PHYSICAL REVIEW VOLUME 176, NUMBER 5 25 DECEMBER 1968

# Theories at Infinite Momentum*†

K. BARDAKCI‡ AND M. B. HALPERN

*Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California 94720*

(Received 25 July 1968)

We construct Galilean-invariant theories (with Schrödinger equations) at infinite momentum that describe interacting relativistic systems. Classes of both first- and second-quantized theories are presented. The formalism provides a general approach to the saturation of current algebra; positivity of the mass spectrum is guaranteed, and as much inelasticity as necessary may be introduced. More generally, however, such theories offer the hope of potential-theoretic intuition for relativistic physics.

## I. INTRODUCTION

THE infinite-momentum limit first found use in the derivation of covariant sum rules from current algebra.[1] It was stressed from the beginning that the applicability of the limit was tantamount to there being no subtractions in the covariant dispersion relations of the invariant amplitudes involved. Although the covariant and infinite-momentum approaches to sum rules are equivalent under this assumption, the infinite-momentum technique carried with it certain notational intrigues—the dependence of matrix elements on longitudinal momenta is washed out in the limit, leaving structures reminiscent of a two-dimensional nonrelativistic quantum mechanics (in the transverse variables). Indeed, this intuition played a central role in the original Dashen–Gell-Mann scheme[2] for the saturation of current algebra.

Somewhat later, Weinberg[3] showed that the "old-fashioned" perturbation expansions of some simple field theories have an infinite-momentum limit, the topological structure of which is nonrelativistic (e.g., nonrelativistic propagators, simplified vacuum structure, etc.). Frye and Susskind[4] pushed the analogy further, using the infinite-momentum frame to focus attention on the (two-dimensional) Galilean subgroup of the Poincaré group.

These suggestions led us to inquire just how far the nonrelativistic analogy can be pushed. In particular, can one write Schrödinger (Galilean-invariant) theories at infinite momentum that completely describe interacting relativistic systems? The advantages of such theories would be numerous: (a) They would be free of certain subtractions, thus perhaps softening the divergence problems of ordinary theories. (b) If representations of current algebra could be constructed, the sum rules would automatically be satisfied. This is true only in a few ordinary theories. (c) Because of their Schrödinger formulation, and, in particular, because of their simplified vacuum structure, such theories could offer potential-theoretic intuition for relativistic physics.

Before sketching our results, we should state what we mean by a theory (at infinite momentum). We shall demand Poincaré invariance, unitarity, and positivity of the mass spectrum, but we shall be more relaxed about locality (crossing symmetry, spin statistics, etc.). We shall feed-in some locality through the requirement of local current algebra and/or Lorentz-invariant $S$ matrix, and in our second-quantized representations we will work with (Schrödinger) fields local in the transverse plane. As a result, we shall find that antiparticles and/or spin statistics are not required, although they may be included if desired. It may be that the universe is no more local than this, but we have no real objection to the reader viewing any of these theories as approximate. Indeed, perhaps our primary objective in writing such theories is to lay the foundations for some approximate (potential-theoretic) models.

The plan of the paper is as follows. In Sec. II, we review the infinite-momentum limit, and emphasize that it can always be viewed either as an integration over the light cone, or a change of variables to a set natural to the infinite-momentum frame, or both. The (free-particle) results of Susskind for the infinite-momentum limit of the Galilean subgroup of the Poincaré group are taken, in this sense, as a starting point. Then we complete the Poincaré algebra in terms of the Galilean variables for $n$ free particles. The Hamiltonians for such systems have the usual nonrelativistic form. The details of the change of variable (necessary to obtain these representations from the usual ones) are given in the Appendix. Another way of stating the results of Sec. II is that the Poincaré group can always be represented in the space of solutions of a free (two-dimensional) Schrödinger equation.

In Sec. III, we give a construction for introducing potentials into the Hamiltonian (interactions between

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* This work was supported in part by the U. S. Atomic Energy Commission.

† This is an extended version of lectures given at the Summer Institute for Theoretical Physics, Boulder, Colorado (1968).

‡ A. P. Sloan Foundation Fellow.

[1] S. Fubini and G. Furlan, Physics **1**, 229 (1965); S. Adler, Phys. Rev. Letters **14**, 1051 (1965); W. Weisberger, *ibid.* **14**, 1047 (1965); R. Dashen and M. Gell-Mann, Phys. Letters **17**, 145 (1965). For a comprehensive discussion of sum rules and infinite momentum, see S. Adler and R. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968).

[2] R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966). Some representative (later) attempts to represent current algebra at infinite momentum are M. Gell-Mann, D. Horn, and J. Weyers, Institute for Advanced Study report (unpublished); K. Bardakci, M. B. Halpern, and G. Segrè, Phys. Rev. **168**, 1728 (1968); H. Leutwyler, Phys. Rev. Letters **20**, 561 (1968).

[3] S. Weinberg, Phys. Rev. **150**, 1313 (1966).

[4] L. Susskind, Phys. Rev. **165**, 1535, (1968); **165**, 1547 (1968); G. Frye and L. Susskind, *ibid.* **165**, 1553 (1968).

the particles), while keeping the nonrelativistic analogy. Among other constraints, it turns out that the potential must be Galilean-invariant in just the usual way. In group-theoretic terms, the construction finds large classes of interacting (two-dimensional) Schrödinger equations whose solutions provide representation spaces for the Poincaré group. At the end of Sec. III, we discuss positivity of the mass spectrum. The necessary and sufficient condition for this is that the Hamiltonian be a self-adjoint operator, bounded from below, just the condition that the potential theory itself be well defined.

Section IV is devoted to second-quantized representations. A free representation in terms of Schrödinger second-quantized fields is given, along with an action principle at infinite momentum to determine the interactions. From the action principle, we recover the usual theories (i.e., the modified Feynman graphs of Weinberg) plus some others. These others are nonlocal in the usual sense, in that they need not have crossing, antiparticles, or the correct connection between spin and statistics.

In Sec. V, we consider the construction of currents and the saturation of current algebra. By Noether's theorem, it turns out that a form for the "good current" is always the probability density of the Schrödinger equation, while the (transverse) spatial currents may be taken as the (transverse) probability flux. These currents automatically satisfy current algebra, but they are not guaranteed to transform like four-vectors at infinite momentum, i.e., to satisfy the "angular condition."[2] In this approach, since the mass spectrum must be positive and the currents do satisfy the algebra, then the angular condition is the crux of the problem. By solving the Galilean part of the Poincaré group explicitly, we give the angular condition as a relatively simple operator condition. Free solutions and solutions for the ordinary field theories are given, but no attempt is made in this paper to see if the angular condition is satisfied for the more interesting cases. It may be that the angular condition is not satisfied for any of these (Noether) currents, in which case one might want to begin thinking about nonlocal currents: Solve the angular condition and take whatever (in general nonlocal) currents result.

In Sec. VI we discuss our results and collect some miscellaneous comments, particularly about the introduction of fermionic representations.

## II. INFINITE MOMENTUM AND FREE PARTICLES

We begin by stating our convention for the Poincaré algebra:

$$[P_\mu,P_\nu]=0,\quad [M_{\mu\nu},P_\rho]=i(g_{\nu\rho}P_\mu-g_{\mu\rho}P_\nu),$$
$$[M_{\mu\nu},M_{\rho\kappa}] =i[g_{\nu\rho}M_{\mu\kappa}+g_{\mu\kappa}M_{\nu\rho}-g_{\nu\kappa}M_{\mu\rho}-g_{\mu\rho}M_{\nu\kappa}], \tag{II.1}$$

where $g_{oo}=-g_{ii}=1$ ($i=1$, 2, 3). The rotations and boosts are, respectively, $\epsilon_{ijk}J_k\equiv M_{ij}$, $K_i\equiv M_{io}$; thus

$$[J_i,J_j]=-[K_i,K_j]=i\epsilon_{ijk}J_k,\quad [J_i,K_j]=i\epsilon_{ijk}K_k. \tag{II.2}$$

For historical reasons, we introduce infinite momentum with currents. Let $V_\mu{}^\alpha(x)$ be a local spin-one current ($\mu=0$, 1, 2, 3), with internal symmetry index $\alpha$. Its commutation relations with the Poincaré group are

$$[M_{\mu\nu},V_\rho{}^\alpha(x)]=-i(x_\mu\partial_\nu-x_\nu\partial_\mu)V_\rho{}^\alpha +i(g_{\nu\rho}V_\mu{}^\alpha-g_{\mu\rho}V_\nu{}^\alpha),$$
$$i[P_\mu,V_\rho{}^\alpha]=\partial_\mu V_\rho{}^\alpha, \tag{II.3}$$

where we take $x^\mu=(t,\mathbf{x})$. The objects of particular interest in current algebra are the time components of the current $V_0{}^\alpha(x)$. To obtain these components at infinite momentum, we construct

$$\rho_\alpha(\mathbf{x}_\perp)\equiv\lim_{\lambda\to\infty} e^{i\lambda K_3}\int dz\; V_0{}^\alpha(\mathbf{x},0)e^{-i\lambda K_3}, \tag{II.4}$$

where $\mathbf{x}_\perp$ is just the transverse part of $\mathbf{x}$. Bardakci and Segrè[5] showed that the limit can be taken explicitly,[6] yielding

$$\rho_\alpha(\mathbf{x}_\perp)=\int dz[V_0{}^\alpha(\mathbf{x}_\perp,z,z)+V_3{}^\alpha(\mathbf{x}_\perp,z,z)]$$
$$=\int dzdt\;\delta(t-z)[V_0{}^\alpha(x)+V_3{}^\alpha(x)]. \tag{II.5}$$

These are the so-called "good currents" at infinite momentum, usually taken to satisfy a two-dimensional algebra of the form

$$[\rho_\alpha(\mathbf{x}_\perp),\rho_\beta(\mathbf{x}_\perp')]=iC_{\alpha\beta\gamma}\rho_\gamma(\mathbf{x}_\perp)\delta^{(2)}(\mathbf{x}_\perp-\mathbf{x}_\perp'). \tag{II.6}$$

A lesson to bear in mind is that the infinite-momentum limit can be achieved simply by doing this integral over the light cone. As constructed, the good currents commute with the "lightlike" subgroup of the Poincaré group

$$\mathfrak{L}=\{K_3,\;J_1+K_2,\;K_1-J_2,\;P_0+P_3\}. \tag{II.7}$$

These are the generators that leave the direction of the lightlike vector $\eta^\mu=(1,0,0,1)$ invariant, so they are in this sense singled out at infinite momentum. Because the lightlike group commutes with the good currents, the simplest representation of the current matrix elements involves boosting the states with $J_1+K_2$, $K_1-J_2$, and $K_3$.[5] We shall have use for this kind of boost in Sec. III, and will return to the subject of currents in Sec. V.

What about the infinite-momentum limit of the generators of the Poincaré group themselves? The limit is defined as in Eq. (II.4), but this in general leads to in-

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[5] K. Bardakci and G. Segrè, Phys. Rev. **159**, 1263 (1967).

