Cook for reading the manuscript. Part of this work was carried out while I was a summer visitor at Brookhaven National Laboratory.

APPENDIX

In Eq. (3) the function Q_N was defined in terms of f_N, the coefficient of s, and Δ_N, the determinant of the potential-theory ladder graph. This Δ_N is obtained from the corresponding function for relativistic ladder diagrams, Eq. (A1) of I, by setting the Feynman parameters for one side of the ladder equal to zero. The potential theory f_N is obtained from its relativistic counterpart, Eq. (A9) of I, by setting the parameters for the two sides of the ladder equal. Thus, nonrelativistically f_N is a quadratic function of y_i. If the x_i of both ends of a potential ladder are set equal to zero (ends contracted), we find that

$$f_N=\Big(\sum_{i=1}^{N} y_i\Big)\Delta_N. \tag{A1}$$

The functions appearing in (6) are defined as follows:

$$J(\alpha,s)=\sum_{N=1}^{\infty}\left(\frac{g}{2\pi}\right)^N\int_0^\infty dy_i dx_i \times\prod_1^{N-1}\left(\frac{x_i^{\alpha+2}+(\alpha+1)-(\alpha+2)x}{(\alpha+1)(\alpha+2)}\right)\frac{\partial^2}{\partial x_i^2} \times\left(\frac{\exp(-\mu^2\sum x_i+s\sum y_i)}{\Delta_N^{\alpha+3/2}}\right). \tag{A2}$$

$J_1(\alpha,s)$ differs from (A2) in that the sum starts with $N=2$, and there is an additional derivative with respect to y_1 acting on the square bracket in the integrand. $J_{1N}(\alpha,s)$ starts with $N=3$ and has derivatives with respect to y_1 and Y_N acting on the square bracket. These extra derivatives can be integrated out immediately and the appropriate y_i set to equal zero. We have used (A1) in deriving (A2). One important property of these functions is that $JJ_{1N}=J_1^2$.

Just as J_1 and J_{1N} differed from J by the presence of extra derivatives, $J_\sigma^{n,m}$ and $J_\delta^{n,m}$ in (17) differ from $J^{n,m}$ given in (18) by the presence of derivatives with respect to $\sigma=y+z$ and $\delta=y-z$.

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Off- and On-Shell Analyticity of Three-Particle Scattering Amplitudes*

D. D. BRAYSHAW†
Rockefeller University, New York, New York
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Off-shell analytic properties are obtained for amplitudes which satisfy approximate three-body equations of the Faddeev-Lovelace type. These properties are used to construct an explicit representation for the off-shell three-particle amplitude in terms of the solution of a new integral equation. The advantages of this new equation for both analytical and computational purposes are discussed, and the representation is used to determine on-shell analyticity for the amplitude. The form of the representation is such that the important three-body singularities are exhibited in a particularly transparent fashion.

I. INTRODUCTION

RECENTLY a great deal of attention has been devoted to the study of nonrelativistic three-body systems. This interest has been stimulated by the derivation of exact integral equations for the three-particle scattering amplitude by Faddeev.[1] However, the direct application of these equations to physically interesting three-body problems is somewhat impractical. Instead, the great majority of applications to date have been based on approximate equations of the type suggested by Lovelace.[2] The latter result from the Faddeev equations when the off-shell two-body amplitudes are approximated by functions separable in the initial and final momenta. They are written in terms of quasi-two-particle amplitudes which describe processes involving the scattering of single particles off two-particle bound or resonant states. This paper is concerned with determining the on- and off-shell analytic properties of such amplitudes, and with the derivation of new integral equations possessing several important advantages over the original (Faddeev-Lovelace) equations.

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† Present address: Department of Physics, Columbia University, New York, New York.

[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1459 (1960) [English transl.: Soviet Phys.—JETP 12, 1014 (1961)]; Dokl. Akad. Nauk SSSR 138, 565 (1961) [English transl.: Soviet Phys.—Doklady 6, 384 (1961)].

[2] C. Lovelace, Phys. Rev. 135, B1225 (1964).

The integral equations of interest are formally similar to the usual two-body partial-wave equations, and are written in terms of variables corresponding to the initial and final off-shell momenta. In this case, however, the "potentials" as well as the "propagators" depend parametrically on the total energy. For fixed energy the solutions of these equations may be shown to have certain analyticity properties as functions of the off-shell momenta. This off-shell analyticity is of importance in the very practical problem of solving these equations by numerical techniques. This is because it is usually impractical to perform such calculations without some type of contour deformation procedure. When the momentum dependence of a solution is fixed at the on-shell value, the resulting amplitude is known to possess certain analytic properties as a function of the total energy which are of direct physical interest. In the familiar two-body problem, for example, the on-shell partial-wave amplitude is analytic in the total energy except for the right-hand elastic cut, left-hand cuts associated with the potentials, and (possibly) poles corresponding to bound or resonant states.

The usual method for determining the on- or off-shell analyticity discussed above is to study contour "pinches" in the multidimensional integrals which occur in the perturbation expansion of the integral equation. This method has been applied to the exact Faddeev equations[3] in order to determine on-shell analyticity for the case where the interparticle potential is a superposition of Yukawas. The discussion to be given in this paper is based on a quite different technique previously developed by the author, and described in a recent paper,[4] hereafter referred to as A. In this approach the integral equation is used directly to determine the off-shell analyticity, and this information is then employed to write an explicit representation for the amplitude. The on-shell analytic properties can be obtained trivially from this representation.

The relevant equations and notation are given in Sec. II, as well as a brief review of their derivation in the Faddeev-Lovelace approach. After obtaining the general form of these equations, the discussion is specialized to the case of three identical particles. The resulting equations are representative of the general class of equations under consideration, and are chosen to simplify the rather detailed analysis to be given in the subsequent sections. Partial-wave equations are obtained for this special case.

The discussion in Sec. III is devoted to determining the off-shell analyticity of the partial-wave amplitudes defined in Sec. II. Using the off-shell approach developed in A, these amplitudes are shown to be analytic in the off-shell variable except for certain specific singularities. These singularities take the form of cuts in the complex plane, and can be divided for convenience into two groups. Those which are purely kinematical in nature are examined in detail in this section; those which are dynamical in origin are treated in the Appendix.

The off-shell analytic properties so obtained provide sufficient information to write an explicit representation for the off-shell amplitude. Such a representation is derived in Sec. IV in terms of the solution of a new integral equation. This new equation is shown to possess several important advantages over the original Faddeev-Lovelace equation. These advantages are due principally to the manner in which the various three-body singularities appear in the kernel and in the inhomogeneous terms, i.e., the singularities appear explicitly in functional form in contrast to the implicit fashion in which they are contained in the original equation, arising in the latter case through pinches of the integration contour. This fact makes the representation especially suitable for investigating the on-shell analytic properties of the amplitudes considered. One consequence of this has already been explored in a joint paper[5] by the author and Peierls. In that paper, referred to hereafter as B, representations of the above type were used to investigate the connection between rescattering singularities and three-body bound or resonant states.

Finally, a modification of the above approach is introduced in Sec. V in order to provide a practical alternative method for solving equations of the Faddeev-Lovelace type numerically. It is shown how the well-known difficulties which arise in solving such equations in the domain of positive total energy can be eliminated. The method thus illustrated is of sufficient generality to be of practical utility in performing many similar computations.

II. THREE-BODY EQUATIONS AND KINEMATICS

In this section we will briefly review the derivation of the equations which will concern us in the remainder of this paper. These equations, which are essentially those obtained by Lovelace,[2] result from employing a separable approximation to the off-shell two-body scattering amplitude in the Faddeev[1] equations. Such equations are by now familiar in the literature[2,6,7] and our main objective will be to establish some notation. After having obtained the general form of these equations, we will specialize to the case of three identical particles whose interaction is characterized by a single bound state between pairs. Although the resulting equation is perhaps the simplest of the Lovelace type, the important features of the general problem are nonetheless present. For clarity, the detailed discussion to be given in sub-

[3] M. Rubin, R. Sugar, and G. Tiktopoulos, Phys. Rev. **146**, 1130 (1966).

[4] D. D. Brayshaw, Phys. Rev. **167**, 1505 (1968).

[5] D. D. Brayshaw and R. F. Peierls, Phys. Rev. (to be published).

[6] J. H. Hetherington and L. H. Schick, Phys. Rev. **137**, B935 (1965).

[7] R. Aaron and R. D. Amado, Phys. Rev. **150**, 857 (1966).

sequent sections of this paper will be confined to this special case.

We denote the momentum of particle α by $\mathbf{k}_\alpha$ and its mass by m_α, where α takes on the values 1, 2, 3. In the three-body center-of-mass (c.m.) system, we have $\sum_\alpha \mathbf{k}_\alpha = 0$, and we can describe the system by giving the value of the total c.m. energy W and any two independent linear combinations of the $\mathbf{k}_\alpha$. It is convenient to introduce the following variables:

$$\mathbf{p}_\alpha = \mu_\alpha\left(\frac{\mathbf{k}_\beta}{m_\beta} - \frac{\mathbf{k}_\gamma}{m_\gamma}\right),\qquad \mathbf{q}_\alpha = M_\alpha\left(\frac{\mathbf{k}_\beta+\mathbf{k}_\gamma}{m_\beta+m_\gamma} - \frac{\mathbf{k}_\alpha}{m_\alpha}\right), \tag{1}$$

where $\alpha\beta\gamma$ are cyclic permutations of 1, 2, 3 and

$$\mu_\alpha^{-1} = m_\beta^{-1} + m_\gamma^{-1},\qquad M_\alpha^{-1} = m_\alpha^{-1} + (m_\beta+m_\gamma)^{-1}. \tag{2}$$

Thus, in the c.m. system, $-\mathbf{q}_\alpha$ is the momentum of particle α, while $\mathbf{p}_\alpha$ is the momentum of particle β in the $\beta\gamma$ c.m. system. Of the six vectors $\mathbf{p}_\alpha$, $\mathbf{q}_\alpha$, any two suffice, with W, to completely specify the three-body state, which we denote by $|\mathbf{p}_\alpha\mathbf{q}_\alpha\rangle$. For a physical state, they must satisfy the on-shell condition

$$W = \sum_{\alpha=1}^{3} (\mathbf{k}_\alpha^2/2m_\alpha) = \omega_\beta + \nu_\beta,$$

for any β, where ω_α is the energy associated with $\mathbf{q}_\alpha$ and ν_α is the energy associated with $\mathbf{p}_\alpha$, namely,

$$\nu_\alpha = p_\alpha^2/2\mu_\alpha,\qquad \omega_\alpha = q_\alpha^2/2M_\alpha. \tag{3}$$

We denote the three-particle scattering operator by $T(W)$, and define operators $T_\alpha(W)$ such that

$$\langle\mathbf{p}_\alpha'\mathbf{q}_\alpha'|T_\alpha(W)|\mathbf{p}_\alpha\mathbf{q}_\alpha\rangle = \delta(\mathbf{q}_\alpha'-\mathbf{q}_\alpha)\langle\mathbf{p}_\alpha'|t_\alpha(W-\omega_\alpha)|\mathbf{p}_\alpha\rangle, \tag{4}$$

where $\langle\mathbf{p}_\alpha'|t_\alpha(\nu)|\mathbf{p}_\alpha\rangle$ is the usual off-shell two-body amplitude for scattering of particles β and γ. The operator $T_\alpha(W)$ evidently represents the amplitude for the scattering of particles β and γ, with particle α being undeflected. For the case where the particles interact through pair forces only, it can be shown that $T(W)$ can be written in the form

$$T(W) = \sum_{\alpha=1}^{3} T_\alpha(W) + \sum_{\alpha,\beta=1}^{3} T_\alpha(W)X_{\alpha\beta}(W)T_\beta(W), \tag{5}$$

where $X_{\alpha\beta}(W)$ satisfies the equation

$$X_{\alpha\beta}(W) = \bar{\delta}_{\alpha\beta}G_0(W) + \sum_{\gamma=1}^{3} \bar{\delta}_{\alpha\gamma}G_0(W)T_\gamma(W)X_{\gamma\beta}(W),$$

and

$$\langle\mathbf{p}_\alpha'\mathbf{q}_\alpha'|G_0(W)|\mathbf{p}_\alpha\mathbf{q}_\alpha\rangle = \frac{\delta(\mathbf{p}_\alpha'-\mathbf{p}_\alpha)\delta(\mathbf{q}_\alpha'-\mathbf{q}_\alpha)}{\omega_\alpha+\nu_\alpha-W-i\epsilon}, \tag{6}$$

$$\bar{\delta}_{\alpha\beta} \equiv (\delta_{\alpha\beta}-1).$$

The problem of determining $T(W)$ thus reduces to that of solving Eq. (6) for the quantities $X_{\alpha\beta}(W)$. However, when inserted between the states $|\mathbf{q}_\alpha\mathbf{p}_\alpha\rangle$, this equation becomes an integral equation in six variables. It is thus much too difficult to solve. The problem becomes greatly simplified if one approximates the two-body scattering amplitude by the separable form[8]

$$\langle\mathbf{p}'|t_\alpha(\nu)|\mathbf{p}\rangle = \sum_{\lambda=1}^{N_\alpha} \frac{g_\lambda^\alpha(\mathbf{p}')g_\lambda^\alpha(\mathbf{p})^*}{D_\lambda^\alpha(\nu)}, \tag{7}$$

where $D_\lambda^\alpha(\nu)$ is some denominator function which vanishes when $\nu=\nu_\lambda^\alpha$, corresponding to a bound or resonant state of particles β and γ of energy ν_λ^α. The function $g_\lambda^\alpha(\mathbf{p}')$ is a vertex form factor for the formation of the two-body bound or resonant state.[9] In the summation above we have assumed N_α such states.

