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Kinematic Singularity Structure of Amplitudes for a Four-Particle Process with One Massless Particle*

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For a two-to-two particle interaction, one of the particles being massless, we discuss and give the kinematic-singularity-free combinations of helicity amplitudes, the constraint equations, and an expression for invariant amplitudes in terms of helicity amplitudes. The constraint equations are obtained from an invariant-amplitude expansion rather than from the crossing matrix.

1. INTRODUCTION

SEVERAL accounts have been given[1–3] of the kinematic singularity structure of helicity amplitudes for four-particle processes (i.e., two-body scattering), and of the construction of regularized helicity amplitudes which do not have kinematic singularities, these regularized amplitudes being simple combinations of helicity amplitudes multiplied by kinematic factors. More recently, further investigations[3–6] have shown that at certain values of the c.m. energy, there are linear relations, known as constraint equations, between different regularized helicity amplitudes and between their derivatives. We give here an extension, to the case of a four-particle process with one of the particles massless, of the results on kinematic-singularity-free amplitudes. The constraint equations are obtained without considering the crossing matrix. We also give some formulas concerning the use of invariant amplitudes for such a process.[7] Although the results on regularized helicity amplitudes would appear to be of interest principally in the discussion of the application of the Regge-pole model to photoproduction, there is no essential difficulty introduced by considering the massless particle to have arbitrary spin, and for the most part we shall do so.

The definition of a kinematic singularity used here is that of Stapp.[2] The helicity amplitudes can be expressed in terms of M-functions (amplitudes with spinor transformation properties).[8] These M-functions are assumed to be analytic functions of the components of the momenta of the particles on the complex mass shell, apart from singularities ("dynamical singularities") required by unitarity. In a c.m. system, the helicity amplitudes are certain functions of scalar invariants s and t. Kinematic singularities are then defined to be the singularities (in s and t) that the helicity amplitudes would have if there were no dynamical singularities in the M-functions.

The expression of the scattering amplitudes in terms of kinematic-singularity-free invariant amplitudes (functions of s and t) has been given by Hepp[9] and by Williams,[10] and extended to the case of massless particles by Zwanziger.[7] In Sec. 2, after defining some notation and conventions, we recall these results, and also show how to invert the expansion of Ref. 7 (for one particle massless), that is, express the invariant amplitudes explicitly in terms of M-functions and thus in terms of helicity amplitudes.

In Sec. 3, we consider the construction of regularized helicity amplitudes, which are in practice much easier to deal with than the invariant amplitudes, except for low-spin processes. We use the method of Cohen-Tannoudji, Morel, and Navelet,[3,11] who obtain the kinematic singularities from the expression of helicity amplitudes in terms of invariant amplitudes. The results differ from those for the case of four massive particles only by a singularity factor at $s=m_2^2$.[12] Section 3 D mentions one obvious dynamical singularity, arising in lowest-order perturbation theory, which can coincide with one of the kinematic singularities; a formula is given for identifying the regularized amplitudes containing this singularity.

In Sec. 4, the constraint equations on the regularized amplitudes are obtained. Although constraint equations can be written down directly by inspection of the expression of helicity amplitudes in terms of invariant amplitudes, some algebra is unfortunately necessary to simplify them, and to put them in a form comparable with those obtained in Ref. 3 (where the constraint

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10 D. N. Williams, Lawrence Radiation Laboratory Report No. UCRL-11113, 1963 (unpublished).

11 J. P. Ader, M. Capdeville, and H. Navelet [Nuovo Cimento **56A**, 315 (1968)] give a treatment of the analyticity properties of two-body helicity amplitudes for reactions involving massless particles. Their results for the case of one massless particle correspond with ours, except that they obtain a slightly different set of kinematic-singularity-free combinations of helicity amplitudes in the case of one fermion in the initial and the final states. Their method is compared with ours, briefly, in Sec. 5 below.

12 We take $s=(p_1+p_2)^2$, $m_1=0$.

equations were obtained by consideration of singularities in the crossing matrix). Again, the constraint equations differ from those for four massive particles only at $s=m_2^2$. Finally, in Sec. 4 D, we show that we have in fact obtained all the constraint equations on the helicity amplitudes, by proving that if the helicity amplitudes have only the kinematic singularities given in Sec. 3, and satisfy the given constraint equations, then the invariant amplitudes are necessarily analytic in s and t.

2. *M*-FUNCTIONS AND INVARIANT AMPLITUDES

A. Notation

For reactions involving massless particles, Zwanziger[7,13] has shown how one may construct M-functions with spinor transformation properties.[8] Zwanziger has also given singularity-free decompositions of such M-functions into invariant amplitudes, in the cases of four-particle reactions involving one or two massless particles. We shall first recall these results, and define for this purpose some notation.[14]

Let the four particles, labelled by i ($i=1, 2, 3, 4$), have four-momenta $\{p_i\}$, $p_i=p_{i\mu}$ ($\mu=0, 1, 2, 3$), in some standard frame. We have $p_i^2=p_{i0}^2-\mathbf{p}_i^2=m_i^2$, and $p_1+p_2=p_3+p_4$. As usual, $s=(p_1+p_2)^2$ and $t=(p_1-p_3)^2$. Let $\{p_i'\}$ be the momenta in an s-channel c.m. frame: $[p_{k1}'+p_{k2}'=0$ $(k=1, 2, 3)$; $p_{2i}'=0$ $(i=1, 2, 3, 4)]$.

We shall take particle 1 to be massless. Unless stated otherwise, "c.m. frame" will mean the s-channel c.m. frame in which the momenta of particles 1 to 4 are, respectively,

$$(k,0,0k);\quad (w_2,0,0,-k);\quad (w_3,p\sin\theta,0,p\cos\theta);\quad (w_4,-p\sin\theta,0,-p\cos\theta)\,. \tag{2.1}$$

We have, when $m_1=0$,

$$\begin{aligned} k&=\tfrac{1}{2}s^{-1/2}(s-m_2^2)\,,\\ w_2&=\tfrac{1}{2}s^{-1/2}(s+m_2^2),\\ w_i&=\tfrac{1}{2}s^{-1/2}(s+m_i^2-m_j^2),\quad [i,\,j=3,4]\\ p&=\tfrac{1}{2}s^{-1/2}S_{34}\,,\\ S_{34}&=\phi_{34}\psi_{34}=\Omega_{34}\Xi_{34}=\Omega_{43}\Xi_{43}\,,\end{aligned}$$

where

$$\phi_{34}=[s-(m_3+m_4)^2]^{1/2},\qquad \psi_{34}[s-(m_3-m_4)^2]^{1/2},$$
$$\Omega_{34}=[(\sqrt{s}+m_3)^2-m_4^2]^{1/2},\quad \Xi_{34}=[(\sqrt{s}-m_3)^2-m_4^2]^{1/2}.$$

Also

$$\cos\theta=\frac{2st+s^2-s(m_2^2+m_3^2+m_4^2)-m_2^2(m_3^2-m_4^2)}{(s-m_2^2)S_{34}}\,,$$

$$\sin\theta=\frac{2s^{1/2}[\phi(s,t)]^{1/2}}{(s-m_2^2)S_{34}}\,,$$

where

$$\phi(s,t)=st(m_2^2+m_3^2+m_4^2-s-t)+sm_3^2m_2^2(-m_4^2)+tm_2^2(m_3^2-m_4^2)-m_2^2m_3^2(m_2^2+m_3^2-m_4^2)$$

and $\phi(s,t)=0$ includes the boundary of the physical region.

In the case of a four-particle reaction involving only massive particles, M-functions may be defined in terms of helicity amplitudes[15] by

$$H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(\{p_i\})=M^{\alpha_3\alpha_4;\alpha_1\alpha_2}(\{p_i\})\bigotimes_i D^{s_i}[B_i(p_i)\epsilon_i]_{\alpha_i}{}^{\lambda_i}. \tag{2.2}$$

Here $B_i(p_i)\in SL(2,C)$ represents the Lorentz transformation taking a state of particle i ($m_i>0$) at rest and with definite spin projection in the 3 direction, into a state with momentum p_i and definite helicity. The matrices ϵ_i are given by

$$\epsilon_1=\epsilon_2=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \epsilon_3=\epsilon_4=i\sigma^2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \tag{2.3}$$

$$D^s(i\sigma^2)_{\alpha\beta}=(-1)^{s+\alpha}\delta_{\alpha,-\beta}\,. \tag{2.4}$$

The matrix $D^s(A)_\alpha{}^\beta$, $A\in L(2,C)$ is given by

$$D^s(A)_\alpha{}^\beta=\sum_{p+q=s-\beta}[(s+\alpha)!(s-\alpha)!(s+\beta)!(s-\beta)!]^{1/2}\times\frac{A_{11}{}^{s+\alpha-p}A_{12}{}^pA_{21}{}^{s-\alpha-q}A_{22}{}^q}{(s-\alpha-p)!p!(s-\alpha-q)!q!}. \tag{2.5}$$

In the case of a boost to velocity $\mathbf{v}=\mathbf{n}\tanh u$, $A\in SL(2,C)$ is given by

$$A=\cosh\tfrac{1}{2}u-\mathbf{n}\cdot\boldsymbol{\sigma}\sinh\tfrac{1}{2}u; \tag{2.6}$$

$\boldsymbol{\sigma}=(\sigma^1,\sigma^2,\sigma^3)$ are the usual Pauli matrices. In the case of a rotation clockwise through angle θ about direction $\mathbf{n}$, $A\in SU(2)$ is given by

$$A=\cos\tfrac{1}{2}\theta-i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\tfrac{1}{2}\theta \tag{2.7}$$

and when $\mathbf{n}=(0,1,0)$, we write the corresponding transformation matrix $D^s(A)$ as $d^s(\theta)$.[16]

Under a Lorentz transformation L, the M-functions transform according to[17]

$$M^{\{\alpha_i\}}(\{p_i\})=M^{\{\alpha_i'\}}(\{Lp_i\})\bigotimes_i D^{s_i}(L)_{\alpha_i'}{}^{\alpha_i} \tag{2.8}$$

[13] D. Zwanziger, Phys. Rev. **133**, B1036 (1964).

[14] Most of this introductory material is given by P. Moussa and R. Stora, in Lectures at Hercegnovi International School of Elementary Particle Physics, 1966 (Gordon and Breach, Science Publishers, Inc., New York, to be published); H. Joos, Fortschr. Physik **10**, 65 (1962); and in Ref. 3. The notation of the latter most closely resembles ours.

[15] The helicity amplitudes are normalized such that for given helicities the s-channel c.m. differential cross section is given by $d\sigma/d\Omega=(64\pi^2s)^{-1}(p/k)|H(s,t)|^2$, k and p being the initial and final 3-momenta in the c.m. frame.

[16] These are related to the rotation matrices given, for example, by A. R. Edmonds [*Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957)] by $d^s(\theta)$[here]$=d^s(-\theta)$[Edmonds].

[17] We write L, R, etc., as shorthand for the 2×2 matrices representing the corresponding Lorentz transformations, rotations, etc.

(the indices $\{\alpha_i\}$ being of the type referred to in the literature as upper undotted spinor indices). In particular, for s-channel c.m. helicity amplitudes

$$H_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{(s)}(s,t) = M^{\{\alpha_i\}}(\{p_i'\}) \bigotimes_i D^{s_i}(B_i(p_i')\epsilon_i)_{\alpha_i}{}^{\lambda_i}. \quad (2.9)$$

$B_i(p_i')$ can be expressed as

$$B(p')=R(\theta)Z_m(p'), \quad (2.10)$$

$Z_m(p')$ being the boost in the 3 direction taking $(m,0,0,0)$ $(m>0)$ to $(p_0',0,0,|\mathbf{p}'|)$ and $R(\theta)$ a rotation through angle θ about the 2 axis, where

$$p_\mu'=(p_0',|\mathbf{p}'|\sin\theta,0,|\mathbf{p}'|\cos\theta).$$

$$D^s(Z_m(p'))_\alpha{}^\beta=\left(\frac{p_0'+m-|\mathbf{p}'|\sigma^3}{2m(p_0'+m)}\right)^{2\alpha}\delta_{\alpha\beta} = \left(\frac{p_0'-|\mathbf{p}'|}{m}\right)^{\alpha}\delta_{\alpha\beta}. \quad (2.11)$$

Our notation differs from that of Jacob and Wick[18] in that we do not introduce their conventional factor $(-1)^{s_i-\lambda_i}$ for $i=2, 4$; this factor is dropped also in Ref. 3.