[6] $V_0$ goes infinite in the limit, but so does $z$. A rescaling of $z$ (change of variable) removes the whole problem.

finities in the limit. For free-particle representations, and in particular for the generators of the (two-dimensional) Galilean subgroup of the Poincaré group, namely,

$$\mathcal{G}=\{P_0+P_3,\ P_0-P_3,\ P_i(i=1,\,2),\ J_1+K_2,\ K_1-J_2,\ J_3\}\,, \quad \text{(II.8)}$$

Susskind[4] showed that these infinities can be scaled and/or subtracted out consistently. For the case of one free particle of mass $m$, that is,

$$M_{\mu\nu}=x_\mu P_\nu-x_\nu P_\mu,\quad [x_\mu,P_\nu]=-ig_{\mu\nu}\,,$$
$$P^\mu=i\frac{\partial}{\partial x_\mu},\quad P^2=m^2\,, \quad \text{(II.9)}$$

Susskind's results are in the limit,

$$\begin{gathered}P_0+P_3=\eta\,,\\ H=P_0-P_3=-\frac{\nabla^2}{\eta}+\frac{m^2}{\eta}=\frac{P_\perp{}^2}{\eta}+\frac{m^2}{\eta}\,,\\ P_i=-i\partial_i\ (i=1,\,2)\,,\\ J_1+K_2=\eta x_2,\quad K_1-J_2=\eta x_1\,,\\ J_3=-i(x_1\partial_2-x_2\partial_1),\quad K_3=\tfrac{1}{2}i[\eta,\partial/\partial\eta]_+\,,\end{gathered} \quad \text{(II.10)}$$

where $\nabla^2=\partial_1{}^2+\partial_2{}^2$, $P_\perp{}^2=P_1{}^2+P_2{}^2$, and all coordinates and derivatives refer to $\mathbf{x}$. In Susskind's derivation, $\eta$ (called $\alpha$) has a definite meaning (being essentially Weinberg's $\eta$), but we shall take this representation as a starting point, considering it as an evident change of variables from the usual set to those which are natural in the infinite-momentum frame. The details of the change of variable (to obtain this representation from the usual one) are given in the Appendix.[7] The representation emphasizes the fact that (up to factors of two which can be fixed if desired) $P_0+P_3$ is (analogous to) the nonrelativistic mass, $H=P_0-P_3$ is the Hamiltonian,[8] $P_\perp=(P_1,P_2)$ are the (transverse) translation operators, $J_1+K_2$ and $K_1-J_2$ are the nonrelativistic (transverse) boosts, $J_3$ generates rotations in the transverse plane, and $K_3$ is the mass scaling operator.

If we are to fully describe the scalar particle, we must complete the representation of the Poincaré group in terms of these nonrelativistic variables. This is accomplished by

$$\begin{gathered}J_1=-\frac{\partial}{\partial\eta}\partial_2+\tfrac{1}{2}[x_2,P_3]_+\,,\\ J_2=\frac{\partial}{\partial\eta}\partial_1-\tfrac{1}{2}[x_1,P_3]_+\,,\end{gathered} \quad \text{(II.11)}$$

where $P_3$ (and $P_0$) is already known from (II.10)

$$P_3=\tfrac{1}{2}\left(\eta-\frac{-\nabla^2+m^2}{\eta}\right),\quad P_0=\tfrac{1}{2}\left(\eta+\frac{-\nabla^2+m^2}{\eta}\right). \quad \text{(II.12)}$$

For the case of $n$ free particles, we need only add the single-particle representation $n$ times. Of special interest in what follows is the representation for two free particles, and, in particular, its form in "center-of-mass" variables. We define these variables in accord with our Schrödinger analogy:

$$\begin{gathered}\mathbf{P}=\mathbf{P}^{(1)}+\mathbf{P}^{(2)},\quad \mathbf{Z}=\frac{1}{M}(\eta_1\mathbf{x}^{(1)}+\eta_2\mathbf{x}^{(2)})\,,\\ \boldsymbol{\pi}=\frac{1}{M}(\eta_2\mathbf{P}^{(1)}-\eta_1\mathbf{P}^{(2)}),\quad \boldsymbol{\omega}=\mathbf{x}^{(1)}-\mathbf{x}^{(2)}\,,\\ M=\eta_1+\eta_2,\quad \mu=\frac{\eta_1\eta_2}{\eta_1+\eta_2},\quad N=\frac{\partial}{\partial\eta_2}-\frac{\partial}{\partial\eta_1}\,,\end{gathered} \quad \text{(II.13)}$$

where $\mathbf{Z}$, $\mathbf{P}$, $\boldsymbol{\pi}$, and $\boldsymbol{\omega}$ are two-dimensional ($i=1$, 2) vectors with the properties

$$\begin{gathered}[Z_i,P_j]=i\delta_{ij},\quad [\omega_i,\pi_j]=i\delta_{ij}\,,\\ [\mathbf{Z},\boldsymbol{\omega}]=[\mathbf{P},\boldsymbol{\omega}]=[\mathbf{Z},\boldsymbol{\pi}]=[\mathbf{P},\boldsymbol{\pi}]=0\,.\end{gathered} \quad \text{(II.14)}$$

The result for the Poincaré group is, after some algebra,

$$\begin{gathered}P_0+P_3=M\,,\\ H=P_0-P_3=(-\nabla_Z{}^2/M)-(\nabla_\omega{}^2/\mu)+(m^2/\mu)\,,\\ J_1+K_2=MZ_2,\quad K_1-J_2=MZ_1\,,\\ K_3=\tfrac{1}{2}i[M,\partial/\partial M]_++\tfrac{1}{2}i[\mu,\partial/\partial\mu]_+\,,\\ J_3=Z_1P_2-Z_2P_1+\omega_1\pi_2-\omega_2\pi_1\,,\\ J_1=-\tfrac{1}{2}[1/M,K_3]_+P_2+\tfrac{1}{2}[Z_2,P_3]_++\tfrac{1}{2}i[N,\pi_2]_+\\ -\tfrac{1}{4}[\omega_2,(2/M)\boldsymbol{\pi}\cdot\mathbf{P}+(\eta_2-\eta_1/\eta_1\eta_2)(\pi^2+m^2)]_+\,,\\ J_2=\tfrac{1}{2}[1/M,K_3]_+P_1-\tfrac{1}{2}[Z_1,P_3]_+-\tfrac{1}{2}i[N,\pi_1]_+\\ +\tfrac{1}{4}[\omega_1,(2/M)\boldsymbol{\pi}\cdot\mathbf{P}+(\eta_2-\eta_1/\eta_1\eta_2)(\pi^2+m^2)]_+\,.\end{gathered} \quad \text{(II.15)}$$

The next question is how to put an interaction into the system. We set ourselves the task of doing this while keeping the nonrelativistic analogy, namely $P_0+P_3$, $P_\perp$, $J_1+K_2$, $K_1-J_2$, $J_3$, $K_3$ should keep the above forms, while $H$, the Hamiltonian, changes to

$$H=\frac{P_\perp{}^2}{M}+\frac{\pi^2}{\mu}+V\,. \quad \text{(II.16)}$$

The problem then is to find the restrictions on $V$ and the forms of $J_1$, $J_2$ such that we still have the Poincaré group. Instead of trying to guess these things, we introduce in Sec. III a fairly general method of construction that does everything automatically.

[7] Because the infinite-momentum limit is equivalent to this change of variable, we can essentially forget the motivation of the representation in terms of infinite momentum. In this sense, one might choose to describe our work in this paper simply as finding Schrödinger-like representations of the Poincaré group.

[8] The term $m^2/\eta$ is an almost trivial constant potential. See Susskind, Ref. 4.

Notice that we do not explicitly include the term $m^2/\mu$ in the potential, as we expect to find the mass spectrum directly by diagonalizing the Hamiltonian. The four-momentum squared has the form

$$P^2=M((\pi^2/\mu)+V)\equiv MH_{\rm int}\,, \tag{II.17}$$

so the eigenvalues of $MH_{\rm int}$ give the mass spectrum, and it need not resemble the "bare" spectrum. We shall leave the discussion of positivity of the mass spectrum ($P^2>0$) until the end of Sec. III.

## III. CONSTRUCTION OF INTERACTING SYSTEMS

By commuting $H$ (including V) with the generators of the Galilean subgroup (and demanding the Poincaré group), one learns immediately that[9]

$$\begin{aligned}[][J_1+K_2,\ V]=[K_1-J_2,\ V]&\\ =[P_0+P_3,\ V]=[J_3,V]&=0\,,\\ [K_3,V]&=-iV\,.\end{aligned} \tag{III.1}$$

Thus the potential must be Galilean-invariant (just as in two-dimensional quantum mechanics), and have scale $(-1)$ with respect to $K_3$. To get the rest of the conditions on $V$, and the form of $J_1$ and $J_2$, we introduce a construction due to Wigner[10] which reduces the problem to internal variables. Although the contruction works with states, it will yield operator representations in the end.

Consider the states of the system at rest, say $|\mathbf{0}\rangle$, where

$$P_i|\mathbf{0}\rangle=0\quad(i=1,\,2,\,3)\,. \tag{III.2}$$

We can boost these states to states with finite $P_i$ $\times(i=1,\,2,\,3)$ in the following way[5]:

$$\begin{aligned}|\mathbf{P}'\rangle=U(\mathbf{P}')|\mathbf{0}\rangle,\quad P_i|\mathbf{P}'\rangle=P_i'|\mathbf{P}'\rangle,&\\ U(\mathbf{P}')=\exp[i\alpha_1(K_1-J_2)+i\alpha_2(J_1+K_2)+i\alpha_3K_3]\,,&\\ \alpha_1=P_1'/P_0'+P_3',\quad \alpha_2=P_2'/P_0'+P_3'\,,&\\ \alpha_3=\ln(P_0'+P_3'/(P^2)^{1/2})\,,&\end{aligned} \tag{III.3}$$

where the primed quantities denote ($c$-number) eigenvalues, and $P^2=(P_0')^2-P_i'P_i'$ is the invariant four-momentum squared. Now consider the action of $\mathbf{J}$ on the boosted state

$$\begin{aligned}\mathbf{J}|\mathbf{P}'\rangle&=U(\mathbf{P}')\hat{\mathbf{J}}|\mathbf{0}\rangle\,,\\ \hat{\mathbf{J}}&\equiv U^\dagger(\mathbf{P}')\mathbf{J}U(\mathbf{P}')\,.\end{aligned} \tag{III.4}$$

One can easily calculate $\hat{\mathbf{J}}$ from the commutation relations of the Poincaré group. For example

$$\begin{aligned}\hat{J}_1=(1/\beta_3)J_1+\alpha_1J_3-\alpha_2K_3&\\ +(1/2\beta_3)[(\alpha_2^2-\alpha_1^2+1)\beta_3^2-1]&\\ \times(J_1+K_2)+\alpha_1\alpha_2\beta_3(K_1-J_2)\,,&\\ \beta_3=\exp(\alpha_3)=P_0'+P_3'/(P^2)^{1/2}.&\end{aligned} \tag{III.5}$$

[9] These conditions are equivalent to guaranteeing that $P^2$ is Galilean-invariant and commutes with $K_3$.

[10] E. Wigner, Ann. Math. 40, 1 (1939).