We define the three-particle states,

$$|\alpha\lambda\mathbf{q}_\alpha\rangle = \int d\mathbf{p}_\alpha\, g_\lambda^\alpha(\mathbf{p}_\alpha)|\mathbf{p}_\alpha\mathbf{q}_\alpha\rangle, \tag{8}$$

and the function

$$\begin{aligned} Z_{\alpha\beta}(\mathbf{q}_\alpha'\lambda|\mathbf{q}_\beta\mu;W) &\equiv \bar{\delta}_{\alpha\beta}\langle\alpha\lambda\mathbf{q}_\alpha'|G_0(W)|\beta\mu\mathbf{q}_\beta\rangle,\\ &= \bar{\delta}_{\alpha\beta}g_\lambda^\alpha(\mathbf{p}_\alpha')^*g_\mu^\beta(\mathbf{p}_\beta)\Big/\left(\frac{q_\alpha'^2}{2\mu_\beta}+\frac{\mathbf{q}_\alpha'\cdot\mathbf{q}_\beta}{m_\gamma}+\frac{q_\beta^2}{2\mu_\alpha}-W-i\epsilon\right). \end{aligned} \tag{9}$$

Here, $\mathbf{p}_\alpha'$, $\mathbf{p}_\beta$ are given in terms of $\mathbf{q}_\alpha'$, $\mathbf{q}_\beta$ by the relations

$$\mathbf{p}_\alpha' = \mp[\mathbf{q}_\beta + (\mu_\alpha/m_\gamma)\mathbf{q}_\alpha'],\qquad \mathbf{p}_\beta = \pm[\mathbf{q}_\alpha' + (\mu_\beta/m_\gamma)\mathbf{q}_\beta], \tag{10}$$

the upper (lower) sign being taken when $\alpha\beta$ is cyclic (anticyclic). In terms of the states (8), Eq. (6) can be

[8] The equations which result from making this approximation are *formally* identical to three-body equations derived under the assumption of separable potentials [see, e.g., A. N. Mitra, Nucl. Phys. **32**, 529 (1962)], i.e., one can perform either exact calculations for separable potentials or approximate calculations for local potentials. The results to be described pertain to both of these possibilities.

[9] In the following development we will assume the form factors to have the properties which follow when the two-body amplitude is taken to satisfy the Lippmann-Schwinger equation with local potentials of the Yukawa or exponential type, or with separable potentials of the Yamaguchi type.

written in the form

$$\langle\alpha\lambda\mathbf{q}_\alpha'|X_{\alpha\beta}(W)|\beta\mu\mathbf{q}_\beta\rangle=Z_{\alpha\beta}(\mathbf{q}_\alpha'\lambda|\mathbf{q}_\beta\mu;W)+\sum_{\gamma=1}^{3}\sum_{\sigma=1}^{N_\gamma}\int d\mathbf{q}_\gamma''\frac{Z_{\alpha\gamma}(\mathbf{q}_\alpha'\lambda|\mathbf{q}_\gamma''\sigma;W)}{D_\sigma^\gamma(W-\omega_\gamma'')}\times\langle\gamma\sigma\mathbf{q}_\gamma''|X_{\gamma\beta}(W)|\beta\mu\mathbf{q}_\beta\rangle. \quad (11)$$

By expanding the various independent amplitudes $\langle\alpha\lambda\mathbf{q}_\alpha'|X_{\alpha\beta}(W)|\beta\mu\mathbf{q}_\beta\rangle$ in terms of states of definite total angular momentum J, we can reduce (11) to a set of coupled one-dimensional integral equations.

In order to study the off- and on-shell analytic properties of such amplitudes we shall consider a special case in some detail; namely, the case where the three particles are identical and have unit mass, and where there is only one two-body bound state of angular momentum l. For this simplified problem, the two-body scattering amplitude takes the form

$$\langle\mathbf{p}'|t_\alpha(\nu)|\mathbf{p}\rangle=\sum_{\mu=-l}^{l}\frac{g_\mu(\mathbf{p}')g_\mu(\mathbf{p})^*}{D(\nu)}, \quad (12)$$

where $D(\nu_0)=0$, $\nu_0<0$, and we have had to consider the various magnetic quantum numbers separately because a third particle is involved. That is, we will have to couple the "spin" l of the two-body bound state with the orbital angular momentum of the third particle to form states of definite total angular momentum J. Instead of the amplitudes $X_{\alpha\beta}$ we may work with the symmetrized sums, defining

$$X_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)=\sum_{\beta=1}^{3}\langle\alpha\lambda q'|X_{\alpha\beta}(W)|\beta\mu\mathbf{q}\rangle,$$
$$Z_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)=\sum_{\beta=1}^{3}Z_{\alpha\beta}(\mathbf{q}'\lambda|\mathbf{q}\mu;W). \quad (13)$$

Thus

$$Z_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)=\frac{-2g_\lambda(\mathbf{p}')^*g_\mu(\mathbf{p})}{q'^2+\mathbf{q}'\cdot\mathbf{q}+q^2-W-i\epsilon}. \quad (14)$$

Equation (11) now takes the simple form

$$X_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)=Z_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)+\sum_{\sigma=-l}^{l}\int d\mathbf{q}''\frac{Z_{\lambda\sigma}(\mathbf{q}',\mathbf{q}'';W)X_{\sigma\mu}(\mathbf{q}'',\mathbf{q};W)}{D(W-\frac{3}{4}q''^2)}. \quad (15)$$

We note that in this case $\omega_\alpha=\frac{3}{4}q_\alpha^2$.

Physically, the function $X_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)$ is the quasi-two-particle amplitude for the scattering of the third particle off the two-body bound state. The on-shell condition for this amplitude is then that

$$W-\tfrac{3}{4}q'^2=W-\tfrac{3}{4}q^2=\nu_0,$$

or

$$q'=q=\sqrt{E},\qquad E\equiv\tfrac{4}{3}(W-\nu_0). \quad (16)$$

The right-hand elastic cut for this amplitude will appear in an explicit dependence on $\sqrt{E}$. There is an additional right-hand inelastic cut, for $W\geqslant 0$, corresponding to possible breakup of the two-body bound state.

If we choose the z axis of our coordinate system to be along the direction of $\mathbf{q}$, the $X_{\lambda\mu}$ become helicity amplitudes. To project out partial waves, we set

$$X_{\lambda\mu}(\mathbf{q}',\mathbf{q};W)=\sum_{J=0}^{\infty}\frac{2J+1}{4\pi}X_{\lambda\mu}{}^J(q',q;W)d_{\lambda\mu}{}^J(z),$$
$$X_{\lambda\mu}{}^J(q',q;W)=2\pi\int_{-1}^{1}dz\,d_{\lambda\mu}{}^J(z)X_{\lambda\mu}(\mathbf{q}',\mathbf{q};W), \quad (17)$$

where $z=\hat{q}'\cdot\hat{q}$, and the $d_{\lambda\mu}{}^J(z)$ are the rotation functions.[10] [In (17) it is understood that we take $d_{\lambda\mu}{}^J(z)\equiv 0$ if $|\lambda|>J$, or if $|\mu|>J$).] We write a similar equation for $Z_{\lambda\mu}{}^J(q',q;W)$, from which it follows that

$$Z_{\lambda\mu}{}^J(q',q;W)=-4\pi\int_{-1}^{1}\frac{dz\,d_{\lambda\mu}{}^J(z)Y_{l\lambda}(\hat{p}'\cdot\hat{q}')Y_{l\mu}(\hat{p}\cdot\hat{q})g^*(p')g(p)}{q'^2+qq'z+q^2-W-i\epsilon}, \quad (18)$$

where we have invoked the known properties of the form factors to write $g_\lambda(\mathbf{p})$ in the form

$$g_\lambda(\mathbf{p})=g(p)Y_{l\lambda}(\theta)e^{i\lambda\phi}, \quad (19)$$

$Y_{l\lambda}(\theta)$ being the normalized associated Legendre function. In evaluating (18) we note that the following relations follow from (10):

$$p'=[q^2+qq'z+\tfrac{1}{4}q'^2]^{1/2},$$
$$p=[\tfrac{1}{4}q^2+qq'z+q'^2]^{1/2},$$
$$\hat{p}'\cdot\hat{q}'=-(zq+\tfrac{1}{2}q')/p',$$
$$\hat{p}\cdot\hat{q}=(\tfrac{1}{2}q+zq')/p. \quad (20)$$

Substituting (17) into (15) we obtain

$$X_{\lambda\mu}{}^J(q',q;W)=Z_{\lambda\mu}{}^J(q',q;W)+\sum_{\sigma=-l}^{l}\int_0^\infty\frac{dq''\,q''^2Z_{\lambda\sigma}{}^J(q',q'';W)X_{\sigma\mu}{}^J(q'',q;W)}{D(W-\frac{3}{4}q''^2)}. \quad (21)$$

In the sections to follow we will consider (21) in some detail. To aid in that discussion we first introduce some notational changes. We define a function $h(p^2)$ by the relation

$$g(p)=p^lh(p^2), \quad (22)$$

where $h(0)$ is finite. This form for $g(p)$ can be justified

[10] M. E. Rose, in *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

on fairly general grounds. We also define the function

$$f_{\lambda\mu}{}^J(z;\,q',q)=-4\pi(pp')^l d_{\lambda\mu}{}^J(z)Y_{l\lambda}(\hat{p}'\cdot\hat{q}')Y_{l\mu}(\hat{p}\cdot\hat{q})\,. \quad (23)$$

It is easily verified that $f_{\lambda\mu}{}^J$ is a polynomial in all its variables. The above relations allow us to rewrite (18) in the form

$$Z_{\lambda\mu}{}^J(q',q;\,W)=\int_{-1}^{1}\frac{dz\; f_{\lambda\mu}{}^J(z;\,q',q)h^*(p'^2)h(p^2)}{q^2+q'^2+qq'z-W-i\epsilon}\,, \quad (24)$$

which will be especially useful in discussing the analytic properties of $Z_{\lambda\mu}{}^J$ as a function of q'.

Instead of working directly with (21), which is written in terms of the amplitudes $X_{\lambda\mu}{}^J(q'q;\,W)$, it will prove more convenient to deal with functions of definite parity. We therefore define

$$X_{\lambda\mu}{}^{J\pm}(q',q;\,W)\equiv X_{\lambda\mu}{}^J(q',q;\,W)\pm(-1)^\lambda X_{-\lambda\mu}{}^J(q',q;\,W)\,, \quad (25)$$

with similar definitions for $Z_{\lambda\mu}{}^{J\pm}(q',q;\,W)$ and $f_{\lambda\mu}{}^{J\pm}\times(z;\,q',q)$. It follows from (21) and (25) that the $X_{\lambda\mu}{}^{J\pm}$ satisfy the equation

$$X_{\lambda\mu}{}^{J\pm}(q',q;\,W)=Z_{\lambda\mu}{}^{J\pm}(q',q;\,W)+\tfrac{1}{2}\sum_\sigma\int_0^\infty\frac{dq''\; q''^2 Z_{\lambda\sigma}{}^{J\pm}(q',q'';\,W)X_{\sigma\mu}{}^{J\pm}(q'',q;\,W)}{D(W-\frac{3}{4}q''^2)}\,. \quad (26)$$

Furthermore, the relations

$$f_{\lambda\mu}{}^J(-z;\,-q',\,q)=(-1)^{J-\lambda}f_{-\lambda\mu}{}^J(z;\,q',q)=(-1)^{J-\mu}f_{\lambda-\mu}{}^J(z;\,q',q)\,, \quad (27)$$

which follow from the definition (23) of $f_{\lambda\mu}{}^J$ and the properties of the $d_{\lambda\mu}{}^J$ and $Y_{l\mu}$ functions, imply that

$$Z_{\lambda\mu}{}^{J\pm}(-q',\,q;\,W)=\pm(-1)^J Z_{\lambda\mu}{}^{J\pm}(q',q;\,W)\,. \quad (28)$$

Finally, (26) and (28) together imply that $X_{\lambda\mu}{}^{J\pm}\times(q'q;\,W)$ may be extended to negative real values of q' by the relation

$$X_{\lambda\mu}{}^{J\pm}(-q',\,q;\,W)=\pm(-1)^J X_{\lambda\mu}{}^{J\pm}(q',q;\,W)\,. \quad (29)$$

III. OFF-SHELL ANALYTICITY

In this section we employ a technique developed by the author in A in order to determine the analytic properties of the amplitude $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)$, for fixed q and W, as a function of the off-shell variable q'. Later we shall use this information on off-shell analyticity to derive a new integral equation for $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)$ which possesses several important advantages over the original equation, Eq. (26). However, this information is also useful in its own right. For example, the off-shell analyticity we shall derive provides the necessary proof for the contour-rotation method devised by Hetherington and Schick,[6] and extended by Aaron and Amado,[7] for solving equations of type (21) numerically.

