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Simultaneous Partial-Wave Expansion in the Mandelstam Variables: The Group $SU(3)$

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The elastic scattering amplitude of two spinless particles of equal mass $\frac{1}{2}$ was expanded elsewhere in a double series of eigenfunctions which "displayed" its dependence on all the Mandelstam variables s, t, u ($s+t+u=1$). The expansion was then used to investigate the crossing properties of partial-wave amplitudes. We show in this paper that these eigenfunctions are certain basis vectors of the representations (σ,σ) of a suitably defined $SU(3)$. The unequal-mass problem is also discussed.

I. INTRODUCTION

In a previous paper,[1] we proposed an eigenfunction expansion for the elastic scattering amplitude F of two spinless particles of equal mass. These eigenfunctions formed a complete set for a certain class of functions of the Mandelstam variables s, t, u and were associated with well-defined values of angular momenta. They were generated by a partial differential operator $\mathcal{O}$ which commuted with the angular momentum in the three channels, and which was invariant under s, t, u permutations. The expansion coefficients were shown to satisfy an infinite sequence of *finite dimensional* crossing relations due to the crossing symmetry of F. Indications for extending the approach to systems with internal symmetry or spin were also given. In a second paper,[2] the eigenvectors of the crossing matrices were constructed.

In the present work, we formulate a plausible group-theoretical basis for this expansion. In Sec. II, the operator $\mathcal{O}$ is identified with the quadratic Casimir operator of a certain $SU(3)$ and the eigenfunctions of $\mathcal{O}$ with a specific subset of base vectors of its irreducible representations (σ,σ). The partial-wave crossing matrices are just Weyl reflections in these representations.

In Sec. III, the difficulties encountered by this method when the particles do not have the same mass are discussed. Some reasonable hypotheses for the action of $SU(3)$ on the scattering variables are made in order that $\mathcal{O}$ may have the requisite features. The resultant constraints are fulfilled only when the particles are degenerate in mass.

In Appendix A, the unequal-mass system is considered once more, and a few rather remarkable properties of an associated Gram determinant are mentioned. These suggest possible generalizations of our equations to such systems, but also raise many unsolved questions.

In Appendix B, we try to identify the new variables we were naturally led to introduce in Sec. II.

The discussion is specialized in much of what follows to the situation where the eigenvalue problem associated with $\mathcal{O}$ is solved on the Mandelstam triangle. (The boundaries of this triangle are $s=0, t=0, u=0$.) As indicated elsewhere,[1] however, the problem can be solved equally well in the physical region. There should be no difficulty in recognizing that the group which underlies the corresponding eigenfunctions is $SU(2,1)$ rather than $SU(3)$. The necessary modifications in the analysis will also be indicated in the text.

II. THE GROUP

We first recall a few pertinent facts from our previous paper. The particles are supposed to have the same mass $\frac{1}{2}$. If s, t, u are the Mandelstam variables, the Casimir operator of the s-channel little group, when it acts on a function of s and t, has the form

$$\text{(a)}\quad X^2=-\frac{\partial}{\partial z_s}(1-z_s^2)\frac{\partial}{\partial z_s},$$
$$\text{(b)}\quad\quad =-(\partial_t-\partial_u)(tu)(\partial_t-\partial_u),\quad z_s=1+\frac{2t}{s-1}. \tag{2.1}$$

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† Supported by NDEA Fellowship.
[1] A. P. Balachandran and J. Nuyts, Phys. Rev. **172**, 1821 (1968). See also A. P. Balachandran and J. Nuyts, Nucl. Phys. (to be published).
[2] A. P. Balachandran, W. J. Meggs, and P. Ramond, Phys. Rev. **175**, 1974 (1968).

The eigenfunctions of X^2 are the Legendre polynomials $P_l(z_s)$. In writing (2.1b), we treat s, t, u as independent variables. This is permissible since $(\partial_t-\partial_u)(s+t+u)=0$. Only spinless systems are considered. The corresponding t- and u-channel operators Y^2 and Z^2 are obtained from (2.1b) by cyclic permutations of s, t, u. The operator $\mathcal{O}$ is constructed in terms of X^2, Y^2, Z^2 by the definition

$$\mathcal{O}=X^2+Y^2+Z^2. \tag{2.2}$$

It is required to identify $\mathcal{O}$ with the Casimir operator of some group of transformations $\mathcal{G}$. We first enumerate the properties which are expected to characterize $\mathcal{G}$:

(i) *$\mathcal{G}$ must leave the surface $s+t+u=1$ invariant.* For if it did not, the kinematical constraints on the system would not be maintained under the action of $\mathcal{G}$. The form (2.2) for $\mathcal{O}$ also suggests the following:

(ii) *$\mathcal{G}$ must contain three $SU(2)$ subgroups such that the corresponding Casimir operators reduce to X^2, Y^2, Z^2 when restricted to functions of s and t.*

(iii) *The quadratic Casimir operator of $\mathcal{G}$ must reduce to $\mathcal{O}$ under a corresponding restriction.*

Next, we show that it is possible to identify $\mathcal{G}$ with the group $SU(3)$. Let us implement the transformations of this group on the complex three-vector (z_1,z_2,z_3) of unit length $(\sum_i|z_i|^2=1)$. Let **I**, **U**, **V** be the generators of the three distinguished $SU(2)$ subgroups which act on the pairs (z_1,z_2), (z_2,z_3), (z_3,z_1).[3] If I_3, U_3, V_3 are the diagonal generators in this basis $(I_3+U_3+V_3=0)$, and if f is a function of z_i such that

$$\text{then}\quad \begin{aligned}&\text{(a)}\ \ I_3f=U_3f=0,\\ &\text{(b)}\ \ C_2f=(\mathbf{I}^2+\mathbf{U}^2+\mathbf{V}^2)f,\end{aligned} \tag{2.3}$$

where C_2 is the quadratic Casimir operator of $SU(3)$. [The action of an element $\alpha\in SU(3)$ on a function h of z_i is taken to be standard:

$$(\alpha h)(z_1,z_2,z_3)=h(\alpha^{-1}z_1,\alpha^{-1}z_2,\alpha^{-1}z_3).] \tag{2.4}$$

The problem is therefore solved if the dependences of z_i on s, t, u are such that I_3, U_3 are zero on these latter variables, and, moreover, if

$$\begin{aligned}&\text{(a)}\ \ X^2=\mathbf{I}^2,\quad Y^2=\mathbf{U}^2,\quad Z^2=\mathbf{V}^2,\\ &\text{(b)}\ \ \textstyle\sum_i|z_i|^2=s+t+u.\end{aligned} \tag{2.5}$$

The restriction to functions of s, t, u is understood in (a). (b) suggests that we set[4]

$$z_1=s^{1/2}e^{i\varphi_1},\quad z_2=t^{1/2}e^{i\varphi_2},\quad z_3=u^{1/2}e^{i\varphi_3}. \tag{2.6}$$

Next, we note that owing to (2.4), the generators of $SU(3)$ can be written in the form

$$B_i{}^j=\left(z_i\frac{\partial}{\partial z_j}-z_j^*\frac{\partial}{\partial z_i^*}\right)-\tfrac{1}{3}\delta_i{}^j\sum_k\left(z_k\frac{\partial}{\partial z_k}-z_k^*\frac{\partial}{\partial z_k^*}\right), \tag{2.7}$$

where the relationship between $B_i{}^j$ and **I**, **U**, **V** is given in Ref. 3. Variables can be changed from z_1, z_2, z_3 to s, t, u, φ_1, φ_2, φ_3 and results like

$$I_3=-\tfrac{1}{2}i\left(\frac{\partial}{\partial\varphi_1}-\frac{\partial}{\partial\varphi_2}\right),\quad U_3=-\tfrac{1}{2}i\left(\frac{\partial}{\partial\varphi_2}-\frac{\partial}{\partial\varphi_3}\right) \tag{2.8}$$

can be used to verify that I_3, U_3 annihilate s, t, u and that (2.5a) is satisfied. Thus, the $\mathcal{O}$ of (2.2) is just the quadratic Casimir operator of this $SU(3)$ when restricted to functions of s and t.[4] (The scattering amplitude, of course, is a function of this sort.) It is easily shown from what follows that the second Casimir operator is not an independent one in the representations associated with $\mathcal{O}$.