In Eq. (III.4), $\hat{\mathbf{J}}$ operates directly on $|\mathbf{0}\rangle$. We denote the angular momentum operators as they operate on states at rest by $\mathbf{j}$, i.e.,

$$\mathbf{J}|\mathbf{0}\rangle=\mathbf{j}|\mathbf{0}\rangle; \tag{III.6}$$

$\mathbf{j}$ is the "internal" angular momentum. Of course, we shall in general take

$$j_3=\omega_1\pi_2-\omega_2\pi_1\,, \tag{III.7}$$

while $j_1$ and $j_2$ are as yet unknown.

Our task basically is now to guess forms for $\mathbf{j}$ and boost appropriately. Wigner guarantees[11] us that the following procedure leads to a $\mathbf{J}$ which satisfies the Poincaré algebra: (1) Construct the rest of $\mathbf{j}$ such that it is Galilean-invariant and satisfies the algebra of $SU(2)$ on states at rest[12]:

$$[j_i,j_j]|\mathbf{0}\rangle=i\epsilon_{ijk}j_k|\mathbf{0}\rangle\,. \tag{III.8}$$

(2) Construct $P^2=MH_{\rm int}$ out of the rotational scalars of this group[13]

$$[\mathbf{j},P^2]|\mathbf{0}\rangle=0\,. \tag{III.9}$$

Physically, this requires that a rotation does not change the energy of a state at rest. (3) Invert the above machinery—i.e., from $\hat{\mathbf{J}}$, now as a function of $\mathbf{j}$, calculate

$$\mathbf{J}(\mathbf{P}')\equiv U(\mathbf{P}')\hat{\mathbf{J}}(\mathbf{j})U^\dagger(\mathbf{P}')\,. \tag{III.10}$$

Put all the factors $\mathbf{P}'$ in this expression to the right, and replace them by operator $\mathbf{P}$. The resulting $\mathbf{J}$ will satisfy the Poincaré algebra.

Thus our problem is mechanical. We need only construct various sets of $\mathbf{j}$, and run the machinery backward. Not surprisingly, $\mathbf{j}$ may be constructed either (a) out of $\boldsymbol{\pi}$, $\boldsymbol{\omega}$ alone or (b) out of $\boldsymbol{\pi}$, $\boldsymbol{\omega}$, $N$, $\eta_2-\eta_1$. The first case realizes garden-variety two-dimensional Schrödinger equations, while the second yields two-dimensional Schrödinger equations with much more general "mass-dependent" potentials. We begin with case (a).

### Two-Dimensional Potentials

Construct a Galilean-invariant $\mathbf{j}$ as

$$\begin{aligned}j_1&=\tfrac14[\omega_1,\pi_1^2-\pi_2^2]_++\tfrac12[\omega_2,\pi_2]_+\pi_1+\tfrac12\omega_1\,,\\ j_2&=-\tfrac14[\omega_2,\pi_1^2-\pi_2^2]_++\tfrac12[\omega_1,\pi_1]_+\pi_2+\tfrac12\omega_2\,,\\ j_3&=\omega_1\pi_2-\omega_2\pi_1.\end{aligned} \tag{III.11}$$

Doing the inverse machinery, we find

$$\begin{aligned}J_1=-\tfrac12[1/M,K_3]_+P_2+\tfrac12[Z_2,P_3]_+&\\ +j_3(P_1/M)+j_1[(P^2)^{1/2}/M]\,,&\\ J_2=\tfrac12[1/M,K_3]_+P_1-\tfrac12[Z_1,P_3]_+&\\ +j_3(P_2/M)+j_2[(P^2)^{1/2}/M].&\end{aligned} \tag{III.12}$$

[11] In general, Wigner's construction is guaranteed for irreducible representations. Our representations are, of course, infinitely reducible, and the results of the construction need be checked.

[12] In general, our $\mathbf{j}$ will satisfy $SU(2)$ as operator conditions.

[13] Because $\mathbf{j}$ is Galilean-invariant, the requirements (III.1) are automatically satisfied except for the scale ($K_3$). In practice, this is fixed up trivially with factors of $M$.

The first three terms in each of these expressions are of course independent of our choice of $j_1$, $j_2$. What about $P^2=MH_{\rm int}$? The only invariant in this two-dimensional representation of $\mathbf{j}$ is $j^2$ itself. Thus $P^2$ may be an arbitrary function of $j^2$. The most general internal Hamiltonian is then

$$(\pi^2/\mu)+V=(1/M)f(j^2). \qquad \text{(III.13)}$$

Note that the conditions (III.1) are automatically satisfied. With this form for $P^2$, the reader is invited to check the Poincaré algebra. The restricted form of the Hamiltonian in this representation means that the spectrum is always discrete (although arbitrary). It is not likely[14] that such representations will be of help in the saturation of current algebra.

Unitary equivalents of the representation (III.11) may easily be constructed, but there is at least one inequivalent representation using only $\boldsymbol{\pi}$ and $\boldsymbol{\omega}$. We take

$$\begin{aligned} j_1&=\tfrac{1}{4}(\pi_1{}^2-\pi_2{}^2+\omega_1{}^2-\omega_2{}^2)\,,\\ j_2&=\tfrac{1}{2}(\pi_1\pi_2+\omega_1\omega_2)\,,\\ j_3&=\tfrac{1}{2}(\omega_1\pi_2-\omega_2\pi_1)\,. \end{aligned} \qquad \text{(III.14)}$$

This is the $SU(2)$ part of the representation of the Lorentz group generated by the two-dimensional harmonic oscillator. That it is not equivalent to (III.11) is evident in the factor $\frac{1}{2}$ in $j_3$.[15] This angular momentum has half-integral eigenvalues—thus being an illustration of our contention in Sec. II that the bosonic substratum of the representation need not be observable. The only invariant of this representation is the familiar form

$$\pi^2+\omega^2=2j+1\,, \qquad \text{(III.15)}$$

so $P^2=M((\pi^2/\mu)+V)$ must be some function of this—again an arbitrary discrete spectrum. The special case that looks most like a two-dimensional harmonic oscillator

$$P^2=(M/\mu)(\pi^2+\omega^2)\to V=\omega^2/\mu \qquad \text{(III.16)}$$

illustrates a general property of all these two-dimensional representations. The potential cannot be turned off (because $\pi^2$ alone is not a rotational scalar). In the more general three-dimensional representations, we shall have a choice in this matter.

### Three-dimensional Potentials

First consider the Galilean-invariant structures

$$\mathbf{W}=\left\{\mu^{1/2}\boldsymbol{\omega},\ \frac{i}{(C)^{1/2}}\left[\mu^{1/2}\frac{\eta_1\eta_2}{\eta_2-\eta_1},\ N+\frac{i}{M}\mathbf{P}\cdot\boldsymbol{\omega}\right]_+ -\left(\frac{\mu}{C}\right)^{1/2}\boldsymbol{\pi}\cdot\boldsymbol{\omega}\right\}, \qquad \text{(III.17)}$$

$$\boldsymbol{\Psi}=\left\{\frac{\boldsymbol{\pi}}{\mu^{1/2}},\ \left(\frac{C}{\mu}\right)^{1/2}\right\},$$

where $C$ is an arbitrary constant. These have the properties ($i$, $j$ range from 1 to 3)

$$[W_i,\Psi_j]=i\delta_{ij},\quad [W_i,W_j]=[\Psi_i,\Psi_j]=0. \qquad \text{(III.18)}$$

We have written $\mathbf{W}$, $\boldsymbol{\Psi}$ as three-dimensional vectors, and indeed they will be if we construct

$$j_i=\epsilon_{ijk}W_j\Psi_k.$$

After a little algebra, we can write $\mathbf{j}$ explicitly as

$$\begin{aligned} j_1&=C^{1/2}\omega_2-\frac{\pi_2}{C^{1/2}}\left[\frac{\eta_1\eta_2}{\eta_2-\eta_1},\ iN-\frac{\mathbf{P}\cdot\boldsymbol{\omega}}{M}\right]_+ +\frac{1}{2C^{1/2}}[\boldsymbol{\pi}\cdot\boldsymbol{\omega},\pi_2]_+\,,\\ j_2&=-C^{1/2}\omega_1+\frac{\pi_1}{C^{1/2}}\left[\frac{\eta_1\eta_2}{\eta_2-\eta_1},\ iN-\frac{P\cdot\omega}{M}\right]_+ -\frac{1}{2C^{1/2}}[\boldsymbol{\pi}\cdot\boldsymbol{\omega},\pi_1]_+\,,\\ j_3&=\omega_1\pi_2-\omega_2\pi_1\,. \end{aligned} \qquad \text{(III.19)}$$

The inverse machinery[16] yields exactly the form Eq. (III.12). $P^2=MH_{\rm int}$ must be constructed out of the invariants $W^2$, $\boldsymbol{\Psi}\cdot\mathbf{W}$, and $\Psi^2=(\pi^2/\mu)+(C/\mu)$, or, more precisely, to get the scale right, out of $M\Psi^2$, $W^2/M$, and $\boldsymbol{\Psi}\cdot\mathbf{W}$. Thus the most general Hamiltonian is $(V=M^{-1}\hat{V})$

$$H=\frac{P_1{}^2}{M}+\frac{\pi^2}{\mu}+\frac{C}{\mu}+\frac{1}{M}\hat{V}(M\Psi^2,W^2/M,\boldsymbol{\Psi}\cdot\mathbf{W}). \qquad \text{(III.20)}$$

We learn that $C$ is the analog of $m^2$ in the free-particle representations, although, even in the limit $\hat{V}=0$, $\mathbf{J}$ does not go over into the free two-particle $\mathbf{J}$ of Eq. (II.14). Notice finally that this representation is not restricted to discrete spectra; indeed the class of potentials is strikingly large.

It would be nice to have a representation which reduces to the two-free-particle case when the potential is turned off. Such can be constructed in the following way. We take the Galilean-invariant $\mathbf{j}$ as

$$\begin{aligned} j_1&=\left\{iN'\pi_2-\frac{\eta_2-\eta_1}{4\eta_2\eta_1}[\omega_2,\pi^2]_+\right\}\left(\frac{\eta_1\eta_2}{\pi^2}\right)^{1/2},\\ j_2&=\left\{-iN'\pi_1+\frac{\eta_2-\eta_1}{4\eta_1\eta_1}[\omega_1,\pi^2]_+\right\}\left(\frac{\eta_1\eta_2}{\pi^2}\right)^{1/2},\\ j_3&=\omega_1\pi_2-\omega_2\pi_1\,, \end{aligned} \qquad \text{(III.21)}$$

[14] I. T. Grodsky and R. F. Streater, Phys. Rev. Letters 20, 695 (1968); H. D. I. Abarbanel and Y. Frishman, Phys. Rev. 171, 1442 (1968).

[15] The inverse machinery yields (III.12) again. Also, of course, $J_3=Z_1P_2-Z_2P_1+j_3$.

