Changes of the Petrov Type of a Space-Time

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A theoretical study is made of the "evolution" of a space-time of one Petrov type into another. The word "evolution" is used to mean that change of type which a point observer would consider his local space-time had experienced as a discontinuous front swept over him. The study is motivated by a desire to better understand the Petrov classification and thus improve its use as a tool to deal with solutions in general relativity. An algorithm is developed that may in principle be used to test the evolution of any closed-form solution known. A flow diagram is found which shows the routes that each Petrov type may use and the Petrov type of the discontinuity causing each change. During this development some results of the study of discontinuous hypersurfaces are found essentially equivalent to Trautman's information approach to waves, thus yielding further motivation for the use of the latter. The Weyl technique is understood as an important method for at least partially separating the coordinate and physical behaviors.

I. INTRODUCTION

A VERY significant device to study solutions of the field equations of general relativity is that of the invariant classification. It is important because it is invariant under changes in coordinate choice, which often trouble attempts to compare known solutions. It is also useful, among other things, in checking the essential nature of singularities in solutions. Perhaps the most useful of these classifications is that of Petrov.[1]

Using the Petrov formulation, any fourth-rank tensor A_{ijkl} which has the symmetries

$$A_{ijkl}=-A_{jikl}=-A_{ijlk}=A_{klij} \tag{1.1}$$

and the property

$$g^{ik}A_{ijkl}=0 \tag{1.2}$$

may be classified. (Note that throughout the paper the convention of Landau and Lifshitz[2] will be used: Latin letters run 0 to 3, Greek letters 1 to 3, and indices repeated in one term are to be summed over.) This is done by grouping the indices by pairs to uniquely define a 6×6 matrix from the independent components of the tensor. This matrix will assume a particular form in terms of two 3×3 matrices:

$$\mathbf{A}=\begin{pmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{N} & -\mathbf{M}\end{pmatrix}. \tag{1.3}$$

The eigenvalues of a combination of these ($\mathbf{P}=\mathbf{M}+i\mathbf{N}$) gives the invariants. The matrix $\mathbf{A}$ will fall into one of the canonical classes (all types will have traceless $\mathbf{M}$ and $\mathbf{N}$).

In relativity the Petrov classification is used to type the Riemann curvature tensor and the Weyl curvature tensor. The classification is thought to characterize any solution in a nontrivial fashion, and a good deal of work has been done which relates to such a procedure one way or another. Notable are the Goldberg-Sachs theorem,[3] the Bialas theorem,[4] the Peeling theorem,[5] and several works which cast the classification into other than bivector form.[3,6,7] Closely related to the present work are those of Pirani,[8] Stellmacher,[9] and Treder.[10] Pirani attempted[11] to define gravitational radiation in terms of the Petrov type of the Riemann tensor, and Stellmacher studied the algebraic properties of discontinuities in solutions to the field equations. Treder has generalized and expanded Stellmacher's work considerably. This work differs from these, however, in that it considers differential, not algebraic, properties of the field theory; it is applied to the Petrov-classification-type changes; and it attempts to be general enough to allow considerations of the nonvacuum gravity field.

The hope of this work is to improve understanding of the relation between the Petrov canonical types and thereby further the classification's usefulness in discussing solutions in general relativity. In order to do this it is necessary to study second-order discontinuities in solutions to the field equations and understand how the curvature tensors change across such a discontinuity, as governed by the field equations and lower-order continuity. The study is then somewhat similar to the Rankine-Hugoniot work in fluid dynamics or to the O'Brien and Synge[12] relativistic counterpart or

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[1] A. Z. Petrov, "Classification of Spaces Supporting Gravitational Fields," Scientific Transactions of the Kazan State University **114**, 1954. Translation No. 29, Astronautics Information, J. P. L., Pasadena, Calif., 1963.

[2] L. Landau and E. Lifshitz, *Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962), pp. 313 ff.

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[4] A. Bialas, Acta Phys. Polon. **23**, 699 (1963).

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[7] F. Pirani, in *Gravitation: Introduction to Current Research*, edited by Louis Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 6, pp. 212–216.

