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Relativistic Corrections to Nonrelativistic Two-Particle Dynamical Calculations: Demonstration of the Validity of the Drell-Hearn-Gerasimov Sum Rule for Weakly Bound Composite Particles

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The problem of the first-order relativistic corrections to the conventional nonrelativistic perturbation treatment of electromagnetic interactions of bound states is discussed. It is shown how the Thomas precession effect is usually mishandled for small excitation energies acting on a bound state. These results are demonstrated by the explicit calculation of the single-particle matrix elements of the charge density and the low-energy Compton scattering amplitude in a simple bound-state model. The amplitudes, which are known by virtue of relativistic invariance and current conservation, are also explicitly calculated, but agreement is obtained only by applying the relativistic modifications of the c.m. variables found in a previous paper. At the same time, the Drell-Hearn-Gerasimov sum rule is shown to be consistent for this model and also for simple models of the hydrogen atom and the deuteron when the constituent particles have arbitrary masses.

I. INTRODUCTION

THE calculation of the effects of electromagnetic interactions acting on particles which are components of bound states while incorporating consistently the requirements of relativistic invariance, at least to a given order in v/c, is an important problem covering the whole realm of atomic, nuclear, and strong-interaction physics. Except for the purely nonrelativistic case, where the calculational procedures to be followed are clear cut, it is perhaps fair to say that only partial solutions have been offered.

The goals that we have in mind are the determination of moderately low-energy Compton scattering or photodisintegration cross sections for bound states, and also the electromagnetic perturbations on such intrinsic bound-state parameters as its mass and coupling constant to its constituents, in terms of the assumed known electromagnetic properties of the composite constituents. In the energy range of interest here these constituents comprise the only coupled channel of any importance. By this we restrict ourselves to bound states with clearly identifiable constituents, i.e., the weakly bound systems of atomic or nuclear physics, naive bootstrap models such as the nucleon composed of itself and a pion, and quark models. (We do not pretend that quarks form the most obviously coupled channel for hadrons, but they do form, if they exist, a deeper level of elementarity: The properties of hadrons are then reducible to properties of quarks, and the calculational procedures used here can be adopted if the hadron spends most of the time in its simplest quark configuration.) In this domain, the full machinery of S-matrix theory involving the choice of invariant analytic variables, crossing, and dispersion relations seems rather complicated and inappropriate. The other method of achieving complete relativistic invariance is by using the off-mass-shell Bethe-Salpeter (BS) wave functions to represent the bound state. While the matrix elements of operators acting between a discrete bound state and other states can, in principle, be calculated in terms of these wave functions, there are problems of practicality[1] unless an instantaneous interaction (e.g., for electromagnetically bound systems using only the Coulomb force in the radiation gauge) is assumed.[1]

However, for relatively weakly bound systems in the region when the excitation energies involved are small compared with the particle masses, an expansion in terms of v/c, or effectively the inverse particle masses, should be valid. The aim here, then, is to study the relativistic corrections required on the standard nonrelativistic perturbation procedure for processes involving the electromagnetic field and bound states. Hence the basic dynamical variables used are the individual particle observables—positions, momentum, and spin—and the electromagnetic field itself. The individual particle operators act directly on the multiparticle states, which are required to transform in the correct relativistic fashion. We do not deal here with representative wave functions for the bound states, such as provided by the BS equation, so that the methods used here lack contact with an explicit field-theoretic basis (such as is contained in Ref. 1), which could be regarded as furnishing an exact treatment of the interactions of a bound state. On the other hand, we have perhaps the merits of simplicity and in the end the results are the same.

To demonstrate this approach, we apply it to testing the consistency of the Drell-Hearn-Gerasimov (DHG)[2,3] sum rule by seeing whether it is satisfied for a weakly bound state if the sum rule is valid for the constituents

[1] This approach has been adopted for the problems tackled here by S. Brodsky and J. R. Primack, Phys. Rev. **174**, 2071 (1968); Ann. Phys. (N.Y.) (to be published).

[2] S. B. Gerasimov. Yadern. Fiz. **2**, 598 (1965) [English transl.: Soviet J. Nucl. Phys. **2**, 430 (1966)]; S. D. Drell and A. C. Hearn, Phys. Rev. Letters **16**, 908 (1966). The sum rule is implicitly contained in L. I. Kapidus and Chou Kuang-Chou, Zh. Eksperim. i Teor. Fiz. **41**, 1546 (1961) [English transl.: Soviet Phys.—JETP **14**, 1102 (1962)].

[3] M. Hosoda and K. Yamamoto, Progr. Theoret. Phys. (Kyoto) **30**, 425 (1966); **30**, 426 (1966).

of the bound state.[4–7] The sum rule expresses the anomalous magnetic moment of a particle as a weighted integral over photon-absorption cross sections. For a simple bound-state model with weak binding, the anomalous magnetic moment is known in terms of the magnetic moments of the constituents, and the integral of the cross section over the low-energy region can be calculated explicitly by closure. In the high-energy region, when production processes would dominate, the photon cross section for the bound state can be taken to be the sum of the photon cross sections for the constituents.[8] The weighted integral over these cross sections is known since the sum rule is assumed to apply to each of the constituents.

This method has been used before for a variety of semirealistic models, applying conventional nonrelativistic perturbation procedures, when the claim was that the DHG sum rule did not apply to weakly bound systems. If true, this result would have cast serious doubt on the general validity of the sum rule, since as is well known, in strong-interaction physics, whether particles are composite and what can be taken as their constituents is a matter of uncertainty and perhaps of convention. But then it is difficult to imagine dynamics such that the sum rule applies to strongly and not to weakly bound states.

Nevertheless, the results of Ref. 4 were very suggestive in that, for all examples studied, there was complete cancellation in terms dependent on the anomalous magnetic moment of the constituent particles. Most remarkably, only in terms independent of the anomalous magnetic moments was the low-energy calculation of the sum rule inconsistent. Furthermore, when the same calculational methods were applied[9] to find the Compton scattering amplitude for the simplest bound-state model of Ref. 4 to first order in the photon energy, the results were in disagreement with the general low-energy theorem[10] on which the sum rule is based. The discrepancy again occurred in a term proportional to α by itself, arising from the spin-orbit part of the conventional electromagnetic-interaction Hamiltonian.

However, the magnitude of the spin-orbit term depends on the relativistic Thomas precession effect[11] which, as shown in Sec. II, is usually incorrectly treated for a bound state. Thus the main substance of this paper is the correct handling of this effect for two-body bound states. Indeed, since the low-energy Compton scattering theorem relies only on current conservation (or gauge invariance) and relativistic invariance, we have only to maintain these requirements to order v^2/c^2 to give the proper low-energy Compton amplitude for the bound state and, incidentally, to show that the DHG sum rule is consistent. The basic idea is to use the relativistic division of the single-particle dynamical variables on which the electromagnetic field acts into overall c.m. variables, which connect the ground state only to unexcited states and internal canonical c.m. variables, which have matrix elements between the ground and excited states. This relativistic c.m. decomposition has been given by us before[12] both exactly, and to first order in M^{-2} (or effectively v^2/c^2), which is the result we shall use here.

The outline of this paper then is as follows. In Sec. II we shall briefly discuss these relativistic c.m. variables and Thomas precession for a bound state. We calculate the matrix elements of the charge density between bound states in the $N\sigma$ model and demonstrate the relativistic corrections. In Sec. III the calculation of the low-energy theorem for the $N\sigma$ model is reproduced and the relativistic remedy which also cures the sum rule is applied. In Sec. IV the panacea is used on the hydrogen atom and deuteron sum rules for arbitrary constituent masses and the isotopic spin sum rules also mentioned. Finally, a few general remarks are offered in a conclusion.

II. THOMAS PRECESSION AND BOUND STATES

In this and the following sections we shall be concerned with the $N\sigma$ model[4,9] as a convenient dynamical framework. This contains three particles with the following properties:

Particle	Mass	Spin	Charge	Magnetic moment
N	M	$\frac{1}{2}$	e	$\mu=(e/2M)(1+\kappa)$
σ	m	0	0	0
N'	$M'\approx M+m$	$\frac{1}{2}$	e	$\mu=(e/2M')(1+\kappa')$

N' is a weakly bound $S_{1/2}$ state of N and σ with binding energy $\beta\ll M$, m. For the N particle the electromagnetic interaction Hamiltonian would have been written before 1926[11] as

$$\begin{aligned} H_{I\,\text{nonrelativistic}}=&\,e\Phi(\mathbf{r}_N)-(e/2M)\\ &\times[\mathbf{A}(\mathbf{r}_N)\cdot\mathbf{p}_N+\mathbf{p}_N\cdot\mathbf{A}(\mathbf{r}_N)]+(e^2/2M)\mathbf{A}^2(\mathbf{r}_N)\\ &-\mu\boldsymbol{\sigma}_N\cdot\mathbf{H}(\mathbf{r}_N)-(\mu/2M)\{\boldsymbol{\sigma}_N\cdot[\mathbf{E}(\mathbf{r}_N)\times\mathbf{p}_N]\\ &-\boldsymbol{\sigma}_N\cdot[\mathbf{p}_N\times\mathbf{E}(\mathbf{r}_N)]+2e\boldsymbol{\sigma}_N\cdot[\mathbf{A}(\mathbf{r}_N)\times\mathbf{E}(\mathbf{r}_N)]\}\,, \end{aligned} \quad (2.1)$$

with $\mathbf{r}_N$, $\mathbf{p}_N$, and $\boldsymbol{\sigma}_N$ as the canonical position, momentum, and spin variables, respectively, for the N particle. Each of the terms in (2.1) has a simple interpretation, the last (spin-orbit term) arising from a moving magnetic dipole in an electric field seeing a magnetic field. As is well known, (2.1) neglects the celebrated Thomas

