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Relativistic Center-of-Mass Variables for Two-Particle Systems with Spin

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The definition of the total momentum, position, and spin for Galilean- or Lorentz-invariant two-free-particle systems is discussed by using the requirement that the generators of the respective invariance groups should have the same form expressed in terms of them as for a single particle. Internal c.m. dynamical variables are introduced by applying the singular transformation due to Gartenhaus and Schwartz on the basic single-particle dynamical variables, the transformation mapping the whole Hilbert space onto the c.m. subspace spanned by states of zero total momentum. The form of the internal c.m. position operator is given for particles with spin, for what appears to be the first time. Using these dynamical variables, it is shown how an interaction can be introduced while maintaining Galilean or Lorentz invariance and satisfying the asymptotic condition of freely propagating particles for large separations.

I. INTRODUCTION

THE role that the c.m. dynamical variables play in nonrelativistic quantum mechanics is well known. The intention here is to derive the corresponding form for the c.m. dynamical variables appropriate to a relativistic quantum two-particle problem when the fundamental dynamical variables are also the individual-particle dynamical observables—momentum, position, and spin. In order to motivate the subsequent relativistic discussion, we retread in this introduction the familiar nonrelativistic ground, paying careful attention to the essential requirement of invariance under the inhomogeneous Galilean group $\mathcal{G}$.[1]

As is also true relativistically for the inhomogeneous Lorentz or Poincaré group $\mathcal{P}$, the irreducible representations of $\mathcal{G}$, which describe a single elementary particle, are characterized by the values of the two Casimir invariants, mass m and spin s. For two noninteracting particles of mass m_1,m_2, and spin s_1,s_2 associated with Hilbert spaces $\mathcal{H}([m_1s_1])$, $\mathcal{H}([m_2s_2])$,[2] the generators of $\mathcal{G}$ acting in the direct product Hilbert space

$$\mathcal{H}_2=\mathcal{H}([m_1s_1])\otimes\mathcal{H}([m_2s_2])$$

are given by the sums of the individual-particle generators acting in their respective Hilbert spaces. With the basic dynamical variables as the momenta, position, and spin operators of each particle, the commutation relations are

$$[r_{\alpha i},p_{\beta j}]=i\delta_{\alpha\beta}\delta_{ij},\quad \alpha,\beta=1,\,2$$
$$[s_{\alpha i},s_{\beta j}]=i\delta_{\alpha\beta}\epsilon_{ijk}s_k,\quad i,j,k=1,2,3 \tag{1.1}$$

all other commutators vanishing, and also

$$\mathbf{s}_\alpha^2=s_\alpha(s_\alpha+1). \tag{1.2}$$

The forms of the total Hamiltonian, momentum, and angular momentum are obvious[3]:

$$H=m_1+m_2+\mathbf{p}_1^2/2m_1+\mathbf{p}_2^2/2m_2,$$
$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2, \tag{1.3}$$
$$\mathbf{J}=\mathbf{r}_1\times\mathbf{p}_1+\mathbf{r}_2\times\mathbf{p}_2+\mathbf{s}_1+\mathbf{s}_2.$$

To complete the algebra of $\mathcal{G}$ (which is not really a Lie algebra unless the total mass m_1+m_2 is included; but these complications are not material here), we need the generator of boosts

$$\mathbf{K}=t(\mathbf{p}_1+\mathbf{p}_2)-m_1\mathbf{r}_1-m_2\mathbf{r}_2. \tag{1.4}$$

Given (1.1), the expressions (1.3) and (1.4) automatically ensure that the algebra of $\mathcal{G}$ is satisfied and hence if these operators are used on $\mathcal{H}_2$ to generate time and space translations, rotations, and boosts, then we obtain a nonrelativistic two-particle theory invariant under $\mathcal{G}$. The total momentum, position, and spin are now introduced by requiring that the generators expressed in terms of these operators have the same form as they would for a single particle:

$$H=M+\Delta M+\mathbf{P}^2/2M,\quad \mathbf{P}=\mathbf{P},$$
$$\mathbf{J}=\mathbf{R}\times\mathbf{P}+\mathbf{S},\quad \mathbf{K}=t\mathbf{P}-M\mathbf{R}. \tag{1.5}$$

By virtue of the algebra of $\mathcal{G}$—$\mathbf{P}$, $\mathbf{R}$, and $\mathbf{S}$ obey the commutation relations (1.1) while they necessarily commutate with M and ΔM. To maintain simplicity ΔM is assumed to vanish as both the individual-particle momenta go to zero; thus $M=m_1+m_2$. [The equivalence of (1.5) to (1.3) and (1.4) does not define M uniquely but since mass is absolutely conserved in Galilean-invariant theories, this is the only natural definition; if the algebra is extended to be a Lie algebra by including the mass term among the elements, then this follows immediately.] Hence $\mathbf{P}$ and $\mathbf{R}$ can be ex-

[1] M. Hammermesh, Ann Phys. (N. Y.) 9, 518 (1960); J. M. Levy-Leblond, J. Math. Phys. 4, 776 (1963).

[2] We confine ourselves throughout to positive-mass representations.

[3] We choose the form of H to coincide with the nonrelativistic limit of relativistic expressions given later. As far as the algebra of $\mathcal{G}$ is concerned, any operator or c number commuting with the generators may be added to H.

pressed in terms of the single-particle variables,

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2, \quad \mathbf{R}=\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}. \tag{1.6}$$

By expressing the generators in terms of **P**, **R**, **S**, and ΔM, we have effectively reduced $\mathcal{H}_2$ to a direct sum over irreducible representations of $\mathcal{G}$,[4]

$$\mathcal{H}_2\sim\sum_{j,\eta}\int_{\oplus} dM'\mathcal{H}_\eta([M',j]), \tag{1.7}$$

where the label η is used for the multiple occurrence of equivalent irreducible representations. The Casimir invariants $M+\Delta M$ and $\mathbf{S}^2$ acting on $\mathcal{H}_\eta([M',j])$ have the values M' and $j(j+1)$; the generators transform states within a given irreducible subspace in the usual fashion. Besides the operators **P**, **R**, and **S** which act between states within irreducible subspaces, the basic dynamical variables contain parts which mix such subspaces. To form a complete set of dynamical variables with **P** and **R** under which $\mathcal{H}_2$ is irreducible, further variables are formed from the basic set which commute with **P** and **R**. Thus the basic algebra generated by $\mathbf{r}_\alpha\, \mathbf{p}_\alpha\, \mathbf{s}_\alpha$, $\alpha=1, 2$, has now to be contracted to a subalgebra which commutes with **P** and **R**.

As is shown in Sec. II, this can be achieved by a singular transformation[5] such that

$$U\mathbf{P}U^*=0, \quad U\mathbf{S}U^*=\mathbf{S}, \quad U^*\mathbf{R}U=0. \tag{1.8}$$

The transformation $U\cdots U^*$ is not defined on the whole algebra but diverges when applied to any operator which does not commute with **P**; otherwise it preserves commutation relations and the transformed operator commutes with **R** and **P**. Defining the transformation on $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{r}_1-\mathbf{r}_2$, $\mathbf{s}_1$, and $\mathbf{s}_2$, we obtain the internal c.m. dynamical variables

$$\begin{aligned}\mathbf{k}&=U\mathbf{p}_1U^*=-U\mathbf{p}_2U^*,\\ \mathbf{r}&=U(\mathbf{r}_1-\mathbf{r}_2)U^*,\\ \mathbf{s}_\alpha'&=U\mathbf{s}_\alpha U^*, \quad \alpha=1, 2.\end{aligned} \tag{1.9}$$

In the nonrelativistic case, the conventional expressions are found (see Ref. 3 for details or the discussion in Sec. III here):

$$\mathbf{k}=\frac{m_2\mathbf{p}_1-m_1\mathbf{p}_2}{m_1+m_2}, \quad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2, \quad \mathbf{s}_\alpha'=\mathbf{s}_\alpha, \tag{1.10}$$

which obey the expected commutation relations (1.1). The form of ΔM and **S** in terms of these variables can now be easily seen from (1.3) and (1.5), using (1.8) and (1.9):

$$\begin{aligned}\Delta M&=U(H-M)U^*=\mathbf{k}^2\tfrac{1}{2}(1/m_1+1/m_2),\\ \mathbf{S}&=U(\mathbf{J}-\mathbf{R}\times\mathbf{P})U^*\\ &=U[(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{p}_1+\mathbf{s}_1+\mathbf{s}_2+(\mathbf{r}_2-\mathbf{R})\times\mathbf{P}]U^*\\ &=\mathbf{r}\times\mathbf{k}+\mathbf{s}_1+\mathbf{s}_2,\end{aligned} \tag{1.11}$$

where we have used the fact that $\mathbf{r}_2-\mathbf{R}$ commutes with **P** and so is a valid operator on which the transformation is well defined. The form of ΔM demonstrates that, for noninteracting particles, the range of intergation over M' in (1.7) extends from $M'=m_1+m_2$ to ∞.