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[9] K. Stellmacher, Math. Ann. **115**, 740 (1938).

[10] H. Treder, *Gravitative Stosswellen* (Akademie-Verlag, Berlin, 1962).

[11] F. Pirani, in *Recent Developments in General Relativity* (The Macmillan Co., New York, 1962), pp. 93 and 94.

[12] S. O'Brien and J. L. Synge, Dublin Institute for Advance Studies Report No. A9, Dublin, 1952 (unpublished).

to the tetrad study of jump conditions by Estabrook and Wahlquist.[13] These changes of Petrov type may be found from partial differential equations. One then knows the behavior of a type under these circumstances. A flow diagram summarizing possible changes comes from the invariance of the classification to a change of sign of the invariants.

II. CHANGE OF TYPE

Bialas[4] has shown that if the Weyl tensor is continuous at a point, then in a neighborhood of that point it is not more special algebraically than at that point. In order to study changes of Petrov type, one investigates the behavior of a space-time supporting a discontinuity. Immediately one must be careful to distinguish between discontinuities that appear as a result of coordinate choice from those which appear for physical reasons. It is known[14] that physically meaningul discontinuities of a space-time can occur only on characteristic hypersurfaces (null hypersurfaces) of the field equations, all others being subject to removal by transformation.

It is, of course, not necessary that every characteristic surface be a surface of discontinuity. In fact, Taniuti[15] has suggested that the existence of a wave can be defined in terms of the existence of a discontinuity. This is in clear contrast to defining wave existence in terms of Petrov types themselves.

The particular order of the discontinuity is, at this point in this paper, undecided. Several authors[12,16,17] have discussed what is to be expected. Something of a physical argument can be given that the accepted second-order discontinuities are the expected ones. In standard cases of discontinuities of an electromagnetic field ($[d\phi/dn]=4\pi\sigma$), one does not expect discontinuities to arise in the potential, since this could seriously affect the uniqueness of the field's definition. We would exclude a dipole-layer case, assuming that the existence of negative gravitational mass is unlikely[18] and might affect general-relativistic theory itself. This implies that one would not expect the g_{ij} to be discontinuous. Also, the first derivatives of g_{ij} which appear through the $\Gamma_{i,jk}$ in the geodesic equation should be continuous. This is implied since transformation to a locally inertial frame is possible and essential discontinuities would seem to imply that the principle of equivalence is invalid. Such a possibility is unacceptable. The remaining possibility, a jump in the second derivatives of the g_{ij}, can be seen to be consistent with the equation of geodesic deviation as simply a difference in relative accelerations of two bodies. Such a possibility is acceptable and represents the presence of a wave front.

Consider, then, a region divided into two parts by a discontinuous hypersurface S described on one side by g_{ij} and on the other by $\bar{g}_{ij}$. We consider the difference between these two if we extend the one solution, say, $\bar{g}_{ij}$, analytically across the surface. This difference η_{ij} will be due to the discontinuity and in general is nonzero:

$$\eta_{ij}=g_{ij}-\bar{g}_{ij}. \tag{2.1}$$

If g_{ij} and $\bar{g}_{ij}$ are to be second-order discontinuous only and are to be fields satisfying the field equations, then we have

$$\eta_{ij}|_S=0, \tag{2.2}$$

$$\eta_{ij,k}|_S=0, \tag{2.3}$$

$$\bar{R}_{ij}-\tfrac{1}{2}\bar{g}_{ij}\bar{R}=K\bar{T}_{ij}, \tag{2.4}$$

$$R_{ij}-\tfrac{1}{2}g_{ij}R=KT_{ij}. \tag{2.5}$$

In the real world, one does not expect a true mathematical discontinuity to exist, but rather such an abrupt change that what is done here would serve as a good model. It is in this way similar to how the hydrodynamic shock is treated. Sufficiently near S, η_{ij} is small compared with g_{ij}, $\bar{g}_{ij}$. This allows us to get a contravariant counterpart of η_{ij} from orthonormality of the g_{ij} and $\bar{g}_{ij}$. If

$$g^{lk}=\bar{g}^{lk}+\sigma^{lk}, \tag{2.6}$$

then

$$\delta_j{}^k=\delta_j{}^k+\bar{g}_{lj}\sigma^{lk}+\eta_{lj}\bar{g}^{lk}+O(\eta^2).$$

