Trajectory and Mass Shift of a Classical Electron in a Radiation Pulse*

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We solve the Hamilton-Jacobi equations of motion of a classical charge in the presence of a traveling pulse of electromagnetic radiation. The orbit solutions for arbitrary radiation pulse shape are given in terms of one-dimensional integrals, and for a particular choice of singly peaked pulse shape are given in detail as a function of the particle's proper time. The question of the particle mass is then investigated, and an expression for the classical interacting mass is given as an explicit function of the time width of the radiation pulse. The average shift in the square of the mass, Δm^2, varies smoothly from zero to a maximum and then to zero again as the pulse overtakes and passes the charged particle. The maximum value is $\Delta m^2=\frac{1}{2}e^2a^2$ (where e and a are the particle charge and the field amplitude). We conclude that for the free-electron–photon scattering experiments currently being contemplated at optical frequencies the maximum figure is likely to be the relevant one.

I. INTRODUCTION

THE problem of electrons described quantum mechanically interacting with very intense and effectively monochromatic radiation fields has received a great deal of attention, experimental as well as theoretical, in the past four or five years.[1] Discrepancies among early published calculations of Compton scattering[2–4] in very intense radiation fields has led to an enduring controversy over the mass of the interacting electron, and the possibility of an extra, radiation-intensity-dependent, Compton wavelength shift.[5]

Much of the controversy stems from the fact that, although almost all questions can be answered exactly in a certain sense, the exact solutions require the assumption of plane wave electron and electromagnetic fields. This results in the impossibility of ever asymptotically decoupling the interacting objects, which fact has led to several different points of view on the question of experimental measurability. Needless to say, it is therefore very interesting to examine the question experimentally. However, it seems unlikely that an experimental answer will be provided soon, and this has encouraged a number of theoretical attempts to reconsider the same problems by introducing wave packets and adiabatic damping explicitly in the quantum-mechanical calculations.[6]

However, as one of us has pointed out (Ref. 1, Sec. 2.2), there is nothing essentially quantum mechanical about the questions being asked. The same time-dependent electron mass that is the root of the Compton scattering controversy occurs in the classical electron-radiation field interaction as well. In the present paper we determine exactly the trajectory of a classical charge, moving in the field of a pulse of electromagnetic radiation, and show the effects of changes in initial conditions, measurement times, and pulse width. Since the charge is a classical object, obeying relativistic equations of motion, and the radiation field is a travelling pulse, there is never any difficulty with asymptotic conditions or with initial and final separation of the charge and the field.

II. HAMILTON'S PRINCIPAL FUNCTION

The most direct method of solution is the Hamilton-Jacobi technique. The relativistic equation for the principal function is a nonlinear second-order equation in three space variables and one time variable:

$$(\partial S/\partial t)^2-(\nabla S-e\mathbf{A})^2=m^2\,, \tag{1}$$

where m is the particle mass. We have assumed the electromagnetic field to be completely described by the transverse vector potential $\mathbf{A}$, and have set the scalar potential Φ equal to zero. We assume the vector potential depends only on the proper time parameter[7] $\tau=t-\mathbf{n}\cdot\mathbf{r}$, where $\mathbf{n}$ is the direction of propagation of the pulse. We will later assume a specific pulse shape.

In the absence of the radiation field, when $\mathbf{A}=0$, the

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[1] For a review of recent developments and principal references in this field see, for example, a contribution by one of us (JE) in *Progress in Optics*, edited by E. Wolf (North-Holland Publishing Co., Amsterdam, 1968), Vol. VII.

[2] N. D. Sengupta, Bull. Math. Soc. (Calcutta) **44**, 175 (1952); Vachaspati, Phys. Rev. **128**, 664 (1962); **130**, 2598 (E) (1963); A. I. Nikishov and V. I. Ritus, Zh. Eksperim. i Teor. Fiz. **46**, 776 (1964) [English transl.: Soviet Phys.—JETP **20**, 757 (1964)]; I. I. Gol'dman, Phys. Letters **8**, 103 (1964).

[3] L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964).

[4] Z. Fried and J. H. Eberly, Phys. Rev. **136**, B871 (1964).