Having obtained the c.m. internal dynamical variables, it is now obvious how the generators of $\mathcal{G}$ may be modified so as to include an interaction between the two particles represented by the Hilbert space $\mathcal{H}_2$. It is only necessary to modify ΔM such that it still commutes with **R**, **P**, and **S**. The most general form for two spin-$\frac{1}{2}$ particles is

$$\begin{aligned}\Delta M=&\tfrac{1}{2}\mathbf{k}^2(1/m_1+1/m_2)+V_1(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\\ &+V_2(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{s}_1\cdot\mathbf{s}_2+V_3(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{l}\cdot\mathbf{s}_1\\ &+V_4(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{l}\cdot\mathbf{s}_2+V_5(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{l}\cdot\mathbf{s}_1\mathbf{l}\cdot\mathbf{s}_2\\ &+V_6(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{r}\cdot\mathbf{s}_1\mathbf{r}\cdot\mathbf{s}_2\\ &+V_7(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{k}\cdot\mathbf{s}_1\mathbf{k}\cdot\mathbf{s}_2,\\ &\mathbf{l}=\mathbf{r}\times\mathbf{k}\end{aligned} \tag{1.12}$$

where the interaction is assumed to be invariant under time reversal and parity. Furthermore, it is easy to apply the condition of separability, that as the distance between the particles becomes large they behave as independent free particles, merely by requiring that the potentials $V_j(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)$ vanish fast enough for large particle separations, i.e., as $|\mathbf{r}|\to\infty$. Then for large $|\mathbf{r}|$, the generators describe two freely propagating particles, each individually transforming according to $\mathcal{G}$.

In Sec. II we exhibit the Lie algebra of $\mathcal{P}$ and show how the total momentum and position are to be defined in the relativistic case. Furthermore, the existence of the singular operators U, U^* with the required properties is demonstrated. In Sec. III these operators are applied to obtain the internal c.m. dynamical variables. Finally in the conclusion (Sec. IV) the introduction of a relativistic separable two-particle interaction and a few other points are discussed. In two Appendices Lorentz transformations of the single-particle dynamical variables are explicitly calculated and the spin dependence of the c.m. position operators are demonstrated by a different procedure than that followed in the rest of this paper.

II. RELATIVISTIC POSITION OPERATOR

The necessary and sufficient condition for a (quantized) theory to possess relativistic invariance is that the generators of time and space translations, rotations, and boosts acting on the Hilbert space of

[4] The decomposition has been studied most closely in the relativistic case in A. J. Macfarlane, J. Math Phys. 4, 490 (1963).
[5] S. Gartenhaus and C. Schwartz, Phys. Rev. 108, 842 (1957).

physical states should obey the Lie algebra of $\mathcal{P}$,

$$\begin{aligned}
&[J_i,J_j]=i\epsilon_{ijk}J_k,\quad [J_i,K_j]=i\epsilon_{ijk}K_k\,,\\
&[J_i,P_j]=i\epsilon_{ijk}P_k,\quad [J_i,H]=0,\quad [K_i,H]=-iP_i\,,\\
&[K_i,K_j]=-i\epsilon_{ijk}J_k,\quad [K_i,P_j]=-i\delta_{ij}H\,,\\
&[P_i,P_j]=0,\quad [P_i,H]=0,\quad i,\,j,\,k=1,\,2,\,3
\end{aligned} \tag{2.1}$$

where the generators are Hermitian so that the Hilbert space supports a unitary representation of $\mathcal{P}$-conserving probability. The signs of the generators are such that the general element of $\mathcal{P}$ infinitesimally close to the identity,

$$1-i\delta\boldsymbol{\phi}\cdot\mathbf{J}-i\delta\mathbf{v}\cdot\mathbf{K}-i\delta\mathbf{x}\cdot\mathbf{P}+i\delta tH\,,$$

operating on a state vector corresponding to a given physical system, at time t, gives a new state vector corresponding to the *same* physical system rotated through an angle $\delta\boldsymbol{\phi}$, given a velocity $\delta\mathbf{v}$ and a displacement $\delta\mathbf{x}$ at a time $t+\delta t$. For a finite boost of a physical system by a velocity $\mathbf{v}$, the transformation of the corresponding state is given by

$$\exp(-i\theta\hat{v}\cdot\mathbf{K}),\quad \theta=\tanh^{-1}|\mathbf{v}|\,.$$

For an irreducible representation of the algebra (2.1) characterized by a mass m and spin s, the form of the generators expressed in terms of the corresponding particle momentum, position, and spin operators obeying the commutation relation (1.1) is

$$\begin{aligned}
&H=(m^2+\mathbf{p}^2)^{1/2}\equiv E\,,\quad \mathbf{P}=\mathbf{p}\,,\\
&\mathbf{J}=\mathbf{r}\times\mathbf{p}+\mathbf{s}\,,\\
&\mathbf{K}=t\mathbf{p}-\tfrac{1}{2}(\mathbf{r}E+E\mathbf{r})-\mathbf{p}\times\mathbf{s}/(E+m)\,,
\end{aligned} \tag{2.2}$$

and also

$$\mathbf{s}^2=s(s+1)\,.$$

These expressions are automatically Hermitian and obey the Lie algebra (2.1); the division by $E+m$ is well defined since E is a positive definite operator. For spin $\frac{1}{2}$ the forms (2.2) are exactly those obtained by applying the Foldy-Wouthuysen transformation to the generators in the Dirac representation and disregarding factors β. Because of its applicability to arbitrary spin, and other simplifying features, (2.2) can be regarded as the canonical form of the generators of $\mathcal{P}$. The single-particle states are especially simple for if $|\sigma\rangle$ represents a state at rest (σ labels the spin), then the state with momentum $\mathbf{q}$ is represented by

$$\begin{aligned}
|\mathbf{q}\sigma\rangle&=[m/E(\mathbf{q})]^{1/2}\exp(-i\theta\hat{q}\cdot\mathbf{K})\,|\sigma\rangle\\
&=e^{i[\mathbf{q}\cdot\mathbf{r}-E(\mathbf{q})t]}\,|\sigma\rangle\,,
\end{aligned} \tag{2.3}$$

$$\tanh\theta=|\mathbf{q}|/E(\mathbf{q}),\quad \langle\mathbf{q}'\sigma'|\mathbf{q}\sigma\rangle=\delta_{\sigma'\sigma}\delta(\mathbf{q}'-\mathbf{q})\,.$$

For two noninteracting relativistic particles 1 and 2, generators in the direct-product Hilbert space $\mathcal{H}_2$ are given by the sum of the individual generators of the form (2.2). The total momentum, position, and spin are now defined, in exact correspondence to the nonrelativistic case, by requiring that the generators of $\mathcal{P}$ in $\mathcal{H}_2$ expressed in terms of them should have exactly the same form as for a single particle, (2.2).[6,7] In this case it is possible to solve uniquely for $\mathbf{P}$, $\mathbf{R}$, and $\mathbf{S}$ as functions of the generators:

$$\begin{aligned}
&\mathbf{P}=\mathbf{P},\quad \mathbf{S}=\mathbf{J}-\mathbf{R}\times\mathbf{P},\quad M=(E^2-\mathbf{P}^2)^{1/2},\\
&\mathbf{R}=t\frac{\mathbf{P}}{E}-\frac{1}{2}\left(\frac{1}{E}\mathbf{K}+\mathbf{K}\frac{1}{E}\right)-\frac{\mathbf{P}\times\mathbf{J}}{M(E+M)}+\frac{\mathbf{P}\times(\mathbf{P}\times\mathbf{K})}{EM(E+M)}.
\end{aligned} \tag{2.4}$$

The Lie algebra (2.1) now ensures that $\mathbf{P}$, $\mathbf{R}$, and $\mathbf{S}$ obey the commutation relations (1.1); M and $\mathbf{S}^2$ are the Casimir invariants.