[4] G. Barton and N. Dombey, Phys. Rev. **162**, 1520 (1967).
[5] S. D. Drell and J. Primack (unpublished).
[6] H. R. Pagels, Phys. Rev. **158**, 1566 (1967), Appendix B.
[7] G. Konisi and K. Yamamoto, Progr. Theoret. Phys. (Kyoto) **37**, 538 (1967).
[8] This additivity hypothesis should be valid for weak binding and to first order in $\alpha=e^2/4\pi=1/137$, which is the accuracy required in the calculations.
[9] G. Barton (unpublished).
[10] F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *ibid.* **96**, 1433 (1954).
[11] L. H. Thomas, Nature **117**, 514 (1926); also J. Frenkel, Z. Physik **37**, 243 (1926).
[12] H. Osborn, preceding paper, Phys. Rev. **176**, 1514 (1968).

precession effect. Because of purely relativistic kinematics alone, the spin of the N particle would not be a constant of motion but obeys the equation[13]

$$\frac{d}{dt}\boldsymbol{\sigma}_N=\frac{1}{1+1/\gamma}(\mathbf{v}_N\times\mathbf{a}_N)\times\boldsymbol{\sigma}_N$$
$$=\tfrac{1}{2}(\mathbf{v}_N\times\mathbf{a}_N)\times\boldsymbol{\sigma}_N+O(v_N^3)\,,\qquad(2.2)$$
$$\gamma=(1-\mathbf{v}_N^2)^{-1/2},\ c=1$$

where $\mathbf{v}_N$ and $\mathbf{a}_N=(d/dt)\mathbf{v}_N$ are the velocity and acceleration of the N particle. This time dependence in our basically nonrelativistic formalism can be ascribed to an additional interaction term in the Hamiltonian,

$$V_T=-\tfrac{1}{4}\boldsymbol{\sigma}_N\cdot(\mathbf{v}_N\times\mathbf{a}_N)+O(v_N^3)\,.\qquad(2.3)$$

Now for a *free* N particle,

$$\mathbf{v}_N=(1/M)[\mathbf{p}_N-e\mathbf{A}(\mathbf{r}_N)]+O(M^{-3})\,,$$
$$\mathbf{a}_N=e\mathbf{E}(\mathbf{r}_N)/M+O(M^{-3})\,,\qquad(2.4)$$

so that inserting this into (2.3) and symmetrizing, we have

$$V_T=(e/8M^2)\{\boldsymbol{\sigma}_N\cdot[\mathbf{E}(\mathbf{r}_N)\times\mathbf{p}_N]-\boldsymbol{\sigma}_N\cdot[\mathbf{p}_N\times\mathbf{E}(\mathbf{r}_N)]$$
$$+2e\boldsymbol{\sigma}_N\cdot[\mathbf{A}(\mathbf{r}_N)\times\mathbf{E}(\mathbf{r}_N)]\}+O(M^{-4})\,.\qquad(2.5)$$

Adding (2.5) to (2.1) changes the coefficient of the spin-orbit term to

$$-\frac{1}{4M}\left(2\mu-\frac{e}{2M}\right)=-\frac{e}{8M^2}(1+2\kappa)\,.$$

The result, if the Darwin term is also inserted, is then just the same as applying the Foldy-Wouthuysen (FW) transformation to the Dirac electromagnetic interaction Hamiltonian to order M^{-2} and is the conventional interaction commonly used,

$$H_I=e\Phi(\mathbf{r}_N)-(e/2M)[\mathbf{A}(\mathbf{r}_N)\cdot\mathbf{p}_N+\mathbf{p}_N\cdot\mathbf{A}(\mathbf{r}_N)]$$
$$+(e^2/2M)\mathbf{A}^2(\mathbf{r}_N)-\mu\boldsymbol{\sigma}_N\cdot\mathbf{H}(\mathbf{r}_N)-(1/2M)$$
$$\times(\mu-e/4M)\{\nabla_{\mathbf{r}_N}\cdot\mathbf{E}(\mathbf{r}_N)+\boldsymbol{\sigma}_N\cdot[\mathbf{E}(\mathbf{r}_N)\times\mathbf{p}_N]$$
$$-\boldsymbol{\sigma}_N\cdot[\mathbf{p}_N\times\mathbf{E}(\mathbf{r}_N)]+2e\boldsymbol{\sigma}_N\cdot[\mathbf{A}(\mathbf{r}_N)\times\mathbf{E}(\mathbf{r}_N)]\}\,.\qquad(2.6)$$

It is now easy to see how (2.6) will give mistaken results when applied to the N' bound state since, at very low excitation energies compared to the binding energy, the N and σ particles can be regarded as rigidly coupled with an effective mass M'. Hence (2.4) is changed to

$$\mathbf{v}_N=(1/M)[\mathbf{p}_N-e\mathbf{A}(\mathbf{r}_N)]+O(M^{-3})\,,$$

but

$$\mathbf{a}_N=e\mathbf{E}(\mathbf{r}_N)/M'+O(M^{-3})\,,\qquad(2.7)$$

for $\omega\ll\beta$, where ω is the energy of excitation supplied by the electromagnetic field. With (2.7), the coefficient of V_T is changed to $e/8MM'$ and the effective interaction Hamiltonian for very low energies should have a coefficient of the spin-orbit term

$$-\frac{1}{4M}\left(2\mu-\frac{e}{2M'}\right)=-\frac{e}{8M^2}\left(1+2\kappa+\frac{m}{M'}\right).$$

While these arguments are sufficient to show the presence of a difficulty in the treatment of Thomas precession for particles in bound states,[14] they are by no means sufficient to permit dynamical calculations with interaction energies of the same order as the binding energies. Before addressing ourselves to the solution of this problem, we first show how the Thomas-precession difficulties occur in a simple observable matrix element. The charge and current densities corresponding to the usual FW interaction Hamiltonian (2.6) are, regarding $\mathbf{A}$, Φ, and $\mathbf{E}$ for the moment as external c-number fields,

$$\rho(\mathbf{x})=\delta H_I/\delta\Phi(\mathbf{x})=e\delta(\mathbf{r}_N-\mathbf{x})-(1/4M)(2\mu-e/2M)\{-\nabla_{\mathbf{r}_N}{}^2\delta(\mathbf{r}_N-\mathbf{x})+\boldsymbol{\sigma}_N\cdot[\mathbf{p}_N\times\nabla_{\mathbf{r}_N}\delta(\mathbf{r}_N-\mathbf{x})]$$
$$-\boldsymbol{\sigma}_N\cdot\nabla_{\mathbf{r}_N}\delta(\mathbf{r}_N-\mathbf{x})\times\mathbf{p}_N-2e\boldsymbol{\sigma}_N\cdot[\mathbf{A}(\mathbf{r}_N)\times\nabla_{\mathbf{r}_N}\delta(\mathbf{r}_N-\mathbf{x})]\}\,,$$
$$\mathbf{j}(\mathbf{x})=-\frac{\delta H_I}{\delta\mathbf{A}(\mathbf{x})}-\frac{d}{dt}\frac{\delta H_I}{\delta\mathbf{E}(\mathbf{x})}=(e/2M)\{\mathbf{p}_N,\,\delta(\mathbf{r}_N-\mathbf{x})\}-(e^2/M)\mathbf{A}^2(\mathbf{r}_N)\delta(\mathbf{r}_N-\mathbf{x})+\mu\boldsymbol{\sigma}_N\times\nabla_{\mathbf{r}_N}\delta(\mathbf{r}_N-\mathbf{x})\qquad(2.8)$$
$$-(e/2M)(2\mu-e/2M)\boldsymbol{\sigma}_N\times\mathbf{E}(\mathbf{r}_N)\delta(\mathbf{r}_N-\mathbf{x})+(1/4M)(2\mu-e/2M)\frac{d}{dt}[\nabla_{\mathbf{r}_N}\delta(\mathbf{r}_N-\mathbf{x})$$
$$-\boldsymbol{\sigma}_N\times\mathbf{p}_N\delta(\mathbf{r}_N-\mathbf{x})-\delta(\mathbf{r}_N-\mathbf{x})\boldsymbol{\sigma}_N\times\mathbf{p}_N+2e\boldsymbol{\sigma}_N\times\mathbf{A}(\mathbf{r}_N)\delta(\mathbf{r}_N-\mathbf{x})]$$

to order M^{-2}. For current conservation to this order we expect

$$i[H_0+H_I,\,\rho(\mathbf{x})]+\frac{\partial\rho(\mathbf{x})}{\partial t}+\nabla_{\mathbf{x}}\cdot j(\mathbf{x})=O(M^{-3})\,.\qquad(2.9)$$

The partial time derivative refers to the explicit time dependence of the external fields; H_0 is the total Hamiltonian of the system in the absence of electromagnetic interactions. In terms proportional to e^2, (2.9)

[13] For a modern derivation see A. Chakrabarti, J. Math. Phys. 5, 1747 (1964). This equation is also implicitly contained in Appendix A of Ref. 12.