This gives, to first order,

$$\sigma^{mk}=-\bar{g}^{jm}\eta_{jl}\bar{g}^{lk}. \tag{2.7}$$

Using the above, we now calculate important quantities in the "non-bar" space-time in terms of those in the "bar" space-time and η_{ij}:

$$\Gamma^i{}_{jk}=\bar{\Gamma}^i{}_{jk}+V^i{}_{jk}, \tag{2.8}$$

where $V^i{}_{jk}$ is defined by

$$V^i{}_{jk}=\Lambda^i{}_{jk}+\Theta^i{}_{jk} \tag{2.9}$$

and $\Lambda^i{}_{jk}$, $\Theta^i{}_{jk}$ are defined by

$$\Lambda^i{}_{jk}\equiv\tfrac{1}{2}\bar{g}^{il}(\eta_{lj,k}+\eta_{lk,j}-\eta_{jk,l}), \tag{2.10}$$

$$\Theta^i{}_{jk}\equiv\sigma^{il}\bar{\Gamma}_{l,jk}. \tag{2.11}$$

By definition, the Ricci tensor may be written to first order in η_{ij} as

$$R_{ij}=\bar{R}_{ij}+\Delta R_{ij}, \tag{2.12}$$

where ΔR_{ij} is defined by

$$\begin{aligned}\Delta R_{ij}\equiv{}&\partial_l V^l{}_{ij}-\partial_j V^l{}_{il}+\bar{\Gamma}^l{}_{ij}V^m{}_{lm}\\&+\bar{\Gamma}^m{}_{lm}V^l{}_{ij}-\bar{\Gamma}^l{}_{im}V^m{}_{jl}-\bar{\Gamma}^m{}_{jl}V^l{}_{im}\\&=V^l{}_{ij;l}-V^l{}_{il;j},\end{aligned} \tag{2.13}$$

[13] F. Estabrook and H. Wahlquist, J. Math. Phys. **8**, 2302 (1967).
[14] F. Pirani, *Lectures on General Relativity* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965), Chap. 3.
[15] T. Taniuti, Progr. Theoret. Phys. (Kyoto) Suppl. **9**, 69 (1959).
[16] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnetisme* (Masson et Cie, Paris, 1955).
[17] F. Estabrook and H. Wahlquist, J. Math. Phys. **8**, 2302 (1967).
[18] F. Witteborn and W. M. Fairbank, Phys. Rev. Letters **19**, 1049 (1967).

where covariant differentiation (;) occurs with respect to $\bar{g}_{ij}$. From this result, the curvature scalar is, to first order,

$$R=\bar{R}+\sigma^{lm}\bar{R}_{lm}+\bar{g}^{lm}\Delta R_{lm}. \quad (2.14)$$

Using these last results, it is possible to write (2.5) in terms of the bar space-time and η_{ij} only [using (2.4) to simplify]:

$$K(T_{ij}-\bar{T}_{ij})=\Delta R_{ij}-\tfrac{1}{2}\{\bar{g}_{ij}(\sigma^{lm}\bar{R}_{lm}+\bar{g}^{lm}\Delta R_{lm}) +\eta_{ij}(\bar{R}+\sigma^{lm}\bar{R}_{lm}+\bar{g}^{lm}\Delta R_{lm})\}. \quad (2.15)$$