M-functions transforming according to the complex conjugate representations of the Lorentz group (i.e., with upper dotted spinor indices, $\{\dot\alpha_i\}$) are similarly given by[19]

$$H_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{(s)}(s,t) = M^{\{\dot\alpha_i\}}(\{p_i'\}) \bigotimes_i D^{s_i}[B_i(p_i')^*\epsilon_i']_{\alpha_i}{}^{\lambda_i}, \quad (2.12)$$

where

$$\epsilon_i'=-i\sigma_2\epsilon_i$$

and the corresponding transformation law is

$$M^{\{\dot\alpha_i\}}(\{p_i\})=M^{\{\dot\alpha_i'\}}(\{Lp_i\}) \bigotimes_i D^{s_i}(L)^*{}_{\alpha_i'}{}^{\alpha_i}. \quad (2.13)$$

Note that in (2.9) and (2.12) the M-functions are unchanged if we replace $\{p_i'\}$ by, say, $\{p_i''\}$, momenta measured in another s-channel c.m. frame related to the first by a rotation about the 2 axis.

The above equations are all familiar in the case $m_i>0$. Now let us consider the case when particle 1 is massless.[20] Let p_1' be $(k,k\sin\theta_1,0,k\cos\theta_1)$. Under a Lorentz transformation L, the helicity amplitude transforms according to

$$H_{\lambda_1}(\{p_i\})=H_{\lambda_1}(\{Lp_i\})D^{s_1}(\mathcal{R})_{\lambda_1\lambda_1} \quad (2.14)$$

(where we have suppressed the helicity labels, and corresponding rotation matrices, of the massive particles). λ_1 has the possible values $\pm s_1$ only, and is not summed. The Wigner rotation $\mathcal{R}$ is given by

$$\mathcal{R}=B_1(Lp_1)^{-1}LB_1(p_1).$$

This differs from the massive-particle case only in that

$$B_1(p_1')=R(\theta_1)Z_0(k), \quad (2.15)$$

$Z_0(k)$ being the boost in the 3 direction taking some standard lightlike vector $(\kappa,0,0,\kappa)$ into $(k,0,0,k)$. We take κ as unity without loss of generality. Then

$$D^s(Z_0(k))_\alpha{}^\beta=k^{-\alpha}\delta_{\alpha\beta}. \quad (2.16)$$

Since $\mathcal{R}$ is of the form

$$\mathcal{R}=\begin{pmatrix} e^{-i\phi/2} & 0 \\ x+iy & e^{i\phi/2}\end{pmatrix}$$

(ϕ, x, y being some real functions of L), we have

$$\delta_{s\lambda'}D^s(\mathcal{R})_{\lambda'\lambda}=\delta_{s\lambda}. \quad (2.17)$$

Equation (2.2) or (2.9) may therefore be used to define M-functions for particle 1 massless and with helicity $\lambda_1=+s_1$. Note that now $B_1(p_1')$ is defined by (2.15) instead of (2.10), and that λ_1 is not summed but has the value s_1 only. These M-functions must satisfy the condition

$$M_+{}^{\{\alpha_i\}}(\{p_i\})D^{s_1}(B_1(p_1))_{\alpha_1}{}^{\lambda_1}=0 \quad \text{if} \quad \lambda_1\neq s_1. \quad (2.18)$$

(The subscript + refers to the helicity of particle 1.)

Equation (2.12) may similarly be used to define M-functions in the case $\lambda_1=-s_1$. The corresponding condition on these M-functions is

$$M_-{}^{\{\dot\alpha_i\}}(\{p_i\})D^{s_1}(B_1(p_1)^*)_{\alpha_1}{}^{\lambda_1}=0 \quad \text{if} \quad \lambda_1\neq s_1. \quad (2.19)$$

An undotted spinor index may be changed into a dotted one by operating with the matrix $D^s[(p_j\cdot\sigma) m_j^{-1}i\sigma^2]_\alpha{}^{\dot\alpha}$, which is nonsingular. Here p_j is some momentum such that $p_j^2=m_j^2>0$, and $(p\cdot\sigma)$ means $p_0\mathbf{1}-\mathbf{p}\cdot\sigma$. Hence we could also have defined (different) M-functions with upper undotted indices for $\lambda_1=-s_1$ by, for example,

$$H_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{(s)}(s,t) = M_-{}^{\{\alpha_i\}}(\{p_i'\})D^{s_1}((p_3'\cdot\sigma)m_3^{-1}B_1(p_1')^{-1\dagger})_{\alpha_1\lambda_1} \times\bigotimes_{i\neq1} D^{s_i}(B_i(p_i')\epsilon_i)_{\alpha_i\lambda_i}, \quad (2.20)$$

where we have used $(i\sigma^2)B(i\sigma^2)^{-1}=B^{-1T}$.

We shall also require the quantities

$$M_\pm(p;j,\eta)^{\alpha\beta}=M_\pm{}^{\alpha_3\alpha_4;\beta\alpha_2}(\{p_i\}) \times\begin{pmatrix}\mu' & s_3 & s_4\\ j' & \alpha_3 & \alpha_4\end{pmatrix}\begin{pmatrix}\alpha & j' & s_2\\ j & \mu' & \alpha_2\end{pmatrix}, \quad (2.21)$$

[18] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

[19] When p_i is complex, $f(p_i)^*$ means a function of p_i related to $f(p_i)$ by $f(p_i)^*=[f(p_i^*)]^*$.

[20] The relevant transformation properties are given by S. Weinberg, Phys. Rev. 134, B882 (1964).

$$M_{\pm}(p;\,j,\eta)^{\mu}=M_{\pm}{}^{\{\alpha_i\}}(\{p_i\})\times\begin{pmatrix}\mu'' & s_3 & s_4\\ \mathbf{j}'' & \alpha_3 & \alpha_4\end{pmatrix}\begin{pmatrix}\mu' & s_1 & s_2\\ \mathbf{j}' & \alpha_1 & \alpha_2\end{pmatrix}\begin{pmatrix}\mu & \mathbf{j}' & \mathbf{j}''\\ \mathbf{j} & \mu' & \mu''\end{pmatrix},\quad(2.22)$$

where the M-functions with subscript — are to be taken as having dotted indices. η represents the set of intermediate angular momenta appearing in the Clebsch-Gordan expansion.

$$\begin{pmatrix}\mathbf{j}_1 & \mathbf{j}_2 & \mathbf{j}_3\\ \mu_1 & \mu_2 & \mu_3\end{pmatrix}$$

is the usual 3-j symbol, whose $\{\mu_i\}$ indices may be raised by operating from the left with the matrix $C_j{}^{\mu'\mu}=(-1)^{j+\mu}\delta_{\mu',-\mu}$.

B. Expansion in Terms of Invariant Amplitudes

The well-known[10,14] singularity-free decomposition of $M(p;\,j\eta)^{\mu}$ into invariant amplitudes is still valid when particle 1 is massless; it is

$$M(p;\,j,\eta)^{\mu}=\sum_{l_1l_2}\begin{pmatrix}\mu & \mathbf{l}_1 & \mathbf{l}_2\\ \mathbf{j} & \mu_1 & \mu_2\end{pmatrix}Y_{l_1}{}^{\mu_1}(\mathbf{q}_1)Y_{l_2}{}^{\mu_2}(\mathbf{q}_2)\times A(s,t;\,l_1,l_2,j,\eta),\quad(2.23)$$

where the sum is over values of l_1, l_2 such that $l_1+l_2=j$ or $l_1+l_2=j+1$, and $|l_1-l_2|\leqslant j\leqslant|l_1+l_2|$. $Y_l{}^{\mu}(\mathbf{q})$ is the usual solid spherical harmonic; the semibivectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are given by

$$\begin{aligned}\mathbf{q}_1&=\mathbf{q}(a,b)=p_{a0}\mathbf{p}_b-p_{b0}\mathbf{p}_a-i(\mathbf{p}_a\times\mathbf{p}_b),\\ \mathbf{q}_2&=\mathbf{q}(c,b)=p_{c0}\mathbf{p}_b-p_{b0}\mathbf{p}_c-i(\mathbf{p}_c\times\mathbf{p}_b),\end{aligned}\quad(2.24)$$

where a, b, and c are three different numbers out of (1,2,3,4) and $p_b{}^2>0$.

"Singularity-free" means here that if the $M(p;\,j\eta)^{\mu}$ are analytic functions of the momenta $\{p_i\}$ in some domain, invariant under complex Lorentz transformations, on the complex mass shell $\{p_i\colon\, p_i{}^2=m_i{}^2,\ p_1+p_2=p_3+p_4\}$, then the $A(s,t;\,l_1l_2j\eta)$ are analytic functions in the image of this domain in the space of the scalar products of the momenta, i.e., in s and t.

Because of the conditions (2.18) or (2.19), the invariant amplitudes $A(s,t;\,l_1l_2j\eta)$ are not all independent. Zwanziger[7] has shown how they may be reduced to an independent set in a singularity-free way, and gives the resulting decomposition as[21]

$$M_+(p;\,j,\eta)^{\alpha\beta}=\sum_{\{j_1,l_2\}js_1}\sum_{l=j_1-s_1}^{j_1+s_1}\begin{pmatrix}\alpha & \mathbf{j}_1 & \mathbf{l}_2\\ \mathbf{j} & \nu & \mu_2\end{pmatrix}\times\begin{pmatrix}\mathbf{j}_1 & \mathbf{s}_1 & \mathbf{l}\\ -s_1 & s_1 & 0\end{pmatrix}\begin{pmatrix}\nu & \beta & \mathbf{l}\\ \mathbf{j}_1 & \mathbf{s}_1 & \mu\end{pmatrix}(2l+1)^{1/2}q_1{}^{j_1+s_1-l}\times Y_l{}^{\mu}(\mathbf{q}_1)Y_{l_2}{}^{\mu_2}(\mathbf{q}_2)A_+(s,t;\,jj_1l_2\eta),\quad(2.25)$$

where $\mathbf{q}_1$ and $\mathbf{q}_2$ are the semibivectors given by (2.24) with $a=1$. q_1 is given by

$$q_1=p_1\cdot p_b,\quad q_1{}^2=|\mathbf{q}_1|^2.\quad(2.26)$$

The notation $\{j_1,l_2\}js$ means that j_1 and l_2 are summed over the set

$$\begin{array}{lll} j_1=s_1, & |j-s_1|\leqslant l_2\leqslant j+s_1 & (\text{any } j)\\ j_1=j-l_2, & 0\leqslant l_2\leqslant j-s_1-1 & (j>s_1)\\ j_1=j-l_2+1, & 1\leqslant l_2\leqslant j-s_1 & (j>s_1).\end{array}$$

$M_-(p;\,j\eta)^{\dot\alpha\dot\beta}$ has an analogous decomposition, differing in that we take the complex conjugate spherical harmonics, i.e., $Y_l{}^{\mu}(\mathbf{q})$ is replaced by $(-1)^{\mu}Y_l{}^{-\mu}(\mathbf{q}^c)$, where

$$\mathbf{q}^c(i,j)=p_{i0}\mathbf{p}_j-p_{j0}\mathbf{p}_i+i(\mathbf{p}_i\times\mathbf{p}_j).\quad(2.27)$$

The corresponding invariant amplitudes are written $A_-(s,t;\,jj_1l_2\eta)$. [Note that in Ref. 7 the invariant amplitudes $A_-(s,t)$ correspond to an expansion of the M-functions defined by an equation similar to (2.20), instead of by (2.12) as here.]