[5] T. W. B. Kibble, Phys. Rev. **138**, B740 (1965); P.G. DeBaryshe, Ph.D. dissertation, University of Pittsburgh, 1965 (unpublished); O. Von Roos, Phys. Rev. **150**, 1112 (1966); Z. Fried, A. Baker, and D. Korff, *ibid.* **151**, 1040 (1966); P. Stehle and P. G. DeBaryshe, *ibid.* **152**, 1135 (1966).

[6] H. R. Reiss, Bull. Am. Phys. Soc. **11**, 96 (1966); **12**, 1054 (1967); F. Ehlotzky, *ibid.* 13, 684 (1968); R. A. Neville, *ibid.* **13**, 685 (1968).

[7] The identification of τ as the proper time may be established in several ways. Perhaps the most straightforward method is to observe that the Lorentz force equation of motion leads to the condition $1-\mathbf{n}\cdot\mathbf{v}=(1-v^2)^{1/2}$, where $\mathbf{v}=d\mathbf{r}/dt$, which is sufficient. See, for example, N. D. Sengupta, Bull. Math. Soc. (Calcutta) **41**, 187 (1949); **41**, 189 (1949).

solution to (1) is obvious:

$$S=\beta t-\boldsymbol{\alpha}\cdot\mathbf{r}\,, \qquad (2)$$

where the parameters $\boldsymbol{\alpha}$, β, and the mass m are related in the following way:

$$\beta^2-\boldsymbol{\alpha}^2=m^2. \qquad (3)$$

Since $\mathbf{A}$ is a function of τ only, we assume the solution to (1) may be written:

$$S=\beta t-\boldsymbol{\alpha}\cdot\mathbf{r}+f(\tau)\,, \qquad (4)$$

where f is also a function of space and time only through τ. By substitution of the assumed solution into Eq. (1), one finds immediately a first-order equation for the unknown function $f(\tau)$:

$$2(\beta-\boldsymbol{\alpha}\cdot\mathbf{n})df/d\tau=2e\boldsymbol{\alpha}\cdot\mathbf{A}(\tau)+e^2\mathbf{A}^2(\tau)\,, \qquad (5)$$

having set $m^2-\beta^2+\boldsymbol{\alpha}^2=0$ again. The solution for $f(\tau)$ is easily found by a single integration with respect to τ:

$$f(\tau)=\tfrac{1}{2}(\beta-\boldsymbol{\alpha}\cdot\mathbf{n})^{-1}\int_{\tau_0}^{\tau}[2e\boldsymbol{\alpha}\cdot\mathbf{A}(\tau')+e^2\mathbf{A}^2(\tau')]d\tau'. \qquad (6)$$

Thus we may write the full solution for the principal function in the following convenient form:

$$S=\beta t-\boldsymbol{\alpha}\cdot\mathbf{r}+\tfrac{1}{2}(\beta-\boldsymbol{\alpha}\cdot\mathbf{n})^{-1}[2e\boldsymbol{\alpha}\cdot\mathbf{I}+e^2J], \qquad (7)$$

where the integrals $\mathbf{I}$ and J depend on the specific shape chosen for the radiation pulse, and on certain initial conditions:

$$\mathbf{I}(\tau,\tau_0)=\int_{\tau_0}^{\tau}d\tau'\mathbf{A}(\tau')\,, \qquad (8)$$

$$J(\tau,\tau_0)=\int_{\tau_0}^{\tau}d\tau'\mathbf{A}^2(\tau'). \qquad (9)$$

III. ELECTRON TRAJECTORY

In the usual way, the derivative of the principal function with respect to r_i gives the ith component of the canonical momentum:

$$\partial S/\partial r_i=p_i=mv_i+eA_i\,, \qquad (10)$$

where v_i is the ith component of the particle's kinetic velocity, dr_i/dt. Also the derivative of the principal function with respect to the integration constant α_i equals a constant, say $-c_i$, which later will be related to the τ_0 appearing in Eqs. (8) and (9):

$$\partial S/\partial\alpha_i=-c_i=\text{constant}. \qquad (11)$$

The straightforward application of Eq. (10) to Eq. (7) leads immediately to an equation which allows the three independent constants α_i to be evaluated:

$$mv_i(\tau)+eA_i(\tau)=-\alpha_i-\tfrac{1}{2}n_i(\beta-\boldsymbol{\alpha}\cdot\mathbf{n})^{-1}\times[2e\boldsymbol{\alpha}\cdot\mathbf{A}(\tau)+e^2\mathbf{A}^2(\tau)]. \qquad (12)$$