[16] The characteristic form $N+(i/M)\mathbf{P}\cdot\boldsymbol{\omega}$ in $\mathbf{j}$ is taken for explicit Galilean invariance. The $\mathbf{P}\cdot\boldsymbol{\omega}$ term can actually be omitted, but then the inverse machinery will put it back in.

where $N'=N+(i/M)\mathbf{P}\cdot\boldsymbol{\omega}$, as introduced above. These $j$'s are Hermitian because $\pi^2/\eta_1\eta_2$ commutes with the (curly) bracketed structures in $j_1$ and $j_2$. The inverse machinery yields

$$J_1=-\tfrac{1}{2}[1/M,K_3]_+P_2+\tfrac{1}{2}[Z_2,P_3]_+ +j_3P_1/M$$
$$+\left[iN'\pi_2-\frac{\eta_2\eta_1}{4\eta_1\eta_2}\left(\omega_2,\frac{\pi^2}{\mu}\right)_+\right]\left(\frac{\mu}{\pi^2}\right)^{1/2}\left(\frac{\pi^2}{\mu}+V\right)^{1/2}, \quad \text{(III.22)}$$

etc. We need the invariants of $\mathbf{j}$ to construct the Hamiltonian. These we find again by exhibiting three-vectors under $j$:

$$\boldsymbol{\Psi}=\left\{\boldsymbol{\pi},\ \tfrac{1}{2}(\eta_1-\eta_2)\left(\frac{\pi^2}{\eta_1\eta_2}\right)^{1/2}\right\},$$
$$\mathbf{W}=\left\{\omega_1-i\frac{\pi_1}{\pi^2M^2}[(\eta_2-\eta_1)\eta_1\eta_2,N']_+,\right.$$
$$\omega_2-i\frac{\pi_2}{\pi^2M^2}[(\eta_2-\eta_1)\eta_1\eta_2,N']_+, \quad \text{(III.23)}$$
$$\left.-2\left[\left(\frac{\eta_1\eta_2}{\pi^2}\right)^{1/2}\frac{\eta_1\eta_2}{M^2},\ iN'\right]_+\right\}.$$

The invariants $W^2$, $\boldsymbol{\Psi}\cdot\mathbf{W}$, and $\Psi^2=(\pi^2/\mu)(\frac{1}{4}\mu)$ are Galilean-invariant, now with zero scale, so we can use them directly to form $P^2$. The most general Hamiltonian is then

$$H=\frac{P_\perp^2}{M}+\frac{\pi^2}{\mu}+\frac{1}{M}\hat{V}(W^2,\boldsymbol{\Psi}\cdot\mathbf{W},\Psi^2). \quad \text{(III.24)}$$

Notice that with $\hat{V}=0$, all the square roots cancel and the representation reduces to the free two-particle representation (II.14). To see this explicitly, one needs to use the identity

$$-\tfrac{1}{4}[\omega_2,(2/M)\boldsymbol{\pi}\cdot\mathbf{P}]_+$$
$$=j_3(P_1/M)-\tfrac{1}{2}[\mathbf{P}\cdot\boldsymbol{\omega}/M,\pi_2]_+ \quad \text{(III.25)}$$

in $J_1$, etc. Of course we attain the $m^2=0$ (free) case, but, if desired, this can be trivally fixed by writing $\pi^2\to\pi^2+m^2$ in (III.21). As a final comment on this representation, we note that it is unitarily equivalent to (III.19), being an evident change of variable.

There is another unitarily equivalent representation of interest because it is simpler than the previous two. We take

$$\mathbf{W}=(\boldsymbol{\omega},\ -iMN'),\quad \boldsymbol{\Psi}=(\boldsymbol{\pi},(\eta_1-\eta_2/2M)), \quad \text{(III.26)}$$
$$j_i=\epsilon_{ijk}W_j\Psi_k.$$

The inverse machinery yields (III.12) again, and the most general Hamiltonian is

$$H=\frac{P_\perp^2}{M}+\frac{|\Psi|^2}{M}+\frac{1}{M}\tilde{V}(W^2,\Psi^2,\boldsymbol{\Psi}\cdot\mathbf{W}). \quad \text{(III.27)}$$

Before passing on to a discussion of positivity of the mass spectrum, it is helpful to relate these representations to the literature. These representations appear to accomplish for the infinite-momentum frame what Foldy[17] accomplished for the center-of-mass frame. Among the differences between our representations and Foldy's, one is particularly notable. While Foldy was not actually able to find representations which reduced (for zero potential and arbitrary frame) to the representation of two free particles (separable representations), we have had no particular difficulty with this. Thus the infinite-momentum frame carries with it a rather more thoroughgoing nonrelativistic analogy.

### Positivity of the Mass Spectrum

For the case of one free particle of mass $m$, we have $P^2=m^2>0$. For two free particles of mass $m$, where

$$P^2=M[(\pi^2/\mu)+(m^2/\mu)], \quad \text{(III.28)}$$

one can easily show by the usual argument that $P^2\geqslant 4m^2$. What about our interacting representations, where $m^2/\mu\to V$?

In the first place, positivity of the mass spectrum is always trivial if one is asking only for a representation of the Poincaré group itself. These operators do not connect different values of $P^2$, so it is easy to stay in the space $P^2>0$. This is of course not the whole story. In general, one would like the individual operators in the representation, say $\mathbf{x}^{(1)}$, $\partial/\partial\eta_2$, etc., to be observables, and hence not to link timelike and spacelike states. There are two reasons why one might like the individual operators to have this property: (1) If the potential fell off rapidly for large distances, one would like the individual particles, sufficiently separated, to be separately observable. Of course, this does not apply to any of the harmonic-oscillator-like potentials for which there is no hope of observing the input particles. (2) One has in mind constructing (observable) currents out of (all) the operators in the representation. Thus we must in general have $P^2>0$ as an operator condition.

Of course it is easy to guarantee $P^2>0$ formally simply by constructing it as a positive-definite function of the invariants of $\mathbf{j}$ (as does Foldy), but then the breakup $H_{\text{int}}=M((\pi^2/\mu)+V)$ is somewhat formal [i.e., $V\equiv(P^2/M)-(\pi^2/\mu)$], and we would have trouble with separability. Since we have a representation which goes to two free particles when $V=0$, it is instructive to discuss the conditions directly on $V$ itself.

Working in the space discussed in the Appendix, with $\eta_1$, $\eta_2$ and hence $M$ positive, we see from (II.16) that $P^2>0$ is guaranteed if $H_{\text{int}}=(\pi^2/\mu)+V$ is a positive (essentially) self-adjoint operator. Strikingly, this is just the condition that the potential theory itself be well defined. Strictly speaking, the potential-theory Hamiltonian must be bounded from below. For our

[17] L. L. Foldy, Phys. Rev. 122, 275 (1961).

relativistic case, this involves a finite shift of the (energy) origin. The problem is then essentially the same as in nonrelativistic quantum mechanics: to find those potentials which, if they analytically dominate $\pi^2/\mu$, then they do so in a positive manner—i.e., singularities of the potential need be in general positive. In the general case, our potentials are more complicated than those of two-dimensional quantum mechanics, in that they involve the "masses" and their derivatives, but this does not appear to hinder one's ability to tell acceptable potentials on inspection. [One need remember that operators like $\omega^2-M^2(N')^2$, etc., are formally positive.] Such points however can be subtle, and a rigorous description of the allowed potentials (with attention to overlap of the domains[18] of $\pi^2/\mu$ and $V$, etc.), although beyond the goals of this paper, is worth investigating.

A comment about currents is in order here. If the mass spectrum is constructed positive definite, then the currents, constructed out of the operators in the representation, cannot connect spacelike with timelike states. That is, either we can find currents in our scheme or not, but if we can, they will not be diseased. Put another way, one can show (Sec. V) that $M$ commutes with the currents at infinite momentum, so if the currents required spacelike states, then, again from (II.16), $H_{\text{int}}$ would not be positive self-adjoint, and hence the potential theory would not have been well defined in the first place.

## IV. SECOND-QUANTIZED REPRESENTATIONS

To allow eventually for creation and annihilation, we will need second-quantized representations. To construct these we introduce the second-quantized (Schrödinger) fields

$$[\Psi(\mathbf{x},\eta),\Psi^\dagger(\mathbf{x}',\eta')]=\delta^{(2)}(\mathbf{x}-\mathbf{x}')\delta(\eta-\eta'), \quad \text{(IV.1)}$$

where, as usual, $\mathbf{x}$ and $\mathbf{x}'$ are the (transverse) two-dimensional position vectors. Later we will append isospin indices, etc., to these fields. The fields have the usual nonrelativistic expansion in terms of creation and annihilation operators

$$\Psi(\mathbf{x},\eta)=\int\frac{d^2q}{2\pi}e^{i\mathbf{q}\cdot\mathbf{x}}a(\mathbf{q},\eta), \quad \text{(IV.2a)}$$

$$\Psi^\dagger(\mathbf{x},\eta)=\int\frac{d^2q}{2\pi}e^{-i\mathbf{q}\cdot\mathbf{x}}a^\dagger(\mathbf{q},\eta), \quad \text{(IV.2b)}$$

$$[a(\mathbf{q},\eta),a^\dagger(\mathbf{q}',\eta')]=\delta^{(2)}(\mathbf{q}-\mathbf{q}')\delta(\eta-\eta'), \quad \text{(IV.2c)}$$

where $\mathbf{q}$, $\mathbf{q}'$ are two-dimensional momentum vectors. In terms of these fields, it is easy to construct a second-quantized representation for free particles. Simply sandwich our first-quantized representation (II.10') and (II.11) for one free particle between $\Psi^\dagger$ and $\Psi$, and integrate over $\mathbf{x}$ and $\eta$. Thus

$$P_0+P_3=\int d^2x\,d\eta\,\Psi^\dagger(\mathbf{x},\eta)\eta\Psi(\mathbf{x},\eta),$$

$$P_0-P_3=H=\int d^2xd\eta\;\Psi^\dagger(\mathbf{x},\eta)\times(-\nabla^2/\eta+m^2/\eta)\Psi(\mathbf{x},\eta),$$

$$P_i=\int d^2xd\eta\;\Psi^\dagger(\mathbf{x},\eta)\left(-i\frac{\partial}{\partial x_i}\right)\Psi(\mathbf{x},\eta),$$

$$J_1+K_2=\int d^2xd\eta\;\Psi^\dagger(\mathbf{x},\eta)(\eta x_2)\Psi(\mathbf{x},\eta), \quad \text{(IV.3)}$$

$$K_3=\int d^2xd\eta\;\Psi^\dagger(\mathbf{x},\eta)(\tfrac{1}{2}i[\eta,\partial/\partial\eta]_+)\Psi(\mathbf{x},\eta),$$

$$J_1=\int d^2xd\eta\;\Psi^\dagger(\mathbf{x},\eta)\times\left(-\frac{\partial}{\partial\eta}\partial_2+\tfrac{1}{4}\left[x_2,\eta+\frac{\nabla^2-m^2}{\eta}\right]_+\right)\Psi(\mathbf{x},\eta),$$

and so on.

### Schrödinger and Heisenberg Pictures

The reader is aware that both in our first- and second-quantized representations, we have been working in a "Schrödinger" picture—in that the variable conjugate to the Hamiltonian has not appeared. In fact, the introduction of a "time" variable $\xi$ conjugate to $H$ allows some geometric intuition about the Lorentz transformations and will help us introduce potentials.[19]

We introduce $\xi$ in the "Schrödinger" picture via an equation for the state vector

$$H|\Psi(\xi)\rangle=i\frac{\partial}{\partial\xi}|\Psi(\xi)\rangle, \quad \text{(IV.4)}$$

where $H$ is given in (IV.3). In the "Heisenberg" picture

$$\Psi(\mathbf{x},\eta,\xi)=e^{iH\xi}\Psi(\mathbf{x},\eta)e^{-iH\xi},$$

$$\frac{\nabla^2+m^2}{\eta}\Psi(\mathbf{x},\eta,\xi)=i\frac{\partial}{\partial\xi}\Psi(\mathbf{x},\eta,\xi). \quad \text{(IV.5)}$$

In terms of creation and annihilation operators, the

[18] For potential theory, see, for example, the review article, T. Kato, Progr. Theoret. Phys. (Kyoto) Suppl. 40, 3 (1967).

