Static, Axially Symmetric Point Horizons*

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Properties of event horizons are examined for static, axially symmetric, vacuum space-times. Israel has shown that under fairly general conditions such horizons are singular. One would expect that a positive-mass point particle would correspond to a pointlike singular horizon. It is shown, however, that the horizons *cannot* be pointlike. The geometry of particular horizons is discussed.

1. INTRODUCTION

THE analysis of Kruskal[1] has provided us with a definitive picture of the regularity of the Schwarzschild surface $r=2m$. Two physically relevant questions, however, have not been completely answered: (i) Can analogs of the Schwarzschild surface appear in nature? (ii) Are the geometrical properties of the Schwarzschild surface stable?

Models of gravitational collapse have been constructed[2] which lend plausibility to an affirmative answer to question (i), but a completely decisive argument has not emerged. Also, various perturbation-type calculations[3] agree to some extent on an affirmative answer to question (ii). It is, however, a troublesome feature of these latter investigations that stability is not maintained if perturbations that become singular on the Schwarzschild surface itself are considered. These investigators have not taken such perturbations seriously because they introduce singularities in what the Kruskal picture describes as a regular space-time region.

In this regard, we have recently pointed out that the exact, asymptotically flat, spherically symmetric, static solution of a massless scalar field interacting with an Einstein field has some very surprising features.[4] For any nonvanishing value of the scalar monopole, the event horizon corresponding to the Schwarzschild surface is a *singular point.* It was emphasized that this result could not have been built up from a perturbation expansion (treating the scalar field as a spherically symmetric, static perturbation of the exterior Schwarzschild background). Passing to the limit of zero scalar monopole gives, in fact, the external Schwarzschild solution with a step-function behavior in the metric at $r=2m$. Moreover, to first order in a perturbation calculation the scalar field would of necessity introduce a singular perturbation at the Schwarzschild surface. In this case, knowledge of the exact solution clearly demands that such singular perturbations should not be overlooked but interpreted as an indication that the exact solution is singularly different from the assumed background.

Since any physical implications to be drawn from this model are obscured by the present lack of positive experimental observation of a massless scalar field, it is natural to ask the question: Which of its features might be exhibited by other types of exact solutions? In particular, we shall concentrate in this paper on the exact, axially symmetric, static, vacuum solutions. Using the methods due to Weyl and Levi-Civita, a family of multipolelike solutions can easily be constructed.[5] We shall then proceed to examine the properties of a generalized Schwarzschild surface (GSS). We define such a surface in the static case in terms of the timelike Killing vector ξ^μ orthogonal to the $t=\text{const}$ hypersurfaces. The GSS is the boundary to the static exterior region on which

$$t=\text{const}\,,\quad \xi^\mu\xi_\mu=0\,.$$

One aspect of our question is then partly answered by a theorem of Israel.[6] This theorem may be recast in the following form:

The exterior Schwarzschild geometry is the only maximally extended, static, asymptotically flat, vacuum space-time with a family of simply connected equipotential surfaces which converge to a nonsingular GSS with finite two-dimensional intrinsic geometry.

The equipotential surfaces are given by $t=\text{const}$, $\xi^\mu\xi_\mu=\text{const}$. The singular behavior refers to a scalar of the Riemann tensor.

For the subclass of static, axially symmetric, vacuum space-times to which this theorem is applicable, it then follows that for any arbitrary deviation of the multipole structure from the Schwarzschild case there will be a

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[1] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960).
[2] See, for example, (a) A. G. Doroshkevich, Ya. B. Zel'dovich, and I. D. Novikov, Zh. Eksperim. i Teor. Fiz. **49**, 170 (1965) [English transl.: Soviet Phys.—JETP **22**, 122 (1966)]; (b) R. Penrose, Adams Prize Essay, 1966 (unpublished).
[3] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957); C. V. Vishveshwara, University of Maryland Report No. 778, 1968 (unpublished).
[4] A. I. Janis, E. T. Newman, and J. Winicour, Phys. Rev. Letters **20**, 878 (1968). Equation (2) of this paper should read

$$\phi=\frac{\sqrt{2}A}{\mu r_0}\ln\left|\frac{2R-r_0(\mu-1)}{2R+r_0(\mu+1)}\right|.$$

Earlier attempts to find solutions of this nature are to be found in I. Z. Fisher, Zh. Eksperim. i Teor. Fiz. **18**, 636 (1948); O. Bergmann and R. Leipnik, Phys. Rev. **107**, 1157 (1957). These two papers apparently contain errors, and do not explicitly give a solution exhibiting all of the features discussed here.
[5] See, for example, J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Co., Amsterdam, 1960), Chap. 8, p. 309.
[6] W. Israel, Phys. Rev. **164**, 1776 (1967).