In order to evaluate the α's let us now impose our first physical condition on the solutions. We will require that $v_i(\tau)\to 0$ for those values of τ for which $\mathbf{A}(\tau)\to 0$. In other words, we assume that in the beginning when the radiation pulse is far away and the vector potential vanishes, the particle has zero velocity. In this way we can be certain that whatever velocity the particle has at other later times will be due solely to the interaction with the pulse. Since the α's are constants they are independent of the limiting process by which v_i and $\mathbf{A}$ approach zero, and so must themselves be zero.

After applying Eq. (11) to the principal function, and using our result that $\alpha_i=0$, one finds finally an expression for the ith component of the electron coordinate that is quite simple:

$$r_i(\tau)-c_i=(e/m)I_i+n_i(e^2/2m^2)J. \qquad (13)$$

Now it is obvious that the constant c_i is simply $r_i(\tau_0)$. At this point we impose the second physical assumption, namely, that in the beginning the particle was not only at rest but located at the origin. We have the option of choosing any particular proper time τ_0 to be the initial instant, but in the present circumstances the choice $\tau_0=-\infty$ is the most satisfactory. Thus $r_i(-\infty)=c_i=0$.

The proper velocity components are easily found also by differentiation with respect to τ:

$$v_i(\tau)=(e/m)A_i(\tau)+n_i(e^2/2m^2)\mathbf{A}^2(\tau). \qquad (14)$$

Here one can make several observations rather easily. Since $\mathbf{A}$ is, apart from the influence of the pulse envelope, a sinusoidally oscillating function, it will have an essentially zero average, whereas $\mathbf{A}^2$ has a nonzero average. As a consequence, after the pulse has passed, the electron will have suffered a net displacement in the $\mathbf{n}$ direction much larger than its displacement in the direction of $\mathbf{A}$. There is no displacement at any time in the direction $\mathbf{n}\times\mathbf{A}$.

One further remark is in order. We must keep in mind that our solutions are only implicit, since the electron position vector $\mathbf{r}$ is known only as a function of the proper time τ which is in turn a function of $\mathbf{r}$. A complete explicit solution as a function of t instead of τ is possible to find only approximately.

At this point it is convenient to choose a particular radiation pulse shape. There are no constraints on the choice, in principle at least, but since we expect the main features of the trajectory to be the same for any shape with a single peak we are motivated strongly to choose the simplest shape that has an adjustable width and allows the integrals $\mathbf{I}$ and J to be done analytically. We therefore will write the vector potential explicitly, in the following form[8]:

$$\mathbf{A}(\tau)=\boldsymbol{\varepsilon}a\cos\omega\tau\, e^{-\gamma|\tau|}. \qquad (15)$$

[8] Kibble (Ref. 5) has also made use of this pulse-shape function in an approximate quantum calculation.

Thus the radiation field is described by a real constant amplitude a, a single oscillation frequency ω, and a space-time pulse-width parameter $1/\gamma$. It is important to note that the exponential dependence is on $|\tau|$ and not on $|t|$. This guarantees that the form chosen for $\mathbf{A}$ describes a travelling pulse and not simply an adiabatically time-damped oscillation. In addition, we should point out that our specializations to linear polarization, and zero phase at $\tau=0$, are inessential and may be trivially generalized.

The integrals $\mathbf{I}$ and J then have the values:

$$\mathbf{I}(\tau<0)=\boldsymbol{\varepsilon}a\frac{e^{\gamma\tau}}{\omega^2+\gamma^2}(\gamma\cos\omega\tau+\omega\sin\omega\tau)\,, \tag{16a}$$

$$\mathbf{I}(\tau>0)=\boldsymbol{\varepsilon}a\left(\frac{2\gamma}{\omega^2+\gamma^2}-\frac{e^{-\gamma\tau}}{\omega^2+\gamma^2}\times(\gamma\cos\omega\tau-\omega\sin\omega\tau)\right), \tag{16b}$$