Let us consider the effect of imposing parity invariance on the helicity amplitudes; this will, of course, reduce by half the number of independent invariant amplitudes. The parity invariance condition reads[3]

$$H_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}(s,t)=(-1)^{\Sigma_i(s_i+\lambda_i)}\eta H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t),\quad(2.28)$$

where $\eta=\prod_i\eta_i$, η_i being the intrinsic parity of particle i. Taking, for example, $b=2$, $c=3$ in (2.24), we have in the c.m. frame (2.1)

$$\begin{aligned}\mathbf{q}_1'&=\mathbf{q}'(1,2)=(0,\,0,\,-k(w_2+k)),\\ \mathbf{q}_2'&=\mathbf{q}'(3,2)\\ &=-(w_2p\sin\theta,\,ipk\sin\theta,\,w_2p\cos\theta+w_3k),\end{aligned}\quad(2.29)$$

$$q_1'=k(w_2+k)=\tfrac{1}{2}(s-m_2{}^2).\quad(2.30)$$

Since, for $i=(1,2,3,4)$, $Z(p_i)$ is Hermitian and

$$D^{s_i}(Z(p_i'))_{\alpha}{}^{\beta}=D^{s_i}(Z(p_i'))_{\alpha}{}^{\alpha}\delta_{\alpha,\beta}=D^{s_i}(Z(p_i')^{-1})_{-\alpha}{}^{-\beta}\quad(2.31)$$

and all the matrices are real, we have

$$D^{s_i}(B_i(p_i')^*\epsilon_i')_{\dot\alpha_i}{}^{-\lambda_i}=D^{s_i}(B_i(p_i')\epsilon_i)_{\alpha_i}{}^{\lambda_i}(-1)^{s_i+\lambda_i}.\quad(2.32)$$

Hence, taking $\lambda_1=s_1$, and substituting (2.9) and (2.12) in (2.28), we have $M_-{}^{\{\dot\alpha_i\}}(\{p_i'\})=\eta M_+{}^{\{\alpha_i\}}(\{p_i'\})$. Since $(q_{11}'\pm iq_{12}')$ and $(q_{21}'\pm iq_{22}')$ (the second index on the q's labels the 3-vector components) are real, we have, dropping angular momentum labels,

$$A_+(s,t)=\eta A_-(s,t)\equiv A(s,t),\quad(2.33)$$

so that it is trivial to make the invariant amplitudes satisfy parity invariance. This is in contrast to the case for reactions involving four massive particles, where no explicit formula has yet been found in the general-spin case to obtain an independent set of invariant amplitudes, satisfying parity invariance, from those in Eq. (2.23).

[21] Our notation differs from that of Ref. 7 in a number of details, principally in that we have used upper rather than lower spinor indices, and in our choice of sign for the second terms in (2.6) and (2.7). Consequently, for example, the sign of the real parts of the semibivectors has been reversed.

C. Inversion of Invariant Amplitude Expansions

[The later derivation of the kinematic-singularity-free helicity amplitudes and of the constraint equations will not require the results of this subsection, apart from Eq. (2.34).]

We now consider how to express the invariant amplitudes $A(s,t)$ in terms of the helicity amplitudes $H_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{(s)}(s,t)$. Since the inversion of (2.9) and (2.21) is trivial, we shall consider in detail only the inversion of expansion (2.25), i.e., we shall express $A(s,t;\, jj_1l_2\eta)$ in terms of $M^{\alpha\beta} \equiv M_+(\{p_i'\},j\eta)^{\alpha\beta}$.

Let us use the c.m. frame (2.1) with $\mathbf{q}_1'$ and $\mathbf{q}_2'$ given by (2.29). Since $\mathbf{q}_1'$ is in the 3 direction,

$$Y_l^\mu(\mathbf{q}_1') = (4\pi)^{-1/2}(2l+1)^{1/2}(-q_1')^l\delta_{\mu,0}$$

and (2.25) reduces to

$$M^{\alpha s_1} = \sum_{\{j_1l_2\}js_1} \begin{pmatrix} \alpha & \mathbf{j}_1 & \mathbf{l}_2 \\ \mathbf{j} & -s_1 & \mu_2 \end{pmatrix} (q_1')^{j_1+s_1} Y_{l_2}{}^{\mu_2}(\mathbf{q}_2') \times (4\pi)^{-1/2} A(s,t;\, jj_1l_2\eta)\,. \quad (2.34)$$

We rewrite this in terms of new invariant amplitudes $B_{(m)l}(s,t)$ [with $m=(1,2,3)$; indices j, η suppressed] as follows, for $j>s_1$.

$$M^{\alpha s_1} = \sum_{l=j-s_1}^{j+s_1} \left(\frac{(j-\alpha)!(l+\mu)!}{(j+\alpha)!(l-\mu)!}\right)^{1/2} C_{(l)}{}^\mu(\mathbf{q}_2') q_1'^{2s_1} B_{(1)l}(s,t)$$
$$+ \sum_{l=0}^{j-s_1-1} \left(\frac{(j+\alpha)!(j-\alpha)!}{(l+\mu)!(l-\mu)!}\right)^{1/2} C_{(l)}{}^\mu(\mathbf{q}_2') q_1'^{s_1+j-l} B_{(2)l}(s,t)$$
$$+ \sum_{l=1}^{j-s_1} \left(\frac{(j+\alpha)!(j-\alpha)!}{(l+\mu)!(l-\mu)!}\right)^{1/2} \mu C_{(l)}{}^\mu(\mathbf{q}_2') q_1'^{s_1+j-l+1} B_{(3)l}(s,t)\,, \quad (2.35)$$

where

$$C_{(l)}{}^\mu \equiv (2l+1)^{-1/2}(4\pi)^{1/2} Y_l^\mu, \quad \alpha \equiv \mu - s_1.$$

For $j \leqslant s_1$, the lower limit on the first sum becomes $s_1 - j$ and the other two sums are omitted. The $B_{(m)l}(s,t)$ are given in terms of the $A(s,t;\, j,j_1,l_2,\eta)$ by

$$\begin{aligned}
B_{(1)l}(s,t) &= a_1 A(s,t;\, j,s_1,l,\eta) && \text{for } j-s_1+1 \leqslant l \leqslant j+s_1\,,\\
B_{(1)l}(s,t) &= a_2 A(s,t;\, j,s_1,l,\eta) + a_4 q_1 A(s,\, t;\, j,\, s_1+1,\, l,\, \eta) && \text{for } l = j-s_1\,,\\
B_{(2)l}(s,t) &= a_2 A(s,\, t;\, j,\, j-l,\, l,\, \eta) + a_4 q_1 A(s,\, t;\, j,\, j-l+1,\, l,\, \eta) && \text{for } 0 \leqslant l \leqslant j-s_1-1\,,\\
B_{(3)l}(s,t) &= a_3 A(s,\, t;\, j,\, j-l+1,\, l,\, \eta) && \text{for } 1 \leqslant l \leqslant j-s_1\,,
\end{aligned} \quad (2.36)$$

where the coefficients a_1, a_2, a_3, a_4 (functions of j, l, s_1) may be obtained by comparison of (2.35) with explicit expressions for the 3-j symbols in (2.34). Note that $q_1' = \frac{1}{2}(s - m_2{}^2)$, and that $a_4(l=0)$ is the only zero coefficient, so that the transformation to amplitudes $B_{(m)l}(s,t)$ is nonsingular. [If $j \leqslant s_1$, the values of l in the first equation of (2.36) are $s_1 - j \leqslant l \leqslant s_1 + j$, and the other equations are dropped.]

Now let us define[22]

$$X_\mu{}^l(\omega) = \tfrac{1}{2}(-\sin\omega)^{-l} \frac{d^l}{d(\cot\omega)^l} \times [(\cot\omega + \csc\omega)^\mu + (\cot\omega - \csc\omega)^\mu]\,. \quad (2.37)$$

Some properties of this function are given in the Appendix. If we write $\mathbf{q}_2 = q_2(\sin\omega\,\cos\chi,\, \sin\omega\,\sin\chi,\, \cos\omega)$, then

$$Y_l^\mu(\mathbf{q}_2) = e^{i\mu\chi} Y_l^\mu(\omega) q_2{}^l\,.$$

[22] G. C. Fox, Phys. Rev. **157**, 1493 (1967). The notation differs in detail.

Then, using results given in the Appendix, we obtain

$$B_{(1)l}(s,t) = \sum_\alpha M^{\alpha s_1} e^{-i\chi(s_1+\alpha)} q_1'^{-2s_1} q_2'^{-l} \left(\frac{(j+\alpha)!}{(j-\alpha)!}\right)^{1/2} \times \left(\frac{X_{s_1+\alpha}{}^l(\omega)}{(l+s_1+\alpha)!} - \frac{X_{s_1+\alpha}{}^{l+1}(\omega)}{(l+s_1+\alpha+1)!}\right),$$

$$B_{(2)l}(s,t) = \sum_\alpha M^{\alpha s_1} e^{-i\chi(s_1+\alpha)} q_1'^{(-s_1-j+l)} \times q_2'^{-l} [(j+\alpha)!(j-\alpha)!]^{-1/2} X_{s_1+\alpha}{}^l(\omega)\,, \quad (2.38)$$

$$B_{(3)l}(s,t) = \sum_{\alpha \neq -s_1} M^{\alpha s_1} e^{-i\chi(s_1+\alpha)} q_1'^{(-s_1-j+l-1)} \times q_2'^{-l} [(j+\alpha)!(j-\alpha)!]^{-1/2} (s_1+\alpha)^{-1} X_{s_1+\alpha}{}^l(\omega)\,.$$

For $s_1 \geqslant j$, only the first of these equations applies. The range of values of l for each of these equations is given in (2.36). From (2.29), we have the values of ω, χ, so that

$$B_{(1)l}(s,t)=\sum_{\alpha} M^{\alpha s_1} q_1'^{-2s_1}\left(\frac{(j+\alpha)!}{(j+\alpha)!}\right)^{1/2} \times \mathcal{C}_{\alpha+s_1}\left(\frac{F_{\alpha+s_1}(\mathbf{q}_2',l)}{(l+s_1+\alpha)!}-q_2\frac{F_{\alpha+s_1}(\mathbf{q}_2',\,l+1)}{(l+s_1+\alpha+1)!}\right),$$

$$B_{(2)l}(s,t)=\sum_{\alpha} M^{\alpha s_1} q_1'^{(-s_1-j+l)}[(j+\alpha)!(j-\alpha)!]^{-1/2} \times \mathcal{C}_{\alpha+s_1}F_{\alpha+s_1}(\mathbf{q}_2',l)\,, \tag{2.39}$$

$$B_{(3)l}(s,t)=\sum_{\alpha\neq -s_1} M^{\alpha s_1} q_1'^{(-s_1-j+l-1)}[(j+\alpha)!(j-\alpha)!]^{-1/2} \times \mathcal{C}_{\alpha+s_1}[\alpha+s_1]^{-1}F_{\alpha+s_1}(\mathbf{q}_2',l)\,,$$

where

$$\mathcal{C}_{\mu}\equiv (s^{1/2}m_2^{-1})^{\mu} \times \{(s-m_2^2)[\phi(s,t)]^{-1/2}m_2^{-1}\}^{|\mu|}(-\mathrm{sgn}\mu)^{\mu} \tag{2.40}$$

and[23]

$$F_{\mu}(\mathbf{q}_2,l)\equiv \sum_{r=0}^{[\frac{1}{2}(|\mu|+l)]} b_r q_{23}{}^{|\mu|-l-2r}q_2{}^{2r}, \tag{2.41}$$

where b_r is the coefficient of

$$(\cos\omega)^{|\mu|-l-2r} \quad \text{in} \quad (\sin\omega)^{|\mu|}X^{l}{}_{|\mu|}(\omega)\,,$$

$$b_r=2^{l-1}|\mu|(|\mu|+l-1)!(r+l-1)! \times[(2r+2l-1)!r!(|\mu|-2r-l)!]^{-1}; \tag{2.42}$$

q_1', q_2', and q_{23}' are given by (2.29) and (2.30).

Note that apparent singularities of the $B_{(m)l}(s,t)$ in (2.39) arise from singularities of the transformation from $\{p_i\}$ to (s,t) in the particular c.m. frame we are using. Let us consider[2] the singularity $(p\sin\theta)^{-|s_1+\alpha|}$ appearing in these equations in the coefficients $\mathcal{C}$. Now if $M(\{p_i\})$ is an analytic function of $\{p_i\}$ in some domain on the complex mass shell, it is an analytic function of $(s,t,\mathbf{q}_1,\mathbf{q}_2)$ in the corresponding domain [$\mathbf{q}_1$, $\mathbf{q}_2$ being given by (2.24) with $a=1$, $b=2$, $c=3$]. Suppose the domain includes $q_{21}=q_{22}=0$. Then M has a series expansion near this section in powers of q_{21} and q_{22}, or equivalently in powers of $(q_{21}+iq_{22})$, $(q_{21}-iq_{22})$. But under an infinitesimal rotation through $\delta\phi$ about the 3 axis, $(q_{21}\pm iq_{22})^n \rightarrow (q_{21}\pm iq_{22})^n(1\pm in\delta\phi)$ and $M^{\alpha s_1}\rightarrow M^{\alpha s_1}(1+i(\alpha+s_1)\delta\phi)$; and in any s-channel c.m. frame with the incident 3 momenta parallel to the 3 direction, the M-functions must be invariant under such rotations at $q_{21}=q_{22}=0$ (where all the particles have parallel momenta in a c.m. frame). Hence, writing ξ for the sign of $s_1+\alpha$, $M^{\alpha s_1}$ goes to zero like $(q_{21}+i\xi q_{22})^{|s_1+\alpha|}$ at $q_{21}=q_{22}=0$. From (2.29) we see that this cancels the $(p\sin\theta)^{-|s_1+\alpha|}$ singularity mentioned above, i.e., this example verifies the known singularity-free nature of the invariant amplitudes.