Keeping only first-order terms in η_{ij}, this becomes

$$K\Delta T_{ij}=\Delta R_{ij}-\tfrac{1}{2}\bar{g}_{ij}\sigma^{lm}\bar{R}_{lm}-\tfrac{1}{2}\bar{g}^{ij}(\bar{g}^{lm}\Delta R_{lm}) -\eta_{ij}\{\bar{R}+\bar{g}^{lm}(\partial_n\Lambda^n{}_{lm}-\partial_m\Lambda^n{}_{ln})\} -\tfrac{1}{2}\bar{g}_{ij}\sigma^{lm}(\partial_n\Lambda^n{}_{lm}-\partial_m\Lambda^n{}_{ln}), \quad (2.16)$$

where

$$\Delta T_{ij}\equiv T_{ij}-\bar{T}_{ij}.$$

This is the restriction sought on the η_{ij} required by the above assumptions. Note in the vacuum case this condition is

$$\Delta R_{ij}=0$$

or

$$V^l{}_{ij;l}-V^l{}_{il;j}=0. \quad (2.17)$$

What is of real interest in the change of Petrov type is the Weyl tensor or the Riemann tensor. These must now be expressed in terms of η_{ij} and $\bar{g}_{ij}$. Again keeping only the terms of first order in η_{ij}, these may be written as

$$C_{ijkl}=\bar{C}_{ijkl}+\Delta C_{ijkl}, \quad (2.18)$$

where

$$\Delta C_{ijkl}\equiv\Delta R_{ijkl}-\tfrac{1}{2}\bar{g}_{ik}\Delta R_{lj}+\tfrac{1}{2}\bar{g}_{il}\Delta R_{kj} +\tfrac{1}{2}\bar{g}_{jk}\Delta R_{li}-\tfrac{1}{2}\bar{g}_{jl}\Delta R_{ki}-\tfrac{1}{6}\bar{g}_{il}\bar{g}_{kj}\Delta R+\tfrac{1}{6}\bar{g}_{ik}\bar{g}_{lj}\Delta R \quad (2.19)$$

and

$$R_{ijkl}=\bar{R}_{ijkl}+\Delta R_{ijkl}, \quad (2.20)$$

where

$$\Delta R_{ijkl}\equiv\partial_k(\bar{g}_{mi}\Lambda^m{}_{jl})-\partial_l(\bar{g}_{mi}\Lambda^m{}_{jk})+V^m{}_{jk}\bar{\Gamma}_{m,il}-V^m{}_{jl}\Gamma_{m,ki} +\bar{g}_{mn}(\Lambda^n{}_{il}\bar{\Gamma}^m{}_{jk}-\Lambda^m{}_{ki}\bar{\Gamma}^m{}_{jl}). \quad (2.21)$$

Now we recognize that there is a special domain of interest. The η_{ij} in the neighborhood of the null hypersurface are what change a Petrov type. In this domain, continuity says

$$\eta_{ij}=0,\quad \eta_{ij,l}=0. \quad (2.22)$$

There, ΔR_{ijkl} will become the discontinuity $[R_{ijkl}]$ itself. Using this, the conditions on the discontinuity are a good deal simpler than those for the extension. For the remainder of the paper, we consider everything to be evaluated on the null hypersurface.

The above results may be restated now, noting that

$$\partial_l V^l{}_{ij}=\partial_l\{\tfrac{1}{2}\bar{g}^{lm}(\eta_{mi,j}+\eta_{mj,i}-\eta_{ij,m})\} =\tfrac{1}{2}\bar{g}^{lm}\partial_l(\eta_{mi,j}+\eta_{mj,i}-\eta_{ij,m}) =\tfrac{1}{2}(\eta_{mi,j}{}^m+\eta_{mj,i}{}^m-\eta_{ij,m}{}^m).$$

Substitution gives

$$[R_{ij}]=\tfrac{1}{2}(\eta_{lj,i}{}^l-\eta_{ij,}{}^l{}_l-\bar{g}^{lm}\eta_{ml,ij}+\eta_{il,}{}^l{}_j). \quad (2.23)$$