To determine the internal c.m. dynamical variables, we now introduce the unitary operators[5]

$$U(\alpha)=e^{i\alpha(\mathbf{R}\cdot\mathbf{P}+\mathbf{P}\cdot\mathbf{R})/2},\quad U^*(\alpha)=e^{-i\alpha(\mathbf{R}\cdot\mathbf{P}+\mathbf{P}\cdot\mathbf{R})/2}, \tag{2.5}$$

where, taking $t=0$ since the time dependence after the transformation can be found by applying the Hamiltonian in the usual way,

$$\tfrac{1}{2}(\mathbf{R}\cdot\mathbf{P}+\mathbf{P}\cdot\mathbf{R})=-\{(|\mathbf{P}|/E)\hat{v}\cdot\mathbf{K}\},\quad \hat{v}=\mathbf{P}/|\mathbf{P}| \tag{2.6}$$

the curly brackets denoting symmetrization so as to maintain a Hermitian result. The operators $U(\alpha)$, $U^*(\alpha)$ are used to generate an automorphism on the algebra of the basic dynamical variables $\mathbf{r}_\beta$, $\mathbf{p}_\beta$, $\mathbf{s}_\beta$, preserving the commutation relations [it should be noted that $U(\alpha)$ and $U^*(\alpha)$ are not elements of the Lie group generated by E, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$]. Applying this transformation first to the total momentum, we define

$$\mathbf{P}(\alpha)=U(\alpha)\mathbf{P}U^*(\alpha)\,.$$

By differentiation,

$$d\mathbf{P}(\alpha)/d\alpha=-\mathbf{P}(\alpha)\,,$$

so that

$$\mathbf{P}(\alpha)=e^{-\alpha}\mathbf{P}. \tag{2.7}$$

By taking the limit $\alpha\to\infty$,[8] it is at once apparent that we have obtained a transformation which fulfills the first of the conditions (1.8),

$$U\mathbf{P}U^*=\mathbf{0},\quad U=U(\infty),\quad U^*=U^*(\infty)\,. \tag{2.8}$$

Of course, in the limit $\alpha\to\infty$, $U(\alpha)$ and $U^*(\alpha)$ can no longer be unitary operators since no inverse exists

[6] For a general discussion see T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949); L. L. Foldy, Phys. Rev. **102**, 568 (1956); C. Fronsdal, *ibid.* **113**, 1367 (1959); R. Acharya and E. C. G. Sudarshan, J. Math. Phys. **1**, 532 (1960); A. S. Wightman, Rev. Mod. Phys. **34**, 845 (1962); A. Chakrabarti, J. Math. Phys. **4**, 1223 (1963).

[7] Definitions of the total position operator for a two-particle system exactly equivalent to ours have appeared in Chou Kuang-Chao and M. I. Shirokov, Zh. Eksperim. i Teor. Fiz. **34**, 1230 (1958) [English transl.: Soviet Phys.—JETP **7**, 851 (1958)]; B. Barsella and E. Fabri, Phys. Rev. **128**, 451 (1962); A. Chakrabarti, J. Math. Phys. **5**, 922 (1964); R. A. Berg, *ibid.* **6**, 34 (1965); D. J. Candlin, Nuovo Cimento **37**, 1396 (1965); K. M. Bitar and F. Gursey, Phys. Rev. **164**, 1805 (1968).

[8] This procedure is very similar to that of contraction of a Lie algebra; see R. Hermann, *Lie Groups for Physicists* (W. A. Benjamin, Inc., New York, 1966), Chap. 11.

for the transformation (2.8). This can also be seen in another way since, from (2.8), U^* maps the whole Hilbert space $\mathcal{H}_2$ onto the c.m. subspace (spanned by states of zero total momentum). However, U and U^*, being the limit of unitary operators, are isometric.

Since the unitary transformations for finite α leave commutation relations invariant, then necessarily U, U^* applied to an operator which does not commute with $\mathbf{P}$ will diverge, e.g.,

$$U(\alpha)\mathbf{R}U^*(\alpha)=e^{\alpha}\mathbf{R}.$$

This requires that U, U^* can only be applied to the subalgebra that commutes with $\mathbf{P}$, i.e., to $\mathbf{p}_\beta$, $\mathbf{s}_\beta$ ($\beta=1,2$), and $\mathbf{r}_1-\mathbf{r}_2$. On this subalgebra, the transformation will preserve the commutation relations even in the limit $\alpha\to\infty$. The results, found by performing the transformation for finite α and then taking the limit, are given in Sec. III.

The rest of the requirements (1.8), besides (2.8), are easily seen to follow from

$$U^*(\alpha)\mathbf{R}U(\alpha)=e^{-\alpha}\mathbf{R},\quad U(\alpha)\mathbf{S}U^*(\alpha)=\mathbf{S}. \tag{2.9}$$

Now suppose X is an operator which commutes with $\mathbf{P}$, so that we can define

$$X'=UXU^*=\lim_{\alpha\to\infty}X(\alpha),\quad X(\alpha)=U(\alpha)XU^*(\alpha). \tag{2.10}$$

From the converse of (2.7), and (2.9), we have

$$[\mathbf{P},X(\alpha)]=e^{\alpha}U(\alpha)[\mathbf{P},X]U^*(\alpha)=\mathbf{0},$$
$$[\mathbf{R},X(\alpha)]=e^{-\alpha}U(\alpha)[\mathbf{R},X]U^*(\alpha).$$

Then as $[\mathbf{P},[\mathbf{R},X]]$ is automatically zero, by applying the Jacobi identity and the assumed condition on X, we see that $[\mathbf{R},X]$ is a valid operator for the transformation, so that

$$[\mathbf{P},X']=\mathbf{0},\quad [\mathbf{R},X']=\mathbf{0}. \tag{2.11}$$

Furthermore, if $[\mathbf{S},X]=\mathbf{0}$ then $[\mathbf{S},X']=\mathbf{0}$.

The internal dynamical variables found by our procedure are thus guaranteed to commute with $\mathbf{P}$ and $\mathbf{R}$ as required.