[23] $[x]$ means the nearest integer less than or equal to x.

3. KINEMATIC SINGULARITIES OF HELICITY AMPLITUDES

A. General Remarks

In dealing with amplitudes for general spin processes, it is generally assumed[2,8] that the M-functions are analytic functions of the momenta on the complex mass shell, except for singularities required by unitarity; these singularities are on Lorentz-invariant surfaces and so the invariant amplitudes given in Sec. 2 B are also analytic (in s and t) apart from such dynamical singularities. If we express the helicity amplitudes in terms of these invariant amplitudes, the coefficient functions (functions of s and t) contain singularities, defined to be the kinematic singularities. [In particular, if we artificially substitute for the invariant amplitudes functions analytic in the regions of (s,t) under consideration, the singularities of the helicity amplitudes are the kinematic singularities.]

By means of the methods of Cohen-Tannoudji, Morel, and Navelet,[3] these singularities are readily identified, and linear combinations of helicity amplitudes are found which do not have these singularities.

Let us consider the case where particle 1 (massless) has helicity $\lambda_1=+s_1$. The singularity structure is evidently the same in the case $\lambda_1=-s_1$, and it is immaterial whether parity is conserved in the reactions or not (Sec. 2 B).

From (2.9), (2.21), and (2.34) we have in the c.m. frame (2.1)

$$H_{\lambda_3\lambda_4;s_1\lambda_2}{}^{(s)}(s,t)=(-1)^{s_2-\lambda_2+s_3+\lambda_3}D^{s_1}(Z(p_1'))_{s_1}{}^{s_1} \times D^{s_2}(Z(p_2'))_{\lambda_2}{}^{\lambda_2}D^{s_3}(Z(p_3'))_{-\lambda_3}{}^{-\lambda_3} \times D^{s_4}(Z(p_4'))_{-\lambda_4}{}^{-\lambda_4}C^{s_1\lambda_2\lambda_3\lambda_4}, \tag{3.1}$$

where $Z(p_i)$ is given by (2.16) $(i=1)$ or (2.11) $(i\neq 1)$, and where[24]

$$C^{s_1\lambda_2\lambda_3\lambda_4}=\sum_{j,j'}\sum_{\{j_1,l\}js_1} d^{s_3}(\theta)_{\alpha_3}{}^{-\lambda_3}d^{s_4}(\theta)_{\alpha_4}{}^{\lambda_4}\begin{pmatrix}\mathbf{j}' & \alpha_3 & \alpha_4\\ \mu' & \mathbf{s}_3 & \mathbf{s}_4\end{pmatrix} \times\begin{pmatrix}\mathbf{j} & \mu' & -\lambda_2\\ \alpha & \mathbf{j}' & \mathbf{s}_2\end{pmatrix}\begin{pmatrix}\alpha & \mathbf{j}_1 & \mathbf{l}\\ \mathbf{j} & -s_1 & \mu\end{pmatrix}q_1'^{j_1+s_1}Y_l{}^{\mu}(\mathbf{q}_2') \times(4\pi)^{-1/2}A_+(s,t;\,jj_1l\eta)\,. \tag{3.2}$$

Commuting some d-matrices and 3-j symbols, we may rewrite this as

$$C^{s_1\lambda_2\lambda_3\lambda_4}=\sum_{j,j'}\sum_{\{j_1,l\}js_1} d^{s_2}(-\theta)_{\alpha_2}{}^{-\lambda_2}d^{j_1}(-\theta)_{\beta}{}^{s_1} \times\begin{pmatrix}\mathbf{j}' & -\lambda_3 & \lambda_4\\ \mu' & \mathbf{s}_3 & \mathbf{s}_4\end{pmatrix}\begin{pmatrix}\mathbf{j} & \mu' & \alpha_2\\ \alpha & \mathbf{j} & \mathbf{s}_2\end{pmatrix}\begin{pmatrix}\alpha & \beta & \mathbf{l}\\ \mathbf{j} & \mathbf{j}' & \mu\end{pmatrix}q_1'^{j_1+s_1} \times Y_l{}^{\mu}(\mathbf{q}_2'')(-1)^{j_1-s_1}(4\pi)^{-1/2}A_+(s,t;\,jj_1l\eta)\,, \tag{3.3}$$

where $\mathbf{q}_2''$ is the vector obtained from $\mathbf{q}_2'$ by a rotation through angle $-\theta$ about the 2 axis.

$$\mathbf{q}_2'=-(\omega_2 p\sin\theta,\, ipk\sin\theta,\, \omega_2 p\cos\theta+\omega_3 k)\,, \tag{2.29'}$$

[24] Summation over repeated spin component indices is implied.

$$\mathbf{q}_2''=(\omega_3 k\sin\theta,\ -ipk\sin\theta,\ -\omega_2 p-\omega_3 k\cos\theta)\,, \qquad (3.4)$$

$$q_1'=k(\omega_2+k)=\tfrac{1}{2}(s-m_2^2)\,. \qquad (2.30')$$

[Momenta are related to (s,t) by formulas at the beginning of Sec. 2 A.]

We also require the expression

$$d^j(\theta)_\lambda{}^\mu=\left(\frac{(j+\lambda)!(j-\lambda)!}{(j+\mu)!(j-\mu)!}\right)^{\xi_1/2}(\cos\tfrac{1}{2}\theta)^{|\mu+\lambda|}\times(\xi_2\sin\tfrac{1}{2}\theta)^{|\lambda-\mu|}P_{j-\Lambda}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta)\,, \qquad (3.5)$$

where

$$\Lambda=\max(|\lambda|,|\mu|),\quad \xi_1=\operatorname{sgn}(|\lambda|-|\mu|),\quad \xi_2=\operatorname{sgn}(\lambda-\mu)\,,$$

and $P_n{}^{(\alpha,\beta)}(x)$ is a polynomial (the Jacobi polynomial) in x of order n. In particular,

$$Y_l{}^\mu(\mathbf{q})=\left(\frac{(2l+1)(l+\mu)!(l-\mu)!}{4\pi}\right)^{1/2}\frac{1}{l!}[-\tfrac{1}{2}\epsilon(q_1+i\epsilon q_2)]^{|\mu|}\times q^{l-|\mu|}P_{l-|\mu|}{}^{(|\mu|,|\mu|)}(q_3/q)\,, \qquad (3.6)$$

where $\epsilon=\operatorname{sgn}\mu$. (In this equation, the suffixes on the q's are 3-vector component labels; q appears only as powers of $q^2=\mathbf{q}^2$.)

The coefficient functions evidently have possible singularities at $\phi(s,t)=0$, $s=m_2^2$, $s=(m_3\pm m_4)^2$, $s=0$. In the following, we shall use the descriptions "nonsingular" or "finite," applied to combinations of helicity amplitudes, to mean kinematically nonsingular or kinematically finite; that is, nonsingular or finite if the invariant amplitudes are replaced by functions analytic at the points under consideration.

B. General-Mass Case

By this we mean $m_3\neq m_4$, and only m_1 is zero.

(a) The singularity at $\phi(s,t)=0$ is dealt with exactly as in Ref. 3. At this surface, $\sin\theta$, $\sin\frac{1}{2}\theta$, and $\cos\frac{1}{2}\theta$ are singular, but not $\cos\theta$. From (3.1), (3.2), (3.5), and (3.6),

$$H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)=\sum_{\alpha_3\alpha_4}(\sin\tfrac{1}{2}\theta)^A(\cos\tfrac{1}{2}\theta)^B R_1(s,t)_{\alpha_3\alpha_4;\lambda_1\lambda_2}\,, \qquad (3.7)$$

where $R_1(s,t)$ is nonsingular and nonzero at $\phi(s,t)=0$. A and B are given by

$$A(\alpha_3\alpha_4)=|\alpha_3+\lambda_3|+|\alpha_4-\lambda_4|+|\mu|\,,$$
$$B(\alpha_3\alpha_4)=|\alpha_3-\lambda_3|+|\alpha_4+\lambda_4|+|\mu|\,,$$

where $\mu=\lambda_1-\lambda_2+\alpha_3+\alpha_4$. Hence the well-known result that

$$\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\equiv(\sin\tfrac{1}{2}\theta)^{-|\lambda-\mu|}(\cos\tfrac{1}{2}\theta)^{-|\lambda+\mu|}H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t) \qquad (3.8)$$

is nonsingular and, in general, nonzero, at $\phi(s,t)=0$. As usual,

$$\lambda=\lambda_1-\lambda_2,\quad \mu=\lambda_3-\lambda_4\,.$$

(b) The singularities at $s=(m_3\pm m_4)^2$ are also dealt with as in Ref. 3, except that treatment of the case of an initial (and final) boson-fermion state is modified. This modification is independent of the masslessness of particle 1, and the results are essentially those of Hara.[1]

$Z(p_3)$ and $Z(p_4)$ are singular here, as are the d-matrices because of the singularities in $\sin\theta$ and $\cos\theta$, but not $\mathbf{q}_2'$. From (3.1), (3.2), (3.8), and (2.11),

$$\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)=(-1)^{s_2-\lambda_2+s_3+\lambda_3}(\sin\tfrac{1}{2}\theta)^{-|\lambda-\mu|}(\cos\tfrac{1}{2}\theta)^{-|\lambda+\mu|}\times[(\omega_3-p)/m_3]^{-\lambda_3}[(\omega_4-p)/m_4]^{-\lambda_4}\times\sum_{\alpha_3\alpha_4}d^{s_3}(\theta)_{\alpha_3}{}^{-\lambda_3}d^{s_4}(\theta)_{\alpha_4}{}^{\lambda_4}R_2(s,t)_{\alpha_3\alpha_4;\lambda_1\lambda_2}\,, \qquad (3.9)$$

where $R_2(s,t)$ is nonsingular and nonzero at $s=(m_3\pm m_4)^2$.

If the amplitude is continued through 2π around the point $s=(m_3\pm m_4)^2$ in the complex s plane, we have $S_{34}\to -S_{34}$. Hence $\theta\to\theta-\zeta\pi$ (ζ is the sign of Im $\tan\theta$ near the singular point), and

$$[(\omega_i-p)/m_i]^{-\lambda_i}\to[(\omega_i-p)/m_i]^{\lambda_i}(\gamma_i)^{2s_i}\,,\quad (i=3,4)$$

where γ_i is the sign of $\omega_i=\pm m_i$ at the singular point. For positive $\sqrt{s}$, γ_i is negative only when $s=(m_3-m_4)^2$ and $m_i<m_j$ $(i,j=3,4)$. For negative $\sqrt{s}$, γ_i has the opposite sign. Then

$$\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)_{\mathrm{II}}=\eta_{34}(\gamma_3)^{2s_3}(\gamma_4)^{2s_4}\bar{H}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t)\,, \qquad (3.10)$$

where $\eta_{34}=(-1)^{s_3-s_4+\lambda}$. (The suffix II indicates that the function is evaluated on another sheet, specified above.)

Consequently[2,3] the following expression has no branch point at $S_{34}=0$:

$$\{\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\pm\eta_{34}\bar{H}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t)\}\mathcal{K}_x(\pm)\,.$$

x is the number of fermions in the final state. The factors $\mathcal{K}_x(\pm)$ are given in Eq. (3.16). These combinations of helicity amplitudes can have poles (or zeros) in $(S_{34})^2$ at the singular point, and from (3.9) these poles[25] are of order $[\frac{1}{2}(s_3+s_4-\Lambda)]$,[23] where

$$\Lambda=\max(|\lambda|,|\mu|)\,.$$

(c) The singularity at $s=m_2^2$ is the only one whose form may not be deduced by taking $m_1\to 0$ in the results of Refs. 1, 2, or 3. At $s-m_2^2=2q_1'=0$, $Z(p_1')$ is singular, and the d-matrices are singular because of poles in $\sin\theta$ and $\cos\theta$, but $Z(p_2')$ and $\mathbf{q}_2''$ are not.

From (3.1), (3.3), (2.11), (2.16), and (3.8),

$$\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)=(\sin\tfrac{1}{2}\theta)^{-|\lambda-\mu|}(\cos\tfrac{1}{2}\theta)^{-|\lambda+\mu|}(s-m_2^2)^{-s_1}\times\sum_{j_1}\sum_{\alpha_2\beta}d^{s_2}(-\theta)_{\alpha_2}{}^{-\lambda_2}d^{j_1}(-\theta)_\beta{}^{s_1}(s-m_2^2)^{j_1+s_1}\times R_3(s,t,j_1)_{\lambda_3\lambda_4;\alpha_2\beta}\,, \qquad (3.11)$$

where $R_3(s,t)$ is nonsingular at $s=m_2^2$. It is not difficult to show that the coefficient of at least one invariant amplitude in $R_3(s,t)$ is nonzero at $s=m_2^2$. The singularity is evidently just a pole in $(s-m_2^2)^{1/2}$ of order

[25] A "pole of order n" (n integer) is a zero of order $-n$ if n is negative and is to be ignored if n is zero.