This completes our identification of $\mathcal{G}$. We note here an identity which was vital in the preceding construction. As the cosines of the scattering angles in the three channels may be defined to be

$$\cos\theta_s=\frac{1+2t}{s-1},\quad \cos\theta_t=\frac{1+2u}{t-1},\quad \cos\theta_u=\frac{1+2s}{u-1}, \tag{2.9}$$

we have

$$(t/u)^{1/2}=\tan\tfrac{1}{2}\theta_s,\quad (u/s)^{1/2}=\tan\tfrac{1}{2}\theta_t,\quad (s/t)^{1/2}=\tan\tfrac{1}{2}\theta_u. \tag{2.10}$$

The eigenfunctions of $\mathcal{O}$ are contained in the basis vectors of a certain class of irreducible representations of $SU(3)$. The harmonic functions of $SU(3)$ have been

[3] The generators **I**, **U**, **V** are formally related to the λ_i of M. Gell-Mann [Phys. Rev. **125**, 1067 (1962)] through

$$[I_1,I_2,I_3]=\tfrac{1}{2}[\lambda_1,\lambda_2,\lambda_3],$$
$$[U_1,U_2,U_3]=\tfrac{1}{2}[\lambda_6,\lambda_7,-\tfrac{1}{2}(\lambda_3-3^{1/2}\lambda_8)],$$
$$[V_1,V_2,V_3]=\tfrac{1}{2}[\lambda_4,\lambda_5,-\tfrac{1}{2}(\lambda_3+3^{1/2}\lambda_8)].$$

The generators $B_i{}^j$ of Eq. (2.7) are related to **I**, **U**, **V** through

$$I_3=\tfrac{1}{2}(B_1{}^1-B_2{}^2),$$
$$I^+=I_1+iI_2=B_1{}^2,$$
$$I^-=I_1-iI_2=B_2{}^1,$$

where the equations for **U**, **V** are obtained by permuting the indices 1, 2, 3 of $B_i{}^j$.

We emphasize that the internal symmetry group $SU(3)$ and the $SU(3)$ of this paper act on different physical variables and should not be identified except by an isomorphism. The correspondence introduced above is purely abstract.

[4] As we have mentioned in the Introduction, the discussion in the body of the paper is largely confined to the interior of the Mandelstam triangle, where the relevant group is $SU(3)$. We shall here give an example of the modifications necessary for the physical region $[s\geq 1, 1-s\leq t\leq 0]$, where the group becomes $SU(2,1)$. The constraint equation $s-(-t)-(-u)=1$ is identified with the invariant quadratic form $|z_1|^2-|z_2|^2-|z_3|^2=1$ of $SU(2,1)$ by the relations $z_1=s^{1/2}\exp(i\varphi_1)$, $z_2=(-t)^{1/2}\exp(i\varphi_2)$, $z_3=(-u)^{1/2}\exp(i\varphi_3)$. As regards the generators **I**, **U**, **V**, **U** refer to the $SU(2)$ and **I** and **V** to the $SU(1,1)$ subgroups [the latter being locally isomorphic to the 2+1 Lorentz group $SO(2,1)$]. Some information regarding $SU(2,1)$ can be found in L. C. Biedenharn, J. Nuyts, and N. Straumann, Ann. de l'Inst. Henri Poincaré **3**, 13 (1965). See also Appendix B of Ref. 1.

constructed by Bég and Ruegg.[5] A comparison of their results with our basis[1]

$$S_{\sigma-l}^{l}(s,t)=(1-s)^{l}P_{\sigma-l}^{(2l+1,0)}(2s-1)P_{l}(\cos\theta_{s}) \quad (2.11)$$

shows that these are the central elements in the weight diagram of the representations (σ,σ). Here, the axes in the weight space are related to a well-known way to the operators[3]

$$I_3=-\tfrac{1}{2}i\left(\frac{\partial}{\partial\varphi_1}-\frac{\partial}{\partial\varphi_2}\right),\quad \lambda_8=\frac{i}{2\times 3^{1/2}}\left(\frac{\partial}{\partial\varphi_1}+\frac{\partial}{\partial\varphi_2}-2\frac{\partial}{\partial\varphi_3}\right). \quad (2.12)$$

When acting on $F(s,t)$, these operators are clearly zero. Hence the basis vectors $S_{\sigma-l}^{l}$ belong only to the center of the diagram.

It may be observed that the elements of the Weyl group on the three variables (z_1,z_2,z_3) induce permutations of s, t, u in the arguments of the amplitude F. The partial-wave crossing matrices which were evaluated in a previous paper[1] are thus the matrix elements of the Weyl operators in the representations (σ,σ).[6]

In the discussion of this section, it was necessary to introduce three arbitrary angles φ_1, φ_2, φ_3 in order to identify our operators with functions of $SU(3)$ generators. In Appendix B, we attempt an interpretation of these variables.

III. UNEQUAL-MASS PROBLEM

The operator $\mathcal{O}$ was singled out to generate eigenfunctions for the expansion of F in the equal-mass configuration because of its following features: (i) It commuted with the total angular momentum in the s, t, u channels (when restricted to functions of s and t). (ii) It was invariant under s, t, u permutations.