singularity on the GSS. It remains to determine whether singular *pointlike* event horizons exist. This is answered in Sec. 4, where it is shown that no GSS can be a true pointlike source for a static, axially symmetric, vacuum space-time. The feature of the interacting scalar-field solution which is characteristic of asymmetric static vacuum fields is the singular nature of the GSS; but the pointlike structure of the GSS cannot be realized in the static, vacuum case. By pointlike, in the present context, we mean that the geometry of a sequence of spacelike two-surfaces enclosing the GSS approaches the geometry of a point. The truncated Schwarzschild solution previously described by the present authors[4] does not have a pointlike GSS in this sense, although if interpreted as the limit of a sequence of nonvacuum solutions, its GSS is pointlike.

In Secs. 2 and 3, we examine some interesting features of a GSS in static, axially symmetric, vacuum space-times. A particular solution (closely related to the interacting scalar-field solution) is given that can be considered to possess a GSS with zero two-dimensional surface area, but it is shown that the GSS cannot be a true point source. In Appendix A, we discuss the connection between alternative treatments of static, axially symmetric space-times. In Appendix B, we give an interesting solution for coupled negative- and positive-mass sources that give rise to a positive total mass.

2. FORMALISM OF WEYL AND LEVI-CIVITA

Weyl and Levi-Civita have shown that the metric of a static, axially symmetric, vacuum gravitational field may be put in the form[5]

$$ds^2=e^{2\psi}dt^2-e^{2\gamma-2\psi}(d\rho^2+dz^2)-\rho^2e^{-2\psi}d\phi^2\,, \quad (2.1)$$

where ψ is an axially symmetric solution of the Newtonian potential equation (in cylindrical coordinates)

$$\psi_{\rho\rho}+(1/\rho)\psi_\rho+\psi_{zz}=0 \quad (2.2)$$

and γ is determined by a path integral of the equations

$$\gamma_\rho=\rho(\psi_\rho{}^2-\psi_z{}^2)\,,\quad \gamma_z=2\rho\psi_\rho\psi_z\,. \quad (2.3)$$

Formally, we may view ψ as a Newtonian gravitational potential in the background Euclidean three-space

$$dl^2=d\rho^2+dz^2+\rho^2d\phi^2\,. \quad (2.4)$$

There is, however, a certain amount of distortion inherent in this viewpoint. The Schwarzschild solution is not generated by the monopole solution of Eq. (2.2) but by the potential of a rod of length $2m$ with uniform half-unit density. The monopole solution of Eq. (2.2) generates the Curzon solution[7] of Einstein's equations, a nonspherically symmetric solution.

In terms of the Weyl–Levi-Civita coordinates, the timelike Killing vector is given by

$$\xi^\mu=(1,0,0,0).$$

The conditions for a GSS are $t=\text{const}$ and $g_{00}=e^{2\psi}=0$, so that *the GSS corresponds to either a point or line source of positive density for the Newtonian potential* ψ. Thus the GSS forms a singular boundary to the background Euclidean manifold. In the four-dimensional curved geometry, the time evolution of the GSS is a null hypersurface generated by null rays with tangents ξ^μ. As in the Schwarzschild case (but, perhaps, in a limiting sense), this null hypersurface is shear-free and divergenceless by virtue of Killing's equation

$$\xi^{(\mu;\nu)}=0.$$

A GSS also serves as a source for an asymptotically flat, exterior Einstein field that has positive total mass. To show this, it is first useful to establish the following lemma.

Lemma. In an asymptotically flat space-time, for a GSS whose domain in the background Euclidean manifold is a single point, the potential ψ is that of a pure positive-mass monopole.

Proof.[8] Asymptotic flatness implies that we can expand ψ in a series

$$\psi=c-\frac{m}{r}+\sum_{l=1}^{\infty}S_l(\theta,\phi)r^{-l-1}\,, \quad (2.5)$$

where the coefficients S_l are linear combinations of the appropriate spherical harmonics and (r,θ,ϕ) are related to (ρ,z,ϕ) by the usual transformation between polar and cylindrical coordinates. The constant term c can be eliminated by the usual coordinate convention that g_{00} approach 1 at spatial infinity. For a GSS, ψ must uniformly approach $-\infty$ for all directions of approach to $r=0$. There are, however, directions of approach for which the term

$$S=\sum_{l=1}^{\infty}S_lr^{-l-1}$$

is always positive (this follows from the observation that the average of S over a sphere $r=\text{const}$ centered about the singularity is zero). For sufficiently small r the term S in ψ will dominate. Hence, for a point singularity in the Euclidean background to give rise to a GSS we must have $S=0$.