$2(s_2-\Lambda)$, i.e.,

$$(s-m_2{}^2)^{s_2-\Lambda}\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)$$

is regular at $s=m_2{}^2$.

(d) Singularities at $s=0$ are found as in Ref. 3, with appropriate modifications for boson-fermion scattering. Consider the singularities in (3.1) and (3.2). At $s=0$, $\sin\theta$ (and either $\cos\frac{1}{2}\theta$ or $\sin\frac{1}{2}\theta$), all the $Z(p_i)$, and $(q_{21}'\pm iq_{22}')$ are singular; $\cos\theta$ and q_{23}' are finite and nonsingular. We have

$$(q_{21}'\pm iq_{22}')=-(s/m_2)^{\mp 1/2}m_2{}^{1/2}p\sin\theta. \quad (3.12)$$

The phase of S_{34} at $s=0$ will be taken as π [e.g., by having the cut associated with the branch points $\phi_{34}=0$ and $\psi_{34}=0$ running along the real s axis from $(m_3-m_4)^2$ to $(m_3+m_4)^2$]. The singularity structure is independent of this choice of sign. Then, with $\epsilon_{34}=\mathrm{sgn}(m_3-m_4)$, the behavior of various functions in (3.1) and (3.2) near $s=0$ is

$$\cos\theta\sim-\epsilon_{34},$$

$$(\sin\tfrac{1}{2}\theta)^{|\alpha-\beta|}(\cos\tfrac{1}{2}\theta)^{|\alpha+\beta|}\sim s^{|\alpha+\epsilon_{34}\beta|/2},$$

$$D^s(Z(p_i'))_{-\lambda_i}{}^{-\lambda_i}=[(w_i-p)/m_i]^{-\lambda_i}\sim(s^{-\epsilon_{ij}/2})^{-\lambda_i}, \quad (i=3,4)$$

$$D^s(Z(p_2'))_{\lambda_2}{}^{\lambda_2}=[(w_2-k)/m_2]^{\lambda_2}=(m_2s^{-1/2})^{\lambda_2},$$

$$D^s(Z(p_1'))_{\lambda_1}{}^{\lambda_1}=k^{-\lambda_1}\sim s^{\lambda_1/2}.$$

Hence,

$$H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)=\sum_{\alpha_3\alpha_4}s^{c/2}R_4(s,t)_{\alpha_3\alpha_4;\lambda_1\lambda_2},$$

where $R_4(s,t)$ is nonsingular and nonzero at $s=0$, and

$$C=\lambda_1-\lambda_2+\epsilon_{34}(\lambda_3-\lambda_4)+|\alpha_3-\epsilon_{34}\lambda_3|+|\alpha_4+\epsilon_{34}\lambda_4|-\mu,$$

$$\mu=\alpha_3+\alpha_4-\lambda_2+\lambda_1.$$

Because $\{C(\alpha_3\alpha_4)\}$ is a set of even integers with minimum value zero, $H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)$ is nonsingular at $s=0$, and

$$[\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\pm\eta_{34}\bar{H}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t)]\times\mathcal{K}_x(\pm)s^{(|\lambda|+|\mu|)/2} \quad (3.13)$$

is nonsingular at $s=0$.

However, if $x=1$ (i.e., initial and final states consist of a boson and a fermion), then (3.13) has a square-root branch point at $s=0$. This singularity arises, firstly because $|\lambda+\mu|$ and $|\lambda-\mu|$ differ by an odd integer, so that the square-root branch point is cancelled in only one of $\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)$ or $\bar{H}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t)$; and secondly because of $s^{1/2}$ singularities in $\mathcal{K}_{(1)}(\pm)$. In this case, therefore, (3.13) is analytic in $\sqrt{s}$ rather than in s. Note that when the amplitudes are continued through 2π about $s=0$,

$$\begin{aligned}&\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}\to(-1)^{|\lambda+\epsilon_{34}\mu|}\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2},\\&\Omega_{34}\to-\epsilon_{34}\Xi_{34},\quad \Xi_{34}\to-\epsilon_{34}\Omega_{34}.\end{aligned} \quad (3.14)$$

(e) The kinematic-singularity-free combinations of helicity amplitudes ("regularized" helicity amplitudes) are

$$\begin{aligned}F^{\pm}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)=&(\sqrt{s})^{|\lambda|+|\mu|}(s-m_2{}^2)^{s_2-\Lambda}(S_{34})^{m_{34}}\mathcal{K}_x(\pm\gamma)\\&\times[\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\pm\eta_{34}\bar{H}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t)], \quad (3.15)\end{aligned}$$

with

$$\lambda=\lambda_1-\lambda_2,\quad \mu=\lambda_3-\lambda_4,\quad \Lambda=\max(|\lambda|,|\mu|),$$

$$m_{34}=s_3+s_4-\Lambda,\quad \eta_{34}=(-1)^{s_3-s_4+\lambda},\quad \gamma=(-1)^{m_{34}},$$

x is the number of fermions in the final state, and

$$\begin{aligned}&\mathcal{K}_0(+)=1, &&\mathcal{K}_0(-)S_{34}{}^{-1},\\&\mathcal{K}_1(+)=\Omega_{FB}{}^{-1}, &&\mathcal{K}_1(-)=\Xi_{FB}{}^{-1},\\&\mathcal{K}_2(+)=\psi_{34}{}^{-1}, &&\mathcal{K}_2(-)=\phi_{34}{}^{-1}.\end{aligned} \quad (3.16)$$

(FB is 34 or 43, depending on which of the two final particles is the fermion.) Evidently,

$$F^{\pm}{}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t)=\pm\eta_{34}F^{\pm}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t).$$

The form of (3.15) is the same, whether $\lambda_1=s_1$ or $\lambda_1=-s_1$.

For $x=1$, we have analyticity in $\sqrt{s}$ rather than in s. However, from (3.14), we obtain the so-called generalized MacDowell reciprocity relation for the amplitudes (3.15),

$$F^{\pm}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-\sqrt{s})=(-1)^{2s_3}[\mathrm{sgn}(\lambda\mu)]F^{\mp}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(\sqrt{s}).$$

(There is no analogous relation for the kinematic-singularity-free amplitudes of Ref. 3.) Hence, in the case of boson-fermion scattering, the amplitudes

$$\{F^{+}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}\pm(-1)^{2s_3}[\mathrm{sgn}(\lambda\mu)]F^{-}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}\}(\sqrt{s})^{a(\pm)} \quad (3.17)$$

have no kinematic singularities at all, where

$$a(+)=0,\quad a(-)=-1.$$

Unlike (3.17), the amplitudes (3.15) correspond at large $\cos\theta$ to definite parity for each J in the partial-wave expansion.

C. Equal-Mass Case

By this, we mean that $m_3=m_4=m$ and only m_1 is zero. In such cases, $s_3=s_4=s_f$.

This differs from the previous case only in that $\psi_{34}=0$ coincides with $s=0$; the kinematic singularities at $\phi(s,t)=0$, at $\phi_{34}=0$ and at $s=m_2{}^2$ are as before. At $s=0$, there are singularities in $\cos\theta$, $\sin\frac{1}{2}\theta$, $\cos\frac{1}{2}\theta$, in all the $Z(p_i)$, and in $(q_{21}'\pm iq_{22}')$, but not in $\sin\theta$ or q_{23}'.

From (3.1), (3.2), and (3.12),

$$\begin{aligned}\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)=&(-1)^{s_2-\lambda_2+s_3+\lambda_3}(\sin\tfrac{1}{2}\theta)^{-|\lambda-\mu|}\\&\times(\cos\tfrac{1}{2}\theta)^{-|\lambda+\mu|}k^{-\lambda_1}(s^{1/2}/m_2)^{-\lambda_2}\\&\times\left(\frac{s^{1/2}-(s-4m^2)^{1/2}}{2m}\right)^{-\lambda_3-\lambda_4}\sum_{\alpha_3\alpha_4}d^{s_3}(\theta)_{\alpha_3}{}^{-\lambda_3}d^{s_4}(\theta)_{\alpha_4}{}^{\lambda_4}\\&\times s^{-|\nu|/2}R(s,t)_{\alpha_3\alpha_4;\lambda_1\lambda_2}, \quad (3.18)\end{aligned}$$

where $R(s,t)$ is nonsingular and nonzero at $s=0$, and where $\nu=\alpha_3+\alpha_4-\lambda_2+\lambda_1$.

If the amplitude is continued through 2π around the point $s=0$ in the complex s plane, we have $\theta\to-\theta-\zeta\pi$ (ζ is the sign of $\sin\theta$ at $s=0$) and

$$\left(\frac{s^{1/2}-(s-4m^2)^{1/2}}{2m}\right)^{-\lambda_3-\lambda_4}\to\left(\frac{s^{1/2}-(s-4m^2)^{1/2}}{2m}\right)^{\lambda_3+\lambda_4}(-1)^{\lambda_3+\lambda_4}.$$

This gives

$$\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)_{\rm II}=(-1)^{\lambda_3+\lambda_4}\bar{H}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t) \quad (3.19)$$

(the suffiix II again denoting that the function is evaluated on another sheet), so that

$$[\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\pm\eta_{34}(-1)^{\lambda_3+\lambda_4+\lambda}\bar{H}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t)]s^{a(\pm)/2}$$

has no branch point at $s=0$; with $a(+)=0$, $a(-)=-1$. The $Z(p_i)$ ($i=3$, 4) and the d-matrices are finite at $s=0$, so that we are left with a pole in $\sqrt{s}$ of order $\max(-s_1+\lambda_2+\nu)=2s_f$. (The d-matrix elements are nonzero for $\alpha_3=\alpha_4=s_f$.)

Hence the kinematic-singularity-free combinations of helicity amplitudes in the case $m_3=m_4=m$, $s_3=s_4=s_f$ are

$$F^{\pm}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)=(s)^{s_f+\beta(\pm)/2}(s-m_2^2)^{s_2-\Lambda}\times(s-4m^2)^{[m_{34}+\alpha(\pm)]/2}\{\bar{H}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\pm\eta_{34}\bar{H}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t)\}. \quad (3.20)$$

Here λ, μ, Λ, η_{34}, and m_{34} are the same as given under (3.15). $\alpha(\pm)$ and $\beta(\pm)$ are given by

$$\alpha(+\xi_1)=0,\quad \alpha(-\xi_1)=-1,\quad \xi_1=(-1)^{m_{34}},$$
$$\beta(+\xi_2)=0,\quad \beta(-\xi_2)=-1,$$
$$\xi_2=(-1)^{2s_f}(-1)^{\lambda_3+\lambda_4+\lambda}.$$

D. Effects of Charged External Particles

When the massless particle has spin 1 or 2, it can couple at zero energy to a "charge."[26] For $s_1=1$ (which would appear to be the only physically interesting case, namely, photoproduction) this charge is the electric charge; for $s_1=2$ (the massless particle being a graviton) the charge has the same value $(8\pi G)^{1/2}$ for all particles, G being the gravitational constant.

This results in the appearance of readily identified poles in the helicity amplitudes in lowest-order perturbation theory, from diagrams in which the massless particle couples at zero energy to the charge carried by one of the external particles. In the c.m. frame (2.1), zero energy of the massless particle corresponds to $(s-m_2^2)\to0$, so that these poles may coincide with the kinematic-singularity factor $(s-m_2^2)^{s_2-\Lambda}$ obtained in Sec. 3 B. These extra singularities have been referred to at times as "kinematic singularities," so that we shall briefly consider them here.

[26] S. Weinberg, Phys. Rev. **135**, B1049 (1964).

According to Ref. 26,

$$H_{\lambda_3\lambda_4;\pm s_1\lambda_2}(s,t\colon p_{10}\to0)\sim\sum_i\eta_ig_i[p_i\cdot\epsilon_\pm^*(p_1)]^{s_1}(p_i\cdot p_1)^{-1}H_{\lambda_2\lambda_3\lambda_4}, \quad (3.21)$$

where s_1 is 1 or 2; $i=(2,3,4)$; η_i is $+1$ for outgoing and -1 for incoming particles; $g_i=e_i$ (electric charge) for particle 1 a photon and $g_i=(8\pi G)^{1/2}$ for particle 1 a graviton. The residue $H_{\lambda_2\lambda_3\lambda_4}$ corresponds to the vertex $(2\to3+4)$, and it also contains the kinematic singularity factors discussed previously.