$$J(\tau<0)=\tfrac{1}{4}a^2e^{2\gamma\tau}\left(\frac{1}{\gamma}+\frac{\gamma\cos2\omega\tau+\omega\sin2\omega\tau}{\omega^2+\gamma^2}\right), \tag{17a}$$

$$J(\tau>0)=\tfrac{1}{4}a^2\left[\frac{2}{\gamma}+\frac{2\gamma}{\omega^2+\gamma^2}-e^{-2\gamma\tau}\left(\frac{1}{\gamma}+\frac{1}{\omega^2+\gamma^2}\times(\gamma\cos2\omega\tau-\omega\sin2\omega\tau)\right)\right]. \tag{17b}$$

Before going on to the question of the mass shift, we should point out that the classical electron trajectory in a given *monochromatic* radiation field is well known,[9] but is not directly comparable with the solution given in (14), or with the results calculated above in the monochromatic limit. This is due to the choice of initial conditions, which specify that at $\tau=-\infty$ both the displacement and the velocity of the electron are zero. In all previous trajectory calculations for monochromatic waves of which we are aware, the electron is chosen to be at rest either at $\tau=0$, or to be at rest on the average for all τ. Both of these sets of conditions can easily be imposed on Eqs. (12) and (13), in which case the standard results are obtained.

However, these monochromatic field trajectories can be misleading for two reasons. First of all, a nearly trivial objection is that no experiment is ever done with truly monochromatic radiation fields. One must always use a pulse with greater or lesser spectral width. Much more important is the experimenter's inability to duplicate either set of initial conditions assumed in the typical monochromatic field calculations. The position and velocity of the electron can be assumed known only well in advance of the interaction with the field, while the monochromatic field calculations assume that the electron is at the origin at rest (or at rest on the average) at the peak of the interaction. Such assumptions are patently unrealistic.

IV. MASS SHIFT

The solution for the principal function may now be used advantageously to discuss the electron mass shift from the classical point of view. In the usual way we make the identification of the interacting mass $M(\tau)$ through the relation

$$M^2=E^2-\mathbf{p}^2=(\partial S/\partial t)^2-(\nabla S)^2. \tag{18}$$

Our solution, in Eq. (7), leads to the result:

$$M^2(\tau)=m^2+e^2\mathbf{A}^2(\tau). \tag{19}$$

Clearly, in the presence of a pulse of radiation, $M^2(\tau)$ is generally not equal to m^2. However, it is certainly equal to m^2 if the pulse is far enough away. Thus the physically desirable asymptotic separation of the electron and the radiation is manifest again in the mass equation.

In most imaginable experiments, especially at optical frequencies, an average rather than instantaneous shift will be important. Thus we calculate the average shift during a time $2T$ symmetric about $\tau=0$:

$$\langle(M^2-m^2)\rangle\equiv\Delta m^2=\tfrac{1}{2}e^2a^2\chi(\gamma,2T)\,, \tag{20}$$

where

$$\chi(\gamma,2T)=\frac{1}{\omega T}\frac{\omega^2}{\gamma^2+\omega^2}\left\{\frac{\omega}{2\gamma}+\frac{\gamma}{\omega}+e^{-2\gamma T}\left[\cos\omega T\left(\sin\omega T-\frac{\gamma}{\omega}\cos\omega T\right)-\frac{\omega}{2\gamma}\right]\right\}. \tag{21}$$

In this simple classical formula we can see both of the conclusions reached in the recent quantum calculations. In the monochromatic limit ($\gamma\to0$) an average over an integral number of periods gives $\chi=1$ and a mass shift $\Delta m^2=\frac{1}{2}e^2a^2$, in agreement with the quantum-mechanical and quantum-field-theoretic results of a number of authors.[2–5,10,11] On the other hand, for arbitrary finite γ, the average mass shift is identically zero if the average is taken over the entire electron trajectory. This is the classical analog to a point raised by Stehle and DeBaryshe[12] in the course of an argument against the existence of a frequency shift. The question to be decided is which of these correct limit results has any relevance.