III. DETERMINATION OF THE INTERNAL C.M. DYNAMICAL VARIABLES

The transformation is first applied to the individual particle momentum, so we define

$$\mathbf{p}_\beta(\alpha)=U(\alpha)\mathbf{p}_\beta U^*(\alpha),\quad \beta=1,\,2. \tag{3.1}$$

By differentiation,

$$\frac{d}{d\alpha}\mathbf{p}_\beta(\alpha)=-iU(\alpha)\left[\left\{\frac{|\mathbf{P}|}{E}\hat{v}\cdot\mathbf{K}\right\},\mathbf{p}_\beta\right]U^*(\alpha)$$
$$=-\frac{|\mathbf{P}(\alpha)|}{E(\alpha)}E_\beta(\alpha)\hat{v}, \tag{3.2}$$

where $\hat{v}$ is an invariant, since the transformation leaves the direction of $\mathbf{P}$ unchanged, (2.7); and since M is also invariant, it commutes with both $\mathbf{R}$ and $\mathbf{P}$, we thus have

$$\begin{aligned}E(\alpha)&=[M^2+\mathbf{P}^2(\alpha)]^{1/2}=E_1(\alpha)+E_2(\alpha),\\ E_\beta(\alpha)&=[m_\beta{}^2+\mathbf{p}_\beta{}^2(\alpha)]^{1/2}.\end{aligned} \tag{3.3}$$

If we define

$$\begin{aligned}\theta(\alpha)&=\tanh^{-1}\left(\frac{|\mathbf{P}|}{E}\right)-\tanh^{-1}\left(\frac{|\mathbf{P}(\alpha)|}{E(\alpha)}\right),\\ \sinh\theta(\alpha)&=[|\mathbf{P}|E(\alpha)-|\mathbf{P}(\alpha)|E]/M^2,\\ \cosh\theta(\alpha)&=[EE(\alpha)-|\mathbf{P}||\mathbf{P}(\alpha)|]/M^2,\end{aligned} \tag{3.4}$$

so that as α goes from 0 to ∞, $\theta(\alpha)$ goes from 0 to $\tanh^{-1}(|\mathbf{P}|/E)$, and (3.2) then becomes

$$d\mathbf{p}_\beta(\alpha)/d\theta(\alpha)=-E_\beta(\alpha)\hat{v}. \tag{3.5}$$

As shown in Appendix A, (3.5) integrates to a conventional Lorentz transformation,

$$\begin{aligned}\mathbf{p}_\beta(\alpha)&=\mathbf{p}_\beta+[\cosh\theta(\alpha)-1]\hat{v}\cdot\mathbf{p}_\beta\hat{v}-\sinh\theta(\alpha)\,E_\beta\hat{v},\\ E_\beta(\alpha)&=\cosh\theta(\alpha)\,E_\beta-\sinh\theta(\alpha)\,\hat{v}\cdot\mathbf{p}_\beta.\end{aligned} \tag{3.6}$$

In the limit $\alpha\to\infty$ we are just transforming these dynamical momentum variables to the instantaneous c.m. frame moving with velocity $\mathbf{v}=\mathbf{P}/E$:

$$\begin{aligned}\mathbf{k}=\mathbf{p}_1(\infty)&=-\mathbf{p}_2(\infty)\\ &=[(E_2+\omega_2)\mathbf{p}_1-(E_1+\omega_1)\mathbf{p}_2]/(E+M),\\ \omega_\beta=E_\beta(\infty)&=(m_\beta{}^2+\mathbf{k}^2)^{1/2},\\ M=E(\infty)&=\omega_1+\omega_2.\end{aligned} \tag{3.7}$$

For the spin operators defining

$$\mathbf{s}_\beta(\alpha)=U(\alpha)\mathbf{s}_\beta U^*(\alpha), \tag{3.8}$$

then

$$\frac{d}{d\alpha}\mathbf{s}_\beta(\alpha)=iU(\alpha)\frac{|\mathbf{P}|}{E}\frac{1}{E_\beta+m_\beta}[\hat{v}\cdot(\mathbf{p}_\beta\times\mathbf{s}_\beta),\,\mathbf{s}_\beta]U^*(\alpha)$$
$$=\frac{|\mathbf{P}(\alpha)|}{E(\alpha)}\frac{1}{E_\beta(\alpha)+m_\beta}[\hat{v}\times\mathbf{p}_\beta(\alpha)]\times\mathbf{s}_\beta(\alpha). \tag{3.9}$$

By using the variable $\theta(\alpha)$ again, (3.9) can be cast into the form of a conventional Lorentz transformation as shown in Appendix A:

$$d\mathbf{s}_\beta(\alpha)/d\theta(\alpha)=[E_\beta(\alpha)+m_\beta]^{-1}[\hat{v}\times\mathbf{p}_\beta(\alpha)]\times\mathbf{s}_\beta(\alpha). \tag{3.10}$$

The integration of (3.10), demonstrated in Appendix A, gives the well-known Wigner rotation or Thomas precession[9] of the spin variable $\mathbf{s}_\beta$ about an axis $\mathbf{p}_\beta\times\hat{v}$ in going to the instantaneous c.m. frame.

$$\mathbf{s}_\beta(\alpha)=\cos\gamma\ \mathbf{s}_\beta+(1-\cos\gamma)\hat{n}\cdot\mathbf{s}_\beta\hat{n}-\sin\gamma\ \hat{n}\times\mathbf{s}_\beta,$$

$$\tan\tfrac{1}{2}\gamma=\frac{|\mathbf{p}_\beta\times\hat{v}|}{(E_\beta+m_\beta)\coth\frac{1}{2}\theta(\alpha)-\hat{v}\cdot\mathbf{p}_\beta},\quad \hat{n}\equiv\frac{\mathbf{p}_\beta\times\hat{v}}{|\mathbf{p}_\beta\times\hat{v}|}. \tag{3.11}$$

Taking the limit $\alpha\to\infty$, and using the relations

$$\tan\tfrac{1}{2}\gamma=\frac{1-\cos\gamma}{\sin\gamma}=\frac{1}{b_\beta}|\mathbf{p}_\beta\times\mathbf{P}|,\quad \sin\gamma=(b_\beta/a_\beta)|\mathbf{p}_\beta\times\mathbf{P}|,$$

$$\cos\gamma=1-\frac{1}{a_\beta}|\mathbf{p}_\beta\times\mathbf{P}|^2,\quad 2a_\beta=b_\beta{}^2+|\mathbf{p}_\beta\times\mathbf{P}|^2, \tag{3.12}$$

we then have

$$\mathbf{s}_\beta'=\mathbf{s}_\beta(\infty)=(1-(|\mathbf{p}_\beta\times\mathbf{P}|^2)/a_\beta)\mathbf{s}_\beta+(1/a_\beta)\mathbf{p}_\beta\times\mathbf{P}\cdot\mathbf{s}_\beta\mathbf{p}_\beta\times\mathbf{P}-(b_\beta/a_\beta)(\mathbf{p}_\beta\times\mathbf{P})\times\mathbf{s}_\beta, \tag{3.13}$$

where

$$a_\beta=(E_\beta+m_\beta)(\omega_\beta+m_\beta)(E+M)M,$$
$$b_\beta=(E_\beta+m_\beta)(E+M)-\mathbf{P}\cdot\mathbf{p}_\beta, \tag{3.14}$$
$$b_{1,2}=(\omega_\beta+m_\beta)(E+M)\pm\mathbf{P}\cdot\mathbf{k}.$$

To determine the internal c.m. (relative) position operator we first study the transformation of $\{\hat{v}\cdot\mathbf{r}_1\}$:

$$\{\hat{v}\cdot\mathbf{r}_1(\alpha)\}=U(\alpha)\{\hat{v}\cdot\mathbf{r}_1\}U^*(\alpha), \tag{3.15}$$

so that

$$\frac{d}{d\alpha}\{\hat{v}\cdot\mathbf{r}_1(\alpha)\}=-U(\alpha)\left\{\hat{v}\cdot\frac{\partial}{\partial\mathbf{p}_1}\frac{|\mathbf{P}|}{E}\hat{v}\cdot\mathbf{K}\right\}U^*(\alpha). \tag{3.16}$$