To investigate the positive-mass aspect of a GSS, consider the following application of the Gauss integral theorem:

$$\int_R\nabla^2(e^{2\psi})d^3V=\int_{\dot R}\nabla(e^{2\psi})\cdot d\mathbf{S}\,, \quad (2.6)$$

where the volume and surface elements are determined here by the background Euclidean geometry [see Eq. (2.4)]. The region of integration R is the Euclidean

[7] H. E. J. Curzon, Proc. London Math. Soc. **23**, 477 (1924).

[8] The following proof is somewhat heuristic. The basis for a mathematically more rigorous proof can be found in O. D. Kellogg, *Foundations of Potential Theory* (Dover Publications, Inc., New York, 1953), p. 270.

three-space with the domain of the singular Newtonian source deleted. The boundary $\dot{R}$ then consists of two limiting surface integrals, one being over an infinite sphere S_∞ and the other over the equipotential surface S_ϵ determined by $e^{2\psi}=\epsilon$ in the limit $\epsilon \to 0$. Carrying out the differentiations in Eq. (2.6) gives

$$2\int_R e^{2\psi}(\nabla\psi\cdot\nabla\psi)d^3V=\int_{S_\infty} e^{2\psi}\nabla\psi\cdot d\mathbf{S} -\int_{S_\epsilon} e^{2\psi}\nabla\psi\cdot d\mathbf{S}. \quad (2.7)$$

If the source of the Newtonian potential ψ corresponds to a point in the background Euclidean manifold, the above lemma tells us that $\psi\sim 1/r$ and that the equipotential surfaces S_ϵ are spheres. Consequently $\nabla\psi\cdot d\mathbf{S}$ is bounded and the surface integral over S_ϵ tends to zero as $\epsilon\to 0$. For an arbitrary line source in the Euclidean background or for a combination of line and point sources this result is easily generalized. The Gauss integral theorem gives

$$\int_{S_\epsilon} e^{2\psi}\nabla\psi\cdot d\mathbf{S}=\epsilon\int_{S_\epsilon}\nabla\psi\cdot d\mathbf{S}=\epsilon\int_{S_\infty}\nabla\psi\cdot d\mathbf{S}.$$

For large r we now use Eq. (2.5) (putting $c=0$) to obtain

$$\int_{S_\infty}\nabla\psi\cdot d\mathbf{S}=\int_{S_\infty} e^{2\psi}\nabla\psi\cdot d\mathbf{S}=4\pi m,$$

where m may be identified as the total mass of the system. In general, therefore, the integral over S_ϵ vanishes as $\epsilon\to 0$, and from the positive definiteness of the left-hand side of Eq. (2.7) we obtain $m>0$.

This result is quite apparent physically. A total positive mass is necessary for the creation of an infinite red-shift surface, negative mass giving rise to blue shifts.

3. EREZ-ROSEN FORMALISM

Erez and Rosen[9] have pointed out that the multipole structure of static axially symmetric solutions takes on a simpler form when elliptical coordinates are used to describe the background Euclidean manifold rather than the cylindrical coordinates of Weyl and Levi-Civita. The connection between the two formalisms is elucidated in Appendix A.

It is illustrative to examine the properties of the GSS in two simple cases. First, consider the case of a Schwarzschild particle with a superimposed quadrupole moment, as constructed by Erez and Rosen. The correct expressions[9] for ψ and γ are

$$\psi=\tfrac{1}{2}[1+\tfrac{1}{4}q(3\lambda^2-1)(3\mu^2-1)]\ln\frac{\lambda-1}{\lambda+1}+\tfrac{3}{4}q\lambda(3\mu^2-1), \quad (3.1)$$

$$\begin{aligned}\gamma=&\tfrac{1}{2}(1+2q+q^2)\ln\frac{\lambda^2-1}{\lambda^2-\mu^2}-\tfrac{3}{2}q(1-\mu^2)\left(\lambda\ln\frac{\lambda-1}{\lambda+1}+2\right)\\&+(9/4)q^2(1-\mu^2)\bigg[(\lambda^2+\mu^2-1-9\lambda^2\mu^2)\\&\times\tfrac{1}{16}(\lambda^2-1)\ln^2\frac{\lambda-1}{\lambda+1}+\tfrac{1}{4}(\lambda^2+7\mu^2-\tfrac{5}{3}-9\mu^2\lambda^2)\\&\times\lambda\ln\frac{\lambda-1}{\lambda+1}+\tfrac{1}{4}\lambda^2(1-9\mu^2)+\mu^2-\tfrac{1}{3}\bigg], \quad (3.2)\end{aligned}$$