Now, the Casimir operators of any $SU(3)$ have to commute with the Casimir operators of its $SU(2)$ subgroups and the group elements which permute these subgroups. Therefore, to obtain a generalization of $\mathcal{O}$ when the particles are not of the same mass, one may try to fit the three little groups of the scattering process together into a suitable $SU(3)$. We discuss the simplest of these possibilities here and show that it does not work. (See also Appendix A.) The considerations are not general enough to rule out $SU(3)$ altogether.

Let (z_1,z_2,z_3) be the labels of a vector in the (3,0) representation of this $SU(3)$. Experience with the equal-mass system and, in particular, Eq. (2.10), suggests that we set

$$|z_2/z_3|^{1/2}=\tan\tfrac{1}{2}\theta_s,\quad |z_3/z_1|^{1/2}=\tan\tfrac{1}{2}\theta_t,\quad |z_1/z_2|^{1/2}=\tan\tfrac{1}{2}\theta_u, \quad (3.1)$$

where θ_s, θ_t, and θ_u are the three scattering angles. But (3.1) implies the identity

$$\tan\tfrac{1}{2}\theta_s\tan\tfrac{1}{2}\theta_t\tan\tfrac{1}{2}\theta_u=1. \quad (3.2)$$

It may be verified that this identity is fulfilled if and only if all the four particles are of equal mass.

It is also of interest to attempt to realize only two of the three little groups correctly rather than all three. (Consider, for instance, pion-nucleon scattering.) So, we may try, instead of (3.1),

$$|z_2/z_3|^{1/2}=\tan\tfrac{1}{2}\theta_s,\quad |z_3/z_1|^{1/2}=\tan\tfrac{1}{2}\theta_t. \quad (3.3)$$

This means that

$$|z_2|^{1/2}\cot\tfrac{1}{2}\theta_s=|z_1|^{1/2}\tan\tfrac{1}{2}\theta_t. \quad (3.4)$$

There are two other constraints to be considered. The partial-wave expansion of F in the s-channel, for example, is its expansion in a series of $P_l(\cos\theta_s)$ when the variable s is held fixed. [This is not the same as its expansion in terms of $P_l(\cos\theta_s)$, where $s(1+\cos\theta_s)$ and $\cos\theta_s$, say, are regarded as independent variables.] Therefore, $|z_1|$ must be a function of s alone, and $|z_2|$ one of t alone:

$$|z_1|=|z_1|(s),\quad |z_2|=|z_2|(t). \quad (3.5)$$

First set $t=0$ and then set $s=0$ in (3.4) to solve for $|z_1|$ and $|z_2|$. Then, verify that these solutions are inconsistent with (3.4) for arbitrary s and t in the unequal-mass problem.

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APPENDIX A: SOME PROPERTIES OF A GRAM DETERMINANT FOR THE UNEQUAL-MASS SYSTEM

Consider the scattering amplitude for the process $1+2\rightarrow 3+4$ which is characterized by the momenta $p^{(i)}$ $(i=1, 2, 3, 4)$. The $p^{(i)}$ satisfy the identities

$$(p^{(i)})^2=m_i^2, \quad (A1)$$

$$p^{(1)}+p^{(2)}=p^{(3)}+p^{(4)}. \quad (A2)$$

We can associate a Gram matrix g with this system as

[5] M. A. B. Bég and H. Ruegg, J. Math. Phys. **6**, 677 (1965). The d functions in their Eq. (3.24) are related to Jacobi polynomials. See, for example, A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), p. 58.

[6] See, in this connection, the discussion of Weyl reflections in $SU(3)$ by A. J. Macfarlane, E. C. G. Sudarshan, and C. Dullemond, Nuovo Cimento **30**, 845 (1963); N. Mukunda and L. K. Pandit, J. Math. Phys. **6**, 746 (1965); K. J. Lezuo, *ibid.* **8**, 1163 (1967). Further references may also be found there.

follows: Define first the three four-vectors

$$\begin{aligned} &\text{(a)}\quad S_\mu = p_\mu^{(1)} + p_\mu^{(2)} = p_\mu^{(3)} + p_\mu^{(4)}, \\ &\text{(b)}\quad T_\mu = p_\mu^{(1)} - p_\mu^{(3)} = p_\mu^{(4)} - p_\mu^{(2)}, \\ &\text{(c)}\quad U_\mu = p_\mu^{(1)} - p_\mu^{(4)} = p_\mu^{(3)} - p_\mu^{(2)}. \end{aligned} \tag{A3}$$