Using

$$\hat{v}\cdot\frac{\partial}{\partial\mathbf{p}_1}\hat{v}=0,\quad \hat{v}\cdot\frac{\partial}{\partial\mathbf{p}_1}\frac{|\mathbf{P}|}{E}=\frac{1}{E^2E_1}(EE_1-\mathbf{P}\cdot\mathbf{p}_1),$$

and the expression for $\mathbf{K}$ as the sum of two single-particle terms of the form given in (2.2), (3.16) becomes

$$\frac{d}{d\alpha}\tfrac{1}{2}\mathbf{r}_1(\alpha)\cdot\hat{v}+\text{H.c.}=\tfrac{1}{2}\mathbf{r}_1(\alpha)\cdot\hat{v}\frac{\hat{v}\cdot\mathbf{p}_1(\alpha)}{E_1(\alpha)}\frac{|\mathbf{P}(\alpha)|}{E(\alpha)}+\text{H.c.}$$
$$+\frac{\hat{v}\cdot\mathbf{p}_1(\alpha)}{E_1(\alpha)}\frac{|\mathbf{P}(\alpha)|}{E(\alpha)}\frac{(\mathbf{p}_1\times\hat{v})\cdot\mathbf{s}_1}{[E_1(\alpha)+m_1]^2}-\left\{\frac{1}{E(\alpha)|\mathbf{P}(\alpha)|E_1(\alpha)}(EE_1-\mathbf{P}\cdot\mathbf{p}_1)\frac{|\mathbf{P}|}{E}\hat{v}\cdot\mathbf{K}\right\}, \tag{3.17}$$

where we have used the result that under the transformation

$$(\mathbf{p}_1\times\hat{v})\cdot\mathbf{s}_1,\quad EE_1-\mathbf{P}\cdot\mathbf{p}_1,\quad\text{and}\quad\{(|\mathbf{P}|/E)\hat{v}\cdot\mathbf{K}\}$$

are invariant. By virtue of these same conditions, and also

$$\hat{v}\cdot\mathbf{p}_1(\alpha)\frac{|\mathbf{P}(\alpha)|}{E(\alpha)}=-\frac{d}{d\alpha}E_1(\alpha),\quad \int_0^\alpha d\alpha'\frac{1}{E(\alpha')|\mathbf{P}(\alpha')|}=\frac{\sinh\theta(\alpha)}{|\mathbf{P}(\alpha)||\mathbf{P}|},$$

(3.17) can be integrated and simplified to yield

$$\tfrac{1}{2}\mathbf{r}_1(\alpha)\cdot\hat{v}+\text{H.c.}=\tfrac{1}{2}\mathbf{r}_1\cdot\hat{v}\frac{E_1}{E}\left(e^\alpha+\frac{E_2}{E_1(\alpha)}\right)+\tfrac{1}{2}\mathbf{r}_2\cdot\hat{v}\frac{E_2}{E}\left(e^\alpha-\frac{E_1}{E_1(\alpha)}\right)+\text{H.c.}$$
$$+(\mathbf{p}_1\times\hat{v})\cdot\mathbf{s}_1\left[\frac{1}{E_1(\alpha)[E_1(\alpha)+m_1]}-\frac{1}{E(E_1+m_1)}\left(e^\alpha+\frac{E_2}{E_1(\alpha)}\right)\right]-(\mathbf{p}_2\times\hat{v})\cdot\mathbf{s}_2\frac{1}{E(E_2+m_2)}\left(e^\alpha-\frac{E_1}{E_1(\alpha)}\right). \tag{3.18}$$

It should be noted that this result diverges as $\alpha\to\infty$, as was to be expected, but if we form the difference $\{\hat{v}\cdot[\mathbf{r}_1(\alpha)-\mathbf{r}_2(\alpha)]\}$ the divergent terms disappear:

$$\tfrac{1}{2}[\mathbf{r}_1(\alpha)-\mathbf{r}_2(\alpha)]\cdot\hat{v}+\text{H.c.}=\tfrac{1}{2}(\mathbf{r}_1-\mathbf{r}_2)\cdot\hat{v}\frac{E_1E_2}{E}\frac{E(\alpha)}{E_1(\alpha)E_2(\alpha)}+\text{H.c.}$$
$$+(\mathbf{p}_1\times\hat{v})\cdot\mathbf{s}_1\left(\frac{1}{E_1(\alpha)[E_1(\alpha)+m_1]}-\frac{E_2}{(E_1+m_1)E}\frac{E(\alpha)}{E_1(\alpha)E_2(\alpha)}\right), \tag{3.19}$$

[9] V. I. Ritus, Zh. Eksperim. i Teor. Fiz. 40, 352 (1961) [English transl.: Soviet Phys.—JETP 13, 240 (1961)]; A. Chakrabarti, J. Math Phys. 5, 1747 (1964).

where the term dependent on $\mathbf{s}_2$ has been omitted since it can easily be formed by subtracting the $\mathbf{s}_1$ term with $1\leftrightarrow 2$. Continuing to apply this convention, we may now directly find $\mathbf{r}_1(\alpha)-\mathbf{r}_2(\alpha)$ by using the same procedure as previously:

$$d[\mathbf{r}_1(\alpha)-\mathbf{r}_2(\alpha)]/d\alpha=\tfrac{1}{2}[\mathbf{r}_1(\alpha)-\mathbf{r}_2(\alpha)]\cdot\hat{v}\left(\mathbf{p}_1(\alpha)\frac{E_2(\alpha)}{E_1(\alpha)}+\mathbf{p}_2(\alpha)\frac{E_1(\alpha)}{E_2(\alpha)}\right)\frac{|\mathbf{P}(\alpha)|}{E(\alpha)}+\text{H.c.}$$
$$-\left(\frac{\mathbf{p}_1(\alpha)}{E_1(\alpha)}\frac{\hat{v}\cdot(\mathbf{p}_1\times\mathbf{s}_1)}{[E_1(\alpha)+m_1]^2}+\frac{\hat{v}\times\mathbf{s}_1(\alpha)}{E_1(\alpha)+m_1}\right)\frac{|\mathbf{P}(\alpha)|}{E(\alpha)}-\left(\frac{\mathbf{p}_1(\alpha)}{E_1(\alpha)}-\frac{\mathbf{p}_2(\alpha)}{E_2(\alpha)}\right)\frac{\hat{v}\cdot(\mathbf{p}_1\times\mathbf{s}_1)}{E_1(\alpha)+m_1}\frac{|\mathbf{P}(\alpha)|}{E(\alpha)^2}. \quad (3.20)$$

Substituting (3.19)

$$d[\mathbf{r}_1(\alpha)-\mathbf{r}_2(\alpha)]=\tfrac{1}{2}(\mathbf{r}_1-\mathbf{r}_2)\cdot\hat{v}\frac{E_1E_2}{E}\left(\frac{\mathbf{p}_1(\alpha)}{E_1(\alpha)^2}+\frac{\mathbf{p}_2(\alpha)}{E_2(\alpha)^2}\right)d\theta(\alpha)+\text{H.c.}$$
$$+(\mathbf{p}_1\times\hat{v})\cdot\mathbf{s}_1\frac{\mathbf{p}_1(\alpha)}{E_1(\alpha)}\left(-\frac{1}{[E_1(\alpha)+m_1]^2}+\frac{1}{E_1(\alpha)[E_1(\alpha)+m_1]}-\frac{1}{E_1(\alpha)(E_1+m_1)}\right)d\theta(\alpha)$$
$$+(\mathbf{p}_1\times\hat{v})\cdot\mathbf{s}_1\frac{1}{E(E_1+m_1)}\left(E_1\frac{\mathbf{p}_1(\alpha)}{E_1(\alpha)^2}-E_2\frac{\mathbf{p}_2(\alpha)}{E_2(\alpha)^2}\right)d\theta(\alpha)-\frac{\hat{v}\times\mathbf{s}_1(\alpha)}{E_1(\alpha)+m_1}d\theta(\alpha). \quad (3.21)$$