[14] It is perhaps fortunate that atomic electrons are so nearly free; otherwise, the g factor of an electron might have appeared even more anomalous in the early days of quantum mechanics.

is immediately satisfied (we take $\mu\sim e/M$), but for the part proportional to e, it is necessary that

$$i[H_0,\,\delta(\mathbf{r}_N-\mathbf{x})]=(1/2M)[\mathbf{p}_N\cdot\nabla_{\mathbf{r}_N}\delta(\mathbf{r}_N-\mathbf{x})+\nabla_{\mathbf{r}_N}\delta(\mathbf{r}_N-\mathbf{x})\cdot\mathbf{p}_N]+O(M^{-3}),$$

or, alternatively,

$$i[H_0,\mathbf{r}_N]=\mathbf{p}_N/M+O(M^{-3}). \qquad (2.10)$$

Equation (2.10) is automatically satisfied for a free N particle when $H_0=M+\mathbf{p}_N{}^2/2M+O(M^{-3})$; otherwise we must require that in the interaction picture $\mathbf{p}_N=M\dot{\mathbf{r}}_N+O(M^{-2})$.

After this disgression we now consider the matrix element of the charge density between single-N'-particle states,

$$(2\pi)^3\langle N'\mathbf{p}_2s_2|\rho(\mathbf{0})|N'\mathbf{p}_1s_1\rangle =u^\dagger(s_2)\bigg[e-\frac{1}{4M'}\bigg(2\mu-\frac{e}{2M'}\bigg)(\mathbf{p}_2-\mathbf{p}_1)^2+\frac{1}{2M'}\bigg(2\mu-\frac{e}{2M'}\bigg)i\boldsymbol{\sigma}\cdot(\mathbf{p}_2\times\mathbf{p}_1)\bigg]u(s_1), \qquad (2.11)$$

where $u(s)$ is an ordinary two-component nonrelativistic spinor. The right-hand side of (2.11) is obtained by taking the usual relativistic form for the matrix element of the charge current density 4-vector between single-spin-$\frac{1}{2}$ particle states and expanding consistently to order M^{-2} [including the invariant normalization factor $(M^2/E_2E_1)^{1/2}$]. The momentum dependence of the Dirac and Pauli form factors F_1 and F_2 has been neglected since these will be unimportant here.

Given the form (2.8) for $\rho(\mathbf{x})$, the matrix element can be explicitly calculated. All that is required is the expansion of the single-particle dynamical variables in terms of c.m. variables. Nonrelativistically,

$$\mathbf{p}_N=(M/M')\mathbf{P}+\mathbf{q}\,,\quad \mathbf{r}_N=\mathbf{R}+(m/M')\mathbf{r}\,,\quad \boldsymbol{\sigma}_N=\boldsymbol{\sigma}. \qquad (2.12)$$

This decomposition has the crucial property that the momentum dependence of a two-particle state can be factored off, e.g., $|N'\mathbf{p}s\rangle=|\mathbf{p}\rangle\otimes|N's\rangle$, where the total momentum and position operator $\mathbf{P}$ and $\mathbf{R}$ act only on the over-all momentum part $|\mathbf{p}\rangle$ and the internal momentum, position, and spin variables $\mathbf{q}$, $\mathbf{r}$, and $\boldsymbol{\sigma}$ act only on the translationally invariant part $|N's\rangle$. Writing (2.8) as

$$\rho(\mathbf{x})=e\delta(\mathbf{r}_N-\mathbf{x})+\delta\rho(\mathbf{x})\,, \qquad (2.13)$$

and then supposing $\Phi=0$ and $\mathbf{E}$, $\mathbf{A}$ are second-quantized fields from now on, applying (2.12) gives

$$(2\pi)^3\langle N'\mathbf{p}_2s_2|\delta\rho(\mathbf{0})|N'\mathbf{p}_1s_1\rangle=\langle N's_2|e^{i(M/M')(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{r}}\times\bigg[-\frac{1}{4M}\bigg(2\mu-\frac{e}{2M}\bigg)(\mathbf{p}_2-\mathbf{p}_1)^2+\frac{1}{2M}\bigg(2\mu-\frac{e}{2M}\bigg)i\boldsymbol{\sigma}\cdot[(\mathbf{p}_2-\mathbf{p}_1)\times\mathbf{q}]+\frac{1}{2M'}\bigg(2\mu-\frac{e}{2M}\bigg)i\boldsymbol{\sigma}\cdot(\mathbf{p}_2\times\mathbf{p}_1)\bigg]|N's_1\rangle$$

$$\approx u^\dagger(s_2)\bigg[-\frac{1}{4M}\bigg(2\mu-\frac{e}{2M}\bigg)(\mathbf{p}_2-\mathbf{p}_1)^2+\frac{1}{2M'}\bigg(2\mu-\frac{e}{2M}\bigg)\times i\boldsymbol{\sigma}\cdot(\mathbf{p}_2\times\mathbf{p}_1)\bigg]u(s_1). \qquad (2.14)$$

In the last line, the exponential factor which generates the form factor has been neglected, and also $\langle N's_2|\mathbf{q}|N's_1\rangle=\mathbf{0}$. Using the same nonrelativistic procedures on the single δ-function term in (2.13),

$$(2\pi)^3\langle N'\mathbf{p}_2s_2|e\delta(\mathbf{r}_N)|N'\mathbf{p}_1s_1\rangle =\langle N's_2|e\,e^{i(m/M')(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{r}}|N's_1\rangle \approx u^\dagger(s_2)eu(s_1). \qquad (2.15)$$

Now the sum of (2.15) and (2.14) does not agree with the general form (2.11). Agreement should not be expected in the spin-independent $\sim M^{-2}$ terms since the form factor F_1 contributes in this order, but the spin-dependent $\sim M^{-2}$ terms should agree, being of zero order in the momentum dependence of F_2. It can be seen that the necessary correction is exactly that corresponding to the effective low-energy modification of the Thomas precession terms in H_I.

Our approach to the required changes in the above calculations has been to study the relativistic corrections to the c.m. decomposition (2.12). In Ref. 12 we have shown that relativistically in the representation of the generators of the Poincaré group where the individual-particle momentum, position, and spin operators are the basic dynamical variables (i.e., for spin-$\frac{1}{2}$ systems after FW transformation and so when H_I is the correct electromagnetic interaction), then for any two-particle state $|f\rangle$ which is an eigenstate of momentum $\mathbf{p}_f$ and of the mass operator with eigenvalue M_f

$$|f\rangle=|\mathbf{p}_f\rangle\otimes|f_{\text{int}}\rangle\,,\ |\mathbf{p}_f\rangle=e^{i\mathbf{p}_f\cdot\mathbf{R}}|\mathbf{0}\rangle. \qquad (2.16)$$

$|f_{\text{int}}\rangle$ is the internal two-particle state which is acted upon by the internal c.m. variables; it is invariant under translation, and boosts in the direction of motion. $|\mathbf{p}_f\rangle$ acts under Lorentz transformations, or the total momentum $\mathbf{P}$ and position operator $\mathbf{R}$, as a single particle of mass M_f.

To order M^{-2} (reckoning interchangeably also in powers of m and M'), we have corresponding to (2.12) from Fq. (4.5) of Ref. 12

$$\mathbf{p}_N=\frac{M}{M'}\mathbf{P}+\mathbf{q}+\left(\frac{m-M}{2M'Mm}\mathbf{q}^2+\frac{1}{2M'^2}\mathbf{P}\cdot\mathbf{q}\right)\mathbf{q},$$

$$\mathbf{r}_N=\mathbf{R}+(m/M')\mathbf{r}+\tfrac{1}{2}\mathbf{r}\left(\frac{M-m}{2M'Mm}\mathbf{q}^2-\frac{1}{2M'^2}\mathbf{P}\cdot\mathbf{q}\right)+\text{H.c.}$$

$$+\tfrac{1}{2}\mathbf{r}\cdot\mathbf{P}\frac{1}{M'^2}\left(\tfrac{1}{2}\mathbf{q}-\frac{m}{M}\mathbf{q}-\frac{m}{2M}\mathbf{P}\right)+\text{H.c.} \quad (2.17)$$

$$-\frac{1}{4M'M}\mathbf{q}\times\boldsymbol{\sigma}+\frac{m}{4M'^2M}\mathbf{P}\times\boldsymbol{\sigma},$$

$$\boldsymbol{\sigma}_N=\boldsymbol{\sigma}+(1/2MM')(\mathbf{q}\times\mathbf{P})\times\boldsymbol{\sigma}.$$