where the radial coordinate λ and the angular coordinate μ are related to the Weyl–Levi-Civita coordinates of Eq. (2.1) by $z=m\lambda\mu$ and $\rho^2=m^2(\lambda^2-1)(1-\mu^2)$, and where positive values of the quadrupole parameter q imply that the body is elongated along the symmetry axis. As noted in Ref. 2(a), for reasonably small q the equipotential surfaces are simply connected, closed, and embedded in one another, quite unlike the situation for a Newtonian monopole plus quadrupole (whose equipotential surfaces bifurcate). The GSS is given by $\lambda=1$. We now examine the shape of the GSS by calculating the arc length of small coordinate circles in the limit $\lambda\to 1$. To determine the azimuthal dimensions, we take coordinate circles about the z axis at some fixed polar direction μ. The arc length depends upon

$$\sqrt{(-g_{\phi\phi})}=me^{-\psi}(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2}.$$

Up to a finite constant of proportionality, we find, for $\lambda\sim 1$,

$$\sqrt{(-g_{\phi\phi})}\sim(\lambda-1)^{-q(3\mu^2-1)/4}(1-\mu^2)^{1/2}.$$

For positive q, the circumferences will be zero for polar directions $3\mu^2<1$ and infinite for polar directions $3\mu^2>1$. Similarly, for the polar dimension, we have

$$\sqrt{(-g_{\mu\mu})}\sim(\lambda-1)^{q(3\mu^2-1)[1+\frac{1}{2}q(3\mu^2-1)]/4}\times(\lambda^2-\mu^2)^{-q-\frac{1}{2}q^2}(1-\mu^2)^{-\frac{1}{2}}.$$

The polar arc length obtained by integrating $\sqrt{(-g_{\mu\mu})}$ from $\mu=-1$ to $\mu=+1$ also tends to infinity as $\lambda\to 1$. On the other hand, the surface area of a coordinate sphere $\lambda=\text{const}$ is determined by

$$\sqrt{(g_{\mu\mu}g_{\phi\phi})}\sim(\lambda-1)^{(1/8)q^2(3\mu^2-1)^2}(\lambda^2-\mu^2)^{-q-\frac{1}{2}q^2}.$$

Integrating over the sphere gives the rather surprising result that for reasonably small values of q the surface area tends to zero as $\lambda\to 1$ [the opposite conclusion was stated in Ref. 2(a)]. Even though the GSS can be enclosed in a sphere of arbitrarily small surface area, it is obviously not a point.

[9] G. Erez and N. Rosen, Bull. Res. Council Israel **8F**, 47 (1959). Their Eq. (15) for γ contains an error that was corrected in Ref. 2(a), Eq. (I.1). This latter equation contains a misprint that is corrected in our Eq. (3.2).

As a second illustration, consider the superposition of a Schwarzschild particle and an Erez-Rosen monopole (this augmented Schwarzschild solution is discussed further in Appendix A):

$$\psi=\tfrac{1}{2}\epsilon\ln[(\lambda-1)/(\lambda+1)], \quad (3.3)$$

with $\epsilon=1$ giving pure Schwarzschild. The path-integral equations for γ give

$$\gamma=\tfrac{1}{2}\epsilon^2\ln[(\lambda^2-1)/(\lambda^2-\mu^2)]. \quad (3.4)$$

Although in the Erez-Rosen coordinates this solution looks harmlessly different from Schwarzschild, this is not actually the case. The geometry is no longer spherically symmetric. For $\lambda\sim1$, we have

$$\sqrt{(-g_{\phi\phi})}\sim(\lambda-1)^{\frac{1}{2}(1-\epsilon)}(1-\mu^2)^{\frac{1}{2}},$$
$$\sqrt{(-g_{\mu\mu})}\sim(\lambda-1)^{-\frac{1}{2}\epsilon(1-\epsilon)}(\lambda^2-\mu^2)^{\frac{1}{2}(1-\epsilon^2)}(1-\mu^2)^{-\frac{1}{2}}.$$