This is quite close to the differential form of the standard Lorentz transformation of a position operator demonstrated in Appendix A. Using integrals such as

$$\int_0^\alpha d\theta(\alpha')\frac{\mathbf{p}_\beta(\alpha')}{E_\beta(\alpha')^2}=\frac{\mathbf{p}_\beta(\alpha)\sinh\theta(\alpha)}{E_\beta E_\beta(\alpha)}+[\cosh\theta(\alpha)-1]\hat{v}/E_\beta,$$

then

$$\mathbf{r}_1(\alpha)-\mathbf{r}_2(\alpha)=\mathbf{r}_1-\mathbf{r}_2+\tfrac{1}{2}(\mathbf{r}_1-\mathbf{r}_2)\cdot\hat{v}\left([\cosh\theta(\alpha)-1]\hat{v}+\frac{\sinh\theta(\alpha)}{E}\left[\mathbf{p}_1(\alpha)\frac{E_2}{E_1(\alpha)}+\mathbf{p}_2(\alpha)\frac{E_1}{E_2(\alpha)}\right]\right)+\text{H.c.}$$
$$+(\mathbf{p}_1\times\hat{v})\cdot\mathbf{s}_1(\mathbf{p}_1-\hat{v}\cdot\mathbf{p}_1\hat{v})\frac{\sinh\theta(\alpha)}{(E_1+m_1)[E_1(\alpha)+m_1]E_1(\alpha)}+(\mathbf{p}_1\times\hat{v})\cdot\mathbf{s}_1\hat{v}\frac{E_1-E_1(\alpha)}{(E_1+m_1)[E_1(\alpha)+m_1]E_1(\alpha)}$$
$$+(\mathbf{p}_1\times\hat{v})\cdot\mathbf{s}_1\sinh\theta(\alpha)\left(\frac{\mathbf{p}_1(\alpha)}{E_1(\alpha)}-\frac{\mathbf{p}_2(\alpha)}{E_2(\alpha)}\right)\frac{1}{E(E_1+m_1)}+\hat{v}\cdot\mathbf{s}_1(\mathbf{p}_1\times\hat{v})\frac{\cosh\theta(\alpha)-1}{(E_1+m_1)[E_1(\alpha)+m_1]}$$
$$-\hat{v}\times\mathbf{s}_1\sinh\theta(\alpha)\frac{(E_1+m_1)[\cosh\theta(\alpha)+1]-\hat{v}\cdot\mathbf{p}_1\sinh\theta(\alpha)}{(E_1+m_1)[E_1(\alpha)+m_1][\cosh\theta(\alpha)+1]}. \quad (3.22)$$

Finally, by letting α go to infinity we obtain the comparatively simple form

$$\mathbf{r}=\mathbf{r}_1(\infty)-\mathbf{r}_2(\infty)$$
$$=\mathbf{r}_1-\mathbf{r}_2+\tfrac{1}{2}(\mathbf{r}_1-\mathbf{r}_2)\cdot\frac{\mathbf{P}}{M}\left[\frac{\mathbf{P}}{E+M}-\frac{\mathbf{P}\cdot\mathbf{k}}{E\omega_1\omega_2}\mathbf{k}+\left(\frac{1}{\omega_1}-\frac{1}{\omega_2}\right)\mathbf{k}\right]+\text{H.c.}$$
$$+(\mathbf{k}\times\mathbf{P})\cdot\mathbf{s}_1\left(\frac{1}{E(E_1+m_1)\omega_1\omega_2}+\frac{E+M}{\omega_1 a_1}\right)\mathbf{k}+\frac{1}{a_1}[\mathbf{P}\cdot\mathbf{s}_1(\mathbf{k}\times\mathbf{P})+(\mathbf{k}\times\mathbf{P})\cdot\mathbf{s}_1\mathbf{P}]-\frac{b_1}{a_1}\mathbf{P}\times\mathbf{s}_1, \quad (3.23)$$

where it should be noted that

$$(\mathbf{k}\times\mathbf{P})\cdot\mathbf{s}_1=(\mathbf{k}\times\mathbf{P})\cdot\mathbf{s}_1'=(\mathbf{p}_1\times\mathbf{P})\cdot\mathbf{s}_1=-(\mathbf{p}_2\times\mathbf{P})\cdot\mathbf{s}_1.$$

This is obvious from the equations, often used in the above calculations, which express the individual particle momentum and energy entirely in terms of the c.m. variables

$$\mathbf{p}_1=\mathbf{k}+\frac{\mathbf{P}\cdot\mathbf{k}}{M(E+M)}\mathbf{P}+\frac{\omega_1}{M}\mathbf{P},\qquad E_1=(E/M)\omega_1+\mathbf{P}\cdot\mathbf{k}/M, \quad (3.24)$$

and also $1\leftrightarrow 2$ when, it is to be remembered, $\mathbf{k}$ changes sign.

IV. CONCLUSION

In the above we have shown how the generators of $\mathcal{P}$ for two noninteracting particles can be written in the canonical form

$$H=E\equiv(M^2+\mathbf{P}^2)^{1/2},\quad \mathbf{P}=\mathbf{P}$$
$$\mathbf{J}=\mathbf{R}\times\mathbf{P}+\mathbf{S},\tag{4.1}$$
$$\mathbf{K}=-\tfrac{1}{2}(\mathbf{R}E+E\mathbf{R})-(\mathbf{P}\times\mathbf{S})/(E+M),$$

where the following commutation relations hold:

$$[R_i,P_j]=i\delta_{ij},\quad [S_i,S_j]=i\epsilon_{ijk}S_k,\quad [\mathbf{R},\mathbf{S}]=[\mathbf{P},\mathbf{S}]=[\mathbf{R},M]=[\mathbf{P},M]=[\mathbf{S},M]=0.\tag{4.2}$$

In terms of the single-particle dynamical variables, we have

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2,\quad M=\omega_1+\omega_2,$$

$$\mathbf{R}=\frac{1}{2}\left[\mathbf{r}_1\frac{\omega_1}{M}+\mathbf{r}_2\frac{\omega_2}{M}+(\mathbf{r}_1-\mathbf{r}_2)\frac{\mathbf{P}\cdot\mathbf{k}}{M(E+M)}+(\mathbf{r}_1-\mathbf{r}_2)\cdot\frac{\mathbf{P}}{M(E+M)}\left(\mathbf{k}-\frac{\mathbf{P}\cdot\mathbf{k}}{E(E+M)}\mathbf{P}\right)\right]+\text{H.c.}$$
$$+\frac{\mathbf{k}\times\mathbf{s}_1}{M(E_1+m_1)}+\frac{\omega_1-m_1}{M(E+M)(E_1+m_1)}\mathbf{P}\times\mathbf{s}_1-\frac{\mathbf{P}}{EM(E+M)(E_1+m_1)}(\mathbf{P}\times\mathbf{k})\cdot\mathbf{s}_1,\tag{4.3}$$

$$\mathbf{S}=\mathbf{r}\times\mathbf{k}+\mathbf{s}_1'+\mathbf{s}_2'.$$

For $\mathbf{R}$ the $\mathbf{s}_2$-dependent terms have been dropped since they have exactly the same form as the $\mathbf{s}_1$ terms with $1\leftrightarrow 2$. A variety of different expressions for $\mathbf{R}$ are possible[10]; we have chosen the form where the momentum-dependent coefficients are given in terms of c.m. momentum variables as far as is consistent with simplicity. The form of $\mathbf{S}$ is derived as simply as in the nonrelativistic case in (1.11).

The internal c.m. dynamical variables, which commute with $\mathbf{R}$ and $\mathbf{P}$, are given in the text by (3.7), (3.13), and (3.14), and (3.23).

To introduce an interaction while retaining relativistic invariance, it is necessary to modify the functional form of the generators E, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ in terms of the basic dynamical variables while retaining the commutation relations (2.1). To obey the condition of separability, it is necessary that the generators should approach the form (4.1) for large particle separations, i.e., large $|\mathbf{r}|$.