The relativistic correction will not affect the calculation of the matrix element of $\delta\rho$ which is itself of order M^{-2}. However, (2.15) is modified to order M^{-2} by using the form (2.17) for $\mathbf{r}_N$:

$$(2\pi)^3\langle N'\mathbf{p}_2s_2|e\delta(\mathbf{r}_N)|N'\mathbf{p}_1s_1\rangle=\langle N's_2|e\, e^{i(M/M')(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{r}}$$
$$\times\{1+O(M^{-2})-(1/4M'M)i\boldsymbol{\sigma}\cdot[(\mathbf{p}_2-\mathbf{p}_1)\times\mathbf{q}]$$
$$+(m/4M'^2M)i\boldsymbol{\sigma}\cdot(\mathbf{p}_2\times\mathbf{p}_1)\}|N's_1\rangle, \quad (2.18)$$

where the $O(M^{-2})$ term in parentheses is spin-independent. Proceeding as before, neglecting form-factor contributions, we obtain

$$(2\pi)^3\langle N'\mathbf{p}_2s_2|e\delta(\mathbf{r}_N)|N'\mathbf{p}_1s_1\rangle\approx u^\dagger(s_2)e$$
$$\times[1+(m/4M'^2M)i\boldsymbol{\sigma}\cdot(\mathbf{p}_2\times\mathbf{p}_1)]u(s_1). \quad (2.19)$$

This, combined with (2.14), gives the desired spin-dependent term found in (2.11), showing how our c.m. variable modification correctly treats Thomas precession for weakly bound states, at least in this simple case.

III. LOW-ENERGY THEOREMS AND THE DHG SUM RULE FOR THE $N\sigma$ MODEL

As is well known, the amplitude for Compton scattering off spin-$\frac{1}{2}$ particle states at rest, to zero and first order in the photon energy ω, is given by a general theorem, at least to order e^2, due to Low, and Gell-Mann and Goldberger,[10] in terms of just the charge, magnetic moment, and mass of the spin-$\frac{1}{2}$ particle. Any method of calculation incorporating the requirements of relativistic invariance and current conservation on which the theorem is based should reproduce this result. We apply conventional second-order perturbation theory using the FW electromagnetic interaction Hamiltonian (2.6) for photon scattering off a N' particle at rest.[15]

$$u^\dagger(s')fu(s)\delta(\mathbf{p}'+\mathbf{k}'-\mathbf{k})$$
$$=-(2\pi)^3(4\omega'\omega)^{1/2}(E'/M')^{1/2}\langle N'\mathbf{p}'s',\mathbf{k}'\lambda'|$$
$$\times[H_I+H_I(M'+\omega-H_0)^{-1}H_I]|N'\mathbf{0}s,\mathbf{k}\lambda\rangle, \quad (3.1)$$
$$(M'^2+\mathbf{p}'^2)^{1/2}=E'=M'+\omega-\omega',\ \omega=|\mathbf{k}|,\ \omega'=|\mathbf{k}'|,$$

with $\mathbf{k},\lambda$ and $\mathbf{k}',\lambda'$ the momentum, helicity of the initial and final photon states. In the nonrelativistic regime in which the FW transformation is expected to be valid, $\omega\ll M$, $\omega'=\omega+\omega O(\omega/M)$. The first-order or direct contribution to the amplitude f results from those parts of H_I quadratic in the radiation field.

$$u^\dagger(s')f_{\text{direct}}u(s)\delta(\mathbf{p}'+\mathbf{k}'-\mathbf{k})=-\langle N'\mathbf{p}'s'|e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_N}$$
$$\times\left[\frac{e^2}{M}\boldsymbol{\varepsilon}'^*\cdot\boldsymbol{\varepsilon}-\frac{e}{M}\left(2\mu-\frac{e}{2M}\right)i\omega\boldsymbol{\sigma}_N\cdot\boldsymbol{\varepsilon}'^*\times\boldsymbol{\varepsilon}\right]|N'\mathbf{0}s\rangle$$
$$+\frac{1}{M}O\left(\frac{\omega^2}{M^2}\right). \quad (3.2)$$

$\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}'$ are the polarization vectors corresponding to the initial and final helicity states λ, λ'. In the exponential factor $\mathbf{r}_N=\mathbf{R}+(m/M')\mathbf{r}$ and in any matrix element the dependence on $\mathbf{R}$ can be factored off, giving rise to the overall momentum conservation δ function. The $\mathbf{r}$ dependence gives rise to retardation. The expectation value of $|\mathbf{r}|$ is $\sim(2M_R\beta)^{-1/2}$, where $M_R=mM/M'$ is the reduced mass. For $\omega\ll(M\beta)^{1/2}$, the exponentials can be expanded and, taking $\langle N's'|\mathbf{r}|N's\rangle=\mathbf{0}$ by parity arguments, we have

$$f_{\text{direct}}=-\frac{e^2}{M}T_1+\frac{e}{M}\left(2\mu-\frac{e}{2M}\right)T_2+\frac{1}{M}O\left(\frac{\omega^2}{M(M\beta)^{1/2}}\right), \quad (3.3)$$
$$T_1=\boldsymbol{\varepsilon}'^*\cdot\boldsymbol{\varepsilon},\quad T_2=i\omega\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}'^*\times\boldsymbol{\varepsilon}.$$

Any relativistic corrections are at least $\sim M^{-2}$, so necessarily by dimensional reasons they must give contributions to f_{direct} of higher than first order in ω. Similarly, it is only necessary to use the FW electromagnetic interaction Hamiltonian H_I to order M^{-2}.

In the second-order perturbation term in (3.1), only parts of H_I linear in the field variables contribute to order e^2. The sum over intermediate states can be separated into unexcited single-particle and excited states of the bound-state system. For unexcited states the denominator in (3.1) is $\sim\omega$, while the single-particle matrix elements of the electric and magnetic parts of H_I are at least $\sim\omega$, the spin-orbit single-particle matrix element is $\sim\omega^2$ and so is neglected. Adding also the intermediate contribution of unexcited bound states plus two photons so as to restore crossing symmetry and using the nonrelativistic c.m. decomposition for the

[15] The nonrelativistic part of the calculation for the low-energy theorem is reproduced from Ref. 9.

same reasons as in f_{direct}, we obtain

$$u^\dagger(s')f_{\text{unexcited}}u(s)=-\frac{1}{\omega}\sum_{s''}\Bigg[\langle N's'|e^{-i(m/M')\mathbf{k}'\cdot\mathbf{r}}\times\left(\frac{e}{M'}\boldsymbol{\varepsilon}'^*\cdot\mathbf{k}+\frac{e}{M}\boldsymbol{\varepsilon}'^*\cdot\mathbf{q}+\mu i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}'^*\times\mathbf{k}\right)|N's''\rangle\times\langle N's''|e^{i(m/M')\mathbf{k}\cdot\mathbf{r}}\left(\frac{e}{M}\boldsymbol{\varepsilon}\cdot\mathbf{q}-\mu i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\times\mathbf{k}\right)|N's\rangle-\begin{pmatrix}\boldsymbol{\varepsilon}\leftrightarrow\boldsymbol{\varepsilon}'^*\\ \mathbf{k}\leftrightarrow-\mathbf{k}'\end{pmatrix}\Bigg]+\frac{1}{M}O\left(\frac{\omega^2}{M^2}\right). \quad (3.4)$$

Neglecting retardation, and using $\langle N's'|\mathbf{q}|Ns\rangle=0$ [16] and the result that the spin operator $\boldsymbol{\sigma}$ (also the total momentum $\mathbf{P}$ and position operator $\mathbf{R}$) do not connect the ground state to excited states, we get

$$f_{\text{unexcited}}=-2\mu^2T_3-\frac{e\mu}{M'}T_4+\frac{1}{M}O\left(\frac{\omega^2}{M(M\beta)^{1/2}}\right),$$
$$T_3=i\omega\boldsymbol{\sigma}\cdot[(\boldsymbol{\varepsilon}'^*\times\hat{k}')\times(\boldsymbol{\varepsilon}\times\hat{k})],$$
$$T_4=i\omega[\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}'^*\times\hat{k}')\boldsymbol{\varepsilon}\cdot\hat{k}'-\boldsymbol{\varepsilon}'^*\cdot\hat{k}\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}\times\hat{k})]. \quad (3.5)$$

The result (3.5) is just that found generally by Low for the single-particle intermediate-state contribution to f. In his argument the remaining terms are dictated by gauge invariance. Here we procede to directly evaluate the excited intermediate-state contribution to f.