Taking $0<\epsilon<1$, the azimuthal circumferences have zero arc length in the limit $\lambda\rightarrow1$; but the polar circumference will be infinite in this limit. The surface areas of the spheres $\lambda=$ const, determined by

$$\sqrt{(g_{\mu\mu}g_{\phi\phi})}\sim(\lambda-1)^{\frac{1}{2}(1-\epsilon)^2}(\lambda^2-\mu^2)^{\frac{1}{2}(1-\epsilon^2)},$$

approach zero in the limit $\lambda\rightarrow1$, but again the GSS cannot be considered to be a point. This augmented Schwarzschild solution is especially significant because the Newtonian potential ψ is formally equivalent to the Newtonian potential obtained in the coupled scalar-field model. In fact, this vacuum Erez-Rosen solution satisfies all the coupled Einstein equations with the appropriate energy-momentum tensor for a spherically symmetric, static, scalar field *except* the coupled analog of Eqs. (2.3) for the path integral of γ. The critical role of the coupled scalar field is to introduce just the right source for these path-integral equations to establish the spherical symmetry and pointlike character of the GSS.

For both the Erez-Rosen Schwarzschild plus quadrupole and augmented Schwarzschild solutions, the Riemann scalar $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ is infinite on the GSS. These results, however, cannot be inferred directly from the formulation of Israel's theorem as given in Sec. 1, because in both cases the two-geometry of the equipotential surfaces becomes singular as $\psi\rightarrow-\infty$. For the Schwarzschild plus quadrupole solution, Ref. 2(a) gives

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\sim q^2(\lambda-1)^{-1}.$$

For the augmented Schwarzschild solution, we find for $0<\epsilon<1$ that

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\sim(\epsilon-1)^2(\lambda-1)^{-2(1-\epsilon+\epsilon^2)}.$$

4. NONEXISTENCE OF A POINT GSS

In Sec. 2 we observed that a GSS consists of points where the potential ψ tends to $-\infty$. From the standpoint of Newtonian potential theory in the background Euclidean manifold, the source for ψ must then be either a point or line distribution of positive density. However, the shape of the GSS in terms of the background Euclidean geometry and its shape in terms of the four-dimensional curved geometry may radically differ. Indeed, as expressed by the lemma in Sec. 2, for a GSS to correspond to a point in the background Euclidean manifold, the source must be a pure positive-mass monopole. The associated monopole potential generates the Curzon solution of Einstein's equations. But it is easy to verify that the equipotential surfaces of the Curzon geometry attain infinite intrinsic dimensions as they approach the GSS.[10] Hence, what does correspond to a point GSS from the Euclidean point of view is distinctly not a point in the four-geometry.

Using the formalism of Weyl and Levi-Civita, we now establish in the general axially symmetric, static, vacuum case that a GSS cannot be a true point. To begin, consider the consequences of being able to surround the GSS with azimuthal circles of arbitrarily small arc length. We immediately obtain from Eq. (2.1) that

$$\sqrt{(-g_{\phi\phi})}=\rho e^{-\psi}$$

must tend to zero at the GSS. But at the same time $e^{-\psi}$ tends to infinity. Therefore the GSS must arise from a line distribution on the z axis such that

$$\rho e^{-\psi}\rightarrow0 \quad \text{as} \quad \rho\rightarrow0. \quad (4.1)$$

This condition again makes it clear that the Newtonian source must be a line distribution and not a point distribution.

We also require that it be possible to surround the GSS with a closed polar curve (along which $\phi=$ const) having arbitrarily small arc length. In terms of the background Euclidean geometry, any such curve will contain a finite segment along which dz does not vanish. Hence, at least for one value of z occupied by the GSS we must have

$$\sqrt{(-g_{zz})}=e^{\gamma-\psi}\rightarrow0 \quad \text{as} \quad \rho\rightarrow0.$$

This requires that $\psi-\gamma$ be unbounded from above for small ρ. From the first of Eqs. (2.3), we have

$$\psi-\gamma+c=-\int_\rho^1(\psi_\rho-\gamma_\rho)d\rho=-\int_\rho^1(\psi_\rho-\rho\psi_\rho{}^2+\rho\psi_z{}^2)d\rho,$$

where c is a finite integration constant. Therefore

$$\psi-\gamma+c\leq-\int_\rho^1\rho\psi_\rho\frac{\partial}{\partial\rho}(\ln\rho-\psi)d\rho. \quad (4.2)$$

The ρ component of the Newtonian force $(-\psi_\rho)$ due to a positive-mass distribution on the z axis can never be repulsive, so that

$$\psi_\rho\geq0. \quad (4.3)$$

[10] This has also been noted by J. Stachel, Phys. Letters 27A, 60 (1968).