The Mandelstam variables are

$$s = S^2, \quad t = T^2, \quad u = U^2, \tag{A4}$$

while

$$\begin{aligned} &\text{(a)}\quad S\cdot T = \tfrac{1}{2}(m_1^2 + m_4^2 - m_2^2 - m_3^2), \\ &\text{(b)}\quad T\cdot U = \tfrac{1}{2}(m_1^2 + m_2^2 - m_3^2 - m_4^2), \\ &\text{(c)}\quad U\cdot S = \tfrac{1}{2}(m_1^2 + m_3^2 - m_2^2 - m_4^2). \end{aligned} \tag{A5}$$

The Gram matrix g is defined to be the real symmetric matrix obtained from the scalar products of S, T, U:

$$g = \begin{bmatrix} s & S\cdot T & S\cdot U \\ S\cdot T & t & T\cdot U \\ S\cdot U & T\cdot U & u \end{bmatrix}. \tag{A6}$$

When the masses are equal, $S\cdot T = T\cdot U = U\cdot S = 0$ and g is diagonal.

The matrix g has some remarkable properties of which we now list a few.

With every vector R_μ of the form $\alpha S_\mu + \beta T_\mu + \gamma U_\mu$, we can identify a row vector $\tilde{r} = (\alpha,\beta,\gamma)$ in the space of g such that

$$R_\mu R^\mu = \tilde{r} g r. \tag{A7}$$

Let N_μ be the normal to the scattering plane:

$$N_\mu = \epsilon_{\mu\nu\lambda\rho} S^\nu T^\lambda U^\rho. \tag{A8}$$

The square of the norm of N is proportional to the determinant of g:

$$N_\mu N^\mu = -\det g. \tag{A9}$$

The equation

$$\det g = 0 \tag{A10}$$

describes the boundaries of the physical region.[7] Further,

$$\operatorname{Trace} g = s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{A11}$$

Let $\bar{s}$, $\bar{t}$, $\bar{u}$ be the three eigenvalues of g, and let ξ, η, ζ be the corresponding real, orthonormal eigenvectors. As explained above, there is a natural mapping of ξ, η, ζ into the space of the four-vectors. It is given by the equation

$$\bar{S}_\mu = \xi_1 S_\mu + \xi_2 T_\mu + \xi_3 U_\mu \tag{A12}$$

and similar ones for $\bar{T}$, $\bar{U}$. We have

$$\begin{aligned} &\text{(a)}\quad \bar{S}\cdot\bar{T} = \bar{T}\cdot\bar{U} = \bar{U}\cdot\bar{S} = 0, \\ &\text{(b)}\quad \bar{S}^2 = \bar{s}, \quad \bar{T}^2 = \bar{t}, \quad \bar{U}^2 = \bar{u}. \end{aligned} \tag{A13}$$

In terms of the variables $\bar{s}$, $\bar{t}$, $\bar{u}$, the physical regions are bounded by $\det g = \bar{s}\bar{t}\bar{u} = 0$, that is, by the three straight lines

$$\bar{s} = 0, \quad \bar{t} = 0, \quad \bar{u} = 0. \tag{A14}$$

If the masses are equal, then $\bar{s} = s$, $\bar{t} = t$, $\bar{u} = u$.

The sign ambiguity of ξ, η, ζ can be resolved where desired by requiring that $\bar{S}$, $\bar{T}$, $\bar{U}$ reduce to S, T, U when the masses are the same.

The change of variables $s, t, u \to \bar{s}, \bar{t}, \bar{u}$ involves square and cube roots and is not trivial. But it has the beauty of formally mapping the unequal-mass scattering regions into the corresponding equal-mass ones. As such, it suggests the following solution to the unequal-mass problem: replace the variables s, t, u in the equal-mass results by $\bar{s}$, $\bar{t}$, $\bar{u}$. We shall not discuss here the physical implications of such an expansion and its relation to the analyticity properties of the scattering amplitude.