To simplify the notation in what follows, we will temporarily drop the superscripts in (26), rewriting it in the form

$$X_{\lambda\mu}(q',q;\,W)=Z_{\lambda\mu}(q',q;\,W)+\tfrac{1}{4}\sum_\sigma\int_{-\infty}^{\infty}\frac{dk\; k^2 Z_{\lambda\sigma}(q',k;\,W)X_{\sigma\mu}(k,q;\,W)}{D(W-\frac{3}{4}k^2)}\,, \quad (30)$$

with

$$Z_{\lambda\mu}(q'q;\,W)=\int_{-1}^{1}\frac{dz\; f_{\lambda\mu}(z;\,q',q)h^*(p'^2)h(p^2)}{q'^2+q^2+qq'z-W-i\epsilon}\,.$$

Equation (30) defines a function of q', $X_{\lambda\mu}(q',q;\,W)$, whose domain is the real axis. However, this equation can also be used to define the analytic continuation of $X_{\lambda\mu}(q',q;\,W)$ into the complex q' plane. To do so we shall proceed in analogy with the approach developed in A; the method is also described in B for three-body equations in a one-dimensional model. For definiteness, we shall fix W and q such that

$$\mathrm{Re}(\sqrt{W})>0\,,\quad 0<\mathrm{Im}(\sqrt{W})<\sqrt{(-\nu_0)}\,,\quad q>0\,. \quad (31)$$

Later we will extend our result to arbitrary values of W.

We must first establish the analytic properties of the function $Z_{\lambda\mu}(q',q;\,W)$ in the complex q' plane. From (30) we note that singularities of $Z_{\lambda\mu}(q',q;\,W)$ can arise from three sources: (1) the vanishing of the denominator $q'^2+q^2+qq'z-W$, (2) singularities of $h^*(p'^2)$, and (3) singularities of $h(p^2)$. The first produces a cut along the curve

$$q'=-\tfrac{1}{2}zq+[W-(1-\tfrac{1}{4}z^2)q^2]^{1/2}\,, \quad (32)$$

where z varies from -1 to $+1$. This cut is plotted in Fig. 1 for the case $q^2<\mathrm{Re}W$. There is also an analogous cut in the lower half-plane, but it will be sufficient for our purposes to consider only singularities in the upper half-plane. Once we know the properties of $X_{\lambda\mu}(q',q;\,W)$ for $\mathrm{Im}q\geqslant 0$, we can trivially obtain its properties for $\mathrm{Im}q\leqslant 0$ by using (29).

From the relation (22) of the function $h(p^2)$ to the form factor $g(p)$, and from the known properties of the form factors, we deduce that $h(p^2)$ has the general form

$$h(p^2)=\int_\mu^\infty\frac{d\alpha\;\sigma(\alpha)}{p^2+\alpha^2}\,, \quad (33)$$

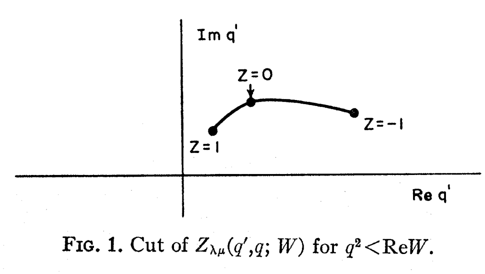

FIG. 1. Cut of $Z_{\lambda\mu}(q',q;\,W)$ for $q^2<\mathrm{Re}W$.

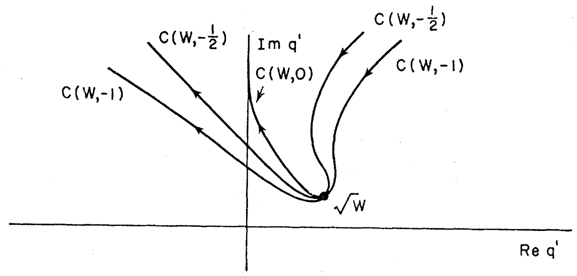

FIG. 2. The curves $C(W,z)$ for $z=0$, $-\frac{1}{2}$, and -1.

i.e., $g(p)$ is known to be analytic except for cuts along the imaginary p axis. The weight function $\sigma(\alpha)$ in (33) must be such that $h(p^2)$ satisfies the asymptotic condition

$$\lim_{|p|\to\infty} h(p^2)<Kp^{-2l-2}, \tag{34}$$

for some constant K, since it is known[9] that $g(p)$ falls off like p^{-l-2} for large p. It is clear from (20), (30), and (33) that $Z_{\lambda\mu}(q',q;W)$ will have cuts in the q' plane due to its dependence on $h^*(p'^2)$ and $h(p^2)$. If q' lies on such a cut and q is real, it is easy to show that $|\mathrm{Im}q'|\geqslant\mu$. These cuts, therefore, all lie outside the strip $|\mathrm{Im}q'|<\mu$ when q is real.

In performing actual calculations, it is necessary to choose some approximate form for $g(p)$, and such calculations will only be meaningful if the results obtained do not depend strongly on the particular form factor chosen. This will clearly not be the case if the form factors produce singularities in $Z_{\lambda\mu}(q',q;W)$ which lie close to the real q' axis, i.e., if μ is small. For $Z_{\lambda\mu}$, and hence $X_{\lambda\mu}$, to be relatively insensitive to the choice of $g(p)$, it is necessary for the singularities induced by $g(p)$ to be far away from the real axis when compared to the other singularities which occur in the problem. As we shall see later, the natural "scale" for the latter is the quantity $|\nu_0|^{1/2}$. The condition for stability under the variation of $g(p)$ is then that $\mu\gg|\nu_0|^{1/2}$. In practice we would expect this to be satisfied, since μ is on the order of a mass, while W and ν_0 are nonrelativistic energies.

Therefore, in determining the off-shell analyticity of $X_{\lambda\mu}(q',q;W)$, it is meaningful to distinguish between the singularities which arise from the cuts of $h(p^2)$ and those which would be present even if $h(p^2)$ were entire. The latter depend only weakly on the specific choice of the form factors, and the above argument suggests that the values of $X_{\lambda\mu}(q',q;W)$ for real q' are largely determined by its behavior at these singularities. Thus, for the remainder of this section, we will restrict our analysis to the case where $h(p^2)$ is entire. The singularities neglected in this way will be considered as a kind of "perturbation," and are discussed in some detail in the Appendix.

Aside from the cuts of $Z_{\lambda\mu}(q',q;W)$, the function $X_{\lambda\mu}(q',q;W)$ will have additional singularities arising from the integral term in (30). We observe that this term can be written in the form

$$\tfrac{1}{2}\sum_\sigma\int_{-1}^{0}dz \times\int_{-\infty}^{\infty}\frac{dk\,k^2 f_{\lambda\sigma}(z;q',k)h^*(p'^2)h(p''^2)X_{\sigma\mu}(k,q;W)}{D(W-\frac{3}{4}k^2)(q'^2+k^2+kq'z-W-i\epsilon)}. \tag{35}$$

Due to the denominator $(q'^2+k^2+kq'z-W)$, it is clear that this term becomes discontinuous when q' lies along the curves $C(W,z)$, defined by

$$C(W,z)=-\tfrac{1}{2}z\alpha+[W-(1-\tfrac{1}{4}z^2)\alpha^2]^{1/2},\quad -1\leqslant z\leqslant 0 \tag{36}$$

as α takes on all real values. Examples of the curves $C(W,z)$ are plotted in Fig. 2. These curves have the property that $C(W,z_1)$ lies above $C(W,z_2)$ if $z_1>z_2$. Thus all the curves $C(W,z)$ lie above $C(W,-1)$. From (32) and (36) it is clear that the cut of $Z_{\lambda\mu}(q',q;W)$ lies entirely on or above $C(W,-1)$. It is also clear from the form of (35) that the integral term of (30) is analytic for q' below $C(W,-1)$. Thus (30) implies that $X_{\lambda\mu}(q',q;W)$ may be analytically continued above the real axis and throughout the region below $C(W,-1)$.

To aid in the subsequent discussion we introduce some geometrical notation. We define the region $R(W,z)$ to be the region in the q' plane bounded from above by $C(W,z)$, and from below by the real axis. We define $\tilde{R}(W,z)$ to be the region above $C(W,z)$, so that $R(W,z)$ and $\tilde{R}(W,z)$ together make up the upper half-plane. We have thus established that $X_{\lambda\mu}(q',q;W)$ is analytic for $q'\in R(W,-1)$.

In order to analytically continue $X_{\lambda\mu}(q',q;W)$ above the line $C(W,-1)$, we define the functions

$$I_{\lambda\sigma\mu}^{\pm}(q',q;z)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dk\,k^2}{D(W-\frac{3}{4}k^2)}\frac{f_{\lambda\sigma}(z;q',k)h^*(p'^2)h(p''^2)X_{\sigma\mu}(k,q;W)}{\{q'+\frac{1}{2}zk+[W-(1-\frac{1}{4}z^2)k^2]^{1/2}\}\{q'+\frac{1}{2}zk-[W-(1-\frac{1}{4}z^2)k^2]^{1/2}\pm i\epsilon\}}. \tag{37}$$

Since $f_{\lambda\sigma}(z;q',k)$ is an entire function of its variables, it follows that $I_{\lambda\sigma}^{-}(q',q;z)$ is analytic for $q'\in R(W,z)$. Again neglecting singularities of the functions $h^*(p'^2)$ and $h(p''^2)$, it follows that $I_{\lambda\sigma\mu}^{+}(q',q;z)$ is analytic for $q'\in\tilde{R}(W,z)$. The function $I_{\lambda\sigma\mu}^{-}$ has been chosen so that, when q' is real, (30) can be written as

$$X_{\lambda\mu}(q',q;W)=Z_{\lambda\mu}(q',q;W)+\sum_\sigma\int_{-1}^{0}dz\,I_{\lambda\sigma\mu}^{-}(q',q;z). \tag{38}$$

It is clear that (38) defines the analytic continuation of $X_{\lambda\mu}(q',q;W)$ throughout $R(W,z)$. For $q'\in\tilde{R}(W,z)$, we define

$$\Delta_{\lambda\sigma\mu}(q',q;z)=\frac{-\pi i q'' f_{\lambda\sigma}(z;q',q'')}{(zq'+2q'')D(W-\frac{3}{4}q''^2)} \times h^*(p'^2)h(p''^2)X_{\sigma\mu}(q'',q;W), \quad (39)$$

where

$$q''=\phi(q',z)\equiv-\tfrac{1}{2}zq'-[W-(1-\tfrac{1}{4}z^2)q'^2]^{1/2}.$$

As we shall show, the function $\Delta_{\lambda\sigma\mu}(q',q;z)$ is analytic for $q'\in\tilde{R}(W,z)$ except for singularities corresponding to the vanishing of $D(W-\frac{3}{4}q''^2)$ and the square-root cut of $\phi(q',z)$. Assuming this for the moment, we define a new function

$$\begin{aligned}F_{\lambda\sigma\mu}(q',q;z)&=I_{\lambda\sigma\mu}^-(q',q;z), \quad \text{if} \quad q'\in R(W,z)\\ &=I_{\lambda\sigma\mu}^+(q',q;z)+\Delta_{\lambda\sigma\mu}(q',q;z), \quad \text{if} \quad q'\in\tilde{R}(W,z).\end{aligned} \quad (40)$$

By construction, the function $F_{\lambda\sigma\mu}(q',q;z)$ is continuous across $C(W,z)$. By the above statements on analyticity, it is also analytic everywhere in the upper half-plane, except for the singular points of $\Delta_{\lambda\sigma\mu}(q',q;z)$ which have yet to be determined. It follows that the analytic continuation of $X_{\lambda\mu}(q',q;W)$ from $R(W,-1)$ into the remainder of the upper half-plane is given by

$$X_{\lambda\mu}(q',q;W)=Z_{\lambda\mu}(q',q;W)+\sum_\sigma\int_{-1}^{0}dz\,F_{\lambda\sigma\mu}(q',q;z). \quad (41)$$

Aside from the cut of $Z_{\lambda\mu}(q',q;W)$, singularities of $X_{\lambda\mu}(q',q;W)$ can only arise from the functions $\Delta_{\lambda\sigma\mu}\times(q',q;z)$ which appear on the right-hand side of (41) when $\mathrm{Im}q'$ is sufficiently large. To determine these singularities it is convenient to adopt the following procedure. We first observe that the regions $\tilde{R}(W,z)$ have the property

$$z_1>z_2\Rightarrow\tilde{R}(W,z_1)\subset\tilde{R}(W,z_2). \quad (42)$$

This is a simple algebraic consequence of the definition (36) of $C(W,z)$, and is illustrated by Fig. 2. It is clear that as z varies from -1 to 0, $\tilde{R}(W,z)$ shrinks down to the curve $C(W,0)$. It follows that we may define a function $\hat{z}(q')$, for $q'\in\tilde{R}(W,-1)$, such that $\hat{z}(q')$ is the largest value of z for which $q'\in\tilde{R}(W,z)$, i.e.,

$$\begin{aligned}&q'\in\tilde{R}(W,\hat{z}(q')),\\ &q'\notin\tilde{R}(W,\hat{z}(q')+\epsilon).\end{aligned} \quad (43)$$

In fact, it is easy to show that

$$\hat{z}(q')=-2\frac{|\mathrm{Re}q'-\mathrm{Im}W/2\,\mathrm{Im}q'|}{||q'|^2+\mathrm{Re}W(\mathrm{Re}q'/\mathrm{Im}q')\,\mathrm{Im}W|^{1/2}}. \quad (44)$$

The function $\hat{z}(q')$ has the property that the point $q''=\phi(q',\hat{z}(q'))$ lies on the real axis.