$\epsilon_\pm(p_1)$ is the polarization of particle 1, and in c.m. frame (2.1) it is

$$\epsilon_\pm(p_1')=2^{-1/2}(0,\,1,\,\pm i,\,0)+a_\pm p_1',$$

where $a_\pm$ is an arbitrary scalar function of the momenta. Equation (3.21) is given in Ref. 26 for $\mathbf{p}_1\to0$, which is not in fact useful for considering the singularity structure in s and t; however, the arguments given there may be applied to the case $\mathbf{p}_1^2\to0$ as well.

For $s_1=1$ (photoproduction), (3.21) introduces first-order poles at (depending on the charge configuration) $s=m_2^2$, $t=m_3^2$, and $u=m_4^2$, i.e., $s+t=m_2^2+m_3^2$. (Of course, lowest-order perturbation-theory diagrams introduce a dynamical pole for every particle or resonance that can be exchanged in the direct channel or in a crossed channel.) Let us consider under what circumstances a pole at $s=m_2^2$ can appear, since this will modify the factor $(s-m_2^2)^{s_2-\Lambda}$ used above to obtain kinematic-singularity-free amplitudes.

The requirements are that

(i) particle 2 must be charged; (ii) the helicities must be such that a final state (of particles 3 and 4) exists with the same total angular momentum and parity as the spin and parity of particle 2; and (iii) the final state must have the same total isospin (and G parity if applicable) as particle 2.

Condition (ii) means that of the regularized helicity amplitudes [(3.15), (3.20)] $F^A{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)$ (A stands for $+$ or $-$), an extra pole factor appears in those with[27]

$$A=\eta_2\eta_3\eta_4(-1)^{s_2-\Lambda},\quad s_2\geqslant|\mu| \quad (3.22)$$

where, as usual, η_i is the intrinsic parity of particle i, and $\Lambda=\max(|\lambda|,|\mu|)$. We have used the fact of parity conservation in the reaction, and that $\cos\theta\to\infty$ as $(s-m_2^2)\to0$.

If particles 3 and 4 belong either to the same isospin multiplet or to charge-conjugate multiplets, the generalized Pauli principle, together with conditions (ii) and (iii), provides conditions which the regularized helicity amplitudes satisfying (3.22) must additionally satisfy in order to have the extra pole factor $(s-m_2^2)^{-1}$. [If the particles are actually identical or charge-conjugate, condition (iii) is not needed here.] Let the above isospin multiplets have isospin T_f. Then these further conditions

[27] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

are

$$\begin{aligned} \lambda_3 &\neq \lambda_4 \quad \text{if} \quad (-1)^{s_2} = -x, \\ \lambda_3 &\neq -\lambda_4 \quad \text{if} \quad \eta_2\eta_3\eta_4(-1)^M = -x, \end{aligned} \tag{3.23}$$

where $M = \min(|\lambda|,|\mu|)$. $x = (-1)^{2T_f - T_2}$ for final particles in the same isospin multiplet, and $x = G_2(-1)^{T_2}$ for final particles in charge-conjugate multiplets, where T_2 and G_2 are, respectively, the isospin and G parity of particle 2.

Condition (iii) also affects the particular isospin-invariant amplitudes in which the extra factor appears. Let us express the helicity amplitudes as (dropping the helicity labels)

$$F_{t_2t_3t_4}(s,t) = \sum_{T_1T'} \begin{pmatrix} \mathrm{T}_1 & \mathrm{T}_2 & \mathrm{T}' \\ 0 & t_2 & t' \end{pmatrix} \begin{pmatrix} \mathrm{T}_3 & \mathrm{T}_4 & \mathrm{T}' \\ t_3 & t_4 & t' \end{pmatrix} \times F(s,t;\, T_1T'), \tag{3.24}$$

where particle i has isospin T_i with third component t_i, and T_1 is 0 or 1. Then the extra pole factor occurs only in the isospin-invariant amplitude $F(s,t;\, T_1T_2)$, and in particular if particle 2 has definite G parity, $T_1 = 1$ only.

For example, in the case of pion photoproduction from nucleons[28] ($T_2 = 1$, $T_3 = T_4 = \frac{1}{2}$), the isospin coefficient of $F(s,t;\, 1T_2)$ is proportional to the usual $\tilde{\chi}\frac{1}{2}[\sigma_\beta,\sigma_3]\chi$, where χ and $\tilde{\chi}$ are are the nucleon and antinucleon isospinors, respectively, and $\beta = t_2$. This is the isospin coefficient of the amplitudes conventionally labelled $A^{(-)}$. From (3.22) and (3.23), only $F_{\frac{1}{2}\frac{1}{2};10}{}^{-(-)}(s,t)$ (and the parity conjugate amplitude) has this extra factor $(s - m_2{}^2)^{-1}$. [The second $(-)$ superscript refers to the isospin coupling.] Now the kinematic-singularity-free helicity amplitudes of Sec. 3 C are in this case related to the conventional invariant amplitudes[29] by

$$\begin{aligned} F^{+}{}_{\frac{1}{2}\frac{1}{2};10} &= \sqrt{2}(2M_N A_4 - A_1), \\ F^{-}{}_{\frac{1}{2}\frac{1}{2};10} &= \sqrt{2}(A_1 + A_2 s), \\ F^{+}{}_{\frac{1}{2}-\frac{1}{2};10} &= \sqrt{2}(sA_4 - 2M_N A_1), \\ F^{-}{}_{\frac{1}{2}-\frac{1}{2};10} &= \sqrt{2}A_3, \end{aligned}$$

and these amplitudes $A_i(s,t)$ are free of kinematic singularities[30]; however, the above considerations show that $A_2{}^{(-)}$ does in fact have a pole at $s = m_2{}^2$, as is well known. Note in particular that for the charge configurations $\gamma\pi^0 \to N\bar{N}$, there is no pole at $s = m_2{}^2$ in the invariant amplitudes in the lowest-order nonvanishing perturbation-theory diagrams[31] (to first order in e and third order in the pion-nucleon coupling constant). In view of its dependence on the isospin configuration, such a pole would not appear to qualify for description as a kinematic singularity.[32]

[28] Our s channel is the reaction $\gamma\pi \to N\bar{N}$.

[29] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[30] J. P. Ader, M. Capdeville, and P. Salin, Nucl. Phys. **B3**, 407 (1967).

[31] S. Minami, Progr. Theoret. Phys. (Kyoto) **7**, 69 (1952).

[32] This matter is also considered by F. S. Henyey, Phys. Rev. **170**, 1619 (1968).

There has been some confusion on this point, since the treatment of Ball[33] proceeds in a different way: The amplitude is first expressed as $\epsilon(p_1)_\mu \bar{u}_f T_\mu u_i$ (u_i and u_f being the nucleon spinors), and T_μ is then decomposed into eight invariant amplitudes $B_i(s,t)$ whose coefficients are constructed from particle momenta and γ matrices. Gauge invariance, namely, $p_{1\mu}T_\mu = 0$, where p_1 is the photon momentum, tells us that six of the $B_i(s,t)$ are related to the four $A_i(s,t)$ by[33]

$$\begin{aligned} &A_1 = B_1 - M_N B_6, \quad A_2 = 2B_2(s - m_\pi{}^2)^{-1}, \\ &A_3 = -B_8, \qquad A_4 = -\tfrac{1}{2}B_6, \\ &(2t + s - 2M_N{}^2 - m_\pi{}^2)B_2 = 2(s - m_\pi{}^2)B_3, \\ &B_5 - \tfrac{1}{4}(2t + s - 2M_N{}^2 - m_\pi{}^2)B_6 + \tfrac{1}{2}(s - m_\pi{}^2)B_8 = 0. \end{aligned}$$

(B_4 and B_7 have vanishing coefficients for physical photons.) These equations indicate kinematic zeros in B_2 and $2B_5 - (t - M_N{}^2)B_6$ at $s = m_\pi{}^2$, and in B_3 and $2B_5 + (s - m_\pi{}^2)B_8$ at $t = u$, but not a kinematic pole in A_2 at $s = m_\pi{}^2$ as has sometimes been stated.

Finally, we note that in the case $s_1 = 2$ (particle 1 a graviton) exactly the same conditions apply [except, of course, that condition (i) above is dropped] to the appearance of an extra factor $(s - m_2{}^2)^{-1}$.

4. CONSTRAINT EQUATIONS ON HELICITY AMPLITUDES

A. Constraint Equation at $s = m_2{}^2$

In Sec. 3, we obtained combinations of helicity amplitudes, multiplied by kinematic factors, which have no kinematic singularities. However, the inverse expansion, expressing the invariant amplitudes in terms of the helicity amplitudes, will in general have kinematic singularities that are not removed by the above process. The inverse expansion will contain apparent kinematic poles whose residues are combinations of helicity amplitudes; these "residues" must vanish so as to remove these poles since the invariant amplitudes are free of kinematic singularities. Hence at certain values of s we shall have relations between different helicity amplitudes, and between derivatives (of a particular order) of helicity amplitudes. These constraint equations will be derived directly from the formulas of Sec. 3 A.

First, let us consider the behavior of the helicity amplitudes at $s = m_2{}^2$. From (3.1) and (3.3), we see that

$$H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)(-1)^{s_2-\lambda_2} D^{s_2}(Z(p_2)^{-1})_{\lambda_2}{}^{\lambda_2} d^{s_2}(\theta)_{-\lambda_2}{}^{\alpha_2} \tag{4.1}$$

has no singularity at $s = m_2{}^2$, for all α_2 ($|\alpha_2| \leqslant s_2$). We are excluding values of t such that $\phi(m_2{}^2,t) = 0$. From (3.11), we know that

$$\mathfrak{F}_{\lambda_2} \equiv H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)(s - m_2{}^2)^{s_2}$$

is nonzero and nonsingular at $s = m_2{}^2$. The equations that follow are independent of λ_1, λ_3, and λ_4, so that we drop these labels. Also, $d^{s_2}(\theta)_{-\lambda_2}{}^{\alpha_2}(s - m_2{}^2)^{s_2}$ is nonzero

[33] J. S. Ball, Phys. Rev. **124**, 2014 (1961).

and nonsingular at $s=m_2^2$. Then

$$\left\{\frac{d^n}{ds^n}\left[\sum_\mu \mathfrak{F}_{-\mu}(-1)^\mu \times\left(\frac{\sqrt{s}}{m_2}\right)^{-\mu}(s-m_2^2)^{s_2}D^{s_2}(\theta)_\mu{}^\alpha\right]\right\}_{s=m_2^2}=0 \quad (4.2)$$

for $0\leqslant n\leqslant 2s_2-1$ (n integer), $|\alpha|\leqslant s_2$.

The presence of boosts and rotation matrices makes this constraint equation unsuitable for practical application, so we proceed to simplify it. First multiply (4.2) for each value of α by $[(s_2+\alpha)!(s-\alpha)!]^{1/2}(\tan\frac{1}{2}\theta)^\alpha$. This factor is nonsingular at $s=m_2^2$, so that we can move it to the right of the differential operator to obtain an equivalent set of equations. Then, taking successive differences between equations with consecutive values of α, we can replace the set of equations (4.2) by

$$\left\{\frac{d^n}{ds^n}\left[\sum_\mu \mathfrak{F}_{-\mu}(-1)^\mu\left(\frac{\sqrt{s}}{m_2}\right)^{-\mu} \times(s-m_2^2)^{s_2}[(s_2+\mu)!(s_2-\mu)!]^{1/2} \times P_{s_2-\mu-m}{}^{(s_2+\mu,\,-s_2+\mu+m)}(\cos\theta)\right]\right\}_{s=m_2^2}=0, \quad (4.3)$$

where $0\leqslant m\leqslant 2s_2$ (m integer), and $P_n{}^{(\alpha,\beta)}(\cos\theta)$ is a Jacobi polynomial. Equation (4.3) can be rewritten as

$$\left\{\frac{d^n}{ds^n}\left[\sum_\mu \mathfrak{F}_{-\mu}\left(\frac{\sqrt{s}}{m_2}\right)^{-\mu}[(s-m_2^2)\sin\theta]^\mu \times[(s-m_2^2)+(s-m_2^2)\cos\theta]^{s_2-\mu-m} \times\frac{(s-m_2^2)^m}{(s_2-\mu-m)!}\left(\frac{(s_2-\mu)!}{(s_2+\mu)!}\right)^{1/2}\right]\right\}_{s=m_2^2}=0. \quad (4.4)$$

First consider $m=n$. $\cos\theta$ and $\sin\theta$ have first-order poles at $s=m_2^2$, with residues related by $\sin\theta\sim i\zeta\cos\theta$. ζ is ±1, the sign depending on the determination of $\sqrt{[\phi(s,t)]}$ which has phase $\pm\frac{1}{2}\pi$ at $s=m_2^2$. The nonvanishing terms in (4.4) are then

$$\sum_\mu \mathfrak{F}_{-\mu}(\zeta i)^\mu\left(\frac{(s_2-\mu)!}{(s_2+\mu)!}\right)^{1/2}\frac{1}{(s_2-\mu-m)!}=0 \quad (4.5)$$

for $0\leqslant m\leqslant 2s_2-1$.