[19] Although we illustrate our remarks below in the second-quantized case, they apply as well, of course, to the first-quantized situation.

free time dependence is

$$\Psi(\mathbf{x},\eta,\xi)=\int \frac{d^2q}{2\pi}\exp(i\mathbf{q}\cdot\mathbf{x})a(\mathbf{q},\eta)\times\exp\left[-i\left(\frac{q^2+m^2}{\eta}\right)\xi\right] \quad (\text{IV}.6)$$

and similarly for $\Psi^\dagger$.

Of course $\xi$ is not the "real" time because $H$ is not $P_0$. Writing

$$P_0x_0-P_3x_3=\tfrac{1}{2}(P_0+P_3)(x_0-x_3)+\tfrac{1}{2}(P_0-P_3)(x_0+x_3) \quad (\text{IV}.7)$$

with all indices covariant, we learn that $\xi=\frac{1}{2}(t-z)$. Later we shall have some use for the variable $l=\frac{1}{2}(t+z)$ conjugate to $P_0+P_3$.

### Action Principle

Toward finding interacting representations, it will be helpful to introduce an action principle. Among other things, this will allow us to give a more familiar geometric interpretation for our generators of the Poincaré group.

We begin with the action for a free system:

$$I=\int_{-\infty}^{+\infty}d\xi\int_0^\infty d\eta\int d^2x\, \Psi^\dagger(\mathbf{x},\eta,\xi)\times\left(\frac{P^2-m^2}{\eta}\right)\Psi(\mathbf{x},\eta,\xi), \quad (\text{IV}.8)$$

where $P^2=(P_0-P_3)(P_0+P_3)-P_\perp{}^2=i\eta(\partial/\partial\xi)+\nabla^2$ is the invariant four-momentum squared of the first-quantized representation. At first, we consider the system as a classical field theory and derive the Poincaré transformations in the usual way via $c$-number invariances of the action. $I$ is invariant under any of the transformations.

$$\{\Psi\to\Psi+i\alpha\partial_i\Psi,\ \Psi^\dagger\to\Psi^\dagger+i\alpha\partial_i\Psi^\dagger\}\quad (P_\perp)$$

$$\{\Psi\to\Psi-i\alpha\eta\Psi,\ \Psi^\dagger\to\Psi^\dagger+i\alpha\eta\Psi^\dagger\}\quad (P_0+P_3)$$

$$\{\Psi\to\Psi-\alpha(-\nabla^2+m^2/\eta)\Psi,\ \Psi^\dagger\to\Psi^\dagger+\alpha(-\nabla^2+m^2/\eta)\Psi^\dagger\}\quad (P_0-P_3=H)$$

$$\{\Psi\to\Psi-\alpha\eta x_2\Psi,\ \Psi^\dagger\to\Psi^\dagger+\alpha\eta x_2\Psi^\dagger\}\quad (J_1+K_2)$$

$$\{\Psi\to\Psi-\alpha\eta x_1\Psi,\ \Psi^\dagger\to\Psi^\dagger+\alpha\eta x_1\Psi^\dagger\}\quad (K_1-J_2)$$

$$\{\Psi\to\Psi+i\alpha(x_1\partial_2-X_2\partial_1)\Psi,\ \Psi^\dagger\to\Psi^\dagger+i\alpha(x_1\partial_2-x_2\partial_1)\Psi^\dagger\}\quad (J_3)$$

$$\left\{\Psi\to\Psi-\tfrac{1}{2}i\alpha\left[\eta,\frac{\partial}{\partial\eta}-\frac{1}{2\eta}\right]_+\Psi,\ \Psi^\dagger\to\Psi^\dagger-\tfrac{1}{2}i\alpha\left[\eta,\frac{\partial}{\partial\eta}-\frac{1}{2\eta}\right]_+\Psi^\dagger\right\}\quad (K_3)$$

$$\left\{\begin{aligned}&\Psi\to\Psi-i\alpha\left[-\left(\frac{\partial}{\partial\eta}-\frac{1}{2\eta}\right)\partial_2-\tfrac{1}{2}ix_2\frac{\partial}{\partial\xi}+\tfrac{1}{2}x_2\eta+i\xi\partial_2\right]\Psi\\&\Psi^\dagger\to\Psi^\dagger+i\alpha\left[-\left(\frac{\partial}{\partial\eta}-\frac{1}{2\eta}\right)\partial_2-\tfrac{1}{2}ix_2\frac{\partial}{\partial\xi}+\tfrac{1}{2}x_2\eta+i\xi\partial_2\right]\Psi^\dagger\end{aligned}\right\}\quad (J_1)$$

$$\left\{\begin{aligned}&\Psi\to\Psi-i\alpha\left[+\left(\frac{\partial}{\partial\eta}-\frac{1}{2\eta}\right)\partial_1+\tfrac{1}{2}ix_1\frac{\partial}{\partial\xi}-\tfrac{1}{2}x_1\eta-i\xi\partial_1\right]\Psi\\&\Psi^\dagger\to\Psi^\dagger+i\alpha\left[\left(\frac{\partial}{\partial\eta}-\frac{1}{2\eta}\right)\partial_1+\tfrac{1}{2}ix_1\frac{\partial}{\partial\xi}-\tfrac{1}{2}x_1\eta-i\xi\partial_1\right]\Psi^\dagger\end{aligned}\right\},\quad (\mathrm{J}_2)\qquad (\text{IV}.9)$$

where $\alpha$ is some constant, different in general of course for each transformation. In this way, one identifies the Poincaré group. In addition to the introduction of $\xi$ dependence, there are two simple differences between this representation and our previous (IV.3). In the first place, wherever $H$ appeared, it has now been replaced by $i(\partial/\partial\xi)$. This is closer to the usual field theoretic way of approaching the Poincaré group, and, not surprisingly, allows geometric interpretation (which is particularly needed for our $J_{1,2}$). For example in the case of $J_1$, going over to Laplace transform space $[i((\partial/\partial\eta)-(1/2\eta))\to l,\ \eta\to -i(\partial/\partial l)]$, the rotation due to $J_1$ takes a more familiar form,[20] namely $i(x_2\partial_t-t\partial_2)$ where $t=\frac{1}{2}(\xi+l)$. The second difference is that, wherever $i(\partial/\partial\eta)$ appeared before, we now have $i((\partial/\partial\eta)-(1/2\eta))$. This is the natural (Hermitian) derivative in a space

[20] This is not the usual $J_1$, but it is an angular momentum nonetheless.

where the $\eta$ metric is $d\eta/\eta$—as is dictated by the action. In the corresponding Hermitian second-quantized representation, say for $J$,

$$J_1=\int d^2x\int_0^\infty d\eta\Psi^\dagger(x,\eta,\xi)\times\left(-\frac{\partial}{\partial\eta}\partial_2-\tfrac{1}{2}ix_2\frac{\partial}{\partial\xi}+\tfrac{1}{2}x_2\eta+i\xi\partial_2\right)\Psi(x,\eta,\xi)\,, \quad \text{(IV.10)}$$

the extra terms in $(1/2\eta)\partial_2$, being anti-Hermitian, cancel out. For convenience we will continue our discussion in Sec. IV via the $c$-number transformations.

Now we want to add an interaction term to the action. In general such a term must be Poincaré-invariant, but we would like also to preserve the Schrödinger analogy. To keep $\Psi^\dagger$ conjugate to $\Psi$, we must demand that the interaction is local in $\xi$. Thus, before imposing Poincaré invariance, the most general number-conserving interaction is of the form

$$\int_{-\infty}^{+\infty}d\xi\int_0^\infty d\eta_1\cdots\int_0^\infty d\eta_4\int d^2x_1\cdots\times\int d^2x_4\Psi^\dagger(\mathbf{x}_1,\eta_1,\xi)\Psi^\dagger(\mathbf{x}_2,\eta_2,\xi)\times\mathcal{V}(\mathbf{x}_1\cdots\mathbf{x}_4,\eta_1\cdots\eta_4)\Psi(\mathbf{x}_3,\eta_3,\xi)\Psi(\mathbf{x}_4,\eta_4,\xi)\,, \quad \text{(IV.11)}$$

where $\mathcal{V}$ is the "potential". Because the interaction is local in $\xi$, it is simple to show from Poincaré invariance that it must also be local in $\mathbf{x}$ and $l$ (the Laplace-transform conjugate of $\eta$); in fact, we are allowed only one number-conserving interaction

$$I=I_{\text{free}}+\lambda\int_{-\infty}^{+\infty}d\xi\int d^2x\times\int_0^\infty d\eta_1\cdots\int_0^\infty d\eta_4\frac{\delta(\eta_1+\eta_2-\eta_3-\eta_4)}{(\eta_1\eta_2\eta_3\eta_4)^{1/2}}\times\Psi^\dagger(\mathbf{x},\eta_1,\xi)\Psi^\dagger(\mathbf{x},\eta_2,\xi)\Psi(\mathbf{x},\eta_3,\xi)\Psi(\mathbf{x},\eta_4,\xi)\,, \quad \text{(IV.12)}$$

where $\lambda$ is a coupling constant. It is instructive to reduce this to a Schrödinger equation in the two-particle subspace. In our first-quantized notation, one obtains in the usual way

$$\left(-\frac{\nabla_{(1)}{}^2}{\eta_1}-\frac{\nabla_{(2)}{}^2}{\eta_2}\right)\Psi(\mathbf{x}_1,\mathbf{x}_2,\eta_1,\eta_2,\xi)+\lambda\delta^{(2)}(\mathbf{x}_1-\mathbf{x}_2)\int\frac{d\eta_1'd\eta_2'\delta(\eta_1+\eta_2-\eta_1'-\eta_2')}{(\eta_1\eta_2\eta_1'\eta_2')^{1/2}}\times\Psi(\mathbf{x}_1,\mathbf{x}_2,\eta_1',\eta_2',\xi)=i\frac{\partial}{\partial\xi}\Psi(\mathbf{x}_1,\mathbf{x}_2,\eta_1,\eta_2,\xi)\,. \quad \text{(IV.13)}$$

Potentials nonlocal in $\eta$ were not explicitly discussed in Sec. III but the reader will verify that this potential meets all the requirements set forth there. This Schrödinger equation is algebraic in the position variables and is almost trivial, corresponding, as we shall note below, to an $S$-wave "chain graph" approximation in the $\lambda\phi^4$ theory.