The matrix elements of $H_M=-\mu\boldsymbol{\sigma}_N\cdot\mathbf{H}(\mathbf{r}_N)$ between the ground and excited states are $\sim\omega^2$ so that if $\omega\ll\beta$, when the denominator in (3.1) is not proportional to ω as the excited state, threshold occurs at $E_f-M'=\beta$, this magnetic term does not contribute to f_{excited} to order ω. Only the electric and spin-orbit interactions are important. For the electric part of H_I, which has matrix elements of zero order in ω, we obtain

$$u^\dagger(s')f_{\text{excited }E}\,u(s)\delta(\mathbf{p}'+\mathbf{k}'-\mathbf{k})=(2\pi)^3(4\omega'\omega)^{1/2}\sum_{f\neq N'}\Bigg[\langle N'\mathbf{p}'s',\mathbf{k}'\lambda'|H_E|f\rangle\times\langle f|H_E|N'\mathbf{0}s,\mathbf{k}\lambda\rangle\left(\frac{1}{E_f-M'}+\frac{\omega}{(E_f-M')^2}\right)+\begin{pmatrix}\omega\leftrightarrow-\omega\\ \mathbf{k}\leftrightarrow-\mathbf{k}'\\ \boldsymbol{\varepsilon}\leftrightarrow\boldsymbol{\varepsilon}'^*\end{pmatrix}\Bigg], \quad (3.6)$$

neglecting terms of relative order ω/β and restricting $\sum_f$ to zero-photon states. The matrix elements in question are given for no retardation by

$$\langle f|H_E|N'\mathbf{p}s,\mathbf{k}\lambda\rangle(2\pi)^{3/2}(2\omega)^{1/2}\approx-(e/M)\times\langle f|\boldsymbol{\varepsilon}\cdot\mathbf{p}_N|N'\mathbf{p}+\mathbf{k}s\rangle. \quad (3.7)$$

To eliminate the summation over intermediate states in (3.6) by closure, we apply to (3.7) the current-conservation requirement (2.10). With zero retardation, to the same degree of approximation, (3.6) becomes

$$u^\dagger(s')f_{\text{excited }E1}u(s)\delta(\mathbf{p}'+\mathbf{k}'-\mathbf{k})=e^2\sum_{f\neq N'}[(i/M)(\langle N'\mathbf{p}'+\mathbf{k}'s'|\boldsymbol{\varepsilon}'^*\cdot\mathbf{p}_N|f\rangle\times\langle f|\boldsymbol{\varepsilon}\cdot\mathbf{r}_N|N'\mathbf{k}s\rangle-\langle N'\mathbf{p}'-\mathbf{k}s'|\boldsymbol{\varepsilon}\cdot\mathbf{r}_N|f\rangle\times\langle f|\boldsymbol{\varepsilon}'^*\cdot\mathbf{p}_N|N'-\mathbf{k}'s\rangle)+\omega(\langle N'\mathbf{p}'+\mathbf{k}'s'|\boldsymbol{\varepsilon}'^*\cdot\mathbf{r}_N|f\rangle\times\langle f|\boldsymbol{\varepsilon}\cdot\mathbf{r}_N|N'\mathbf{k}s\rangle-\langle N'\mathbf{p}'-\mathbf{k}s'|\boldsymbol{\varepsilon}\cdot\mathbf{r}_N|f\rangle\times\langle f|\boldsymbol{\varepsilon}'^*\cdot\mathbf{r}_N|N'-\mathbf{k}'s\rangle)]. \quad (3.8)$$

Since the summation in (3.8) is only over excited states, those parts of $\mathbf{p}_N$ and $\mathbf{r}_N$ which connect the ground state to unexcited states must be neglected before closure is applied and the summation eliminated. In the first set of terms in (3.8), the nonrelativistic c.m. decomposition is sufficient since they are already $\sim M^{-1}$. The second set of terms is zero nonrelativistically, but the relativistic c.m. decomposition (2.17) contributes to order M^{-2}. To first order in ω, (3.8) reduces to a set of commutators, the nonzero terms being

$$u^\dagger(s')f_{\text{excited }E1}u(s)=e^2\langle N's'|\{(m/MM')i[\boldsymbol{\varepsilon}'^*\cdot\mathbf{q},\boldsymbol{\varepsilon}\cdot\mathbf{r}]-\omega(m/4M'^2M)([\boldsymbol{\varepsilon}'^*\cdot(\mathbf{q}\times\boldsymbol{\sigma}),\boldsymbol{\varepsilon}\cdot\mathbf{r}]+[\boldsymbol{\varepsilon}'^*\cdot\mathbf{r},\boldsymbol{\varepsilon}\cdot(\mathbf{q}\times\boldsymbol{\sigma})])\}|N's\rangle. \quad (3.9)$$

It is not difficult to show that the once retarded matrix elements are zero in the first term of (3.6), so that

$$f_{\text{excited }E}=e^2\frac{m}{MM'}T_1+e^2\frac{m}{2MM'^2}T_2+\frac{1}{M}O\left(\frac{\omega^2}{M\beta}\right). \quad (3.10)$$

There remains only to evaluate that part of f_{excited} which results from the spin-orbit interaction term in H_I. The matrix element between the ground state and excited states for zero retardation is

$$\langle f|H_{\text{s-o}}|N'\mathbf{p}s,\mathbf{k}\lambda\rangle(2\pi)^{3/2}(2\omega)^{1/2}\approx-(1/2M)\times(2\mu-e/2M)i\omega\langle f|\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}\times\mathbf{p}_N)|N'\mathbf{p}+\mathbf{k}s\rangle. \quad (3.11)$$

This is already of order ω, so that for $\omega\ll\beta$ in an expansion of the form (3.6), only the first term is necessary here. Further, only the cross terms between the spin-orbit matrix element (3.11) and the $E1$ matrix element (3.7) are of the required first order in ω. This expression is $\sim M^{-2}$, so that a nonrelativistic c.m. decomposition

[16] Lest it should be thought that the once retarded terms might give a contribution $\sim\omega$ here, $\langle N's'|\epsilon\cdot\mathbf{q}\mathbf{k}\cdot\mathbf{r}|N's\rangle=0$ since the N' orbital wave function is in a $S_{1/2}$ configuration which does not define a direction. This matrix element must be proportional to $\epsilon\cdot\mathbf{k}=0$.

is sufficient.

$$u^\dagger(s')f_{\text{excited }s-o}u(s)=(e/2M)(2\mu-e/2M)(m/M')$$
$$\times\omega\langle N's'|\{[\boldsymbol{\varepsilon}'^*\cdot\mathbf{r},\boldsymbol{\sigma}(\boldsymbol{\varepsilon}\times\mathbf{q})]-[\boldsymbol{\varepsilon}\cdot\mathbf{r},\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}'^*\times\mathbf{q})]\}|N's\rangle,$$

$$f_{\text{excited s-o}}=-\frac{e}{M}\left(2\mu-\frac{e}{2M}\right)\frac{m}{M'}T_2+\frac{1}{M}O\left(\frac{\omega^2}{M\beta}\right). \quad (3.12)$$

Hence for $\omega\ll\beta$ to first order in ω, adding (3.3), (3.5), (3.10), and (3.12),

$$f=-\frac{e^2}{M'}T_1+\frac{e}{M'}\left(2\mu-\frac{e}{2M'}\right)T_2-2\mu^2T_3-\frac{e\mu}{M'}T_4, \quad (3.13)$$

in accordance with the low-energy theorem.[17] If relativistic corrections had not been introduced the coefficient of T_2 would have been altered to $(e/M')\times(2\mu-e/2M)$, which would have had to be corrected by the effective low-energy modification of the Thomas precession term in the interaction.

However, this modification would not suffice to validate the DHG sum rule for this model. Indeed, this should not be expected since the sum rule involves an integral over all photon energies. Following Ref. 4, the sum rule can be written

$$\frac{2\pi^2\alpha}{M'^2}\kappa'^2=\int_0^\infty\frac{d\omega}{\omega}[\sigma_{N'}{}^P(\omega)-\sigma_{N'}{}^A(\omega)]$$
$$\equiv J_{N'}\approx K+J_N+J_\sigma, \quad (3.14)$$

where $\sigma_{N'}{}^{P,A}(\omega)$ are the total absorption cross sections for photons of energy ω striking an N' particle at rest with the photon spin parallel, antiparallel to the initial spin of the N' particle. The integral over these cross sections in (3.14), $J_{N'}$, is decomposed into (i) a low-energy part K due to photodisintegration of the bound state into its constituents, for which the threshold is $\omega=\beta$, and (ii) a basically high-energy part J_N+J_σ resulting from the independent, effectively free-particle, scattering of the photons by the particles N and σ in the bound state. J_N and J_σ are then integrals of the same form as $J_{N'}$ over the N- and σ-particle photon-absorption cross sections. Since the σ particle is spinless, $J_\sigma=0$, and assuming the sum rule to hold for the N particle, $J_N=(2\pi^2\alpha/M^2)\kappa^2$. Then (3.14) becomes

$$\frac{2\pi^2\alpha}{M^2}\left(2\frac{m}{M'}\kappa+\frac{m^2}{M'^2}\right)\approx K. \quad (3.15)$$

To calculate K, the same procedure as used for the ow-energy theorem is followed working consistently to zero order in β/M [or rather $(\beta/M_R)^{1/2}$ since retardation effects are being neglected]. Defining

$$K^{s,\lambda}=\int_0^\infty\frac{d\omega}{\omega}\sigma_{N'}{}^{s,\lambda}(\omega), \quad (3.16)$$

where $\sigma^{s,\lambda}$ is the cross section for an N' of spin state s and photon helicity λ, then to order e^2,

$$K^{s,\lambda}\delta(\mathbf{p})=(2\pi)^4\sum_{f\neq N'}\frac{1}{\omega_f}\langle N'\mathbf{p}s,\mathbf{k}\lambda|H_{\text{e.m.}}|f\rangle$$
$$\times\langle f|H_{\text{e.m.}}|N'\mathbf{0}s,\mathbf{k}\lambda\rangle, \quad (3.17)$$
$$\omega_f=E_f-M'=|\mathbf{k}|.$$

