and by Freedman and Wang.[3] A few models have also been solved in some approximation, as the Van Hove model[4] and the Bethe-Salpeter

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[1] D. S. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).

[2] M. Toller, Nuovo Cimento **53A**, 671 (1968), and references therein.

[3] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

[4] R. L. Sugar and J. D. Sullivan, Phys. Rev. **166**, 1515 (1968).