The condition on these second derivatives becomes

$$(\eta_{lj,i}{}^l-\eta_{ij,}{}^l{}_l-\bar{g}^{lm}\eta_{ml,ij}+\eta_{il,}{}^l{}_j) +\bar{g}_{ij}(\eta^{lm}{}_{,lm}-\eta_{ml,n}{}^n\bar{g}^{ml})=2K[T_{ij}], \quad (2.24)$$

which, in the case $[T_{ij}]=0$, reduces to

$$\eta_{lj,i}{}^l-\eta_{ij,}{}^l{}_l-\bar{g}^{lm}\eta_{ml,ij}+\eta_{il,}{}^l{}_j=0. \quad (2.25)$$

Also under the former conditions we get

$$[R_{ijkl}]=\tfrac{1}{2}(\eta_{ik,jl}-\eta_{kj,il}-\eta_{il,jk}+\eta_{lj,ik}) \quad (2.26)$$

and

$$[C_{ijkl}]=[R_{ijkl}]-\tfrac{1}{3}(\bar{g}_{kj}\bar{g}_{il}-\bar{g}_{ki}\bar{g}_{jl})\bar{g}^{mn}[R_{mn}] +\tfrac{1}{2}K(\bar{g}_{il}[T_{kj}]-\bar{g}_{ik}[T_{lj}] -\bar{g}_{jl}[T_{ki}]+\bar{g}_{jk}[T_{li}]). \quad (2.27)$$

It is worth mentioning that these results also follow without the assumption that $|\eta_{ij}|\ll|\bar{g}_{ij}|$ if one transforms to a freely falling observer in the $\bar{g}_{ij}$ system. Also notable is that the invariant $\bar{g}^{ij}\eta_{ij}$ is zero by continuity on the hypersurface. This implies the continuity conditions on η_{ij} give the Trautman information-theory result $\eta_{ij}=F(\sigma)h_ih_j$ and shows the consistency of this definition of a wave with a discontinuous description of a front.

Further simplification of the restriction on $\eta_{ij,kl}$ may be made. The procedure is similar to what Weyl[19] did with the linearized field equations. First consider the vacuum equation (2.25). This may be written

$$-\Box^2\eta_{ij}+\tau_{ij}+\tau_{ji}=0, \quad (2.28)$$

where

$$\tau_{ij}\equiv\bar{g}^{lm}(\eta_{il,m}-\tfrac{1}{2}\eta_{ml,i})_{,j} \quad (2.29)$$

and

$$\Box^2\eta_{ij}\equiv\eta_{ij,l}{}^l=\bar{g}^{lm}\eta_{ij,lm}. \quad (2.30)$$

Now suppose ϕ_{ij} exist such that

$$\tau_{ij}=\Box^2\phi_{ij}. \quad (2.31)$$

This being so, $\eta^{(W)}{}_{ij}=\phi_{ij}+\phi_{ji}$ is the "Weyl solution." Note that the condition (2.31) is a gauge condition on ϕ_{ij}, namely,

$$\phi_{il,}{}^l{}_j+\phi_{li,}{}^l{}_j-\bar{g}^{lm}\phi_{ml,ij}=\phi_{ij,l}{}^l. \quad (2.32)$$

This is a gauge condition in the sense that it may be shown that $\eta^{(W)}{}_{ij}$ has no consequences in the curvature of the space-time (it being the observable). This will be because the $[R_{ijkl}]$ will in this case depend on the $\bar{\Gamma}^i{}_{jk}$ in such a way that the tensor will be zero in one coordinate system and hence in all. The relation of the τ_{ij} to Killing vectors does not carry over from the linearized case.

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[19] H. Weyl, *Space, Time, Matter* (Dover Publications, Inc., New York, 1950), pp. 248 ff.

We now define

$$\bar{\eta}_{ij} \equiv \eta_{ij} - \eta^{(W)}{}_{ij}, \qquad (2.33)$$

and the physically significant requirement is that

$$\Box^2 \bar{\eta}_{ij} = 0. \qquad (2.34)$$

For the nonvacuum case the Weyl-solution procedure is also possible. It results in a condition on the $\bar{\eta}_{ij}$:

$$\Box^2 \bar{\eta}_{ij} - \tfrac{1}{2} \bar{g}_{ij} \bar{g}^{lm} \Box^2 \bar{\eta}_{lm} = 2K[T_{ij}]. \qquad (2.35)$$

Does this approach separate out all the coordinate dependence? In the linear case one is likely to say so.[20] It is clear, nontheless, that there is still a possibility of a conformal transformation (which does not alter the curvature tensor). It is then only by analogy and the simple (though somewhat generalized) familiar form of the resulting equations that one believes that a good deal of the coordinate dependence is extracted.