Only parts of $H_{\text{e.m.}}$ linear in the field variables are required here which is very similar to the integral over excited states required for the low-energy theorem. In Ref. 4 it was shown, taking $H_{\text{e.m.}}$ as the FW interaction for the N particle (2.6), that the magnetic part did not contribute and that the cross terms between the spin-orbit and electric parts of H_I with zero retardation and using nonrelativistic c.m. decomposition gave

$$K_{\text{s-o}}=\frac{2\pi^2\alpha}{M^2}\frac{m}{M'}(1+2\kappa)+\frac{1}{M^2}O\left(\left(\frac{\beta}{M}\right)^{1/2}\right). \quad (3.18)$$

There remains the contribution to K which is of second order in the electric part of the interaction H_E. Taking the difference between photon states of opposite helicity,

$$(K_E{}^{s,\lambda}-K_E{}^{s,-\lambda})\delta(\mathbf{p})=(2\pi)^4\sum_{f\neq N'}\frac{1}{\omega_f}[\langle N'\mathbf{p}s,\mathbf{k}\lambda|H_E|f\rangle$$
$$\times\langle f|H_E|N'\mathbf{0}s,\mathbf{k}\lambda\rangle-(\boldsymbol{\varepsilon}\leftrightarrow-\boldsymbol{\varepsilon}^*)], \quad (3.19)$$

since for helicity $\lambda\to-\lambda$, the polarization vector $\boldsymbol{\varepsilon}\to-\boldsymbol{\varepsilon}^*$. For zero retardation, which is justified in the limit of vanishing binding energy, the summation in (3.19) and the second term of (3.6) are exactly the same, so that from (3.9)

$$K_E{}^{s,\lambda}-K_E{}^{s,-\lambda}=e^2\pi(m/2M'^2M)u^\dagger(s)i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}^*\times\boldsymbol{\varepsilon}u(s)$$
$$=-2\pi^2\alpha(m/M'^2M)u^\dagger(s)\boldsymbol{\sigma}\cdot\hat{k}u(s). \quad (3.20)$$

Choosing the N' spin to be in the direction of incident photon $\hat{k}$, then $\sigma^{s,\lambda}=\sigma^P$, $\sigma^{s,-\lambda}=\sigma^A$, and

$$K_E=-2\pi^2\alpha\frac{m}{M'^2M}+\frac{1}{M^2}O\left(\left(\frac{\beta}{M}\right)^{1/2}\right). \quad (3.21)$$

This term arises solely by use of the relativistic c.m. variables, in the nonrelativistic case $K_E=0$. It is at once apparent that the sum of (3.21) and (3.18) is equal to the left-hand side of (3.15), so that the DHG sum rule is valid for this simple bound-state model.

[17] It is interesting to note that in Ref. 9 for $\beta\ll\omega\ll(M_R\beta)^{1/2}$ it is shown that f has the form (3.13) with $M'\to M$, to first order in ω, which is the form of the low-energy photon scattering amplitude for a free N particle. This demonstrates roughly as $\omega\to\infty$ in the sense described that the N particle in the bound state scatters independently.

IV. SUM RULE FOR HYDROGEN ATOM AND DEUTERON; ISOTROPIC-SPIN SUM RULES

A simple model of a hydrogenlike atom is obtained from the $N\sigma$ model by giving the σ particle, which now corresponds to the nucleus, a charge Ze. The N particle corresponds to the electron but we retain the anomalous magnetic moment $e\kappa/2M$. The total charge is $(Z+1)e$ while the magnetic moment is still $\mu=e(1+\kappa)/2M$. The sum rule reads

$$2\pi\left(\mu-\frac{(Z+1)e}{2M'}\right)^2=\frac{2\pi^2\alpha}{M^2}\left(\kappa+\frac{m}{M'}-Z\frac{M}{M'}\right)^2=\int_0^\infty\frac{d\omega}{\omega}[\sigma_{N'}{}^P(\omega)-\sigma_{N'}{}^A(\omega)]\approx K+J_N+J_\sigma. \quad (4.1)$$

It is still true that for the integrals over the corresponding free-particle cross sections $J_\sigma=0$, since a spinless particle does not define a direction and $J_N=2\pi^2\alpha\kappa^2/M^2$, by assumption. The electromagnetic interaction $H_{\mathrm{e.m.}}$ is now the sum of the FW Hamiltonian for the N particle, H_{I_N}, and the interaction Hamiltonian for the spin-zero σ,

$$H_{I\sigma}=-(Ze/2m)[\mathbf{p}_\sigma\cdot\mathbf{A}(\mathbf{r}_\sigma)+\mathbf{A}(\mathbf{r}_\sigma)\cdot\mathbf{p}_\sigma]+(Z^2e^2/2m)\mathbf{A}^2(\mathbf{r}_\sigma). \quad (4.2)$$

Defining $K^{s,\lambda}$ again by (3.16), it still has the form (3.17). The magnetic part of H_{I_N} once more does not contribute since with zero retardation the matrix elements (already $\sim\omega$) between the ground state and excited states vanish. The required matrix elements, arranging them to be proportional to ω_f, or ω, are

$$\langle f|H_{I\sigma}|N'\mathbf{0}s,\mathbf{k}\lambda\rangle(2\pi)^{3/2}(2\omega)^{1/2}=iZe(M/M')\omega_f\delta(\mathbf{p}_f-\mathbf{k})\langle f_{\mathrm{int}}|\left(\boldsymbol{\varepsilon}\cdot\mathbf{r}+\frac{m-M}{2M^2m}\tfrac{1}{2}\{\boldsymbol{\varepsilon}\cdot\mathbf{r},\mathbf{q}^2\}+\frac{1}{4M^2}\boldsymbol{\varepsilon}\cdot(\mathbf{q}\times\boldsymbol{\sigma})\right)|N's\rangle,$$

$$\langle f|H_{EN}|N'\mathbf{0}s,\mathbf{k}\lambda\rangle(2\pi)^{3/2}(2\omega)^{1/2}=-ie(m/M')\delta(\mathbf{p}_f-\mathbf{k})\omega_f\langle f_{\mathrm{int}}|\left(\boldsymbol{\varepsilon}\cdot\mathbf{r}+\frac{M-m}{2m^2M}\tfrac{1}{2}\{\boldsymbol{\varepsilon}\cdot\mathbf{r},\mathbf{q}^2\}-\frac{1}{4Mm}\boldsymbol{\varepsilon}\cdot(\mathbf{q}\times\boldsymbol{\sigma})\right)|N's\rangle, \quad (4.3)$$

$$\langle f|H_{\mathrm{s-o}}|N'\mathbf{0}s,\mathbf{k}\lambda\rangle(2\pi)^{3/2}(2\omega)^{1/2}=-ie[(1+2\kappa)/4M^2]\omega\delta(\mathbf{p}_f-\mathbf{k})\langle f_{\mathrm{int}}|\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}\times\mathbf{q})|N's\rangle,$$

neglecting terms of higher order in ω and the inverse masses on the right-hand side. In the first two matrix elements of (4.3), the current-conservation requirements of the form (2.10) have been imposed separately for the N and σ particles so as to bring out a factor ω_f for use in cancelling the energy denominators in the sum rule. To the required order,

$$K_{\mathrm{s-o}}{}^{s,\lambda}=\frac{\pi^2\alpha}{M^2}(1+2\kappa)\left(\frac{m}{M'}-Z\frac{M}{M'}\right)\langle N's|[\boldsymbol{\varepsilon}^*\cdot\mathbf{r}\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}\times\mathbf{q})+\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}^*\times\mathbf{q})\boldsymbol{\varepsilon}\cdot\mathbf{r}]|N's\rangle,$$

$$K_{\mathrm{s-o}}{}^{s,\lambda}-K_{\mathrm{s-o}}{}^{s,-\lambda}=\frac{2\pi^2\alpha}{M^2}(1+2\kappa)\left(\frac{m}{M'}-Z\frac{M}{M'}\right)u^\dagger(s)\boldsymbol{\sigma}\cdot\hat{k}u(s), \quad (4.4)$$

and also

$$K_E{}^{s,\lambda}=4\pi^2\alpha\sum_{f\,\mathrm{int}}\left|\frac{m}{M'}\langle f_{\mathrm{int}}|\left(\boldsymbol{\varepsilon}\cdot\mathbf{r}+\frac{M-m}{2m^2M}\tfrac{1}{2}\{\boldsymbol{\varepsilon}\cdot\mathbf{r},\mathbf{q}^2\}-\frac{1}{4Mm}\boldsymbol{\varepsilon}\cdot(\mathbf{q}\times\boldsymbol{\sigma})\right)|N's\rangle-Z\frac{M}{M'}\langle f_{\mathrm{int}}|\left(\boldsymbol{\varepsilon}\cdot\mathbf{r}+\frac{m-M}{2M^2m}\tfrac{1}{2}\{\boldsymbol{\varepsilon}\cdot\mathbf{r},\mathbf{q}^2\}+\frac{1}{4M^2}\boldsymbol{\varepsilon}\cdot(\mathbf{q}\times\boldsymbol{\sigma})\right)|N's\rangle\right|^2, \quad (4.5)$$

$$K_E{}^{s,\lambda}-K_E{}^{s,-\lambda}=-\frac{2\pi^2\alpha}{M^2}\frac{M}{M'}(1+Z)\left(\frac{m}{M'}-Z\frac{M}{M'}\right)u^\dagger(s)\boldsymbol{\sigma}\cdot\hat{k}u(s).$$

From (4.4) and (4.5), to zero order in $(\beta/M)^{1/2}$, K in (4.1) is given by

$$K=\frac{2\pi^2\alpha}{M^2}\left[2\kappa\left(\frac{m}{M'}-2\frac{M}{M'}\right)+\left(\frac{m}{M'}-Z\frac{M}{M'}\right)^2\right]. \quad (4.6)$$

This is the expression required to automatically satisfy the sum rule (4.1) (for $M\ll m$ see also Ref. 5).