Now consider (4.4) for $m=n-p$, where p is an integer $0<p\leqslant 2s_2-1$. Since $0\leqslant n\leqslant 2s_2-1$, we have $0\leqslant m\leqslant 2s_2-1-p$. [If p is negative, (4.4) obviously contains no terms nonvanishing at $s=m_2^2$.] Equation (4.4) becomes

$$\sum_\mu\left[\sum_{r=0}^{p}\mathfrak{F}_{-\mu}{}^{(r)}(\zeta i)^\mu \times\left(\frac{(s_2-\mu)!}{(s_2+\mu)!}\right)^{1/2}\frac{U(\mu)}{(s-\mu-n+p)!}\right]_{s=m_2^2}=0, \quad (4.6)$$

where $\mathfrak{F}_\mu{}^{(r)}$ means $(d^r/ds^r)[\mathfrak{F}_\mu(s,t)]$, and $U(\mu)$ is some polynomial of order $(p-r)$ in μ, whose coefficients are functions of s and t. Now suppose we have, for all r in $0\leqslant r\leqslant p-1$,

$$\sum_\mu \mathfrak{F}_{-\mu}{}^{(r)}(\zeta i)^\mu\left(\frac{(s_2-\mu)!}{(s_2+\mu)!}\right)^{1/2}\frac{1}{(s_2-\mu-m)!}=0 \qquad (0\leqslant m\leqslant 2s-1-r). \quad (4.7)$$

From this we may obtain

$$\sum_\mu \mathfrak{F}_{-\mu}{}^{(r)}(\zeta i)^\mu\left(\frac{(s_2-\mu)!}{(s_2+\mu)!}\right)^{1/2}\frac{V(\mu)}{(s_2-\mu-m)!}=0, \qquad (0\leqslant m\leqslant 2s-1-p) \quad (4.8)$$

where $V(\mu)$ is any polynomial in μ of order $p-r>0$. Substituting (4.8) into (4.6), we see that (4.7) is true for $r=p$ if it is true for $0\leqslant r\leqslant p-1$, and from (4.5) it is true for $r=0$.

Hence the constraint equation on the helicity amplitudes at $s=m_2^2$ is

$$\sum_{\lambda_2}\left(\frac{d^r}{ds^r}[(s-m_2^2)^{s_2}H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)]\right)_{s=m_2^2} \times(\zeta i)^{-\lambda_2}\left(\frac{(s_2+\lambda_2)!}{(s_2-\lambda_2)!}\right)^{1/2}\frac{1}{(s_2+\lambda_2-m)!}=0 \quad (4.9)$$

for $0\leqslant r\leqslant 2s_2-1$, $0\leqslant m\leqslant 2s_2-1-r$.

By multiplying by suitable functions of m, and adding these equations for values of m from 0 to $(s_2-\alpha)$, we get the equivalent constraint equations

$$\sum_{\lambda_2}\left\{\frac{d^r}{ds^r}[(s-m_2^2)^{s_2}H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t) \times(i\zeta)^{-\lambda_2}d^{s_2}(\tfrac{1}{2}\pi)_{\lambda_2\alpha}]\right\}_{s=m_2^2}=0 \quad (4.10)$$

for $0\leqslant r\leqslant 2s_2-1$, $-s_2+1+r\leqslant\alpha\leqslant s_2$.

One may then express the helicity amplitudes in (4.10) in terms of the regularized helicity amplitudes (3.15) or (3.20), to obtain the required constraint equations on these regularized amplitudes. The factor $(\cos\frac{1}{2}\theta)^{|\lambda+\mu|}$ $(\sin\frac{1}{2}\theta)^{|\lambda-\mu|}$ $(s-m_2^2)^\Lambda$ will then remove the apparent $(\zeta)^{\lambda_2}$ sign ambiguity. Since the regularized helicity amplitudes are nonsingular at $\phi(s,t)=0$ and at $s=m_2^2$, the resulting constraint equations are valid for all t at $s=m_2^2$.

B. Other Constraint Equations

Constraint equations at $s=(m_3\pm m_4)^2$ are found similarly. From (3.1) and (3.2),

$$\sum_{\lambda_3\lambda_4}H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)(-1)^{s_3-\lambda_3}D^{s_3}(Z(p_3)^{-1})_{-\lambda_3}{}^{-\lambda_3} \times D^{s_4}(Z(p_4)^{-1})_{-\lambda_4}{}^{-\lambda_4}d^{s_3}(-\theta)_{-\lambda_3}{}^{\alpha_3}d^{s_4}(-\theta)_{\lambda_4}{}^{\alpha_4} \quad (4.11)$$

has no singularity at $s=(m_3\pm m_4)^2$, for all α_3, α_4. From (3.9), we know that $H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)[s-(m_3\pm m_4)^2]^{(s_3+s_4)/2}$ is finite at $s=(m_3\pm m_4)^2$, where its kinematic singularity is a square-root branch point. First consider the case $s-(m_3+m_4)^2\equiv\phi_{34}{}^2=0$. Put

$$G_{\lambda_3\lambda_4}\equiv\phi_{34}{}^{s_3+s_4}H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\,,\quad G_{\lambda_3\lambda_4}{}^{(n)}\equiv\frac{d^n}{d(\phi_{34})^n}G_{\lambda_3\lambda_4}.$$

Then, as at $s=m_2{}^2$,

$$[G_{\mu\nu}{}^{(n)}(i\zeta)^{\mu+\nu}d^{s_3}(\tfrac{1}{2}\pi)_{\mu\alpha}d^{s_4}(\tfrac{1}{2}\pi)_{\nu\beta}]_{\phi_{34}=0}=0 \quad (4.12)$$

for

$$0\leqslant n\leqslant 2(s_3+s_4)-1$$

and

$$-s_3-s_4+1+n\leqslant\alpha+\beta\leqslant s_3+s_4.$$

But from (3.10), continuation of $G_{\mu\nu}$ along a circuit of 2π about $\phi_{34}=0$ gives

$$(G_{\mu\nu})_{\rm II}=(-1)^{-\mu-\nu}G_{-\mu,-\nu}$$

so that, at $\phi_{34}=0$,

$$r!G_{\mu\nu}{}^{(2r)}=(2r)!\frac{d^r}{ds^r}[G_{\mu\nu}+(-1)^{-\mu-\nu}G_{-\mu,-\nu}]\,,$$
$$r!G_{\mu\nu}{}^{(2r+1)}=(2r+1)!\frac{d^r}{ds^r}\{[G_{\mu\nu}-(-1)^{-\mu-\nu}G_{-\mu,-\nu}]\phi_{34}{}^{-1}\}\,. \quad (4.13)$$

From the symmetry properties of the d-matrices, we obtain the constraint equation

$$\left\{\frac{d^r}{ds^r}[\sum_{\lambda_3\lambda_4}\phi_{34}{}^{s_3+s_4-a}H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\times(i\zeta)^{\lambda_3+\lambda_4}d^{s_3}(\tfrac{1}{2}\pi)_{\lambda_3\alpha_3}d^{s_4}(\tfrac{1}{2}\pi)_{\lambda_4\alpha_4}]\right\}_{\phi_{34}=0}=0 \quad (4.14)$$

for $0\leqslant r\leqslant[\frac{1}{2}(s_3+s_4+\alpha_3+\alpha_4)-1]$,[23] with

$$a=0 \text{ for } (s_3+s_4+\alpha_3+\alpha_4) \text{ even},\qquad a=1 \text{ for } (s_3+s_4+\alpha_3+\alpha_4) \text{ odd}. \quad (4.15)$$

The behavior at $s-(m_3-m_4)^2\equiv\psi_{34}{}^2=0$ (for $m_3\neq m_4$) is almost the same as that at $\phi_{34}=0$. However, the term $(\omega_i-p_i)/m_i$ appearing in $D^{s_i}(Z(p_i))$ has the value -1 rather than $+1$ at $\psi_{34}=0$ if $m_i<m_j$ (i, $j=3$, 4). This introduces an extra factor $(-1)^{\lambda_i}$ into the equation corresponding to (4.12), and we obtain

$$\left\{\frac{d^r}{ds^r}[\sum_{\lambda_3\lambda_4}\psi_{34}{}^{s_3+s_4-b}H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\times(i\zeta)^{\lambda_3+\lambda_4}d^{s_3}(\tfrac{1}{2}\pi)_{\lambda_3\alpha_3}d^{s_4}(\tfrac{1}{2}\pi)_{\lambda_4\alpha_4}]\right\}_{\psi_{34}=0}=0 \quad (m_3\neq m_4)\quad (4.16)$$

for[23] $0\leqslant r\leqslant[\frac{1}{2}(s_3+s_4+\epsilon_{34}(\alpha_3-\alpha_4))-1]$, with

$$\epsilon_{34}=\mathrm{sgn}(m_3-m_4)$$

and

$$b=0 \text{ for } s_3+s_4+\epsilon_{34}(\alpha_3-\alpha_4) \text{ even},\qquad b=1 \text{ for } s_3+s_4+\epsilon_{34}(\alpha_3-\alpha_4) \text{ odd}. \quad (4.17)$$

As one might have expected, the constraint equations (4.14) and (4.16) are the same as those obtained in Ref. 3, in the case of all nonzero masses, by consideration of poles in the crossing matrix for transversity amplitudes (amplitudes in which the spins are quantized along the normal to the reaction plane).

There is no constraint equation for the helicity amplitudes at $s=0$ if $m_3\neq m_4$, as will be shown in Sec. 4 D. However, there is a constraint of a more trivial nature affecting the regularized helicity amplitudes at $s=0$, resulting from singularities in the $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$ factors introduced.[34] From (3.8) and (3.15), we see that

$$F^{\pm}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)=[H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)(\sqrt{s})^{|\lambda|+|\mu|-|\lambda+\epsilon_{34}\mu|}\pm\eta_{34}H_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(s,t)(\sqrt{s})^{|\lambda|+|\mu|-|\lambda-\epsilon_{34}\mu|}]K^{\pm}(s,t)\,, \quad (4.18)$$

where $K^{\pm}(s,t)$ is regular at $s=0$, as are the helicity amplitudes. Then near $s=0$,

$$F^{+}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)[\mathcal{K}_x(\gamma)]^{-1}-[\mathrm{sgn}(\mu\lambda\epsilon_{34})]F^{-}{}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)\times[\mathcal{K}_x(-\gamma)]^{-1}\sim s^M, \quad (4.19)$$

where $M=\min(|\lambda|,|\mu|)$, other symbols being defined in (3.16) and following (3.15). Appropriate derivative conditions are readily formulated; note that when M is half an odd integer (boson-fermion scattering), (4.19) should first be divided by $s^{1/2}$ so that both sides of the relation become regular in s at $s=0$. In particular, provided that λ and μ are both nonzero, the nonderivative conditions at $s=0$ for various numbers of fermions in the final state are

$$(x=0):\ F^+(0,t)=-[\mathrm{sgn}(\lambda\mu)](m_3{}^2-m_4{}^2)^\gamma F^-(0,t)\,,$$
$$(x=1):\ F^+(0,t)=(-1)^{2s_3}[\mathrm{sgn}(\lambda\mu)]F^-(0,t)\,,$$
$$(x=2):\ F^+(0,t)=[\mathrm{sgn}(\lambda\mu)]\{(m_3+m_4)/(m_3-m_4)\}^\gamma\times F^-(0,t)\,,$$

where the helicity labels $(\lambda_3\lambda_4;\lambda_1\lambda_2)$ have been dropped, and where $\gamma=(-1)^{s_3+s_4-\Lambda}$.