APPENDIX B: INTERPRETATION OF ANGLES φ_1, φ_2, φ_3

If we are to find a physical interpretation of the group generators and not just of operators like $\mathbf{I}^2$, $\mathbf{U}^2$, $\mathbf{V}^2$, it is necessary to specify the action of the group on the four-momenta of the system. In particular, the dependences of the momenta on the angles φ_i, which were introduced in the main part of the text, have to be identified. In the familiar theory of angular momentum analysis where only one of the three channels is relevant at a time, φ_2-φ_3, for instance, can be thought of as the azimuthal angle of the spatial momentum in the s-channel final state in a special sort of orthogonal coordinate system. This system has one of its axes in the direction of the incident momentum. The polar angle of the final momentum is defined by this exceptional axis and coincides with the scattering angle θ_s. We refer always to the center-of-mass of the process. Such a choice of coordinates is not unique. In our problem, the three channels are simultaneously involved and a consistent interpretation of the φ's along these lines[1] has proved to be difficult. We therefore present two alternative explanations of these variables in what follows.

(1) We shall continue to work within the Mandelstam triangle. The spatial parts of the four-momenta are pure imaginary here. It is understood until (β) below that an i has been factored out of the space parts of the momenta and hence that all four-vectors are Euclidean. The requisite modifications for the physical region and Minkowski metric will be occasionally indicated within parentheses.

We use the notation of Appendix A with the additional proviso that the masses are equal. The sixteen variables $p_\mu^{(i)}$ are constrained by the eight equations (A1), (A2). In the absence of spin, the scattering manifold is thus characterized by eight independent variables. The first suggestion that comes to mind is to imbed φ_1, φ_2, φ_3 in this manifold in a suitable manner. We shall now characterize the latter in terms of familiar geometrical entities.

[7] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

If S, T, U are introduced as in (A3), then (A2) is fulfilled without further demands since

$$\begin{aligned}&\text{(a)}\ p^{(1)}=\tfrac{1}{2}(S+T+U),\\&\text{(b)}\ p^{(2)}=\tfrac{1}{2}(S-T-U),\\&\text{(c)}\ p^{(3)}=\tfrac{1}{2}(S+U-T),\\&\text{(d)}\ p^{(4)}=\tfrac{1}{2}(S+T-U),\end{aligned}\qquad\text{(B1)}$$

while (A1) assumes the form

$$\begin{aligned}&\text{(a)}\ s+t+u=4m^2,\\&\text{(b)}\ S\cdot T=T\cdot U=U\cdot S=0.\end{aligned}\qquad\text{(B2)}$$

In the body of the paper, we set $m^2=\frac{1}{4}$.

Let us next define the normalized vectors $\hat{S}$, $\hat{T}$, $\hat{U}$ through

$$\begin{aligned}&\text{(a)}\ \hat{S}=S/s^{1/2},\\&\text{(b)}\ \hat{T}=T/t^{1/2},\\&\text{(c)}\ \hat{U}=U/u^{1/2}.\end{aligned}\qquad\text{(B3)}$$

The vectors $\hat{S}$, $\hat{T}$, $\hat{U}$ and the unit vector

$$\hat{N}_\mu=\epsilon_{\mu\nu\lambda\rho}\hat{S}_\nu\hat{T}_\lambda\hat{U}_\rho,\quad \hat{N}_\mu\hat{N}_\mu=+1\qquad\text{(B4)}$$

normal to the plane of the momenta $p^{(i)}$ form an orthonormal tetrad in the four-dimensional Euclidean space. [For the physical region, $\hat{S}=S/s^{1/2}$, $\hat{T}=T/(-t)^{1/2}$, $\hat{U}=U/(-u)^{1/2}$, $\hat{N}_\mu=\epsilon_{\mu\nu\lambda\rho}\hat{S}^\nu\hat{T}^\lambda\hat{U}^\rho$ form an orthonormal tetrad in Minkowski space with $\hat{N}_\mu\hat{N}^\mu=-1$.] Such a tetrad depends on six parameters. Let $\hat{S}_0$, $\hat{T}_0$, $\hat{U}_0$, $\hat{N}_{0\mu}=\epsilon_{\mu\nu\lambda\rho}\hat{S}_{0\nu}\hat{T}_{0\lambda}\hat{U}_{0\rho}$ be a given fixed set of orthonormal vectors (the reference coordinate system). Then, there is a unique element g of $SO(4)$ which maps $\hat{S}_0$, $\hat{T}_0$, $\hat{U}_0$, $\hat{N}_0$ onto $\hat{S}$, $\hat{T}$, $\hat{U}$, $\hat{N}$:

$$g\hat{S}_0=\hat{S},\quad g\hat{T}_0=\hat{T},\quad g\hat{U}_0=\hat{U},\quad g\hat{N}_0=\hat{N}.\qquad\text{(B5)}$$

Conversely, every $g\in SO(4)$ defines a unique orthonormal set $\hat{S}$, $\hat{T}$, $\hat{U}$, $\hat{N}$ through (B5). Thus the set $(\hat{S},\hat{T},\hat{U},\hat{N})$ is in one-to-one correspondence with the group manifold of $SO(4)$. (Similarly, in the physical region, the set $(\hat{S},\hat{T},\hat{U},\hat{N})$ is in one-to-one correspondence with the group manifold of $SO(3,1)$.)