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[9] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1951), p. 120; N. D. Sengupta, Bull. Math. Soc. (Calcutta) **41**, 187 (1949); Vachaspati, Proc. Natl. Inst. Sci. India **29**, A138 (1963); L. S. Brown and T. W. B. Kibble, Ref. 3 (where the solution for a nonmonochromatic field is indicated in implicit form).

[10] F. Ehlotzky, Acta Phys. Austriaca **23**, 95 (1966).

[11] J. H. Eberly and H. R. Reiss, Phys. Rev. **145**, 1035 (1966); H. R. Reiss and J. H. Eberly, *ibid.* **151**, 1058 (1966).

[12] P. Stehle and P. G. DeBaryshe, University of Pittsburgh report, 1964 (unpublished).

We may test the relevance of these estimates of the mass shift by comparing them with the exact classical mass shift in a "reasonable" experimental situation. We assume $\omega \gg \gamma$ so that the field oscillates many times within its pulse envelope; and then calculate Δm^2 over a time interval $\Delta T = 4/\gamma$ centered at $\tau = 0$. This choice of averaging time means that we have included more than 98% of the pulse. We find $\Delta m^2 = \frac{1}{2}e^2a^2\chi(\gamma, 4/\gamma)$, where

$$\chi(\gamma,4/\gamma) = \tfrac{1}{4} - \tfrac{1}{4}e^{-4}[1 - \sin(2\omega/\gamma) + O(\gamma/\omega)]. \quad (22)$$

So, in this situation, the average mass shift is roughly one quarter of the monochromatic result.

The mass shift changes very rapidly in the neighborhood of $\tau = 0$, of course; and decreases sharply outside the 98% interval. In an earlier or later neighboring, but nonoverlapping, time interval of equal size, $\Delta T = 4/\gamma$, the mass shift is less than 1% of the monochromatic result.

V. DISCUSSION

We have solved exactly the classical equations of motion for a charge interacting with a pulse of electromagnetic radiation. In the solutions for the electron trajectory we recover some of the features of earlier calculations which assumed monochromatic fields. However, our choice of more realistic initial conditions than are possible with a monochromatic field allow new aspects of the trajectory to be investigated. We have found, for example, that the charge is carried a finite distance in the direction of pulse travel. This displacement is proportional to the time width of the pulse, and it is easy to show that the ratio of this displacement to the time width is identical with the effective velocity of propagation calculated by Sanderson.[13] There is also a net finite displacement of the charge in the direction which the electric field takes at $\tau = 0$, but it is inversely proportional to the pulse time width. In general, there appear to be no effects which are important for long times or broad pulse shapes which are proportional to the field amplitude a. As in the earlier quantum calculations the important dimensionless parameter is $\Delta m^2/m^2$, which is proportional to the square of the amplitude.

In addition we have calculated the change in the mass of the charge due to its interaction with the field as the pulse of radiation passes over it. On the basis of this classical calculation we think that it is possible to resolve the conflicting claims that have arisen from various quantum estimates of the same quantity. We have exhibited the average mass shift for arbitrary pulse width and arbitrary averaging time, and have shown that if the averaging time corresponds roughly to the time width of the pulse, then the average mass shift is only a few times smaller than the full monochromatic mass shift. Insofar as averaging times of this order of magnitude are the experimentally appropriate ones, we can assert that a mass shift can be expected to occur and to have a value within an order of magnitude of the monochromatic value. On the other hand, the opposite contention that the mass shift is zero is also supported by our calculation if the appropriate averaging time is sufficiently long. We think, however, that this latter situation would arise only in the unlikely event that the experimenter did not know very precisely when the radiation pulse was present. The transition between two maximum and minimum values of the mass shift is continuous, of course.

We should remark finally that the interest in the mass shift has arisen recently because of its direct connection with radiation-intensity-dependent effects in photon-electron scattering at optical frequencies. The mass shift is especially relevant to the question of an additional intensity-dependent contribution to the Compton wavelength shift in Compton scattering. It is in connection with observations of effects such as these that one can assert with greatest confidence that the mass shift *will* be present. The reason is that only the electrons which interact with the radiation pulse near its peak will be scattered strongly enough to be detected. Thus, those electrons which are detected will necessarily have experienced a mass shift which is nearly equal to the maximum.

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[13] J. J. Sanderson, Phys. Letters **18**, 114 (1965).