Also, $\ln\rho-\psi$ is the superposition of the Newtonian potentials of an infinite rod of half-unit density and $-\psi$. From Eq. (4.1), we have

$$\ln\rho-\psi\to-\infty \quad \text{as} \quad \rho\to 0$$

for all values of z. Hence the source of the Newtonian potential $\ln\rho-\psi$ must also consist of a line distribution of *positive* density. Again, the resulting ρ component of the associated Newtonian force cannot be repulsive, so that

$$(\partial/\partial\rho)(\ln\rho-\psi)\geq 0. \tag{4.4}$$

Combining Eqs. (4.2)–(4.4) now gives

$$\psi-\gamma+c\leq 0.$$

For all values of z, $\psi-\gamma$ is bounded from above, in violation of the requirements necessary for the existence of a point GSS.

5. DISCUSSION

The preceding results support the conclusion of Israel[6] that a singular GSS is the rule for static vacuum solutions and that the Schwarzschild case is the exception. Unlike the situation in the presence of a scalar field, the GSS cannot be a localized point source in the static, axially symmetric, vacuum case.[11] As illustrated by the two examples in Sec. 3, however, it is possible for the GSS to have zero surface area. In the augmented Schwarzschild case, this feature of the GSS can be pictured in terms of an infinitely long rod having zero cross section. In the Schwarzschild plus quadrupole case, we believe that there is no such simply connected global picture of the GSS in terms of the limiting member of a family of surfaces embedded in a Euclidean three-space.

The singular nature of the GSS suggests a possible extension of a theorem due to Penrose.[2(b),12] This very important theorem states that, once a trapped surface is formed, either singularities are unavoidable or some basic notions of relativity physics must break down. A trapped surface is a compact two-space all of whose normal null rays converge. The compact Schwarzschild surface is *almost trapped* in the sense that one of its families of normal null rays converges and the other is divergenceless. It appears that under a wide variety of circumstances singularities are also unavoidable after the formation of an almost-trapped surface. In the static vacuum case, the singularities would exist on the almost-trapped surface itself, except in the Schwarzschild case. The same applies to the scalar-field solution[4] and to a wide class of static Einstein-Maxwell fields.[13]

What physical implications these results should bear upon the process of gravitational collapse is not at all clear. It is reasonable to expect that upon collapse to an almost-trapped surface, the dynamics of a body as viewed by external observers will appear to slow down because of the gravitational red shift. This, however, does not guarantee the conclusion that after a long time the system will asymptotically approach an exact static state (or stationary state, when rotation persists). In fact, some theoretical results caution against too hasty an acceptance of such a conclusion. In particular, certain newly discovered conservation laws[14] impose severe restrictions on the possibility of an asymptotically flat system possessing a quadrupole moment reaching a static configuration after a finite time. The attainment of an equilibrium state can be expected, in general, to take place only in a limiting sense after an infinite time. Because from our present mathematical standpoint there are essential difficulties in the description of temporal infinity in general relativity,[15] it is problematical whether one can answer the key question: After a long time does an asymptotically (in time) static state differ in essential features from an exact static state? (Also related to this question is the appearance of terms with secular time behavior in the various treatments of asymptotically flat space-times.[16] These terms correspond to what one would normally expect from objects that are freely separating, so that their multipole moments increase secularly in time. The same secular terms might also arise, however, from the tail of an outgoing radiation field emitted by a single body.)

With these reservations, it is of interest to note certain implications that *would* hold if an asymptotically static state *did* approach an exact static state. In that case, it has been pointed out that if collapse through a nonsingular almost-trapped surface takes place, then the asymptotic state must be spherically symmetric.[17] All multipole moments (and, presumably, all traces of a scalar field) must be radiated away. If, on the other hand, an asymmetric equilibrium state was reached just before the evolution of an almost-trapped surface, the strange properties of the intrinsic spatial geometry would contradict our intuitive notion of the interaction distances between the constituents of a body decreasing as collapse progressed. Whether these implications are of physical consequence or manifestations of a breakdown in classical general relativity is again problematical.

APPENDIX A

The metric of an axially symmetric, static space-time can always be put into the form (with no use of field equations)

$$ds^2=\alpha^2dt^2-\beta^2[(dx^1)^2+(dx^2)^2]-\gamma^2d\phi^2, \tag{A1}$$

[11] Considerable progress toward eliminating the requirement of axial symmetry has been made by J. Stachel (private communication).