More complicated potentials can be constructed if creation and annihilation is allowed. The general prescription for constructing a scalar interaction is simply to take any local product of $\Psi$'s and $\Psi^\dagger$'s with a factor $\eta^{-1/2}$ for each, and the $\eta$-conserving $\delta$ function. Of particular interest is the interaction

$$\lambda\int_{-\infty}^{+\infty}d\xi\int d^2x\int_0^\infty\frac{d\eta}{(\eta)^{1/2}}\int_0^\infty\frac{d\eta'}{(\eta')^{1/2}}\int_0^\infty\frac{d\eta''}{(\eta'')^{1/2}}$$
$$\times[\Psi(\mathbf{x},\eta,\xi)\Psi(\mathbf{x},\eta',\xi)\Psi^\dagger(\mathbf{x},\eta'',\xi)\delta(\eta+\eta'-\eta'')+\Psi(\mathbf{x},\eta,\xi)\Psi^\dagger(\mathbf{x},\eta',\xi)\Psi(\mathbf{x},\eta'',\xi)\delta(\eta-\eta'+\eta'')$$
$$+\Psi^\dagger(\mathbf{x},\eta,\xi)\Psi(\mathbf{x},\eta',\xi)\Psi(\mathbf{x},\eta'',\xi)\delta(\eta-\eta'-\eta'')+\Psi(\mathbf{x},\eta,\xi)\Psi^\dagger(\mathbf{x},\eta',\xi)\Psi^\dagger(\mathbf{x},\eta'',\xi)\delta(\eta-\eta'-\eta'')$$
$$+\Psi^\dagger(\mathbf{x},\eta,\xi)\Psi(\mathbf{x},\eta',\xi)\Psi^\dagger(\mathbf{x},\eta'',\xi)\delta(\eta-\eta'+\eta'')+\Psi^\dagger(\mathbf{x},\eta,\xi)\Psi^\dagger(\mathbf{x},\eta',\xi)\Psi(\mathbf{x},\eta'',\xi)\delta(\eta+\eta'-\eta'')] \quad \text{(IV.14)}$$

consisting of every combination of three fields, but omitting terms which are purely creation or purely annihilation. As we shall note below, this turns out to be exactly the $\lambda\phi^3$ theory. The analogous quartic interaction (14 terms, being all possible quartics, omitting purely creation or annihilation terms) similarly turns out to be the $\lambda\phi^4$ theory.

### $S$ Matrix

Just as for ordinary theories, each picture has its characteristic form for the $S$ matrix. Suppose we wanted to calculate in the Schrödinger picture. The potential $V(H=H_0+V)$ can be read off from the action

$$I=I_{\text{free}}+\int_{-\infty}^{+\infty}V(\xi)d\xi=I_{\text{free}}+\int_{-\infty}^{+\infty}d\xi\,e^{iH\xi}Ve^{-iH\xi}. \quad \text{(IV.15)}$$

Thus, for example, in the case of the number-conserving interaction (IV.12)

$$V=\lambda\int d^2x\int_0^\infty d\eta_1\cdots\int_0^\infty d\eta_4\frac{\delta(\eta_1+\eta_2-\eta_3-\eta_4)}{(\eta_1\eta_2\eta_3\eta_4)^{1/2}}\times\Psi^\dagger(\mathbf{x},\eta_1)\Psi^\dagger(\mathbf{x},\eta_2)\Psi(\mathbf{x},\eta_3)\Psi(\mathbf{x},\eta_4)\,, \quad \text{(IV.16)}$$

and so on. From $V$, the $T$ matrix may be constructed in the usual way via the Lippmann-Schwinger equation

$$T_E=V+VG_+T_E,\quad G_+=1/E-H_0+i\epsilon \quad \text{(IV.17)}$$

taken between states constructed by creation operators on the vacuum. $E$ is of course the eigenvalue of the Hamiltonian. In an infinite-momentum interaction picture, constructed in the usual way, the $S$ matrix is given by

$$S=\Xi\exp\left(i\int_{-\infty}^{+\infty}d\xi\;V(\xi)\right), \quad \text{(IV.18)}$$

where $\Xi$ denotes time ordering with respect to $\xi$, and the fields in $V$ are free fields.

These structures will in general guarantee unitarity, but Lorentz invariance is more difficult. Evidently, $\int_{-\infty}^{+\infty}V(\xi)d\xi$ must be a four-scalar, but this is probably not sufficient. In the higher-order terms of (IV.18), locality may play a role in making the $\xi$ ordering covariant. It is an interesting problem to find the necessary and sufficient conditions on $V$ such that $S$ and $T$ are scalars, but, except for the following paragraph, this is beyond the scope of the present paper.

Using the $T$-matrix formulation in the Schrödinger picture, we have examined the perturbation expansion for the potentials given above, with these results: The potential (III.14) yields exactly the graphs found by Weinberg[3] for the infinite-momentum limit of the $\lambda\phi^3$ theory. To make direct contact with Weinberg, one calculates $T$ between states with $P_0+P_3=1$. There is no loss of generality here because the $S$-matrix explicitly commutes with this generator[4]. Similarly, the 14-term quartic potential mentioned above gives Weinberg's $\lambda\phi^4$ graphs. The interaction (IV.12) yields a set of chain graphs in Weinberg's notation, that is, the $S$-wave, number-conserving part of the $\lambda\phi^4$ theory. This is then a Zachariasen model[21] at infinite momentum.

### Infinite-Momentum Limit of $\lambda\phi^3$ and $\lambda\phi^4$ Theories

In this section we want to show directly by boosting that the theories exhibited above correspond to the $\lambda\phi^3$ and $\lambda\phi^4$ theories. We shall content ourselves with boosting the potentials themselves, leaving the rest as an exercise similar to that of the Appendix. We begin with the interaction Hamiltonian for the $\lambda\phi^3$ theory at zero time: $\lambda\int d^3x\,\phi^3(\mathbf{x},0)$. Boosting this as in Sec. II, and doing an obvious rescaling, we reach

$$\lambda\int d^2x\,dz\,dt\;\delta(t-z)\phi^3(x)\,. \quad \text{(IV.19)}$$

We will evaluate this integral with $\phi$ a free field, namely

$$\phi(x)=\int\frac{d^3k}{[(2\pi)^3\omega_k]^{1/2}}[b(\mathbf{k})e^{-ik.x}+b^\dagger(\mathbf{k})e^{+ik.x}],\qquad [b(\mathbf{k}),b^\dagger(\mathbf{k}')]=\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,, \quad \text{(IV.20)}$$

where $k\cdot x$ is the invariant four-product; that is, one imagines working either in the (ordinary) interaction picture, or to lowest order in perturbation theory. There are eight different terms (with different cubics in $b,b^\dagger$) over which to integrate. Consider one of them, say the term in $bbb^\dagger$. Using $\delta(t-z)$, one can do the $dz\,dt$ integrations, obtaining another $\delta$ function $\delta[(k_0+k_3)-(k_0'+k_3')-(k_0''+k_3'')]$ (all indices covariant, as above) which one rewrites as

$$\int_0^\infty d\eta\int_0^\infty d\eta'\delta[\eta-(k_0+k_3)]\delta[\eta'-(k_0'+k_3')]\times\delta[\eta+\eta'-(k_0''+k_3'')]. \quad \text{(IV.21)}$$

These $\delta$ functions can be used to do the $dk_3$ integrations. Then make the identification

$$a(\mathbf{k}_\perp,\eta)\equiv b\left(\mathbf{k}_\perp,\frac{k_\perp^2+m^2-\eta^2}{2\eta}\right)\left(\frac{1}{\eta}\right)^{1/2}\left(\frac{k_\perp^2+m^2+\eta^2}{2\eta}\right)^{1/2}=\left(\frac{\omega_k}{k_0+k_3}\right)^{1/2}b(\mathbf{k}) \quad \text{(IV.22)}$$

and similarly for $b^\dagger$. These new quantities satisfy the commutation relations (IV.2c), and are in fact the creation and annihilation operators at infinite momentum.[22] With this identification, and the relation between $a,a^\dagger$ and $\Psi,\Psi^\dagger$, we find finally for this term

$$\lambda\int d^2x\int_0^\infty d\eta\int_0^\infty d\eta'\int_0^\infty d\eta''\frac{\delta(\eta+\eta'-\eta'')}{(\eta\eta'\eta'')^{1/2}}\times\Psi(\mathbf{x},\eta)\Psi(\mathbf{x},\eta')\Psi^\dagger(\mathbf{x},\eta''). \quad \text{(IV.23)}$$

The other terms may be treated similarly. Terms which are purely creation operators or purely annihilation operators (pair-production terms) integrate to zero. They involve $\delta[(k_0+k_3)+(k_0'+k_3')+(k_0''+k_3'')]$ whose argument is always positive. Thus out of the eight terms, six survive and these are exactly (the Schrödinger picture form of) Eq. (IV.14). An entirely similar calculation can be done for the $\lambda\phi^4$ theory, leading to the 14-term quartic potential described above.

[21] F. Zachariasen, Phys. Rev. **121**, 1851 (1961).

[22] The relation (IV.22) is the second-quantized form of the results sketched in the Appendix.

The number-conserving potential (IV.12) is just the boosted form of the term in $b^\dagger b^\dagger bb$ of the $\lambda\phi^4$ interaction, that is, in terms of the positive and negative frequency components of $\phi$, it is $\lambda\int d^3x\, \phi^+\phi^+\phi^-\phi^-$. At finite momentum we do not really know how to formulate a theory with such a nonlocal interaction. At infinite momentum, on the other hand, such an interaction generates, quite straight forwardly, a covariant unitary (not crossing symmetric) $S$ matrix. It will be interesting to study this case more fully, especially with reference to the $\xi$-ordering [Eq. (IV.18)] and locality.

To reemphasize that our boosting is essentially a change of variable, we note finally that the (free) field $\phi$ can be written directly in terms of $\Psi$ and $\Psi^\dagger$. From Eqs. (IV.20) and (IV.22),

$$\phi(x)=-(2\pi)^{1/2}\int_0^\infty \frac{d\eta}{\eta^{1/2}} \times[\Psi(\mathbf{x},\eta,\xi)e^{-i\eta(t+z)/2}+\Psi^\dagger(\mathbf{x},\eta,\xi)e^{i\eta(t+z)/2}]. \quad \text{(IV.24)}$$

A formal comment is in order here. We have noticed that our limit procedure yields the infinite-momentum results in the "Schrödinger" picture. This happens of course because of the factor $\delta(t-z)=\frac{1}{2}\delta(\xi)$ that appears in the limit. We can get some feeling for the different pictures via the following argument. We have shown that if $S(x)$ is the scalar interaction of an ordinary field theory, then

$$V=\int d^2x dz dt\ \delta(t-z)S(x) \quad \text{(IV.25)}$$

is the potential in the Schrödinger picture at infinite momentum. There is a simple relation between $\int d^4x\, S(x)$ and the interaction in the "interaction" picture $\int_{-\infty}^{+\infty} V(\xi)d\xi$. In fact, they are equal: Changing variables, using translational invariance, and ignoring factors of 2,

$$\begin{aligned}\int d^4x\, S(x) &= \int d^2x d\xi dl\ S(\mathbf{x}_\perp,\xi,l)\\ &= \int_{-\infty}^{+\infty} e^{iH\xi}\left(\int d^2x dl\ S(\mathbf{x}_\perp,0,l)\right)e^{-iH\xi} \qquad \text{(IV.26)}\\ &= \int_{-\infty}^{+\infty} d\xi\ e^{iH\xi}\left(\int d^2x dz dt\ \delta(t-z)S(x)\right)e^{-iH\xi}\\ &= \int_{-\infty}^{+\infty} d\xi\ V(\xi).\end{aligned}$$

This is then another way of seeing that this structure must be a four-scalar.