We have proved that *the scattering manifold within the Mandelstam triangle (in the physical region) can be identified with the product of the surface* $s+t+u=4m^2$ *and the group manifold of* $SO(4)$ [$SO(3,1)$].

The remaining problem is to imbed the topological product of the circles labelled by the angles φ_i (a torus) in the $SO(4)$ manifold. There are clearly an infinite number of distinct ways of realizing such a map. None of them, however, seems very satisfactory.

(2) We shall finally mention a rather amusing though far-fetched possibility.

A particle in a relativistic theory is abstractly associated with a basis of a representation of the Poincaré group. In a scattering problem with four particles, there is a separate Poincaré group for each separate particle. Let the corresponding Poincaré generators be labelled as $P_\mu^{(i)}$, $M_{\mu\nu}^{(i)}$ ($i=1, 2, 3, 4$). Let f be a function of the four-momenta $p_\mu^{(i)}$. If the particles are spinless, the action of the generators on f can be taken to be

$$\begin{aligned}&\text{(a)}\quad P_\mu^{(i)}f=p_\mu^{(i)}f,\\&\text{(b)}\ M_{\mu\nu}^{(i)}f=i\left[p_\mu^{(i)}\frac{\partial}{\partial p^{\nu(i)}}-p_\nu^{(i)}\frac{\partial}{\partial p^{\mu(i)}}\right]f.\end{aligned}\qquad\text{(B6)}$$

If f is the scattering amplitude F, then

$$\begin{aligned}&\text{(a)}\quad P_\mu F=0,\\&\text{(b)}\ M_{\mu\nu}F=0,\end{aligned}\qquad\text{(B7)}$$

where

$$\begin{aligned}&\text{(a)}\quad P_\mu=P_\mu^{(1)}+P_\mu^{(2)}-P_\mu^{(3)}-P_\mu^{(4)},\\&\text{(b)}\ M_{\mu\nu}=\sum_i M_{\mu\nu}^{(i)}.\end{aligned}\qquad\text{(B8)}$$

As a consequence, F is a function of s and t alone.

The operators $P_\mu^{(3)}+P_\mu^{(4)}$, $M_{\mu\nu}^{(3)}+M_{\mu\nu}^{(4)}$ generate an algebra isomorphic to that of the Poincaré group. This algebra has two Casimir invariants. The first is the operator $(P^{(3)}+P^{(4)})^2$ with eigenvalue $(p^{(3)}+p^{(4)})^2$, and the second is the square of the s-channel Pauli-Lubanski operator[8]

$$X_\mu'=-\{1/2[(P^{(3)}+P^{(4)})^2]^{1/2}\}\times\epsilon_{\mu\nu\lambda\rho}(P^{(3)}+P^{(4)})^\nu(M^{(3)}+M^{(4)})^{\lambda\rho}\qquad\text{(B9)}$$

which acts only on particles 3 and 4. When restricted to a scattering amplitude, X_μ' is invariant under $3\to1$, $4\to2$. Let Y_μ', Z_μ' be defined analogously in the t and u channels. It is immaterial for us here whether they act on the initial or the final particles in the channels. Finally, define

$$\begin{aligned}&\text{(a)}\ X'^2=X_\mu'X'^\mu,\quad Y'^2=Y_\mu'Y'^\mu,\quad Z'^2=Z_\mu'Z'^\mu,\\&\text{(b)}\quad \mathcal{O}'=X'^2+Y'^2+Z'^2.\end{aligned}\qquad\text{(B10)}$$

It is easily checked that:

(i) When restricted to functions of s and t alone [that is, functions which satisfy (B7)], the primed operators in (B10) reduce to the corresponding unprimed operators in the text,

(ii) Their commutators with P_μ, $M_{\mu\nu}$ vanish identically and without any such restriction.

It follows from (ii) that the vector

$$\xi_\mu=p_\mu^{(1)}+p_\mu^{(2)}-p_\mu^{(3)}-p_\mu^{(4)}\qquad\text{(B11)}$$

[the eigenvalue of the operator P_μ of Eq. (B.8a)] can be regarded as a *fixed* nonzero vector in the problem

[8] See, for example, J. Strathdee, J. F. Boyce, R. Delbourgo, and Abdus Salam, ICTP, Trieste, Report No. IC/67/9, 1967 (unpublished).