We make use of $\hat{z}(q')$ to rewrite (41) in the form

$$X_{\lambda\mu}(q',q;W)=Z_{\lambda\mu}(q',q;W)+R_{\lambda\mu}(q',q;W)+S_{\lambda\mu}(q',q;W) \quad (45)$$

for $q'\in\tilde{R}(W,-1)$, where

$$R_{\lambda\mu}(q',q;W)=\sum_\sigma\int_{\hat{z}(q')}^{0}dz\,I_{\lambda\sigma\mu}^-(q',q;z)+\sum_\sigma\int_{-1}^{\hat{z}(q')}dz\,I_{\lambda\sigma\mu}^+(q',q;z), \quad (46)$$

$$S_{\lambda\mu}(q',q;W)=\sum_\sigma\int_{-1}^{\hat{z}(q')}dz\,\Delta_{\lambda\sigma\mu}(q',q;z).$$

The functions appearing in (45) have been chosen in such a way that $R_{\lambda\mu}(q',q;W)$ is analytic for $q'\in\tilde{R}(W,-1)$. Therefore, all singularities of $(X_{\lambda\mu}-Z_{\lambda\mu})$ arise from the function $S_{\lambda\mu}(q',q;W)$. To determine these it is convenient to change the variable of integration in the definition of $S_{\lambda\mu}$ from z to $q''=\phi(q',z)$, so that

$$\begin{aligned}z&=(W-q'^2-q''^2)/q'q'',\\ dz/(zq'+2q'')&=-dq''/q'q''.\end{aligned} \quad (47)$$

We denote the endpoints of the q'' integration by

$$a_0(q')=\phi(q',\hat{z}(q'))$$

and

$$a_-(q')=\phi(q',-1)=\tfrac{1}{2}q'-(W-\tfrac{3}{4}q'^2)^{1/2}. \quad (48)$$

We then obtain

$$S_{\lambda\mu}(q',q;W)=\frac{\pi i}{q'}h^*(W-\tfrac{3}{4}q'^2)\times\sum_\sigma\int_{a_-(q')}^{a_0(q')}dq''\,\rho_{\lambda\sigma}(q',q'';W)X_{\sigma\mu}(q'',q;W), \quad (49)$$

where

$$\rho_{\lambda\sigma}(q',q'';W)=\frac{q''h(W-\frac{3}{4}q''^2)}{D(W-\frac{3}{4}q''^2)}f_{\lambda\sigma}\left(\frac{W-q'^2-q''^2}{q'q''};q',q''\right).$$

Here the contour of integration is along the curve $C(q')$ defined by $q''=-\frac{1}{2}zq'-[W-(1-\frac{1}{4}z^2)q'^2]^{1/2}$ as z varies from -1 to $\hat{z}(q')$. As noted above, the endpoint $a_0(q')$ is always on the real axis.

We now observe a property of the mapping $q''=\phi(q',z)$, which will be very important for our considerations:

$$q'\in\tilde{R}(W,z)\Rightarrow q''\in R(W,z). \quad (50)$$

This has the consequence that the lower limit of the integral in (49), i.e., the point $a_-(q')$, always lies in $R(W,-1)$.

The function $S_{\lambda\mu}(q',q;W)$ will have singularities when q' is such that the integration contour runs into singularities of the integrand. The only such singularity is when $D(W-\frac{3}{4}q''^2)$ vanishes, i.e., when $q''=\sqrt{E}$. This produces a logarithmic branch point at the value of q' such that $\sqrt{E}=a_-(q')$, or $q'=\frac{1}{2}\sqrt{E}+\sqrt{\nu_0}$. We take the associated cut to be along the line

$$q'=\tfrac{1}{2}\sqrt{E}+\sqrt{\nu_0}+ix\,, \tag{51}$$

as x takes on all positive real values. For q' along this cut, the discontinuity

$$\delta_{\lambda\mu}(q',q;\,W)=\lim_{\epsilon\to 0}[S_{\lambda\mu}(q'-\epsilon,\,q;\,W)-S_{\lambda\mu}(q'+\epsilon,\,q;\,W)] \tag{52}$$

is simply the difference between integrating along the two contours $C_\pm$ shown in Fig. 3. It follows that

$$\delta_{\lambda\mu}(q',q;\,W)=\frac{4\pi^2h(\nu_0)}{3D'(\nu_0)q'}\sum_\sigma f_{\lambda\sigma}\left(\frac{W-q'^2-E}{q'\sqrt{E}};\,q',\sqrt{E}\right)\times h^*(W-\tfrac{3}{4}q'^2)X_{\sigma\mu}(\sqrt{E},q;\,W)\,, \tag{53}$$

where

$$D'(\nu_0)\equiv[dD(\nu)/d\nu]_{\nu=\nu_0}.$$

The appearance of the half-on-shell amplitude $X_{\sigma\mu}\times(\sqrt{E},q;\,W)$ on the right side of (53) is to be expected from analogy to our work in A.

We note that singularities of $S_{\lambda\mu}(q',q;\,W)$ do not arise from singularities of the functions $D(W-\frac{3}{4}q''^2)$ and $X_{\sigma\mu}(q'',q;\,W)$, which occur in the integrand. This is due to the fact, noted above, that the integration points q'' all lie in $R(W,\,-1)$. As we observed earlier, the function $X_{\sigma\mu}(q'',q;\,W)$ is analytic for $q''\in R(W,\,-1)$. We take $D(\nu)$ to have the usual right-hand cut of the "D functions," i.e., $\nu\geqslant 0$. It is easily verified that the points q'' such that $W-\frac{3}{4}q''^2\geqslant 0$ all lie in $\bar{R}(W,\,-1)$. Thus singularities of the integrand contribute only the cut (51) to $S_{\lambda\mu}(q',q;\,W)$.

However, $S_{\lambda\mu}(q',q;\,W)$ has an additional cut due to the square root which appears in the endpoint function $a_-(q')$. We take all square-root cuts to run along the positive real axis of the argument; thus this cut runs along the curve $K(W)$:

$$K(W)=\{q'\,|\,\mathrm{Im}(W-\tfrac{3}{4}q'^2)=0;\,\mathrm{Re}(W-\tfrac{3}{4}q'^2)\geqslant 0\}\,. \tag{54}$$

The curve $K(W)$ is plotted in Fig. 4. As q' varies along $K(W)$ the $(W-\frac{3}{4}q'^2)^{1/2}$ takes on all positive real values.

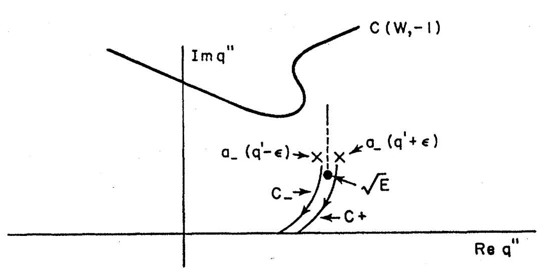

FIG. 3. The contours $C\pm$ for computing the discontinuity defined in (52).

The discontinuity of $S_{\lambda\mu}(q',q;\,W)$ across $K(W)$ is simply the integrand of (49) integrated along the contour $C(W)$, plotted in Fig. 5. Thus, for $q'\in K(W)$,

$$\lim_{\epsilon\to 0}[S_{\lambda\mu}(q'-\epsilon,\,q;\,W)-S_{\lambda\mu}(q'+\epsilon,\,q;\,W)]=\frac{i\pi}{q'}h^*(W-\tfrac{3}{4}q'^2)\sum_\sigma\int_{\frac{1}{2}q'-(W-\frac{3}{4}q'^2)^{1/2}}^{\frac{1}{2}q'+(W-\frac{3}{4}q'^2)^{1/2}}dq''\times\rho_{\lambda\sigma}(q',q'';\,W)X_{\sigma\mu}(q'',q;\,W)\,. \tag{55}$$

Because of the dependence of the right side of (55) on $X_{\sigma\mu}(q'',q;\,W)$, we do not know this discontinuity explicitly; it will be necessary to introduce a new integral equation to compute it.

Since $S_{\lambda\mu}$ has no other singularities, we conclude from (45) that $X_{\lambda\mu}(q',q;\,W)$, in the approximation where we neglect the singularities caused by the functions $h(p^2)$, is analytic for $\mathrm{Im}q'\geqslant 0$, except for the cut of $Z_{\lambda\mu}$ and the two cuts of $S_{\lambda\mu}$ discussed above. If we now restore the superscripts $(J^\pm)$ which we dropped before, we may deduce the analyticity of $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)$ for $\mathrm{Im}q'\leqslant 0$ from (29). All the functions which appear in (45) can be shown to vanish as $|q'|\to\infty$, so we have sufficient information to write a representation for $X_{\lambda\mu}(q',q;\,W)$ based on its analytic properties.

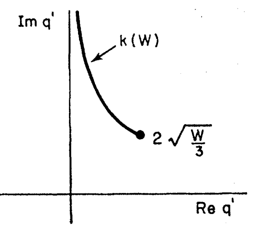

FIG. 4. The curve $K(W)$, where $W-\frac{3}{4}q'^2\geqslant 0$.

IV. DERIVATION OF THE NEW EQUATION

In Sec. III we made use of the integral equation (26) to determine the analytic properties of the amplitude $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)$ in the off-shell variable q'. Neglecting the higher-lying cuts arising from the function $h(p^2)$, we found $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)$ to be analytic except for certain cuts, the discontinuities across which we were able to obtain in terms of values of $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)$ itself. In the Appendix it is shown how this approximation can be improved by successively including the neglected cuts. In this section we make use of this information in order to derive a representation of $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)$ in terms of the solution of a new integral equation. This new equation is then studied, and it is shown to possess several important advantages over (26). This is due to the explicit manner in which the various three-body singularities appear in it. Although our representation has been derived for W and q fixed as in (31), we extend it to all real q and all complex values of W. We also study its behavior when ν_0 is allowed to vary from a two-body bound state energy to a two-body resonant energy.

In both cases we discuss the analytic properties of $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)$ as a function of W.

We first define M to be the largest integer such that $M\leqslant\frac{1}{2}|J-l|$. From (26) and the definition of $Z_{\lambda\mu}{}^{J\pm}\times(q',q;\,W)$, we can show that $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)/(q'^2)^M$ is finite at $q'=0$; from the definitions of the functions appearing in (45) it is simple to show that this ratio vanishes as $|q'|\to\infty$. We define a new function

$$\beta^{J\pm}(q,q')=\frac{1}{q'-q}\pm\frac{(-1)^J}{q'+q}. \tag{56}$$

Applying Cauchy's theorem to the function $X_{\lambda\mu}{}^{J\pm}$

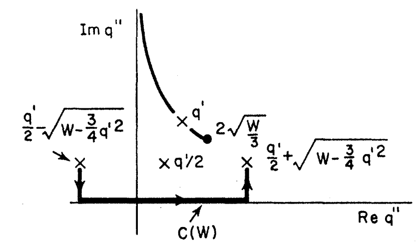

FIG. 5. Path of integration $C(W)$ for computation of the discontinuity across $K(W)$.

$\times(q',q;\,W)/q'^{2M}$, we obtain

$$\begin{aligned}X_{\lambda\mu}{}^{J\pm}(q',q;\,W)&=Z_{\lambda\mu}{}^{J\pm}(q',q;\,W)\\&+\frac{(q'^2)^M}{2\pi i}\int_{\frac{1}{2}\sqrt{E}+\sqrt{\nu_0}}^{i\infty}\frac{dq''}{(q''^2)^M}\beta^{J\pm}(q',q'')\delta_{\lambda\mu}{}^{J\pm}(q'',q;\,W)\\&+\tfrac{1}{2}(q'^2)^M\int_{2\sqrt{(W/3)}}^{i\infty}\frac{dq''\,h^*(W-\frac{3}{4}q''^2)}{q''^{2M+1}}\beta^{J\pm}(q',q'')\\&\times\sum_\sigma\int_{b^-(q'',W)}^{b^+(q'',W)}dQ\,\rho_{\lambda\sigma}{}^{J\pm}(q'',Q;\,W)\\&\qquad\times X_{\sigma\mu}{}^{J}(Q,q;\,W),\end{aligned}\tag{57}$$

where

$$b^\pm(q'',W)=\tfrac{1}{2}q''\pm(W-\tfrac{3}{4}q''^2)^{1/2}.$$

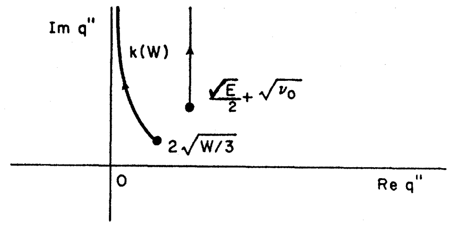

FIG. 6. Integration contours for Eq. (57).