## V. CURRENTS

One of the main motivations for introducing dynamics at infinite momentum is the problem of finding relativistic currents that satisfy current algebra

$$[\rho^\alpha(\mathbf{x}_\perp),\rho^\beta(\mathbf{x}_\perp')]=iC^{\alpha\beta\gamma}\rho^\gamma(\mathbf{x}_\perp)\delta^{(2)}(\mathbf{x}_\perp-\mathbf{x}_\perp'). \quad \text{(V.1)}$$

Our formalism provides a fairly general approach to the problem: (a) Because we have Schrödinger equations, simple currents that satisfy (V.1) are always provided. In general, e.g., the "good" current is the probability density $\Psi^\dagger\Psi$. (b) As discussed in Sec. III, any reasonable Hamiltonian will guarantee a positive mass spectrum. (c) We can allow as much inelasticity as necessary in the saturation. That solutions to the scheme exist, given enough inelasticity, is obvious: The $\phi^3$ and $\phi^4$ theories (with isospin, rewritten at infinite momentum) solve the problem mathematically. The task is to find more interesting solutions, preferably in smaller spaces, so they can be handled nonperturbatively.

In our formalism then, the difficult part of the current algebra problem is to insure that the current transforms like a four-vector at infinite momentum. In what follows, we examine this requirement.

The currents at infinite momentum are given by

$$\rho_\alpha{}^\mu(\mathbf{x}_\perp)\equiv\int dz dt\ \delta(t-z)V_\alpha{}^\mu(x), \quad \text{(V.2)}$$

where $V_\alpha{}^\mu(x)$ is a local four-vector current. The "good" current [that combination which appears in current algebra, Eq. (V.1)] is $\rho_\alpha=\rho_\alpha{}^0-\rho_\alpha{}^3$. From the integral representation (V.2), one deduces the following lightlike (or Galilean) transformation properties of the currents

$$\begin{aligned}&[K_3,\rho^0]=[K_3,\rho^3]=i(\rho^0+\rho^3),\\ &[K_3,\boldsymbol{\rho}^\perp]=-i\boldsymbol{\rho}^\perp,\quad [P_0+P_3,\rho^\mu]=0,\\ &[J_1+K_2,\rho^0]=[J_1+K_2,\rho^3]=-i\rho^2, \qquad \text{(V.3)}\\ &[J_1+K_2,\rho^1]=0,\quad [J_1+K_2,\rho^2]=i\rho,\\ &[K_1-J_2,\rho^0]=[K_1-J_2,\rho^3]=-i\rho^1,\\ &[K_1-J_2,\rho^1]=-i\rho,\quad [K_1-J_2,\rho^2]=0,\end{aligned}$$

where the internal symmetry label $\alpha$ has been suppressed, and $\boldsymbol{\rho}^\perp\equiv(\rho^1,\rho^2)$. As mentioned in Sec. II, the good current then commutes with the lightlike group

$$\begin{aligned}[K_3,\rho]&=[P_0+P_3,\rho]\\ &=[J_1+K_2,\rho]=[K_1-J_2,\rho]=0. \qquad \text{(V.4)}\end{aligned}$$

In the first-quantized notation,[23] one can verify that the general solution to this algebra is

$$\begin{aligned}&\rho^0=(1/2M)[f,P_0]_+-(1/2M^2)[P_i,g_i]_++(\kappa/M^2),\\ &\boldsymbol{\rho}^\perp=-(1/2M)[f,\mathbf{P}]_++(1/M)\mathbf{g}, \qquad \text{(V.5)}\\ &\rho=\rho^0-\rho^3=f,\end{aligned}$$

[23] The second-quantized solution is essentially the same (sandwiched between $\Psi^\dagger$ and $\Psi$, and integrated $d\eta$).

where $f$, $\mathbf{g}=(g_1,g_2)$ and $\kappa$ are as yet undetermined functions that commute with $J_1+K_2$, $K_1-J_2$, $P_0+P_3$, and $K_3$. In the two-particle case, these functions must then involve only $\mathbf{Z}$, $\boldsymbol{\pi}$, $\boldsymbol{\omega}$, and scale-invariant combinations of $N'$, $\eta_1$, and $\eta_2$. By further commuting the currents with $J_3$, one learns that $f$ and $\kappa$ must transform as scalars under this rotation, while $\mathbf{g}$ is a two-vector.

There is one final condition that must be met to insure correct Lorentz transformation properties for the current.

$$[P_i,[J_i,\rho_\alpha{}^0(\mathbf{x}_\perp)]]=0 \quad (i \text{ summed from 1 to 3}). \tag{V.6}$$

This "scalar angular condition" follows immediately from the definition of $\rho^\mu$ and the commutation relations of $V_\alpha{}^\mu(x)$ with $\mathbf{J}$ and $\mathbf{P}$. In fact, the scalar angular condition has a very simple physical interpretation: The fourth component of the current carries no spin, its angular momentum being entirely orbital. Thus, on $\rho^0$, we must have $\mathbf{J}\cdot\mathbf{P}=0$ ($i=1$, 2, 3), which is the content of (V.6). Equations (V.5) and (V.6) are necessary and sufficient for the correct Lorentz transformation properties of the infinite-momentum current.[24]

In general the scalar angular condition is a set of coupled linear differential equations for $f$, $\mathbf{g}$, and $\kappa$. The easiest way to project out these equations is by commuting (V.6) with the generators of the lightlike group. As an example, we give the results for the case of two free (first-quantized) particles.

$$\begin{aligned}
&[\mathbf{J}\cdot\mathbf{P},f/M]-\frac{i}{M^2}([P_1,g_2]-[P_2,g_1])=0\,,\\
&\left[\mathbf{J}\cdot\mathbf{P},\frac{1}{M}[P_i,g_i]\right]+\frac{i}{2M^2}[H_{\text{int}},([P_1,g_2]-[P_2,g_1])]=0\,,\\
&\left[\mathbf{J}\cdot\mathbf{P},\frac{1}{M}([P_2,g_1]-[P_1,g_2])\right]+\frac{i}{M^2}[P_2,[P_2,\kappa]]\\
&\qquad+\frac{i}{M^2}[P_1,[P_1,\kappa]]+\frac{i}{2M}[H_{\text{int}},[P_i,g_i]]=0\,,\\
&[\mathbf{J}\cdot\mathbf{P},\kappa/M]+\frac{i}{4M^2}[P^2,[P_1,g_2]-[P_2,g_1]]\\
&\qquad+\frac{i}{2M}[[P_1,g_2]-[P_2,g_1],H_{\text{int}}]=0\,,
\end{aligned} \tag{V.7}$$

where

$$\begin{aligned}
&[\mathbf{J}\cdot\mathbf{P},\mathcal{O}]\equiv[P_i,[J_i,\mathcal{O}]] \quad (i \text{ summed from 1 to 3})\\
&[P^2,\mathcal{O}]\equiv[P_i,[P_i,\mathcal{O}]] \quad (i \text{ summed from 1 to 3})
\end{aligned} \tag{V.8}$$

[24] Actually the scalar angular condition can be simplified somewhat by using translational invariance. Call the position label of the current $\mathbf{a}=(a_1,a_2)$, i.e., $\rho^\mu=\rho^\mu(\mathbf{a})$. Then we can write

$$\rho^\mu(\mathbf{a})=\exp(i\mathbf{a}\cdot\mathbf{P}_\perp)\rho^\mu(0)\exp(-i\mathbf{a}\cdot\mathbf{P}_\perp),$$

with which it is easy to show that $[P_i,[J_i,\rho_\alpha{}^0(0)]]=0$ ($i$ summed over 1 and 2) is equivalent to (V.6). In this form, of course,

$$[J_3,f_\alpha(0)]=[J_3,\kappa_\alpha(0)]=0,$$
$$[J_3,g_1{}^\alpha(0)]=ig_2{}^\alpha(0),\quad [J_3,g_2{}^\alpha(0)]=-ig_1{}^\alpha(0).$$

and $H_{\text{int}}=\pi^2/\mu$. These equations commute with the lightlike group and hence have no hanging derivatives. In fact, they are relatively simple equations. For any function, say $A$, which has scale $(-1)$,

$$\begin{aligned}
[\mathbf{J}\cdot\mathbf{P},A]=&-\tfrac{1}{2}[H_{\text{int}},[j_3,A]]+[j_i,[P_i,A]]\\
&+\frac{1}{2M}[j_3,[P_1,[P_1,A]]+[P_2,[P_2,A]]]_+.
\end{aligned} \tag{V.9}$$

This applies to each of the equations in (V.7). We shall not attempt to find the general solution to this system, although a particular solution will be noted below. One remark is instructive however: If $f$ is known, then (from these equations) $\mathbf{g}$ and $\kappa$ are known up to constants. This feature is independent of the free case, and, in fact, perfectly general.

One can also write a "vector" angular condition on the good current $\rho$. Either by combining (V.5) and (V.6), or directly one can verify that

$$[\mathbf{J}\cdot\mathbf{P},[\mathbf{J}\cdot\mathbf{P},[\mathbf{J}\cdot\mathbf{P},\rho_\alpha(\mathbf{x}_\perp)]]]=[J\cdot P,[P^2,\rho_\alpha(\mathbf{x}_\perp)]]. \tag{V.10}$$

This relation is essentially, but not quite equivalent to the scalar angular condition: Evidently it does not determine $\mathbf{g}$ and $\kappa$, although these can always be determined from equations like (V.7), once $\rho=f$ is known. This "vector" angular condition is equivalent to the "angular condition" popular in the literature, the usual form being derived for certain matrix elements of the currents, thereby involving the masses of the states in a complicated way. We believe that the operator formalism, and in particular the scalar angular condition, is simpler and more transparent.

It is also useful to know the statement of current conservation at infinite momentum. Again from the integral representation (V.2), it can be shown that $\partial_\mu V_\alpha{}^\mu(x)=0$ implies[4]

$$[H,\rho_\alpha]+2[P_i,\rho_\alpha{}^i]=0 \quad (i \text{ summed over 1, 2}). \tag{V.11}$$

This is in fact just the statement of probability conservation for a two-dimensional Schrödinger equation ($\rho$ being the probability density and $\rho^i$ being the two-dimensional probability flux). In terms of the functions $f^\alpha$, $g_i{}^\alpha$, $\kappa^\alpha$ introduced above, current conservation becomes

$$[H_{\text{int}},f^\alpha]+\frac{2}{M}[P_i,g_i]=0. \tag{V.12}$$

### Simple Solutions

Although we shall not attempt to solve the current algebra problem in this paper, we can at least give solutions for the free cases, and for the $\lambda\phi^3$ and $\lambda\phi^4$ theories at infinite momentum (with internal symmetry).