(fixed, that is, under the little group operations) which is allowed to vanish at the end of the calculations. The Mandelstam invariants in such a case are six in number and are given by

$$s=(p^{(1)}+p^{(2)})^2, \quad s'=(p^{(3)}+p^{(4)})^2 \tag{B12}$$

and similar equations for t, t', u, u'. They are subject to the one constraint

$$s+s'+t+t'+u+u'=\xi^2+8m^2. \tag{B13}$$

The angles φ_i can now be imbedded in this system if we set

$$z_1=2^{-1/2}(s^{1/2}+is'^{1/2})=2^{-1/2}(s+s')^{1/2}e^{i\varphi_1} \tag{B14}$$

and likewise for z_2, z_3. The group $SU(3)$ is allowed to act on (z_1,z_2,z_3) in the standard way [cf. Eq. (2.4) and Ref. 5]. When ξ_μ goes to zero, ξ^2 goes to zero, s', t', u' go, respectively, to s, t, u, and (B13) reduces to the familiar constraint (B.2a).

The situation can be expressed in a picturesque form by saying that our group $SU(3)$ is the remaining ghost group of the five-particle problem when one of the particles collpases to the vacuum.

We conclude with a minor explanation. The three differential operators in (B.10a) are *a priori* defined in their corresponding physical regions and the latter are mutually disjoint. In writing (B.10b) (and in many other remarks in this paper), the analytic continuation of these operators to suitable domains is understood.

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Scalar Density Terms, $K\pi$ and KK Scattering Lengths, and a Symmetry-Breaking Parameter*

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An estimation of s-wave $K\pi$ and KK scattering lengths is made, and a discussion is given of the relation of these quantities to scalar density matrix elements and the relevant symmetry-breaking parameter.

INTRODUCTION

USING low-energy theorems from the $SU(2)\times SU(2)$ current algebra and Adler's partial conservation of axial-vector current (PCAC) self-consistency conditions, Weinberg[1] has estimated the π-π scattering lengths, assuming that a linear expansion of the amplitude into Mandelstam invariants is approximately valid up to threshold. In order to determine some of the coefficients in the above expansion, the isospin properties had to be specified of the scalar density matrix element, or "σ term" $\langle\pi|[A^i,\partial^\mu A_\mu{}^j]|\pi\rangle$, which involves the commutator of an axial-vector charge and divergence of the axial-vector current (i,j are isospin indices or, more generally, unitary spin indices). It turns out that, within the parametrization of Ref. 1, specification alone of the pion vector-charge matrix element and the requirement of no $I=2$ contributions in $[A^i,\partial^\mu A_\mu{}^j]$ completely determine[2] the value of this matrix element:

$$i\langle\pi^l|[A^i,\partial^\mu A_\mu{}^j]|\pi^k\rangle=(m_\pi{}^2)\delta_{ij}\delta_{kl}. \tag{1}$$

On the other hand, Khuri[3] arrived at the same value by applying a low-energy limit directly to the scalar density vertex defined by (1) upon using [the $SU(2)\times SU(2)$ version of] the algebra of scalar and pseudoscalar densities[4] which transform as $(3,3^*)+(3^*,3)$ under $SU(3)\times SU(3)$.

In this paper, an estimation of s-wave $K\pi$ and KK scattering lengths will be made using similar current-algebra techniques, and the relation of these quantities to scalar density terms will be noted. In particular, the scalar density matrix elements appearing in the KK case are estimated by established $SU(3)\times SU(3)$ commutators and PCAC principles, just as in Weinberg's $\pi\pi$ estimation described above. For the $K\pi$ case, additional assumptions are needed and the scalar density matrix elements are determined by: (a) applying a low-energy limit to the scalar density vertex, and (b) using an (approximate) determining relation for the relevant symmetry-breaking parameter κ. The scattering-length determination for the KK case is consistent with this last procedure. Finally, the consistency of our use of the relation for κ itself, Eq. (15), is discussed.

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[1] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).
[2] Reference 1; Eqs. (16) and (20) determine the coefficient of $\delta_{ab}\delta_{cd}$. In the text, we use the state normalization $\langle p|p\rangle=(2\pi)^3 2p_0\delta(0)$.
[3] N. Khuri, Phys. Rev. **153**, 1477 (1967).
[4] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).