Finally, we treat the deuteron for constituents of arbitrary mass. Generalization of the low-energy theorems to the spin-1 case and unsubtracted dispersion relations yield[3,4,6]

$$\pi\left(\mu_d-\frac{e}{M_d}\right)^2=\int_0^\infty\frac{d\omega}{\omega}[\sigma_d{}^P(\omega)-\sigma_d{}^A(\omega)]\equiv J_d. \quad (4.7)$$

Assumptions of the same nature as applied previously give, for the integral of the same form as J_d over the low-energy cross sections for photodisintegration of the deuteron into a proton and neutron,

$$K=J_d-J_p-J_n$$
$$=4\pi^2\alpha\left(\frac{1+\kappa_p}{2M_p}+\frac{\kappa_n}{2M_n}-\frac{1}{M_p+M_n}\right)^2$$
$$-2\pi^2\alpha\left(\frac{\kappa_p^2}{M_p^2}+\frac{\kappa_n^2}{M_n^2}\right). \quad (4.8)$$

In Ref. 4, using nonrelativistic methods and the electromagnetic FW interaction Hamiltonians for the neutron and proton, K is explicitly calculated. All terms dependent on κ_p and κ_n cancel with those on the right-hand side of (4.8), while the term merely proportional to α is

$$-\pi^2\alpha(1/M_p^2)+2\pi^2\alpha(M_n/M_dM_p^2)\,,\quad M_d=M_n+M_p. \quad (4.9)$$

From the experience gained in previous calculations, relativistic corrections to the order required here can only come from those terms in the explicit calculation of K which are of second order in the electric part of the interaction H_E. Since only the proton is charged

$$H_E=-(e/2M_p)[\mathbf{p}_p\cdot\mathbf{A}(\mathbf{r}_p)+\mathbf{A}(\mathbf{r}_p)\cdot\mathbf{p}_p], \quad (4.10)$$

forgetting the unimportant e^2 term. The matrix element in question, by using our c.m. variables for the case of two particles with spin to order M^{-2} and the current-conservation requirement, may be put in the form

$$\langle f|H_E|d\mathbf{0}s,\mathbf{k}\lambda\rangle(2\pi)^{3/2}l(2\omega)^{1/2}=-ie(M_n/M_d)\omega_f\delta(\mathbf{p}_f-\mathbf{k})$$
$$\langle f_{\text{int}}|\bigg(\boldsymbol{\varepsilon}\cdot\mathbf{r}+\frac{M_p-M_n}{2M_n^2M_d}\tfrac{1}{2}\{\boldsymbol{\varepsilon}\cdot\mathbf{r},\mathbf{q}^2\}$$
$$-\frac{1}{4M_nM_p}\boldsymbol{\varepsilon}\cdot\mathbf{q}\times\boldsymbol{\sigma}_p'+(1/4M_n^2)\boldsymbol{\varepsilon}\cdot\mathbf{q}\times\boldsymbol{\sigma}_n'\bigg)|ds\rangle,$$
$$\omega_f=E_f-M_d, \quad (4.11)$$

neglecting terms of higher order in the photon energy ω; $\boldsymbol{\sigma}_p'$ and $\boldsymbol{\sigma}_n'$ are the internal proton and neutron c.m. spin operators. Defining $K^{s,\lambda}$ by (3.16) again, the contribution due to H_E acting twice is

$$K_E^{s,\lambda}=4\pi^2\alpha(M_n^2/M_d^2)\sum_{f\ \text{int}}\bigg|\langle f_{\text{int}}|\bigg(\boldsymbol{\varepsilon}\cdot\mathbf{r}-\frac{1}{4M_nM_p}\boldsymbol{\varepsilon}$$
$$\cdot(\mathbf{q}\times\boldsymbol{\sigma}_p')+\frac{1}{4M_n^2}\boldsymbol{\varepsilon}\cdot(\mathbf{q}\times\boldsymbol{\sigma}_n')\bigg)|ds\rangle\bigg|^2, \quad (4.12)$$

where the $\sim M^{-2}$ spin-independent term is not included since it does not appear when we evaluate to this order the difference

$$K_E^{s,\lambda}-K_E^{s,-\lambda}=-2\pi^2\alpha(M_n/M_d^2)u_d^\dagger(s)$$
$$\times\left(\frac{1}{M_p}\boldsymbol{\sigma}_p'\cdot\hat{k}-\frac{1}{M_n}\boldsymbol{\sigma}_n'\cdot\hat{k}\right)u_d(s). \quad (4.13)$$

The spin-1 deuteron spinor is $u_d(\pm 1)=u_p(\pm\frac{1}{2})u_n(\pm\frac{1}{2})$. Taking the deuteron to have spin-1 projection in the direction of the incident photon, then (4.13) gives the relativistic correction contribution to be

$$-2\pi^2\alpha(M_n-M_p)M_d^2M_p. \quad (4.14)$$

Adding (4.14) and (4.9) is equal to the κ_p, κ_n-independent part of the right-hand side of (4.8). Hence we have demonstrated the DHG sum rule for the deuteron constituted of a neutron and proton of arbitrary mass to zero order in the deuteron binding energy.

Besides the sum rule studied here, which is expressed entirely in terms of observable cross sections, isoscalar and isovector DHG sum rules have been derived[18] which express the isoscalar and isovector magnetic moments in terms of similar integrals to (3.14) over isoscalar and isovector photon-absorption cross sections. Following Ref. 4, the $N\sigma$ model can describe a bound-state isodoublet p' and n' by letting N be either of two elementary members of an isodoublet p and n while σ remains isoscalar. Since the sum rule is satisfied for particles p' and n' separately, then the isoscalar and isovector sum rules are automatically satisfied. For the deuteron, since when $M_p=M_n$ (isospin symmetry) the relativistic correction term (4.14) vanishes, the conclusion of Barton and Dombey that the deuteron satisfies the isospin DHG sum rules is not altered. In any case it should be noted that corrections to order M^{-2} in a photodisintegration cross section beyond the leading term will not alter the isovector character of $M1$ and $E1$ disintegrations (for the deuteron both the spin-orbit term and relativistic effects have the nature of such a correction).

Finally in Ref. 4 Barton and Dombey endeavored to show that the H^3-He^3 isodoublet flouted both the ordinary and isospin sum rules. Without explicit calculation, since we cannot yet deal with the three-body nature of these systems, as a consequence of the above results we may confidently assert that relativistic corrections should effect agreement. The sum rule would indeed be confirmed if we were to assume that H^3-He^3 could be treated as a bound state of a nucleon and a two-nucleon bound state, since then our calculational procedures could easily be sequentially applied to the two-nucleon bound state and then to the effectively two-particle three-nucelon system. By such means the DHG sum rule could be verified for any multiparticle bound state.

[18] M. A. B. Bég, Phys. Rev. Letters **17**, 333 (1966); K. Kawarabayashi and W. W. Wada, Phys. Rev. **152**, 1286 (1966); see also Ref. 6.

V. CONCLUSION

We have endeavored to show that in order to calculate relativistically electromagnetic processes in a conventional noncovariant fashion, it is necessary to modify the nonrelativistic form of the c.m. variables. The procedure adopted here is not very rigorous since, as is well known, it is not sufficient in order to introduce an interaction while maintaining relativistic invariance just to add on an extra term in the Hamiltonian. In general, the generator of boosts has to be modified as well. For a free particle, the FW electromagnetic interaction H_I is justifiably added onto the free relativistic Hamiltonian $H_0=(m^2+\mathbf{p}^2)^{1/2}$, since the form H_0+H_I, to given order in the inverse mass, can be derived by a unitary transformation on the Dirac Hamiltonian incorporating electromagnetic interaction, which can be cast into a manifestly covariant formalism.