For the case of two free (first-quantized) particles, a solution is

$$\rho_\alpha{}^{(\mathbf{a})} = f_\alpha{}^{(\mathbf{a})} = \lambda_\alpha{}^{(1)}\delta^{(2)}(\mathbf{x}_\perp{}^{(1)}-\mathbf{a}) + \lambda_\alpha{}^{(2)}\delta^{(2)}(\mathbf{x}_\perp{}^{(2)}-\mathbf{a}) \quad \text{(V.13)}$$
$$g_i{}^\alpha = \kappa^\alpha = 0\,,$$

where $\mathbf{a}=(a_1,a_2)$ is the position label of the current, and $\lambda_\alpha$ is a representation of $SU(2)$, $SU(3)$, etc. In fact, each term of $\rho_\alpha{}^0$ satisfies the angular condition separately (similarly for $\rho_\alpha$). It is gratifying to note that, at least in the free case, the nonrelativistic analogy is preserved: $\rho_\alpha$ is precisely the probability density for two Schrödinger particles in a two-dimensional world. Similarly, $\rho_\alpha{}^\perp$ is the (transverse) probability flux. Moreover, of course, this current satisfies the two-dimensional continuity equation (V.11). It is not likely that this is the most general solution to (V.7), but it may be that it is the only solution that satisfies a local current algebra.

The situation for the second-quantized theories is quite analogous. For the free case or the $\lambda\phi^3$ and $\lambda\phi^4$ interactions discussed in Sec. IV, we have directly from Noether's theorem and the action principle

$$\rho_\alpha(\mathbf{x}_\perp) = C^{\alpha\beta\gamma}\int_0^\infty d\eta\; \Psi_\beta{}^\dagger(\mathbf{x},\eta)\Psi_\gamma(\mathbf{x},\eta)\,,$$
$$\rho_\alpha{}^\perp(x) = \tfrac{1}{2}C^{\alpha\beta\gamma}\int_0^\infty \frac{d\eta}{\eta}\; \Psi_\beta{}^\dagger(\mathbf{x},\eta)(-i\overleftrightarrow{\partial}_\perp)\Psi_\gamma(\mathbf{x},\eta)\,, \quad \text{(V.14)}$$

where we have trivally addended internal symmetry to the basic fields

$$[\Psi_\alpha(\mathbf{x},\eta),\Psi_\beta{}^\dagger(\mathbf{x}',\eta')] = \delta_{\alpha\beta}\delta^{(2)}(\mathbf{x}-\mathbf{x}')\delta(\eta-\eta')\,. \quad \text{(V.15)}$$

From this, one might guess

$$\rho_\alpha{}^\mu(\mathbf{x}_\perp) = \tfrac{1}{2}C^{\alpha\beta\gamma}\int_0^\infty \frac{d\eta}{\eta}\Psi_\beta{}^\dagger(\mathbf{x},\eta)\overleftrightarrow{P}_\mu\Psi_\gamma(\mathbf{x},\eta)\,, \quad \text{(V.16)}$$

where $P_\mu$ is just the usual first-quantized form in (II.10). This form is entirely analogous to (V.13) and indeed solves the angular condition. This form may also be derived directly by boosting the analogous field theories; that is, take $V_\mu{}^\alpha = C^{\alpha\beta\gamma}\phi^\beta\partial_\mu\phi^\gamma$ and evaluate the integral representation for $\rho_\alpha{}^\mu$ (in the fashion of Sec. IV).

Before passing on, we can make a few comments about the general case. A form for $\rho$ and $\rho^\perp$ is always given (by Noether's theorem) that satisfies the local current algebra (V.1). Because of the Schrödinger formalism, this $\rho$ always has the characteristic form displayed above, while the form of $\rho^i$ will change in the presence of derivative coupling. One approach then is to check directly whether $\rho$ satisfies the triple (vector) angular condition. Alternately, one can try to use the (simpler) scalar angular condition to calculate $\rho^0$. For the free case and the simple field theories discussed, the angular condition is satisfied, but it is doubtful that this is true for very many of the first-quantized potentials. If it turned out that none of the (nontrivial) Noether currents transformed correctly, one might be tempted to solve the angular condition anyway and take whatever (in general nonlocal) currents resulted.

## VI. DISCUSSION

There are a number of loose ends that we would like to discuss in Sec. VI. The first is half-integral spin.

In the first place, it is easy to construct half-integral spin representations (at infinite momentum) while keeping the nonrelativistic analogy. For example, we take the one-free-particle forms of $P_0+P_3$, $P_0-P_3$, $P_\perp$, $J_1+K_2$, $K_1-J_2$, and $K_3$, plus

$$J_1 = -\frac{\partial}{\partial\eta}\partial_2 + \tfrac{1}{2}[x_2,P_3]_+ + \tfrac{1}{2}\sigma_3\frac{P_1}{\eta} + \frac{m}{2\eta}\sigma_1\,,$$
$$J_2 = \frac{\partial}{\partial\eta}\partial_1 - \tfrac{1}{2}[x_1,P_3]_+ + \tfrac{1}{2}\sigma_3\frac{P_1}{\eta} + \frac{m}{2\eta}\sigma_2\,, \quad \text{(VI.1)}$$
$$J_3 = -i(x_1\partial_2 - x_2\partial_1) + \tfrac{1}{2}\sigma_3\,,$$

where $\sigma$ is the set of Pauli matrices. This is a representation for a particle of spin $\frac{1}{2}$, positive energy, and mass $m$. The representation can be constructed via the method of Sec. III, taking $j_i=\frac{1}{2}\sigma_i$, or alternately by boosting the usual representation. Having this representation, the paper could be repeated for it. Here we only want to make some comments about second-quantization, and spin statistics: A free second-quantized representation may be constructed, as in Sec. IV, either with commuting or anticommuting $\Psi$'s. Thus, although the correct connection between spin and statistics can be maintained, it is not required. (Although it was not explicitly mentioned in Sec. IV, the spin-zero particles can also be second-quantized with anticommuting $\Psi$'s). Similarly, antifermions may be included but are not required.

On the other hand, it may be possible to include half-integral spin without such direct methods. That the bosonic substructure of the previous representations need not be observable was already evident in the representation (III.14), which contains half-integral spin. More generally, however, it is well known that the two-dimensional Schrödinger equation for two free particles admits half-integral solutions,[25] in the space of which all operators are self-adjoint and all observables single-valued. At infinite momentum, we are dealing with exactly this situation, and there may be both integral and half-integral solutions to many of the first-quantized theories of Sec. III. Whether this is in fact true depends on whether the rest of the Poincaré generators are self-adjoint in the space of the half-integral solutions. This is presently under investigation.

A second comment concerns our second-quantized theories. Evidently, our unfortunately brief list of

[25] We thank Professor C. Schwartz for a discussion on this subject.

theories follows from our strict adherence to the usual second-quantization scheme—which guarantees that all particles are on an equal footing. In Sec. III, we obtained much larger classes of theories by manipulating the internal variables freely. On the other hand, the interactions of the second-quantized theories do not disturb the current algebra, while most of the potentials of Sec. III probably do not admit (relativistic) current algebra. This gives rise to two possible lines of thought: (a) Can one write other kinds of creation-annihilation theories, say that are not derivable from an action? Such a question is of course relevant to ordinary theories and is probably too much to ask at the present time. (b) More practically, can one find first-quantized representations that more closely resemble the field-theoretic representations? These are likely to satisfy the angular condition, etc., but are difficult to find. This is also under investigation.

## ACKNOWLEDGMENTS

We would like to thank R. Dashen, C. Schwartz, J. Weyers, E. Wichmann, and D. N. Williams for helpful discussion.

## APPENDIX: INFINITE-MOMENTUM LIMIT AS VARIABLE CHANGE

Our purpose here is to spell out the change of variable necessary to achieve the infinite momentum limit representations of Susskind, and our completion of the Poincaré group. In this way, we can also learn the appropriate inner product and suitable dense sets of test functions. We confine our discussion here to the case of one free particle of mass $m$.

We begin with the Poincaré group

$$\mathbf{J}=-i\mathbf{P}\times\nabla_P,\quad \mathbf{K}=i(P_0)^{1/2}\nabla_P(P_0)^{1/2},\qquad (A.1)$$

where $P_0=(P_1^2+P_2^2+P_3^2+m^2)^{1/2}$. The inner product appropriate for this representation is

$$\mathcal{O}[f,g]=\int d^3P\; f^*(\mathbf{P})\mathcal{O}g(\mathbf{P})=0,\qquad (A.2)$$

where the integration is over all momentum space and $\mathcal{O}$ is some operator. A suitable dense set of test functions for obtaining relations between unbounded operators is, for example, the usual set of Gaussian-smeared functions

$$\lim_{|\mathbf{P}|\to\infty}\exp(+|\mathbf{P}|^2)f(\mathbf{P})=0.\qquad (A.3)$$

Now we want to make a change of variable from the set $(P_1,P_2,P_3)$ to $(P_1,P_2,\eta=P_0+P_3)$. The metric of the new inner product differs by the Jacobian of the transformation.

$$\mathcal{O}[f,g]=\mathcal{O}_\infty[f_\infty,g_\infty]=\int d^2P\int_0^\infty d\eta\left(\frac{P_0}{\eta}\right)f_\infty{}^*\mathcal{O}_\infty g_\infty,$$
$$P_0=\frac{1}{2\eta}(\eta^2+P_\perp{}^2+m^2),\qquad (A.4)$$

where $f_\infty$, $g_\infty$ and $\mathcal{O}_\infty$ are the same functions and operator as before, now in terms of the new variables. To complete the change of variable, we list

$$P_3=\frac{1}{2\eta}(\eta^2-P_\perp{}^2-m^2)$$
$$\frac{\partial}{\partial P_\perp}\to\frac{\partial}{\partial P_\perp}+\frac{P_\perp}{P_0}\frac{\partial}{\partial\eta}\qquad (A.5)$$
$$\frac{\partial}{\partial P_3}\to\frac{\eta}{P_0}\frac{\partial}{\partial\eta}.$$

With this in hand, we can rewrite the generators themselves. For example, in the case of $J_1$, we find

$$J_1=-iP_2\frac{\eta}{P_0}\frac{\partial}{\partial\eta}+iP_3\left(\frac{\partial}{\partial P_2}+\frac{P_2}{P_0}\frac{\partial}{\partial\eta}\right).\qquad (A.6)$$

With a little algebra and the fact that $(P_0,J_1)=0$, we can express this as

$$J_1=\left(\frac{P_0}{\eta}\right)^{-1/2}\left(-i\frac{\partial}{\partial\eta}P_2+i\left[P_3,\frac{\partial}{\partial P_2}\right]_+\right)\left(\frac{P_0}{\eta}\right)^{1/2}.\qquad (A.7)$$

In fact, one learns that all the generators can be put in the form

$$M_{\mu\nu}=(P_0/\eta)^{-1/2}M_{\mu\nu}{}^T(P_0/\eta)^{1/2},$$

where $M_{\mu\nu}{}^T$ are (the transverse momentum space counterparts of) the generators given in the text. If new test functions

$$\hat{f}_\infty\equiv(P_0/\eta)^{1/2}f_\infty\qquad (A.9)$$

are defined, the generators can thus be taken as $M_{\mu\nu}{}^T$, with the simple metric $d^2p d\eta$. Note finally the typical behavior of the test functions for large and small $\eta$ ($P_3$ going to plus and minus infinity respectively)

$$\exp(-|P|^2)=\exp[-P_\perp{}^2-(1/4\eta^2)(\eta^2-P_\perp{}^2-m^2)^2].\qquad (A.10)$$

This is certainly adequate to drop boundary terms.