C. Equal-Mass Case

The constraint equation at $\psi_{34}=0$ for $m_3=m_4=m$ needs separate consideration, because $\psi_{34}=0$ now coincides with $s=0$. Let $s_3=s_4=s_f$. From (3.1), (3.2), and (3.18), we have

$$\sum_{\lambda_3\lambda_4}H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)(-1)^{s_3+\lambda_3}\left(\frac{\omega-p}{m}\right)^{\lambda_3+\lambda_4}d^{s_f}(-\theta)_{-\lambda_3}{}^{\alpha_3}\times d^{s_f}(-\theta)_{\lambda_4}{}^{\alpha_4}s^{(\alpha_3+\alpha_4)/2} \quad (4.20)$$

has no singularity at $s=0$, and $H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)s^{s_f}$ is finite

[34] Such constraints were first noticed by S. Frautschi and L. Jones, Phys. Rev. 167, 1335 (1968).

and nonsingular in $\sqrt{s}$ at $s=0$. Then

$$\frac{d^n}{d(\sqrt{s})^n}\Bigg[\sum_{\lambda_3\lambda_4} H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)s^{s_f}(-1)^{\lambda_3} \times\left(\frac{\omega-p}{m}\right)^{\lambda_3+\lambda_4} d^{s_f}(-\theta)_{-\lambda_3}{}^{\alpha_3}d^{s_f}(-\theta)_{\lambda_4}{}^{\alpha_4}\Bigg]_{s=0}=0 \quad (4.21)$$

for $0\leqslant n\leqslant 2s_f-\alpha_3-\alpha_4$.

When $s\to 0$, $\sin\theta\to\zeta=\pm 1$, and $(\omega-p)/m\to\pm i$ [the sign in the latter case depending on the arrangement of the cuts associated with the $(s-4m^2)^{1/2}$ branch point]. By the same method as used to obtain (4.9) from (4.4), we have, substituting $\theta(s=0)=(\frac{1}{2}\pi)\zeta$,

$$\left(\frac{d^n}{d(\sqrt{s})^n}[H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)s^{s_f}]\right)_{s=0} \times(i\zeta)^{\lambda_3+\lambda_4}d^{s_f}(\tfrac{1}{2}\pi)_{\lambda_3\alpha_3}d^{s_f}(\tfrac{1}{2}\pi)_{\lambda_4\alpha_4}=0 \quad (4.22)$$

for $0\leqslant n\leqslant 2s_f+\zeta(\alpha_3-\alpha_4)+1$. From (3.19), equations analogous to (4.13) are obtained, and the constraint equations finally obtained are

$$\left(\frac{d^r}{ds^r}[H_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)s^{s_f-c}(i\zeta)^{\lambda_3+\lambda_4} \times d^{s_f}(\tfrac{1}{2}\pi)_{\lambda_3\alpha_3}d^{s_f}(\tfrac{1}{2}\pi)_{\lambda_4\alpha_4}]\right)_{s=0}=0, \quad (m_3=m_4) \quad (4.23)$$

for $0\leqslant r\leqslant[\frac{1}{2}\zeta(\alpha_3-\alpha_4)+s_f-1]$,[23] with

$$c=0 \quad \text{for} \quad \alpha_3+\alpha_4 \text{ even},$$
$$c=\tfrac{1}{2} \quad \text{for} \quad \alpha_3+\alpha_4 \text{ odd}. \quad (4.24)$$

As expected, this result is the same as that implied by the results of Ref. 3.

D. Completeness of Constraint Equations

It remains to be shown that we have obtained all possible constraint equations on the helicity amplitudes that are imposed by the original assumptions on the analyticity properties of the M-functions. This is achieved by showing that the constraint equations obtained above, together with the analyticity properties of the helicity amplitudes derived in Ref. 3 and Sec. 3, are sufficient for the analyticity of the invariant amplitudes.

Evidently, the constraint equations are exactly equivalent to the absence of singularities in (4.1), (4.11), and (4.20) at, respectively, $s=m_2^2$, $s=(m_3\pm m_4)^2$, and $s=0$. We shall consider explicitly only the case $\lambda_1=+s_1$.

Analyticity of (4.11) at $s=(m_3+m_4)^2$ implies analyticity, at this value of s, of $M^{\alpha s_1}$ [expressed as a function of s and t in the c.m. frame (2.1)] in (2.39), from which it is immediately apparent that the invariant amplitudes are not singular there either. The same applies to $s=(m_3-m_4)^2$ if $m_3\neq m_4$.

Next consider possible singularities at $s=0$ in the case $m_3\neq m_4$. From (2.39), (2.21), (2.9), and the result that the helicity amplitudes are analytic at $s=0$, one finds that

$$B_l(s,t)=\sum_{\alpha\alpha_3\alpha_4} s^{x/2}R(s,t,l)_{\alpha\alpha_3\alpha_4},$$

where $R(s,t,l)$ is nonsingular at $s=0$ and

$$x=\alpha+s_1-\lambda_1+\lambda_2-\epsilon_{34}(\lambda_3-\lambda_4)+|\alpha_3-\epsilon_{34}\lambda_3|+|\alpha_4+\epsilon_{34}\lambda_4|,$$
$$\alpha=\alpha_3+\alpha_4-\lambda_2.$$

[The behavior of the kinematic factors in (2.9) near $s=0$ is given in Sec. 3 B (d).] Evidently x is an even integer with lowest value zero, so that no constraints on the helicity amplitudes are required by analyticity of the invariant amplitudes at $s=0$.

It is somewhat less trivial to show that, at $s=m_2^2$, analyticity of the invariant amplitudes follows from the analyticity of expression (4.1). We require invariant-amplitude expansions of the types (2.23) and (2.25), using

$$\mathbf{q}_1=\mathbf{q}(1,3), \quad \mathbf{q}_2=\mathbf{q}(4,3)$$

[i.e., $a=1$, $b=3$, $c=4$, in (2.24)], in the s-channel c.m. frame with the spatial momentum of particle 3 parallel to the 3 axis. As before, $q_1=\frac{1}{2}(s-m_2^2)$. In Ref. 7 it is shown that, at $q_1=0$, the analyticity of the invariant amplitudes of (2.23) implies the analyticity of those of (2.25). Now the expansion (2.23) can be inverted,[22]

$$B_{(m)l}(s,t)=\sum_\mu M^\mu[(j+\alpha)!(j-\alpha)!]^{-1/2} \times\mathcal{C}'_\mu\mu^{-m}F_j(\mathbf{q}_1,l)q_2^{-j+l-m}, \quad (4.25)$$

where $m=(0,1)$ and the $B_{(m)l}(s,t)$ differ only by numerical constants from the $A(s,t;\, l_1l_2,j,\eta)$ of (2.23) with $l_1+l_2=j+m$. For the notation, see (2.41). $\mathcal{C}_\mu'$ is given by

$$\mathcal{C}_\mu'=[(\omega_3+\epsilon p)k\sin\theta]^{-|\mu|}, \quad \epsilon=\text{sgn}(\mu).$$

None of the coefficients of M^μ in (4.25) is singular at $s=m_2^2$. Since $D^{s_1}(B_1(p_1)^{-1})_{s_1}{}^{s_1}$ is not singular at $s=m_2^2$, the analyticity, at $s=m_2^2$, of (4.1) implies analyticity of the M^μ in (4.25). This gives the required result.

If $m_3=m_4$, absence of a singularity in (4.20) at $s=0$ implies that

$$M^{\alpha s_1}\sim s^{(-\lambda_1+\lambda_2-\alpha_3-\alpha_4)/2} \text{ near } s=0,$$

$M^{\alpha s_1}$ being the M-function appearing in (2.39), and in (2.39) $\alpha=\alpha_3+\alpha_4-\lambda_2$. It is again apparent that this is sufficient for analyticity of the invariant amplitudes at $s=0$.

We note that the well-known existence of singularity-free inversion formulas[10] of the form (4.25), together with the results of Ref. 7 and Sec. 3, is in fact sufficient to obtain all the results of this section; the explicit inversion formulas of Sec. 2 C are not necessary for our present purpose.

5. SUMMARY OF FORMULAS OBTAINED

Kinematic-singularity-free combinations of helicity amplitudes for a four-particle process, one particle being massless, are given in (3.15) and (3.17) (masses of final two particles unequal), and in (3.20) (masses of final particles equal). Linear relationships between helicity amplitudes (and between some of their derivatives) at certain values of s are given in (4.10), (4.14), (4.16), and (4.23). A further relation involving the kinematic-singularity-free amplitudes, which does not correspond to a constraint on the helicity amplitudes, is given in (4.19). In addition, an explicit formula (2.39) is given for the inversion of a certain expansion of the M-function in terms of invariant amplitudes.

These results agree with those obtained by Ader, Capdeville, and Navelet.[11] These authors obtain the kinematic-singularity-free combinations of helicity amplitudes for one or more massless particles in a two-particle to two-particle reaction by using the same invariant-amplitude expansion as in Refs. 3 and 10; a gauge-invariance condition on the M-functions then produces additional kinematic factors corresponding to the presence of the massless particles. For one massless particle, as here, their treatment of the case of odd-fermion-number initial and final states differs from ours. In such a case, their regularized helicity amplitudes have kinematic branch points not only at $s=0$ [as do our amplitudes (3.15) with $x=1$] but also at thresholds and pseudothresholds on the sheet with negative $\sqrt{s}$. Of course, in the study of kinematic singularities of scattering amplitudes, such extraneous singularities on unphysical sheets are usually not important, but they do have the effect that the MacDowell symmetry of the helicity amplitudes is not expressible in a simple form for the regularized helicity amplitudes of Refs. 3 and 11. Consequently, one cannot readily obtain from them amplitudes without any kinematic singularities (even at $s=0$), corresponding to our amplitudes (3.17). (Davies[35] has also shown how the method of Wang[1] may similarly be modified.) The authors of Ref. 11 also obtain the constraint equations, using a method involving consideration of the crossing matrix for transversity amplitudes, as was done in Ref. 3. (Their method is slightly indirect compared with that used here, in that it has not previously been shown that *all* the kinematic constraints implied by analyticity of the M-functions are in fact also implied by singularities in the crossing matrix between regularized helicity amplitudes.)

[35] W. E. A. Davies, Nuovo Cimento **53A**, 828 (1968).

APPENDIX

This is a description of some properties of the function $X_\mu{}^l(\omega)$, defined by (2.37), which is useful in certain cases for the inversion of invariant amplitude expansions. Let

$$X_\mu{}^l(\omega)=\tfrac{1}{2}(-\sin\omega)^{-l}\frac{d^l}{d(\cot\omega)^l}\times[(\cot\omega+\csc\omega)^\mu+(\cot\omega-\csc\omega)^\mu] \quad (2.37')$$

for which μ, l, are integers and $l\geqslant 0$. We have

$$X_{-\mu}{}^l(\omega)=(-1)^\mu X_\mu{}^l(\omega)\,,\qquad X_\mu{}^l(\omega)=0 \text{ when } |\mu|<l. \quad (A1)$$

Evidently,

$$Y_l{}^\mu(\omega)X_\mu{}^{l'}(\omega)=0 \text{ if } l'>l. \quad (A2)$$

If we substitute $z=(\cot\omega'\pm\csc\omega')$ into the formula

$$\sum_\mu C_{(l)}{}^\mu(\omega)z^\mu[(l+\mu)!(l-\mu)!]^{-1/2}l! =[\cos\omega-\tfrac{1}{2}(z-z^{-1})\sin\omega]^l, \quad (A3)$$

where

$$C_{(l)}{}^\mu(\omega)=(2l+1)^{-1/2}(4\pi)^{1/2}Y_l{}^\mu(\omega)\,,$$

we see that

$$\sum_\mu C_{(l)}{}^\mu(\omega)X_\mu{}^{l'}(\omega)[(l+\mu)!(l-\mu)!]^{-1/2}=\delta_{ll'}. \quad (A4)$$

The function $X_\mu{}^l(\omega)$ may also be shown to satisfy the relations

$$\sum_\mu C_{(l)}{}^\mu(\omega)X_\mu{}^{l'}(\omega)[(l+\mu)!]^{1/2}[(l-\mu)!]^{-1/2}\times[(L+\mu)!]^{-1}=\delta_{Ll'} \text{ if } l\geqslant L\geqslant l', \quad (A5)$$

$$\sum_{\mu\neq 0} C_{(l)}{}^\mu(\omega)X_\mu{}^{l'}(\omega)\mu^{-1}[(l+\mu)!]^{1/2}[(l-\mu)!]^{-1/2}\times[(L+\mu)!]^{-1}=0 \text{ if } l\geqslant L\geqslant l'. \quad (A6)$$

Further, it is obvious from the symmetry of the terms under $\mu\to-\mu$ that

$$\sum_{\mu\neq 0} C_{(l)}{}^\mu(\omega)X_\mu{}^{l'}(\omega)\mu^{\pm1}=0. \quad (A7)$$