These conditions can easily be satisfied in the Bakamjian-Thomas form of the Hamiltonian[11] which is obtained by modifying the form of M in (4.1) while maintaining the commutation relations (4.2). [It should be noted that the functional form of the c.m. variables $\mathbf{R}$, $\mathbf{P}$, $\mathbf{r}$, $\mathbf{k}$, $\mathbf{s}_1'$, $\mathbf{s}_2'$ in (4.3), (3.7), (3.13), (3.14), and (3.23) is not to be altered; $M=\omega_1+\omega_2$. Equation (2.4) still holds, but with $M\to M'$.] Thus letting $M\to M'$, the most general expression for a relativistic two-spin-$\frac{1}{2}$ particle theory with a parity- and time-reversal-invariant interaction is given, similarly to the nonrelativistic case in (1.12), by

$$M'=(m_1^2+\mathbf{k}^2)^{1/2}+(m_2^2+\mathbf{k}^2)^{1/2}+V_1(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)$$
$$+V_2(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{s}_1'\cdot\mathbf{s}_2'+V_3(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{l}\cdot\mathbf{s}_1'$$
$$+V_4(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{l}\cdot\mathbf{s}_2'+V_5(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{l}\cdot\mathbf{s}_1'\mathbf{l}\cdot\mathbf{s}_2'$$
$$+V_6(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{r}\cdot\mathbf{s}_1'\mathbf{r}\cdot\mathbf{s}_2'$$
$$+V_7(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)\mathbf{k}\cdot\mathbf{s}_1'\mathbf{k}\cdot\mathbf{s}_2',$$
$$\mathbf{l}=\mathbf{r}\times\mathbf{k}.\tag{4.4}$$

The separability condition is then achieved by ensuring that the $V_i(|\mathbf{r}|,|\mathbf{k}|,|\mathbf{l}|)$ vanish sufficiently rapidly for large $|\mathbf{r}|$.[12]

The expressions obtained here for the c.m. dynamical variables are much too complicated to be used in actual computation. However, we intend to show elsewhere[13] that the modifications from the nonrelativistic form are necessary in order to achieve correct relativistic results in two-particle dynamical calculations. Since our treatment of the two-particle system neglects such explicitly field-theoretic effects as pair creation and crossing, we expect it to be valid in the relatively low-energy region where an expansion in the inverse masses (or effectively v/c) is permissable. To lowest order beyond the nonrelativistic case the results for the basic single-particle variables in terms of the c.m. variables

[10] B. Barsella and E. Fabri, Phys.Rev. **126**, 1561 (1962); and also Ref. 7, where the internal c.m. dynamical variables are given for the spinless case.

[11] B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953); R. Fong and J. Sucher, J. Math. Phys. **5**, 456 (1964).

[12] Relativistically there is some difficulty in the interpretation of $\mathbf{r}_1$ and $\mathbf{r}_2$ as the position operators for particles 1 and 2 since they do not transform covariantly under Lorentz transformation even for free particles (see Appendix A). However, all that is required here is that, for states in which the physical particle distances of separation become increasingly large, the expectation value of $|\mathbf{r}|$ or $|\mathbf{r}_1-\mathbf{r}_2|$ should increase without limit. For any interpretation of $\mathbf{r}_1$ and $\mathbf{r}_2$ this would seem very likely to be true.

[13] H. Osborn, following paper, Phys. Rev. **176**, 1523 (1968).

are

$$\mathbf{p}_1=\frac{m_1}{\mathfrak{M}}\mathbf{P}+\mathbf{k}+\left(\frac{m_2-m_1}{2m_1m_2\mathfrak{M}}\mathbf{k}^2+\frac{1}{2\mathfrak{M}^2}\mathbf{P}\cdot\mathbf{k}\right)\mathbf{P},$$

$$\mathbf{s}_1=\mathbf{s}_1'+(1/2m_1\mathfrak{M})(\mathbf{k}\times\mathbf{P})\times\mathbf{s}_1',\quad \mathfrak{M}=m_1+m_2,$$

$$\mathbf{r}_1=\mathbf{R}+\frac{m_2}{\mathfrak{M}}\mathbf{r}+\tfrac{1}{2}\mathbf{r}\left(\frac{m_1-m_2}{2m_1m_2\mathfrak{M}}\mathbf{k}^2-\frac{1}{2\mathfrak{M}^2}\mathbf{P}\cdot\mathbf{k}\right)+\tfrac{1}{2}\mathbf{r}\cdot\mathbf{P}\frac{1}{\mathfrak{M}^2}\left(\tfrac{1}{2}\mathbf{k}-\frac{m_2}{m_1}\mathbf{k}-\frac{m_2}{2\mathfrak{M}}\mathbf{P}\right)+\text{H.c.}$$
$$-\frac{1}{2\mathfrak{M}m_1}\mathbf{k}\times\mathbf{s}_1'+\frac{1}{2\mathfrak{M}m_2}\mathbf{k}\times\mathbf{s}_2'+\frac{m_2}{2\mathfrak{M}^2m_1}\mathbf{P}\times\mathbf{s}_1'-\frac{1}{2\mathfrak{M}^2}\mathbf{P}\times\mathbf{s}_2',\tag{4.5}$$

and also $1\leftrightarrow 2$. It may be verified that these expressions obey the commutation relations (1.1) to order M^{-2}.

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APPENDIX A

We here discuss the transformation of the basic dynamical variables in canonical form (2.2) of the generators by the boost elements of the Lie group $\mathcal{P}$. For any operator X in a reference frame $\mathcal{S}$,

$$X'=\exp(i\theta\hat{v}\cdot\mathbf{K})X\exp(-i\theta\hat{v}\cdot\mathbf{K}),\quad \tanh\theta=|\mathbf{v}|\tag{A1}$$

represents the same dynamical quantity referred to a reference frame moving with velocity $-\mathbf{v}$ relative to $\mathcal{S}$. In contrast to Sec. III, θ and $\hat{v}$ are now ordinary c numbers. Even in the two-particle Hilbert space $\mathcal{H}_2$, there is no mixing of the dynamical variables for the different particles; thus the subscripts 1 and 2 are dropped here.

For the momentum

$$\frac{d}{d\theta}\mathbf{p}'=-i\exp(i\theta\hat{v}\cdot\mathbf{K})[\{E\mathbf{v}\cdot\mathbf{r}\},\mathbf{p}]\exp(-i\theta\hat{v}\cdot\mathbf{K})$$
$$=E'\hat{v},\quad E'=(m^2+\mathbf{p}'^2)^{1/2}.\tag{A2}$$

Letting

$$\mathbf{p}'=\mathbf{p}-\hat{v}\cdot\mathbf{p}\hat{v}+f(\theta)\hat{v},\quad f(0)=\hat{v}\cdot\mathbf{p}$$
$$E'=(E^2-\hat{v}\cdot\mathbf{p}^2+f^2)^{1/2},\quad E=(m^2+\mathbf{p}^2)^{1/2},\tag{A3}$$

then

$$df/d\theta=E'.\tag{A4}$$

This can be solved to give

$$f(\theta)=(E^2-\hat{v}\cdot\mathbf{p}^2)^{1/2}\sinh\left[\sinh^{-1}\frac{\hat{v}\cdot\mathbf{p}}{(E^2-\hat{v}\cdot\mathbf{p}^2)^{1/2}}+\theta\right]$$
$$=E\cosh\theta+\hat{v}\cdot\mathbf{p}\sinh\theta.\tag{A5}$$

With this result, we obtain

$$\mathbf{p}'=\mathbf{p}+(\cosh\theta-1)\hat{v}\cdot\mathbf{p}\hat{v}+\sinh\theta\, E\hat{v},$$
$$E'=\cosh\theta\, E+\sinh\theta\,\hat{v}\cdot\mathbf{p}.\tag{A6}$$

For the spin, following the same method as above, we have

$$\frac{d\mathbf{s}'}{d\theta}=\frac{1}{E'+m}(\mathbf{p}'\times\hat{v})\times\mathbf{s}'.\tag{A7}$$

This describes a rotation around an axis $\hat{n}$ where

$$\mathbf{s}'=\cos\gamma\,\mathbf{s}+(1-\cos\gamma)\hat{n}\cdot\mathbf{s}\hat{n}+\sin\gamma\hat{n}\times\mathbf{s},$$
$$\hat{n}=\mathbf{p}\times\hat{v}/|\mathbf{p}\times\hat{v}|.\tag{A8}$$

From (A6)–(A8),

$$\frac{d\gamma}{d\theta}=\frac{|\mathbf{p}\times\hat{v}|}{E\cosh\theta+\hat{v}\cdot\mathbf{p}\sinh\theta+m}.\tag{A9}$$