The contours of the q'' integrations are plotted in Fig. 6.

In order to simplify (57), it is helpful to introduce some notational changes. We define

$$\begin{aligned}V_{\lambda\sigma}{}^{J\pm}(q',W)&=\frac{-2\pi ih(\nu_0)(q'^2)^M}{3D'(\nu_0)}\int_{\sqrt{E}+\sqrt{\nu_0}}^{i\infty}\frac{dq\,h^*(W-\frac{3}{4}q^2)}{q^{2M+1}}\\&\times\beta^{J\pm}(q',q)f_{\lambda\sigma}{}^{J\pm}\left(\frac{W-q^2-E}{\sqrt{E}q};\,q,\sqrt{E}\right).\end{aligned}\tag{58}$$

Thus

$$\frac{(q'^2)^M}{2\pi i}\int_{\frac{1}{2}\sqrt{E}+\sqrt{\nu_0}}^{i\infty}\frac{dq''}{(q''^2)^M}\beta^{J\pm}(q',q'')\delta_{\lambda\mu}{}^{J\pm}(q'',q;\,W)=\sum_\sigma V_{\lambda\sigma}{}^{J\pm}(q',W)X_{\sigma\mu}{}^{J\pm}(\sqrt{E},q;\,W).\tag{59}$$

We also define the function $A_{\lambda\sigma}{}^{J\pm}(q',q'';\,\epsilon)$, $\epsilon=\pm1$, such that

$$\begin{aligned}A_{\lambda\sigma}{}^{J\pm}(q',q'';\,\epsilon)&=\tfrac{1}{4}(q'^2)^M\{(W-\tfrac{3}{4}q''^2)^{1/2}-\tfrac{3}{2}\epsilon q''\}\\&\times\int_{q''}^{i\infty}\frac{dQ}{Q^{2M+1}}h^*(W-\tfrac{3}{4}Q^2)\beta^{J\pm}(q',Q)\\&\times\rho_{\lambda\sigma}{}^{J\pm}(Q,(W-\tfrac{3}{4}q''^2)^{1/2}+\tfrac{1}{2}\epsilon q'';\,W).\end{aligned}\tag{60}$$

In terms of $A_{\lambda\sigma}{}^{J\pm}$ we can do one integration in the last term of (57) by parts, rewriting that equation as

$$\begin{aligned}X_{\lambda\mu}{}^{J\pm}(q',q;\,W)&=Z_{\lambda\mu}{}^{J\pm}(q',q;\,W)+\sum_\sigma V_{\lambda\sigma}{}^{J\pm}(q',W)X_{\sigma\mu}{}^{J\pm}(\sqrt{E},q;\,W)\\&+\sum_\sigma\int_{2\sqrt{(\frac{1}{3}W)}}^{i\infty}\frac{dq''}{(W-\frac{3}{4}q''^2)^{1/2}}\{A_{\lambda\sigma}{}^{J\pm}(q',q'';\,1)X_{\sigma\mu}{}^{J\pm}((W-\tfrac{3}{4}q''^2)^{1/2}+\tfrac{1}{2}q'',\,q;\,W)\\&\qquad+A_{\lambda\sigma}{}^{J\pm}(q',q'';\,-1)X_{\sigma\mu}{}^{J\pm}((W-\tfrac{3}{4}q''^2)^{1/2}-\tfrac{1}{2}q'',\,q;\,W)\}.\end{aligned}\tag{61}$$

We now make the change of variables:

$$q''=q''(y)=2[\tfrac{1}{3}(W-y^2)]^{1/2},\qquad y=(W-\tfrac{3}{4}q''^2)^{1/2},\quad 0\leqslant y<\infty.\tag{62}$$

We define

$$\hat{A}_{\lambda\sigma}{}^{J\pm}(q',y;\,W)\equiv A_{\lambda\sigma}{}^{J\pm}(q',q''(y),\,1),\tag{63}$$

so that

$$A_{\lambda\sigma}{}^{J\pm}(q',q''(y);\,-1)=\pm(-1)^J\hat{A}_{\lambda\sigma}{}^{J\pm}(q',\,-y;\,W).$$

From (60) we find that

$$\begin{aligned}\hat{A}_{\lambda\sigma}{}^{J\pm}(q',y;\,W)&=\tfrac{1}{4}(q'^2)^M\{y-[3(W-y^2)]^{1/2}\}\\&\times\int_{2[\frac{1}{3}(W-y^2)]^{1/2}}^{i\infty}\frac{dQ}{Q^{2M+1}}h^*(W-\tfrac{3}{4}Q^2)\beta^{J\pm}(q',Q)\\&\times\rho_{\lambda\sigma}{}^{J\pm}(Q,y+[\tfrac{1}{3}(W-y^2)]^{1/2};\,W).\end{aligned}\tag{64}$$

It is easy to verify that by substituting the values $q'=y\pm[\frac{1}{3}(W-y^2)]^{1/2}$ into (61), it becomes an integral equation in the real variable y for the function $X_{\lambda\mu}{}^{J\pm}\times(y+[\frac{1}{3}(W-y^2)]^{1/2}, q, W)$, $-\infty<y<\infty$. We find that (61) may be written in the form

$$X_{\lambda\mu}{}^{J\pm}(q',q;W)=Z_{\lambda\mu}{}^{J\pm}(q',q;W)+N_{\lambda\mu}{}^{J\pm}(q',q;W)+\sum_\sigma[V_{\lambda\sigma}{}^{J\pm}(q',W)+M_{\lambda\sigma}{}^{J\pm}(q',W)]\times X_{\sigma\mu}{}^{J\pm}(\sqrt{E},q;W),\quad(65)$$

where

$$M_{\lambda\mu}{}^{J\pm}(q',W)\equiv\frac{-2}{\sqrt{3}}\sum_\sigma\int_{-\infty}^{\infty}\frac{dy}{(W-y^2)^{1/2}}\times\hat{A}_{\lambda\sigma}{}^{J\pm}(q',y;W)I_{\sigma\mu}{}^{J\pm}(y;W),$$

$$N_{\lambda\mu}{}^{J\pm}(q',q;W)\equiv\frac{-2}{\sqrt{3}}\sum_\sigma\int_{-\infty}^{\infty}\frac{dy}{(W-y^2)^{1/2}}\times\hat{A}_{\lambda\sigma}{}^{J\pm}(q',y;W)K_{\sigma\mu}{}^{J\pm}(y,q;W),\quad(66)$$

and $I_{\sigma\mu}{}^{J\pm}(y;W)$, $K_{\sigma\mu}{}^{J\pm}(y,q;W)$ are defined to be solutions of the integral equations

$$I_{\lambda\sigma}{}^{J\pm}(y,W)=V_{\lambda\sigma}{}^{J\pm}(y+[\tfrac{1}{3}(W-y^2)]^{1/2},W)-\frac{2}{\sqrt{3}}\sum_\mu\int_{-\infty}^{\infty}\frac{dy'}{(W-y'^2)^{1/2}}\hat{A}_{\lambda\mu}{}^{J\pm}(y+[\tfrac{1}{3}(W-y^2)]^{1/2},y';W)I_{\mu\sigma}{}^{J\pm}(y';W),$$

$$K_{\lambda\mu}{}^{J\pm}(y,q;W)=Z_{\lambda\mu}{}^{J\pm}(y+[\tfrac{1}{3}(W-y^2)]^{1/2},q;W)-\frac{2}{\sqrt{3}}\sum_\sigma\int_{-\infty}^{\infty}\frac{dy'}{(W-y'^2)^{1/2}}\hat{A}_{\lambda\sigma}{}^{J\pm}(y+[\tfrac{1}{3}(W-y^2)]^{1/2},y';W)K_{\sigma\mu}{}^{J\pm}(y',q;W).\quad(67)$$

The half-on-shell amplitudes $X_{\sigma\mu}{}^{J\pm}(\sqrt{E},q;W)$ are determined by setting $q'=\sqrt{E}$ in (65) and solving the simultaneous linear (algebraic) equations

$$\sum_\sigma[\delta_{\lambda\sigma}-V_{\lambda\sigma}{}^{J\pm}(\sqrt{E},W)-M_{\lambda\sigma}{}^{J\pm}(\sqrt{E},W)]\times X_{\sigma\mu}{}^{J\pm}(\sqrt{E},q;W)=Z_{\lambda\mu}{}^{J\pm}(\sqrt{E},q;W)+N_{\lambda\mu}{}^{J\pm}(\sqrt{E},q;W).\quad(68)$$

We have thus derived a method of obtaining $X_{\lambda\mu}{}^{J\pm}\times(q',q;W)$ alternative to the direct solution of (26), i.e., one instead solves (67) and computes $X_{\lambda\mu}{}^{J\pm}(q',q;W)$ from (65), (66), and (68). Before discussing the advantages of this procedure, we want first to remove the restrictions (31) on the values of q and W for which our result is valid.

The dependence of our formulas on q is such that they are clearly valid for arbitrary real q, but in order to extend them to all complex W it is necessary to investigate the properties of our new integral equations (67).

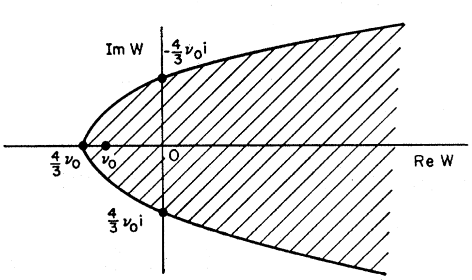

FIG. 7. The region $\mathrm{Im}(\frac{3}{2}\sqrt{E}-\sqrt{\nu_0})<0$ for ν_0 real and negative (shaded area).

It is clear from the explicit dependence of the kernel and inhomogeneous terms of these equations on $(W-y^2)^{1/2}$ that the solutions $I_{\lambda\sigma}{}^{J\pm}(y,W)$ and $K_{\lambda\mu}{}^{J\pm}(y,q;W)$ will have a right-hand cut for $\mathrm{Im}W=0$, $\mathrm{Re}W\geqslant0$. In addition, the function $I_{\lambda\sigma}{}^{J\pm}(y,W)$ has a right-hand cut for $W\geqslant\nu_0$, due to the explicit dependence of $V_{\lambda\sigma}{}^{J\pm}(q',W)$ on $\sqrt{E}\equiv[\frac{4}{3}(W-\nu_0)]^{1/2}$. More precisely, it may be shown that for real q and q', these solutions, the function $\hat{A}_{\lambda\sigma}{}^{J\pm}(q',y;W)$, and hence $M_{\lambda\mu}{}^{J\pm}(q',W)$ and $N_{\lambda\mu}{}^{J\pm}\times(q',q;W)$, are meromorphic functions of W in the parabolic region $\mathrm{Im}(\frac{3}{2}\sqrt{E}-\sqrt{\nu_0})<0$, except for the right-hand cuts. This assertion follows simply from the analyticity and square-integrability of the kernel of (67), and the analyticity of the inhomogeneous terms. The region $\mathrm{Im}(\frac{3}{2}\sqrt{E}-\nu_0)<0$ is plotted in Fig. 7.

The reason for the restriction to this region is due to the fact that the function $\hat{A}_{\lambda\mu}{}^{J\pm}(q',y;W)$ is not defined for all real y when W lies on the boundary curve, $\mathrm{Im}(\frac{3}{2}\sqrt{E}-\sqrt{\nu_0})=0$. This may be seen from (64), which defines $\hat{A}_{\lambda\mu}{}^{J\pm}(q',y;W)$, and from (49), which defines $\rho_{\lambda\sigma}(q',q'';W)$. The latter function has a pole when $q''=\sqrt{E}$, which may be realized for $q''=y+[\frac{1}{3}(W-y^2)]^{1/2}$, for some real y, when W lies on the above-mentioned boundary curve.