It is trivial to show that the discontinuity of the two curvature tensors may be Petrov-typed. Knowing this and the addition theorem for types,[21] the number of possible changes of a particular space-time is limited considerably. Note that the Petrov classification is invariant to a change of sign of the entire tensor. Noting this fact, it is apparent that any allowed evolution of the form (indices suppressed)

$$A + B \rightarrow C$$

must imply that

$$B' + C \rightarrow A$$

will also be allowed, where B' differs from B in sign. The type of B and B' will be the same. It is thus apparent that whatever the most general type may be in such an evolution, it must occur twice in the relation or an impossible evolution would be implied. Using this, a flow diagram of allowed evolution may be obtained quickly (see Fig. 1).

The procedure for testing a space-time is then: Knowing the metric $\bar{g}_{ij}$ of that space-time, solve the appropriate one of the two equations for $\bar{\eta}_{ij}$ [(2.34) or (2.35)] to find the expressions for the discontinuities. These are then used to give $[C_{ijkl}]$, which may be typed. Comparison with the flow diagram indicates the consistent, or actually allowed, evolutions. It is necessary in the process of solving for the $\bar{\eta}_{ij}$ to know also what the equation for the null hypersurface in the $\bar{g}_{ij}$ space-time is. This is necessary to satisfy the conditions on the solution that η_{ij} and its first partial derivatives vanish there.

In order to check the results, the author has solved the vacuum flat-space case using the above procedure and finds, by separating the equations and assuming the procedure to be local to the observer, that only N-type discontinuities occur, as is known. In general, the complex form of $\bar{g}_{ij}$ will probably make calculations using this procedure arduous.

[20] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill Book Co., New York, 1965), pp. 250 and 251.

[21] P. Jordan, J. Ehlers, and R. Sachs, Akad. Wiss. Lit. (Mainz) Abhandl. Math.-Nat. Kl. **1**, 3 (1961), especially p. 25.

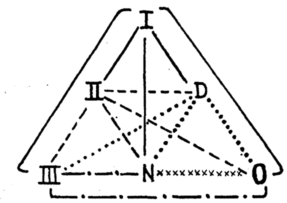

FIG. 1. Flow diagram summarizing possible changes of Petrov type. The lines connecting the various types indicate the Petrov type of the second-order discontinuity producing the change. Solid line implies type I; dashed line, type II; alternating dash and dot, type III; dotted line, type D; and the line of X's, type N.

III. SUMMARY

First, it might be pointed out that we now have equations which allow us to find exactly what type of discontinuous (or wave) front a known space-time will allow locally. Their results are also known from the flow diagram. These results specify the manner in which an observer's space-time may change its Petrov type. The existence of gravitational waves has been touched on slightly and apparently does not depend on the presence of an N-type discontinuity, but rather on whether a space-time of a given type will admit a type-changing discontinuous front of any sort. The latter is limited by the flow diagram and existence of solutions to the equations for $\bar{\eta}_{ij}$. Further, this study seems to indicate that, of the several ways to define a wave behavior, the best in such a nonlinear theory as relativity is apparently that based on front behavior and the concept of information. Some results which were previously known also follow from this approach (for example, if η_{ij} is conformal, C_{ijkl} is unchanged; and if $[T_{ij}]=0$, then $[R_{ij}]=0$).

There are several areas where related further work could be interesting. This approach might be useful (as the Petrov classification itself) in studying singularities in space-times. It might be possible to gain by studying the conditions for existence of solutions $\bar{\eta}_{ij}$ to the equations. Further understanding of the discontinuous-source case ($[T_{ij}]\neq 0$) could prove interesting.

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