This can be integrated to

$$\tfrac{1}{2}\gamma=\tan^{-1}\left(\frac{e^\theta(E+\hat{v}\cdot\mathbf{p})+m}{|\mathbf{p}\times\hat{v}|}\right)$$
$$-\tan^{-1}\left(\frac{E+\hat{v}\cdot\mathbf{p}+m}{|\mathbf{p}\times\hat{v}|}\right).\tag{A10}$$

Substitution in (A8) gives

$$\mathbf{s}'=[1-(1/a)|\mathbf{p}\times\mathbf{u}|^2]\mathbf{s}+(1/a)(\mathbf{p}\times\mathbf{u})\cdot\mathbf{s}(\mathbf{p}\times\mathbf{u})$$
$$+(b/a)(\mathbf{p}\times\mathbf{u})\times\mathbf{s},$$
$$a=(E+m)(E'+m)(\cosh\theta+1),$$
$$b=(E+m)(\cosh\theta+1)+\mathbf{p}\cdot\mathbf{u},\quad \mathbf{u}=\sinh\theta\,\hat{v}.\tag{A11}$$

For the position operator, we first use the obvious result $\hat{v}\cdot\mathbf{K}=\hat{v}\cdot\mathbf{K}'$ to obtain

$$\hat{v}\cdot\mathbf{r}'=t\frac{\hat{v}\cdot(\mathbf{p}'-\mathbf{p})}{E'}+\frac{1}{2}\left(\hat{v}\cdot\mathbf{r}\frac{E}{E'}+\frac{E}{E'}\hat{v}\cdot\mathbf{r}\right)+(\mathbf{p}\times\hat{v})\cdot\mathbf{s}\frac{1}{E'}\left(\frac{1}{E'+m}-\frac{1}{E+m}\right), \quad \text{(A12)}$$

where for generality we do not take $t=0$; it is only the transformed position operator which is time-dependent. The usual procedure gives

$$\frac{d\mathbf{r}'}{d\theta}=t\hat{v}-\frac{1}{2}\left(\hat{v}\cdot\mathbf{r}'\frac{\mathbf{p}'}{E'}+\frac{\mathbf{p}'}{E'}\hat{v}\cdot\mathbf{r}'\right)+\frac{\hat{v}\times\mathbf{s}'}{E'+m}-\frac{\mathbf{p}'}{E'}\frac{\mathbf{p}\times\hat{v}\cdot\mathbf{s}}{(E'+m)^2}. \quad \text{(A13)}$$

With the substitution of (A12), we find

$$\frac{d\mathbf{r}'}{d\theta}=t\left(\hat{v}-\frac{\hat{v}\cdot(\mathbf{p}'-\mathbf{p})}{E'^2}\mathbf{p}'\right)-\frac{1}{2}\left(\hat{v}\cdot\mathbf{r}E\frac{\mathbf{p}'}{E'^2}+\frac{\mathbf{p}'}{E'^2}E\hat{v}\cdot\mathbf{r}\right)-(\mathbf{p}\times\hat{v})\cdot\mathbf{s}\frac{\mathbf{p}'}{E'}\left(\frac{1}{(E'+m)^2}+\frac{1}{E'(E'+m)}-\frac{1}{E'(E+m)}\right)+\frac{\hat{v}\times\mathbf{s}'}{E'+m}. \quad \text{(A14)}$$

The evaluation of the integrals in (A14) is too tedious to relate here but the result, which may be checked by differentiation, is

$$\mathbf{r}'=\mathbf{r}+(\cosh\theta-1)\hat{v}\cdot\mathbf{r}\hat{v}+\sinh\theta\, t\hat{v}-(\cosh\theta-1)\frac{\mathbf{p}'}{E'}t-\sinh\theta\,\frac{1}{2}\left(\hat{v}\cdot\mathbf{r}\frac{\mathbf{p}'}{E'}+\frac{\mathbf{p}'}{E'}\hat{v}\cdot\mathbf{r}\right)-\frac{1}{a}\left((\cosh\theta+1)\frac{\mathbf{p}'}{E'}-\mathbf{u}\right)(\mathbf{p}\times\mathbf{u})\cdot\mathbf{s}+\frac{1}{a}(\mathbf{p}\times\mathbf{u})\mathbf{u}\cdot\mathbf{s}+\frac{b}{a}\mathbf{u}\times\mathbf{s}. \quad \text{(A15)}$$

The first line of (A15) is the same as for the conventional Lorentz transformation; the other terms are an example of the result that a position operator does not transform covariantly.

APPENDIX B

In momentum space it is usual to represent the total position and relative position operators $\mathbf{R}$ and $\mathbf{r}$ by the differential operators $i\partial/\partial\mathbf{P}$ and $i\partial/\partial\mathbf{k}$. It is then not obvious that these operators should be spin-dependent. The necessity for spin dependence arises since the spin operator undergoes a momentum-dependent transformation $\mathbf{s}_1\to\mathbf{s}_1'$ (we neglect $\mathbf{s}_2$ here). Since $\mathbf{R}$ and $\mathbf{r}$ are required to commute with the transformed spin $\mathbf{s}_1'$, we have

$$\mathbf{R}=i\frac{\partial}{\partial\mathbf{P}}\bigg|_{\mathbf{k},\mathbf{s}_1'},\quad \mathbf{r}=i\frac{\partial}{\partial\mathbf{k}}\bigg|_{\mathbf{P},\mathbf{s}_1'}. \quad \text{(B1)}$$

In terms of the relative position operator for spinless particles $\mathbf{r}_0$, it is possible to write $\mathbf{r}$ as

$$\mathbf{r}=i\frac{\partial}{\partial\mathbf{k}}\bigg|_{\mathbf{P},\mathbf{s}_1}+i\frac{\partial\mathbf{s}_1}{\partial\mathbf{k}}\bigg|_{\mathbf{k},\mathbf{s}_1'}\cdot\frac{\partial}{\partial\mathbf{s}_1}\bigg|_{\mathbf{k},\mathbf{P}}=\mathbf{r}_0+x. \quad \text{(B2)}$$

The expression for $\mathbf{r}_0$ has been calculated before.[10] Given that

$$s_1'^j=a_1^{jn}s_1^n,\quad s_1^k=a_1^{jk}s_1'^j, \quad \text{(B3)}$$

where a_1 is an orthogonal spin-independent matrix, then

$$x^i=i\frac{\partial}{\partial k^i}a_1^{jk}\big|_{\mathbf{P}}\, a_1^{jn}\frac{1}{2}\left(s_1^n\frac{\partial}{\partial s_1^k}-s_1^k\frac{\partial}{\partial s_1^n}\right). \quad \text{(B4)}$$

We have used the result

$$\frac{\partial}{\partial k^i}a_1^{jn}\,a_1^{jk}=-\frac{\partial}{\partial k^i}a_1^{jk}\,a_1^{jn} \quad \text{(B5)}$$

to ensure that (B4) is explicitly Hermitian. Now for any spin operators,

$$s^n\partial/\partial s^k-s^k\partial/\partial s^n=i\epsilon^{mnk}s^m, \quad \text{(B6)}$$

so that (B4) can be written

$$x^i=-\tfrac{1}{2}\epsilon^{mnk}a_1^{jn}\partial a_1^{jk}/\partial k^i\big|_{\mathbf{P}}s^m. \quad \text{(B7)}$$

Writing the total position operator $\mathbf{R}$ as

$$\mathbf{R}=\mathbf{R}_0+y, \quad \text{(B8)}$$

with $\mathbf{R}_0$ spin-independent, then $\mathbf{y}$ is given by exactly the same reasoning by

$$y^i=-\tfrac{1}{2}\epsilon^{mnk}a_1^{jn}\partial a_1^{jk}/\partial P^i\big|_{\mathbf{k}}\,s^m. \quad \text{(B9)}$$

With a_1 given by (3.13), we have verified that (B7) and (B9) produce the correct spin-dependent terms shown in (3.23) and (4.3). The calculations are straightforward but tedious, so that we do not discuss them here.