By making use of the explicit expressions (30) and (58) for $Z_{\lambda\mu}{}^{J\pm}(q',q;W)$ and $V_{\lambda\sigma}{}^{J\pm}(q',W)$, respectively, it is easily shown that these functions are also analytic in W within the domain shown in Fig. 7, except for the $\sqrt{E}$ cut of $V_{\lambda\sigma}{}^{J\pm}(q',W)$ and right-hand cuts for $W\geqslant0$. It follows from the above that (65) and (68) are valid for W in this domain, and that within this domain $X_{\lambda\mu}{}^{J\pm}(q',q;W)$ is analytic in W except for the right-hand cuts, and except for possible poles in the range

$\frac{4}{3}\nu_0 < W < \nu_0$ corresponding to three-body bound states. The latter can only occur if the function $D^{J\pm}(W)$, defined as the determinant of the system of linear equations (68), has a zero for W in this range. That is,

$$D^{J\pm}(W) \equiv \det|1 - V^{J\pm}(\sqrt{E},W) - M^{J\pm}(\sqrt{E},W)|. \quad (69)$$

It is straightforward to verify that the analytic continuation of $X_{\lambda\mu}{}^{J\pm}(q',q;W)$, for fixed q' and q, to values of W outside the above-mentioned parabolic region, is given by the equations

$$X_{\lambda\mu}{}^{J\pm}(q',q;W) = Z_{\lambda\mu}{}^{J\pm}(q',q;W) - \frac{2}{\sqrt{3}}\sum_\sigma \int_{-\infty}^{\infty} \frac{dy}{(W-y^2)^{1/2}} \hat{A}_{\lambda\sigma}{}^{J\pm}(q',y;W) \times K_{\sigma\mu}{}^{J\pm}(y,q;W), \quad (70)$$

where $K_{\lambda\mu}{}^{J\pm}$ is the solution of

$$K_{\lambda\mu}{}^{J\pm}(y,q;W) = Z_{\lambda\mu}{}^{J\pm}(y+[\tfrac{1}{3}(W-y^2)]^{1/2},q;W) - \frac{2}{\sqrt{3}}\sum_\sigma \int_{-\infty}^{\infty} \frac{dy'}{(W-y'^2)^{1/2}} \times \hat{A}_{\lambda\sigma}{}^{J\pm}(y+[\tfrac{1}{3}(W-y^2)]^{1/2},y';W) \times K_{\sigma\mu}{}^{J\pm}(y',q;W), \quad (71)$$

$\mathrm{Im}(\frac{3}{2}\sqrt{E}-\sqrt{\nu_0})>0$. In this region poles of $X_{\lambda\mu}{}^{J\pm} \times(q',q;W)$ are associated with eigenvalues of the kernel of (71).

It follows that a unified expression for $X_{\lambda\mu}{}^{J\pm}(q',q;W)$, which is valid for all complex W, can be obtained by making the replacement

$$V_{\lambda\sigma}{}^{J\pm}(q',W) \to \tilde{\Theta}(W,\nu_0) V_{\lambda\sigma}{}^{J\pm}(q',W),$$

where

$$\tilde{\Theta}(W,\nu_0) = 1, \quad \text{for } \mathrm{Im}(\tfrac{3}{2}\sqrt{E}-\sqrt{\nu_0})<0$$
$$= 0, \quad \text{for } \mathrm{Im}(\tfrac{3}{2}\sqrt{E}-\sqrt{\nu_0})>0 \quad (72)$$

everywhere in Eqs. (65)–(67). The resulting expression, for fixed real q' and q, can be used to show that $X_{\lambda\mu}{}^{J\pm} \times(q',q;W)$ is analytic in W except for the right-hand cuts, and except for possible poles for $W<\nu_0$ corresponding to three-particle bound states.

The practical advantages of the expressions for $X_{\lambda\mu}{}^{J\pm}(q',q;W)$ developed in this section are due to the explicit manner in which the three-body singularities appear in them. For example, the complicated singularities above the breakup threshold ($W \geqslant 0$) are quite difficult to analyze by the usual technique, which involves studying contour pinches in the multidimensional integrals arising in the perturbation expansion of (26). In the formulation above, the right-hand singularities arise through the explicit dependence on such terms as $\sqrt{E}$ and $(W-y^2)^{1/2}$, instead of by contour pinches. This greatly simplifies not only the analytic study of (26), but also its numerical solution for $W>0$. With respect

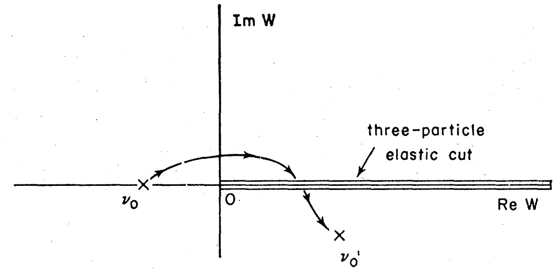

FIG. 8. Continuation path for ν_0 from the bound-state region to the resonance region.

to numerical solution, however, we recall that the above result is not exact, but was obtained by neglecting certain off-shell singularities. In the Appendix it is shown how to modify the above to take the neglected terms into account order by order, while, in the next section, we employ a modified version of the above analysis to obtain a practical method for performing exact numerical calculations.

The expressions above have proven especially useful in investigating the possible relationship between three-body rescattering singularities and three-body resonances, as discussed in B. For this purpose we take our two-body system to have a resonance of energy ν_0, where $\mathrm{Re}\sqrt{\nu_0}>0$, $\mathrm{Im}\sqrt{\nu_0}<0$. Since our expressions for $X_{\lambda\mu}{}^{J\pm}(q',q;W)$ were obtained for ν_0 real and negative, we must analytically continue those expressions in the variable ν_0. This is done by letting ν_0 vary along the path shown in Fig. 8. Thus ν_0 is taken to pass above the three-particle elastic threshold at $W=0$ and down through the right-hand cut, passing onto the second sheet of W where $\mathrm{Im}(\sqrt{W})<0$.

The resulting expressions for $X_{\lambda\mu}{}^{J\pm}(q',q;W)$ are exactly as before, formally. However, in this case $\tilde{\Theta}(W,\nu_0)=0$ everywhere on the first sheet of W. Thus (70) and (71) are valid when ν_0 is a resonance energy for all W on the physical sheet, $\mathrm{Im}(\sqrt{W})>0$. What has happened, of course, is that the parabolic boundary shown in Fig. 7 has followed ν_0 down onto the second sheet. The singularities of $X_{\lambda\mu}{}^{J\pm}$ in this case are plotted in Fig. 9.

It is therefore clear that if we analytically continue $X_{\lambda\mu}{}^{J\pm}(q',q;W)$, for fixed q' and q, as a function of W along the path shown in Fig. 9, down through the

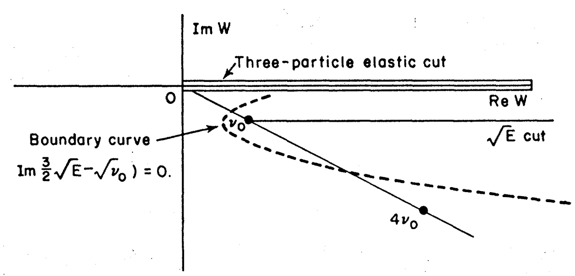

FIG. 9. Singularities of $X_{\lambda\mu}{}^{J}(q',q;W)$ when ν_0 is a resonance energy.

right-hand cuts, we will cross the boundary curve $\mathrm{Im}(\frac{3}{2}\sqrt{E}-\sqrt{\nu_0})=0$. When we do so we will again pick up the terms containing $V_{\lambda\mu}{}^{J\pm}(q',W)$ and Eqs. (65)–(68) will again be valid. It follows that if $D^{J\pm}(W)$, defined in (69), has a zero in this region it will correspond to a three-particle resonant state. As shown in B, this is significant because the function $V_{00}{}^{J\pm}(\sqrt{E},W)$, which occurs in the expression (69) for $D^{J\pm}(W)$, has a logarithmic singularity at the rescattering point, $W=4\nu_0$. A possible connection is then made between this singularity and a nearby zero of $D^{J\pm}(W)$.

The expressions above are also useful for studying the on-shell amplitude $X_{\lambda\mu}{}^{J\pm}(\sqrt{E},\sqrt{E};W)$. Returning to the case where ν_0 is a bound-state energy, we recall that this quantity is the quasi-two-body elastic amplitude for scattering of the third particle off the two-body bound state. For example, $X_{\lambda\mu}{}^{J\pm}(\sqrt{E},\sqrt{E};W)$ may be the amplitude for nucleon-deuteron scattering. As is the case with ordinary two-body partial-wave amplitudes, the on-shell amplitude has left-hand cuts as well as the right-hand cuts discussed above. These left-hand cuts, with one exception, are associated with the potentials in the usual fashion. It is easy to verify that these "potential cuts" arise in the on-shell amplitude from the off-shell cuts associated with the function $h(p^2)$. Since the latter were neglected in this section, the on-shell amplitude given by the above expressions does not have the "potential cuts." These appear, one by one, as the off-shell corrections discussed in the Appendix are added to the formulas of this section.

However, a left-hand cut of a uniquely three-body type does appear in $X_{\lambda\mu}{}^{J\pm}(\sqrt{E},\sqrt{E};W)$ as calculated from the above expressions. This appears solely in the term $Z_{\lambda\mu}{}^{J\pm}(\sqrt{E},\sqrt{E};W)$, i.e., the "Born approximation" to the on-shell amplitude. The cut is of a logarithmic type, with endpoints at $W=4\nu_0$ and $W=\frac{4}{3}\nu_0$. This singularity is associated with the "exchange diagram," shown diagramatically in Fig. 10. We observe that when the condition $\mu\gg|\nu_0|^{1/2}$ [discussed prior to Eq. (35)] is satisfied, this singularity is much more important than the true "potential cuts" in determining the value of $X_{\lambda\mu}{}^{J\pm}(\sqrt{E},\sqrt{E};W)$ for physical values of W, i.e., $W\geqslant\nu_0$. Thus the expressions derived in this section define a kind of "N/D" approximation to the on-shell amplitude. It is clear that, for $W>\frac{4}{3}\nu_0$, the "D function" for this amplitude is the determinant $D^{J\pm}(W)$ of the system of linear equations (68).

Finally, we note that for $W>\nu_0$, the on-shell amplitude as calculated from (68) satisfies the unitarity relation

$$\begin{aligned}X_{\lambda\mu}{}^{J\pm}&((E+i\epsilon)^{1/2},(E+i\epsilon)^{1/2};W)\\&-X_{\lambda\mu}{}^{J\pm}((E-i\epsilon)^{1/2},(E-i\epsilon)^{1/2};W)\\&=-\frac{2\pi i\sqrt{E}}{3D'(\nu_0)}\sum_\sigma X_{\lambda\sigma}{}^{J\pm}((E+i\epsilon)^{1/2},(E+i\epsilon)^{1/2};W)\\&\qquad\times X_{\sigma\mu}{}^{J\pm}((E-i\epsilon)^{1/2},(E-i\epsilon)^{1/2};W).\qquad(73)\end{aligned}$$

For $W>0$, (73) is to be interpreted as just the $\sqrt{E}$ portion of the right-hand cut, i.e., the terms $X_{\lambda\mu}{}^{J\pm}\times((E-i\epsilon)^{1/2},(E-i\epsilon)^{1/2};W)$ are to be evaluated below the $\sqrt{E}$ cut, but above the inelastic cut beginning at $W=0$. To prove (73), it is convenient to put in the $\sqrt{E}$ dependence of the functions $V_{\lambda\sigma}{}^{J\pm}(q',W)$ and $M_{\lambda\sigma}{}^{J\pm}(q',W)$; explicitly,

$$\begin{aligned}V_{\lambda\sigma}{}^{J\pm}(q',W;\sqrt{E})&\equiv V_{\lambda\sigma}{}^{J\pm}(q',W),\\M_{\lambda\sigma}{}^{J\pm}(q',W;\sqrt{E})&\equiv M_{\lambda\sigma}{}^{J\pm}(q',W).\end{aligned}\qquad(74)$$

From (30) and (58) it is easily verified that

$$\begin{aligned}V_{\lambda\sigma}{}^{J\pm}&((E+i\epsilon)^{1/2},W;(E+i\epsilon)^{1/2})\\&-V_{\lambda\sigma}{}^{J\pm}((E-i\epsilon)^{1/2},W;(E-i\epsilon)^{1/2})\\&=-[2\pi i\sqrt{E}/3D'(\nu_0)]Z_{\lambda\sigma}{}^{J\pm}(\sqrt{E},\sqrt{E};W).\qquad(75)\end{aligned}$$

From (66), (67), and (75), one then finds that

$$\begin{aligned}M_{\lambda\sigma}{}^{J\pm}&((E+i\epsilon)^{1/2},W;(E+i\epsilon)^{1/2})\\&-M_{\lambda\sigma}{}^{J\pm}((E-i\epsilon)^{1/2},W;(E-i\epsilon)^{1/2})\\&=-[2\pi i\sqrt{E}/3D'(\nu_0)]N_{\lambda\sigma}{}^{J\pm}(\sqrt{E},\sqrt{E};W).\qquad(76)\end{aligned}$$

Equation (73) then follows trivially from (75) and (76).