[12] R. Penrose, Phys. Rev. Letters **14**, 57 (1965).

[13] W. Israel, Commun. Math. Phys. 8, 245 (1968).

[14] E. T. Newman and R. Penrose, Phys. Rev. Letters **15**, 231 (1965); Proc. Roy. Soc. (London) **A305**, 175 (1968).

[15] R. Penrose, Proc. Roy. Soc. (London) **A284**, 159 (1965).

[16] See, for example, H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. (London) **A269**, 21 (1962); W. B. Bonnor and M. A. Rotenberg, *ibid.* **A289**, 247 (1965).

[17] See Refs. 2(a) and 6.

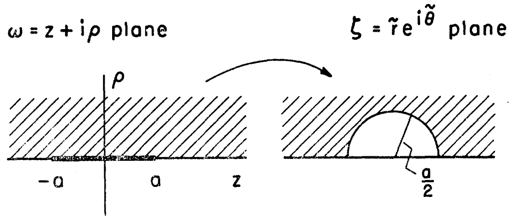

FIG. 1. Mapping from Weyl–Levi-Civita manifold to Erez-Rosen-type manifold, $\omega=\zeta+a^2/4\zeta$.

where α, β, and γ are functions of x^1 and x^2. A more useful alternative form is

$$ds^2=e^{2\psi}dt^2-e^{2(\gamma-\psi)}[(dx^1)^2+(dx^2)^2]-h^2e^{-2\psi}d\phi^2. \quad \text{(A2)}$$

Aside from the trivial coordinate freedom for the origin and scale on t and ϕ, the only freedom that preserves the form (A2) is

$$\tilde{\omega}\equiv\tilde{x}^1+i\tilde{x}^2=f(x^1+ix^2)=f(\omega). \quad \text{(A3)}$$

Under this transformation ψ and h transform as scalars, and

$$e^{2\tilde{\gamma}}=\frac{df}{d\omega}\frac{d\bar{f}}{d\bar{\omega}}e^{2\gamma}. \quad \text{(A4)}$$

The field equations $R_{\mu\nu}=0$ imposed on (A2) yield, with $\Delta\equiv\partial^2/(\partial x^1)^2+\partial^2/(\partial x^2)^2$,

$$\Delta h=0, \quad \text{(A5a)}$$

$$\Delta\psi+\frac{h_{,1}}{h}\psi_{,1}+\frac{h_{,2}}{h}\psi_{,2}=0, \quad \text{(A5b)}$$

$$\frac{h_{,1}}{h}\gamma_{,2}+\frac{h_{,2}}{h}\gamma_{,1}=2\psi_{,1}\psi_{,2}+\frac{h_{,12}}{h}, \quad \text{(A5c)}$$

$$\frac{h_{,1}}{h}\gamma_{,1}-\frac{h_{,2}}{h}\gamma_{,2}=\psi_{,1}{}^2-\psi_{,2}{}^2+\frac{h_{,11}}{h}. \quad \text{(A5d)}$$

The choice of a solution to (A5a) is not a restriction on the general solution but only a coordinate condition.[5] This is most easily seen by remembering that h must transform like a scalar under (A2) and that it is also a harmonic function, and hence can be chosen as the imaginary part of the analytic transformation leading to a new allowed coordinate system. This will be illustrated shortly.

The most common choice of h is $h=x^1$, which leads to the Weyl–Levi-Civita coordinates. If we call $x^1=\rho$ and $x^2=z$, (A2) becomes

$$ds^2=e^{2\psi}dt^2-e^{2(\gamma-\psi)}(d\rho^2+dz^2)-\rho^2e^{-2\psi}d\phi^2, \quad \text{(A6)}$$

with the ψ and γ [from (A5)] satisfying

$$\psi_{,\rho\rho}+(1/\rho)\psi_{,\rho}+\psi_{,zz}=0, \quad \text{(A7a)}$$

$$\gamma_{,\rho}=\rho(\psi_{,\rho}{}^2-\psi_{,z}{}^2), \quad \gamma_{,z}=2\rho\psi_{,\rho}\psi_{,z}. \quad \text{(A7b)}$$

If we now introduce polar coordinates by $r^2=z^2+\rho^2$, $\tan\theta=\rho/z$ (we shall call these Curzon coordinates—later other polar coordinates called Erez-Rosen-type coordinates will be used), the form (A2) will be violated. However, the violation can be simply remedied by introducing a relabeling of the $r=\text{const}$ surfaces, namely, by the transformation $r=e^R$. Then (A6) becomes

$$ds^2=e^{2\psi}dt^2-e^{2(\gamma-\psi)}e^{2R}(dR^2+d\theta^2)-e^{-2\psi}e^{2R}\sin^2\theta\, d\phi^2.$$