In our case, for a two-particle system we have implicitly imagined for H_0, a Hamiltonian of the Bakamjian-Thomas form (see Ref. 12 for more details and references), where the instantaneous interaction is incorporated in the mass operator $\hat{M}$. This could be of the form

$$\hat{M}=(m_1^2+\mathbf{q}^2)^{1/2}+(m_2^2+\mathbf{q}^2)^{1/2}+V(|\mathbf{r}|)\,, \qquad (5.1)$$

where $\mathbf{q}$ and $\mathbf{r}$ are the internal c.m. momentum and position dynamical variables and $V(|\mathbf{r}|)$ is the interaction. With (5.1) we would have the result

$$i[\hat{M},\mathbf{r}]=\frac{m_1m_2}{m_1+m_2}\mathbf{q}+O(M^{-3})\,. \qquad (5.2)$$

Now adding on H_I, expressed in terms of the canonical single-particle variables, to H_0 and proceeding through our previous calculations but using (5.2) in place of (2.10) would not produce the required relativistic correction, although the relativistic c.m. decomposition was still applied. This demonstrates the need for care in adding H_I to H_0 as the valid electromagnetic interaction. The form of H_I has been taken here to be unchanged from that for a free particle, but the dynamical variables appearing to represent the momentum and position operator of the interacting particle must have the requirement (2.10) imposed, justified on the grounds of current conservation. Whether merely imposing current conservation is sufficient for relativistic, and also gauge, invariance of the interaction to higher order than discussed here is uncertain.

There is perhaps one other point to note. In Ref. 12 and here we have used the relativistic c.m. variables for a free two-particle system. Consequently, these variables were appropriate to dynamical calculation with weakly bound states. If we were dealing with a strong-binding situation it would probably be more suitable to let the c.m. variables become dependent on the mass operator.

Note added in proof. There are some points in the text which are oversimplified and some further elucidation is given here. The statement at the end of Sec. IV that the isotopic Drell-Hearn-Gerasimov sum rules are automatically satisfied for the modified $N\sigma$ model since the ordinary sum rule is satisfied for each member of the isodoublet is fallacious. A little more calculation is necessary. The isoscalar (S) and isovector (V) sum rules are

$$2\pi\left(\mu_{S,V}-\frac{e_{S,V}}{2M'}\right)^2=\int_0^\infty\frac{d\omega}{\omega}\times[\sigma_{S,V}{}^{(P)}(\omega)-\sigma_{S,V}{}^{(A)}(\omega)]=J_{S,V}(N')\,, \qquad (X.1)$$

with the same basic notation as before. Using $e_S=(Z+\frac{1}{2})e$, $e_V=\frac{1}{2}e$, and $\mu_S=(e/4M)(1+\kappa_p+\kappa_n)$, $\mu_V=(e/4M)(1+\kappa_p-\kappa_n)$, Eq. (X.1) can be rewritten

$$\frac{\pi^2\alpha}{2M^2}\left(\kappa_p+\kappa_n+\frac{m}{M'}-2Z\frac{M}{M'}\right)^2=J_S(N')\,,\qquad \frac{\pi^2\alpha}{2M^2}\left(\kappa_p-\kappa_n+\frac{m}{M'}\right)^2=J_V(N')\,. \qquad (X.2)$$

Assuming the validity of the isotopic sum rules for the N isodoublet, the high-energy contributions to $J(N')$ are

$$(\pi^2\alpha/2M^2)(\kappa_p+\kappa_n)^2\quad (S)\,,\qquad (\pi^2\alpha/2M^2)(\kappa_p-\kappa_n)^2\quad (V)\,. \qquad (X.3)$$

The low-energy parts can be found as usual, remembering that the electromagnetic field acting on the σ particle only causes isoscalar contributions. The spin-orbit parts are

$$\frac{\pi^2\alpha}{2M^2}[1+2(\kappa_p+\kappa_n)]\left(\frac{m}{M'}-2Z\frac{M}{M'}\right)\quad (S)\,,\qquad \frac{\pi^2\alpha}{2M^2}[1+2(\kappa_p-\kappa_n)]\frac{m}{M'}\quad (V)\,, \qquad (X.4)$$

while relativistic corrections to the electric part give

$$-\frac{\pi^2\alpha}{2M^2}\frac{M}{M'}(1+2Z)\left(\frac{m}{M'}-2Z\frac{M}{M'}\right)\quad (S)\,,\qquad -\frac{\pi^2\alpha}{2M^2}\frac{Mm}{M'^2}\quad (V)\,. \qquad (X.5)$$

Adding Eqs. (X.3)–(X.5) is sufficient to validate the two sum rules (X.2).

The justification for the use of the electromagnetic interaction (2.6) for a relativistic (to order M^{-2}) two-particle theory with our choice of the c.m. decomposition is not completely satisfactory. It is strictly correct only for free particles. In general, a two-body interaction will necessitate a modified electromagnetic interaction (e.g., exchange currents). The condition (2.10) can then be regarded as the requirement for an unchanged electromagnetic interaction even in the preserve of two-body forces.

In Ref. 1 an alternative electromagnetic-interaction Hamiltonian has been derived to take account of the relativistic modification of the spin-orbit forces for a two-particle system while maintaining the conventional nonrelativistic c.m. decomposition of the dynamical variables. The extra terms that appear can also be very simply derived using the approach considered in this paper. For two particles 1 and 2 we define, in terms of the c.m. variables,

$$\mathbf{r}_1^{\text{N.R.}}=\mathbf{R}+\frac{m_1}{\mathfrak{M}}\mathbf{r},\quad \mathbf{r}_2^{\text{N.R.}}=\mathbf{R}-\frac{m_2}{\mathfrak{M}}\mathbf{r},$$
$$\mathbf{p}_1^{\text{N.R.}}=\frac{m_1}{\mathfrak{M}}\mathbf{P}+\mathbf{q},\quad \mathbf{p}_2^{\text{N.R.}}=\frac{m_2}{\mathfrak{M}}\mathbf{P}-\mathbf{q},\quad \mathfrak{M}=m_1+m_2. \tag{X.6}$$

Considering only spin-dependent corrections, the c.m. decomposition from Eq. (4.5) of Ref. 12 forces us to write for the individual-particle dynamical variables

$$\mathbf{r}_1=\mathbf{r}_1^{\text{N.R.}}+\frac{1}{2\mathfrak{M}}\mathbf{p}_2^{\text{N.R.}}\times\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right),$$
$$\mathbf{r}_2=\mathbf{r}_2^{\text{N.R.}}-\frac{1}{2\mathfrak{M}}\mathbf{p}_1^{\text{N.R.}}\times\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right). \tag{X.7}$$

Using (X.7), the potential terms in the Hamiltonian can be expanded to order M^{-2}:

$$e_1\Phi(\mathbf{r}_1)+e_2\Phi(\mathbf{r}_2)=e_1\Phi(\mathbf{r}_1^{\text{N.R.}})+e_2\Phi(\mathbf{r}_2^{\text{N.R.}})$$
$$+\frac{1}{2\mathfrak{M}}\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right)\cdot[e_1\nabla\Phi(\mathbf{r}_1^{\text{N.R.}})\times\mathbf{p}_2^{\text{N.R.}}-e_2\nabla\Phi(\mathbf{r}_2^{\text{N.R.}})\times\mathbf{p}_1^{\text{N.R.}}]. \tag{X.8}$$

To obtain a Hamiltonian that satisfies gauge invariance we let $\nabla\Phi\rightarrow-\mathbf{E}$ and $\mathbf{p}_i^{\text{N.R.}}\rightarrow\mathbf{p}_i^{\text{N.R.}}-e_i\mathbf{A}(\mathbf{r}_i^{\text{N.R.}})$. Since the correction term is itself of order M^{-2}, the superscripts N.R. can be dropped, so that

$$\Delta H_{\text{e.m.}}=-\frac{1}{2\mathfrak{M}}\left(\frac{\mathbf{s}_1}{m_1}-\frac{\mathbf{s}_2}{m_2}\right)\cdot\{e_1\mathbf{E}(\mathbf{r}_1)\times[\mathbf{p}_2-e_2\mathbf{A}(\mathbf{r}_2)]-e_2\mathbf{E}(\mathbf{r}_2)\times[\mathbf{p}_1-e_1\mathbf{A}(\mathbf{r}_1)]\}. \tag{X.9}$$

Using (X.9) in addition to the usual electromagnetic interaction is sufficient to derive the low-energy theorems and the Drell-Hearn-Gerasimov sum rules for arbitrary two-particle bound states, in the limit of vanishing binding energy and with conventional treatment of the c.m. motion.

The effect of the extra terms in (X.8) may possibly be observable in the polarization in the low-energy (<20-MeV) proton-proton scattering. A recent experiment is in quite strong disagreement with the behavior expected from current phase-shift analyses which are unlikely to be in error to the required extent. J. S. C. McKee and T. Osborn [Phys. Letters **28B**, 7 (1968), where other references are given] have suggested an explanation of the discrepancy due to the spin-orbit forces acting on the incident proton in the Coulomb field of the target. However, these authors used only the proton magnetic moment term which is larger than the Thomas precession term in the ratio 2.79:0.5. Hence, if these effects are to be calculated to better than ~25% accuracy, then relativistic corrections are important, since the proton masses are equal.

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