V. NUMERICAL CALCULATIONS ABOVE THE BREAKUP THRESHOLD

Equations of the Faddeev-Lovelace type have been applied by many authors to a variety of physical problems. Comparison of the numerically obtained solutions with experimental scattering data has generally confirmed the ability of this approach to account for the major features of the three-body processes studied. In the absence of experiments involving three incoming particles, one is typically interested in the quasi-two-particle amplitudes $X_{\lambda\mu}{}^J(q',q;W)$ with one or both of q' and q on-shell, i.e., q' or (and) $q=\sqrt{E}$. Thus the amplitude $X_{\lambda\mu}{}^J(\sqrt{E},q;W)$ describes the reaction in which the scattering single particle breaks up the two-particle bound state; the amplitude $X_{\lambda\mu}{}^J(\sqrt{E},\sqrt{E};W)$ describes elastic scattering of the single particle from the bound state. In the case where (q',q) are not on-shell, they are restricted to the range $0<(q',q)<2\sqrt{(\frac{1}{3}W)}$; this follows from $W=p^2+\frac{3}{4}q^2$ with $W>0$.

Examples of processes amenable to such treatment include K-d or N-d elastic scattering, and the breakup reaction $n+d\to n+n+p$. The necessary numerical calculations, however, are complicated by some technical difficulties which arise when $W>0$, i.e., when W is

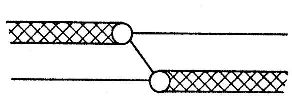

FIG. 10. The exchange diagram. The cross-hatched line is the full propagator for the bound-state pair, and the circle is the vertex for disassociation into two free particles. The single lines represent free-particle propagation.

above the breakup threshold. This is apparent in our particular equation of this type, (21), in which q'-dependent singularities of the integrand pinch the contour of integration, obviating straightforward numerical inversion. One can attempt to circumvent this difficulty by giving W a finite imaginary part, calculating the amplitude for this complex W and extrapolating back to real W. However, the uncertainty introduced by this procedure will be quite large unless ImW is small, and this requires a large number of mesh points for accuracy. This is turn requires large matrix inversions and much computer time.

A considerably better method is the previously mentioned contour rotation procedure of Hetherington and Schick. The integral equation is then written in terms of the function $X_{\lambda\mu}{}^J$ evaluated on the rotated contour, and this equation has a much less singular kernel. However, the maximum angle of rotation must be less than $\theta=\arg(\frac{1}{2}\sqrt{E}+\sqrt{\nu_0})$, and thus θ decreases as W increases, requiring more mesh points to obtain the same accuracy. This is an especially serious defect when ν_0 is a resonance energy. Another disadvantage of this method is that different procedures must be used to determine the physical on-shell and half-on-shell amplitudes from the values of $X_{\lambda\mu}{}^J$ on the rotated contour. Finally, it is not suitable for calculating the fully off-shell amplitude, which is a physical amplitude for elastic three-body scattering.

In this section we employ a modified version of the analysis of Secs. III and IV in order to obtain a new integral equation exactly equivalent to (26). This new equation possesses all of the virtues and none of the above defects of the contour rotation method. To simplify the discussion we will consider the two-body bound state to be in an s wave ($l=0$), and we will make the specific choice $h(p^2)=(p^2+\mu^2)^{-1}$. However, the formulas we will obtain are also valid for any $h(p^2)$ of the form (33), and generalization to arbitrary l is not difficult.

For this case (21) reduces to

$$X^J(q',q;W)=Z^J(q',q;W)+\int_0^\infty \frac{dk\,k^2Z^J(q',k;W)X^J(k,q;W)}{D(W-\frac{3}{4}k^2)}, \quad (77)$$

with

$$Z^J(q',q;W)=-\int_{-1}^1 \frac{dz\,P_J(z)h(p'^2)h(p^2)}{q^2+q'^2+qq'z-W-i\epsilon}.$$

Here

$$h(p'^2)=(q^2+qq'z+\tfrac{1}{4}q'^2+\mu^2)^{-1},$$
$$h(p^2)=(\tfrac{1}{4}q^2+qq'z+q'^2+\mu^2)^{-1}. \quad (78)$$

The principal difficulty in solving (77) numerically for positive W arises from the denominator $(q^2+q'^2+qq'z-W)$ in the integral defining $Z^J(q',q;W)$. There is a range of positive values of q' and q for which this denominator can vanish for z in the interval $(-1,1)$. However, the off-shell approach discussed in the previous sections suggests that one can avoid this difficulty by adopting the following procedure. We split $Z^J(q',q;W)$ into distinct parts, defining new functions

$$B^J(q',q;W)\equiv-\int_{-1}^0 \frac{dz\,P_J(z)\{h(p'^2)h(p^2)-h(W-\frac{3}{4}q'^2)h(W-\frac{3}{4}q^2)\}}{q^2+q'^2+qq'z-W-i\epsilon},$$
$$C^J(q',q;W)\equiv-h(W-\tfrac{3}{4}q^2)\times\int_{-1}^1\frac{dz\,P_J(z)}{q^2+q'^2+qq'z-W-i\epsilon}, \quad (79)$$

so that

$$Z^J(q',q;W)=B^J(q',q;W)+(-1)^JB^J(q',-q;W)+h(W-\tfrac{3}{4}q'^2)C^J(q',q;W).$$

The troublesome singularity now appears only in the function $C^J(q',q;W)$. Our purpose is to substitute this sum of functions for the $Z^J(q',k;W)$ appearing in the integral on the right side of (77), and then to treat the part containing $C^J(q'q;W)$ analytically by the method of Sec. III. In doing so it is convenient to work with the function

$$\tilde{X}^J(q',q;W)\equiv X^J(q',q;W)-Z^J(q',q;W). \quad (80)$$

Interating (77) once, we obtain

$$\tilde{X}^J(q',q;W)=\Omega^J(q',q;W)+\int_0^\infty\frac{dk\,k^2}{D(W-\frac{3}{4}k^2)}Z^J(q',k;W)\tilde{X}^J(k,q;W),$$

with

$$\Omega^J(q',q;W)\equiv\int_0^\infty\frac{dk\,k^2Z^J(q',k;W)Z^J(k,q;W)}{D(W-\frac{3}{4}k^2)}. \quad (81)$$

By making use of the symmetry property

$$\tilde{X}^J(-q',q;W)=(-1)^JX(q',q;W) \quad (82)$$

and the functions B^J and C^J defined in (79), we can rewrite (81) in the form

$$\tilde{X}^J(q',q;W)=\Omega^J(q',q;W)+\int_{-\infty}^\infty\frac{dk\,k^2B^J(q',k;W)\tilde{X}^J(k,q;W)}{D(W-\frac{3}{4}k^2)}+h(W-\tfrac{3}{4}q'^2)F^J(q',q;W), \quad (83)$$

where

$$F^J(q',q;W)\equiv\frac{1}{2}\int_{-\infty}^\infty\frac{dk\,k^2C^J(q',k;W)\tilde{X}^J(k,q;W)}{D(W-\frac{3}{4}k^2)}. \quad (84)$$

We now wish to obtain a representation for $F^J\times(q',q;W)$ by analytically continuing it into the q' plane and determining the singularities of the continued function. Following the method of Sec. III, it is clear that the continuation of (84) is analytic in the upper half-plane except for the singularities of

$$S^J(q',q;W)\equiv\frac{2i\pi}{q'}\int_{a_0(q')}^{\frac12 q'-(W-\frac34 q'^2)^{\frac12}}\frac{dQ\,Qh(W-\frac34Q^2)}{D(W-\frac34Q^2)}\times P_J\left(\frac{W-q'^2-Q^2}{Qq'}\right)\tilde{X}^J(Q,q;W)\quad(85)$$

at points $q'\in\tilde{R}(W,-1)$. Corresponding singularities in the lower half-plane are deduced from (82), which is also satisfied by $F^J(q',q;W)$.

Equation (85) is to be compared with (49), the singularities of which we have already discussed. However, the analysis of (49) neglected singularities of the function $h(p^2)$. The function $h(W-\frac34Q^2)$, which in this case has poles at $Q=\pm2[\frac13(W+\mu^2)]^{1/2}$, contributes an additional singularity to $S^J(q',q;W)$ in the form of a cut beginning at $q'=[\frac13(W+\mu^2)]^{1/2}+i\mu$. We take this cut to be along the line

$$q'=[\tfrac13(W+\mu^2)]^{1/2}+i\mu+ix,\quad 0\leqslant x<\infty\,.\quad(86)$$

We note that the amplitude X^J does not have the cut (86); this is because of a cancellation which occurs when one continues the integral containing B^J in (83) analytically. We observe this without proof, but the statement is easily verified by considering (49) and the corresponding formulas in the Appendix.

It is straightforward to proceed as in Sec. IV, obtaining the following representation for $F^J(q',q;W)$:

$$\begin{aligned}F^J(q',q;W)&=V^J(q',W)\tilde{X}^J(\sqrt{E},q;W)\\&+U^J(q',W)\tilde{X}^J(2[\tfrac13(W+\mu^2)]^{1/2},q;W)\\&+\int_{2(\frac13W)^{1/2}}^{i\infty}\frac{dQ}{(W-\frac34Q^2)^{1/2}}\\&\times[A^J(q',Q;1)\tilde{X}^J((W-\tfrac34Q^2)^{1/2}+\tfrac12Q,\,q;W)\\&+A^J(q',\,Q;-1)\tilde{X}^J((W-\tfrac34Q^2)^{1/2}-\tfrac12Q,\,q;W)],\end{aligned}\quad(87)$$

where

$$\begin{aligned}V^J(q',W)&=\frac{8\pi ih(\nu_0)q'^J}{3D'(\nu_0)}\times\int_{\frac12\sqrt{E}+\sqrt{\nu_0}}^{i\infty}\frac{dq}{q^J}\frac{1}{q^2-q'^2}P_J\left(\frac{W-q^2-E}{\sqrt{E}q}\right),\\U^J(q',W)&=\frac{8\pi iq'^J}{3D(-\mu^2)}\int_{[\frac13(W-\mu^2)]^{1/2}}^{i\infty}\frac{dq}{q^J}\times\frac{1}{q^2-q'^2}P_J\left(\frac{W+4\mu^2+3q^2}{-6q[(W+\mu^2)/3]^{1/2}}\right),\end{aligned}\quad(88)$$

and

$$A^J(q',Q;\epsilon)=-q'^J\left[\frac{(q-2\epsilon Q)qh(W-\frac34q^2)}{D(W-\frac34q^2)}\times\int_Q^{i\infty}\frac{dk}{k^J}\frac{1}{k^2-q'^2}P_J\left(\frac{W-k^2-q^2}{kq}\right)\right]_{q=(W-\frac34Q^2)^{1/2}+\frac12\epsilon Q}.$$

The contour for the Q integration in (87) is the curve $K(W)$, along which $W-\frac34Q^2\geqslant0$. However, (87) can be greatly simplified if we deform this contour such that it lies along the curve where $W-\frac34Q^2$ is pure imaginary; this means rotating onto the second sheet of the square-root function. The new Q contour and the corresponding contours for the curves $\pm(W-\frac34Q^2)^{1/2}+\frac12Q$ are plotted in Fig. 11. It is clear that as we perform this deformation we must cross two poles of the function $A^J(q',Q;1)$, i.e., the functions $h(W-\frac34q^2)$ and $D^{-1}(W-\frac34q^2)$ each have a pole on the real q axis. It is easy to verify that the additional terms we pick up from crossing these poles exactly cancel the first two terms on the right side of (87). It is convenient to make the change of variable

$$(W-\tfrac34Q^2)^{1/2}=(1-i)y,\qquad Q(y)=2[\tfrac13(W+2iy^2)]^{1/2}\quad(89)$$

for $y\geqslant0$ and to introduce the function

$$\tilde{A}^J(q',y;W)\equiv\frac{dQ(y)}{dy}\frac{A^J[q',Q(y);1]}{(1-i)y}.\quad(90)$$

Defining

$$\omega(y)=(1-i)y+[\tfrac13(W+2iy^2)]^{1/2},\quad-\infty<y<\infty\quad(91)$$

$$\psi^J(y,q;W)=\tilde{X}^J[\omega(y),q;W],$$

we can combine (83) and (87) to read

$$\begin{aligned}\tilde{X}^J(q',q;W)&=\Omega^J(q',q;W)+\int_{-\infty}^{\infty}\frac{dk\,k^2B^J(q',k;W)\tilde{X}^J(k,q;W)}{D(W-\frac34k^2)}\\&+h(W-\tfrac34q'^2)\int_{-\infty}^{\infty}dy\,\tilde{A}^J(q',y;W)\psi^J(y,q;W).\end{aligned}\quad(92)$$

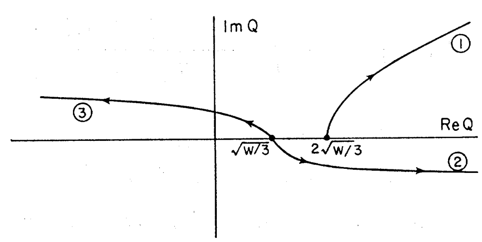

FIG. 11. Curve (1) is the contour for which $W-\frac34Q^2$ is purely imaginary. Curves (2) and (3) represent the values of the functions $\frac12Q+(W-\frac34Q^2)^{1/2}$ and $\frac12Q-(W-\frac34Q^2)^{1/2}$, respectively, as Q runs along curve (1). Curves (2) and (3) together make up the contour $\omega(y)$.

As we observed in Sec. III, we are normally interested in values of $W \ll \mu^2$. This is because μ is on the order of a mass, while W is a nonrelativistic energy. From the definition (79) of $B^J(q',q;\,W)$, it can easily be shown that the k-integration in (92) can be taken along the contour $k=\omega(y)$, providing that $W<\mu^2$ and $q'=\omega(y')$, $q'=\sqrt{E}$, or q' is real and $\leqslant 2(\frac{1}{3}W)^{1/2}$. Thus (92) may be written as an integral equation for the function $\psi^J(y,q;\,W)$:

$$\psi^J(y',q;\,W)=\Omega^J[\omega(y'),q;\,W]+\int_{-\infty}^{\infty}dy\,\tilde{K}^J(y',y;\,W)\psi^J(y,q;\,W),$$

$$\tilde{K}^J(y',y;\,W)\equiv K^J[\omega(y'),y;\,W], \tag{93}$$

$$K^J(q',y;\,W)=h(W-\tfrac{3}{4}q'^2)\tilde{A}^J(q',y;\,W)+\left(\frac{d\omega(y)}{dy}\right)\frac{\omega^2(y)B^J[q',\omega(y);\,W]}{D[(W-\frac{3}{4}\omega^2(y)]}.$$

From the definitions and discussion given above it is not difficult to check that the kernel and inhomogeneous term of (93) are smooth functions of y and y' when W is positive. Thus (93) is much more tractable than (77) to numerical methods of solution. After the solution $\psi^J(y',q;\,W)$ to (93) is obtained, the physical amplitude is calculated by the relation

$$X^J(q',q;\,W)=Z^J(q',q;\,W)+\Omega^J(q',q;\,W)+\int_{-\infty}^{\infty}dy\,K^J(q',y;\,W)\psi^J(y,q;\,W). \tag{94}$$

Equations (93) and (94) enable one to obtain any of the physically interesting amplitudes from a single computational procedure. Therefore, in our opinion, this approach has significant advantages over the calculational methods discussed earlier. Also, there is no difficulty in extending this method to the situation where ν_0 is a two-body resonance energy, and in fact the resulting equations are identical to the above.

We therefore conclude that the off-shell approach introduced in A, and developed further in this paper, may be of some very practical use in the numerical solution of scattering equations, as well as providing a versatile theoretical tool in their analysis.

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APPENDIX: CORRECTIONS DUE TO FORM-FACTOR SINGULARITIES

In determining the off-shell analyticity of the amplitudes $X_{\lambda\mu}{}^{J\pm}(q',q;\,W)$ in Sec. III, we neglected singularities which arise due to the form factors and hence are dependent on the dynamics. In our notation this meant treating the function $h(p^2)$ as if it were entire, although we know it to have the general form (33). As a result, terms were omitted in the representation for $X_{\lambda\mu}{}^{J\pm}\times(q',q;\,W)$ and the new integral equations obtained in Sec. IV. In this Appendix it is shown how the previously neglected singularities can be determined and the corresponding corrections obtained to the equations of Sec. IV.

To avoid complicating the discussion we will assume that

$$\begin{aligned}h^*(p'^2)&=(\tfrac{1}{4}q'^2+zkq'+k^2+\alpha^2)^{-1},\\ h(p^2)&=(q'^2+zkq'+\tfrac{1}{4}k^2+\beta^2)^{-1},\end{aligned} \tag{A1}$$

where α, $\beta\geqslant\mu$. It will be clear that our procedure can easily be generalized to handle any $h(p^2)$ of type (33). The presence of the two denominators of (A1) in (35) necessitates two additional continuation procedures analogous to the procedure described in (37)–(41) for the denominator $(q'^2+zkq'+k^2-W)$. Application of these procedures leads to a generalized version of (45), which we write in the form

$$X_{\lambda\mu}(q',q;\,W)=Z_{\lambda\mu}(q',q;\,W)+\hat{R}_{\lambda\mu}(q',q;\,W)+\hat{S}_{\lambda\mu}(q'q;\,W) \tag{A2}$$

for $q'\in\tilde{R}(W,\,-1)$. Here the function $\hat{R}_{\lambda\mu}(q',q;\,W)$ is analytic for $q'\in\tilde{R}(W,\,-1)$ and $\hat{S}_{\lambda\mu}(q',q;\,W)$ is given by

$$\hat{S}_{\lambda\mu}(q',q;\,W)=S_{\lambda\mu}(q',q;\,W)+\theta(\mathrm{Im}q'-2\alpha)S_{\lambda\mu}{}^1(q',q;\,W)+\theta(\mathrm{Im}q'-\beta)S_{\lambda\mu}{}^2(q',q;\,W), \tag{A3}$$

where

$$S_{\lambda\mu}{}^1(q',q;\,W)=\frac{-\pi i}{q'}(W-\tfrac{3}{4}q'^2+\alpha^2)^{-1}\times\sum_\sigma\int_{\frac{1}{2}q'-i\alpha}^{a'(q',\alpha)}\frac{dk\,kh(\frac{3}{4}(q'^2-k^2)-\alpha^2)}{D(W-\frac{3}{4}k^2)}\times f_{\lambda\sigma}\left(\frac{\frac{1}{4}q'^2+k^2+\alpha^2}{-kq'};\,q',k\right)X_{\sigma\mu}(k,q;\,W), \tag{A4}$$

$$S_{\lambda\mu}{}^2(q',q;\,W)=\frac{-\pi i}{q'}\sum_\sigma\int_{2q'-i2\beta}^{a^2(q',\beta)}\frac{dk\,k(W-\frac{3}{4}k^2+\beta^2)^{-1}}{D(W-\frac{3}{4}k^2)}\times h^*(\tfrac{3}{4}(k^2-q'^2)-\beta^2)f_{\lambda\sigma}\left(\frac{q'^2+\frac{1}{4}k^2+\beta^2}{-kq'};\,q',k\right)\times X_{\sigma\mu}(k,q;\,W),$$

and $S_{\lambda\mu}(q',q;\,W)$ is given by (49). The endpoints $a'(q',\alpha)$ and $a^2(q',\beta)$ lie on the real axis.

From (49) and (A4) we may deduce all singularities of $\hat{S}_{\lambda\mu}(q',q;\,W)$ in $\tilde{R}(W,\,-1)$. It is clear that the two cuts we found for $S_{\lambda\mu}(q',q;\,W)$ in Sec. III are included among these singularities. However, the singularity structure in the q' variable which we find from (A4) is far more complicated than that simple two-cut structure. Analyzing (A4) by the method of Sec. III, it is straightforward to show that $X_{\lambda\mu}(q',q;W)$ has cuts in the upper-half q' plane with branch points at [1]$\pm\frac{1}{2}q+(W-\frac{3}{4}q^2)^{1/2}$, [2] $2(\frac{1}{3}W)^{1/2}$, [3] $\frac{1}{2}\sqrt{E}+\sqrt{\nu_0}$, [4] $\pm\frac{1}{2}q+i\mu$, [5] $\pm 2q+i2\mu$, [6] $\frac{1}{2}\sqrt{E}+i\mu$, [7] $2\sqrt{E}+i2\mu$, [8] $2i\mu$, and [9] an infinite sequence of points generated from the previous 11 by successive applications of the two transformations

$$\hat{Q}_1 q\equiv\tfrac{1}{2}q+i\mu\,,\qquad \hat{Q}_2 q\equiv 2(q+i\mu)\,. \tag{A5}$$

That is, $\hat{Q}_1$ is the operator which takes the point q into the point $\frac{1}{2}q+i\mu$. Thus, if q is one of the 11 branch points enumerated above, there are branch points at the points Q related to q by

$$Q=\hat{Q}_1{}^n\hat{Q}_2{}^m\hat{Q}_1{}^l\cdots\hat{Q}_2{}^k q\,. \tag{A6}$$

Here $\hat{Q}_1{}^n$ means n successive applications of the operator $\hat{Q}_1$, etc. It is clear that the arguments of the branch points generated by successive applications of the operations $\hat{Q}_1$, $\hat{Q}_2$ move closer and closer to $\frac{1}{2}\pi$. We can thus conclude that, with the exception of the inhomogeneous term cuts [1], [4] and [5], and the cuts [2] and [3] discussed in Sec. III, the function $X_{\lambda\mu}(q',q;W)$ is analytic in the upper half-plane except for cuts which lie in the region R, where R consists of the points q' which satisfy

$$\begin{gathered}\operatorname{Im}q'\geqslant\mu\,,\\ \arg(\sqrt{E}+i\mu)\leqslant\arg(q')\leqslant\arg(-\sqrt{E}+i\mu)\,,\end{gathered} \tag{A7}$$

when $q\leqslant\sqrt{E}$.

The discontinuities associated with the above cuts can be calculated from (A4). For example, the discontinuity across the cut with branch point at $q'=\frac{1}{2}\sqrt{E}+i\beta$ is given by

$$\begin{aligned}\lim_{\epsilon\to 0}\,&[S_{\lambda\mu}{}^2(q'-\epsilon,q;W)-S_{\lambda\mu}{}^2(q'+\epsilon,q;W)]\\ &=\frac{-4\pi^2h^*(\frac{3}{4}(E-q'^2)-\beta^2)}{3D'(\nu_0)q'(\nu_0+\beta^2)}\\ &\quad\times\sum_\sigma f_{\lambda\sigma}\left(\frac{q'^2+\frac{1}{4}E+\beta^2}{-q'\sqrt{E}};q',\sqrt{E}\right)\\ &\qquad\times X_{\sigma\mu}(\sqrt{E},q;W)\,.\end{aligned} \tag{A8}$$

By successively adding terms to the expressions of Sec. IV corresponding to the above singularities, one can successively improve the approximate formulas developed in that section. As indicated in the text, such a "perturbation" procedure is meaningful because these singularities are relatively far away from the real axis as compared to the singularities treated in Sec. III. This argument is borne out by some numerical calculations performed by the author for the three-nucleon problem in which the terms corresponding to cuts [1], [2], and [3] accounted for all but about 3% of the exact value of $X_{\lambda\mu}{}^J$. The lowest-order corrections to the formulas of Sec. IV are obtained by replacing the function $V_{\lambda\sigma}{}^{J\pm}(q',W)$, defined by (58), by the sum $V_{\lambda\sigma}{}^{J\pm}(q',W)+U_{\lambda\sigma}{}^{J\pm}(q',W)+W_{\lambda\sigma}{}^{J\pm}(q',W)$, where

$$\begin{aligned}U_{\lambda\sigma}{}^{J\pm}(q',W)&=\int_\mu^\infty d\beta\,\sigma(\beta)U_{\lambda\sigma}{}^{J\pm}(q',W;\beta)\,,\\ W_{\lambda\sigma}{}^{J\pm}(q',W)&=\int_\mu^\infty d\alpha\,\sigma(\alpha)W_{\lambda\sigma}{}^{J\pm}(q',W;\alpha)\,,\\ U_{\lambda\sigma}{}^{J\pm}(q',W;\beta)&=\frac{2\pi iq'^{2M}}{3D'(\nu_0)(\nu_0+\beta^2)}\\ &\quad\times\int_{\frac{1}{2}\sqrt{E}+i\beta}^{i\infty}\frac{dk\,h^*(\frac{3}{4}(E-k^2)-\beta^2)\beta^{J\pm}(q',k)}{k^{2M+1}}\\ &\qquad\times f_{\lambda\sigma}{}^{J\pm}\left(\frac{\frac{1}{4}E+k^2+\beta^2}{-k\sqrt{E}};k,\sqrt{E}\right),\end{aligned} \tag{A9}$$

and

$$\begin{aligned}W_{\lambda\sigma}{}^{J\pm}(q',W;\alpha)&=\frac{2\pi iq'^{2M}}{3D'(\nu_0)}\\ &\quad\times\int_{2\sqrt{E}+i2\alpha}^{i\infty}\frac{dk\,h(\frac{3}{4}(k^2-E)-\alpha^2)\beta^{J\pm}(q',k)}{k^{2M+1}(W-\frac{3}{4}k^2+\alpha^2)}\\ &\qquad\times f_{\lambda\sigma}{}^{J\pm}\left(\frac{E+\frac{1}{4}k^2+\alpha^2}{-k\sqrt{E}};k,\sqrt{E}\right).\end{aligned}$$

Here we have assumed the general form (33) for $h(p^2)$. The functions $U_{\lambda\sigma}{}^{J\pm}$ and $W_{\lambda\sigma}{}^{J\pm}$ are associated with the cuts [3] and [4], respectively.

By adding the corrections (A9) to the expressions of Sec. IV, and by making a "Born approximation" to the integral equations (66), one can obtain an approximate solution to (26). The degree of accuracy of this solution is equivalent to that generated by a second-order determinantal approximation. The main point, however, is not that one can employ this procedure to calculate numbers in some model, but that the expressions of Sec. IV are valid for investigating the major qualitative features of the amplitudes, such as their behavior at the breakup threshold.