Thus we have $h=e^R\sin\theta$, which is easily seen to be a solution to (A5a) if $x^1=R$ and $x^2=\theta$. A third choice of h is now made. Calling $x^1=\tilde{R}$ and $x^2=\tilde{\theta}$, (A5a) yields as a solution

$$\begin{aligned} h&=(e^{\tilde{R}}-a^2/4e^{\tilde{R}})\sin\tilde{\theta}\\ &=2e^{-\delta}\sinh(\tilde{R}+\delta)\sin\tilde{\theta}\\ &=(\tilde{r}-a^2/4\tilde{r})\sin\tilde{\theta}, \end{aligned} \quad \text{(A8)}$$

with $\frac{1}{2}a=e^{-\delta}$, and with $\tilde{r}=e^{\tilde{R}}$ and $\tilde{\theta}$ being the Erez-Rosen-type[18] polar coordinates. The three choices of h appear to be the only ones that lead to the separation of variables in (A5b), and hence are the only ones to be discussed here. Note that the three choices of h can be considered to be the same function,

$$h=\rho=e^R\sin\theta=(e^{\tilde{R}}-\tfrac{1}{4}a^2e^{-\tilde{R}})\sin\tilde{\theta},$$

if we connect the three coordinate systems by the analytic transformations

$$z+i\rho=re^{i\theta}=e^{R+i\theta}=e^{\tilde{R}+i\tilde{\theta}}+\tfrac{1}{4}a^2e^{-(\tilde{R}+i\tilde{\theta})} =\tilde{r}e^{i\tilde{\theta}}+(a^2/4\tilde{r})e^{-i\tilde{\theta}} \quad \text{(A9)}$$

or

$$\omega=e^{\omega'}=e^{\tilde{\omega}}+\tfrac{1}{4}a^2e^{-\tilde{\omega}}.$$

The transformation between the Weyl–Levi-Civita coordinates and the Erez-Rosen-type polar coordinates is then simply

$$z+i\rho=\tilde{r}e^{i\tilde{\theta}}+(a^2/4\tilde{r})e^{-i\tilde{\theta}},$$

or

$$z=(\tilde{r}+a^2/4\tilde{r})\cos\tilde{\theta}, \quad \rho=(\tilde{r}-a^2/4\tilde{r})\sin\tilde{\theta}.$$

It is convenient to look upon this transformation as a mapping of the upper half (the physical portion, $\rho\geq0$) of the complex $z+i\rho$ plane onto the $\tilde{r}e^{i\tilde{\theta}}$ complex plane (see Fig. 1). The line segment $\rho=0$, $-a\leq z\leq a$ gets mapped into the semicircle of radius $\frac{1}{2}a$. (It is this mapping that explains why the Schwarzschild solution, when expressed in Weyl–Levi-Civita coordinates, appears to be the metric of a rod.)

It is now a relatively easy task to express any solution in either of the coordinate systems.

We now wish to make some comments on several of the known solutions.

[18] Strictly speaking, these coordinates are not identical to the ones used by Erez and Rosen, Their r is related to our $\tilde{r}$ by $r=(1+m/2\tilde{r})^2\tilde{r}$. The Schwarzschild solution in our coordinates is thus in the isotropic form, while in the Erez-Rosen coordinates it is in the usual form.

The most interesting solutions to Eq. (A7a) are those that are singular on a finite portion of the line $\rho=0$; for example, the Curzon-type multipoles $\psi=(\partial^n/\partial z^n)\times(1/r)$, $n\geq 0$, are singular at the origin, the Schwarzschild solution is singular on the line between $-m\leq z\leq m$, with mass density $\frac{1}{2}$, and all the Erez-Rosen multipoles can be given as solutions to (A7a) with a variable mass density on the same finite portion of the line [for the mass plus quadrupole, the density is $\sigma=\frac{1}{2}+\frac{3}{4}(q/m^2)(z^2-\frac{1}{3}m^2)$].

There is one obvious solution with unusual properties, which appears to have been overlooked in the literature. This is the solution to Eq. (A7a) due to a uniform mass density *different from* $\frac{1}{2}$ on a finite portion of the line $\rho=0$. It appears as if it should be the Schwarzschild solution, but on further analysis it turns out not to be spherically symmetric, becoming so only when $\sigma=\frac{1}{2}$.

This solution, when transformed to the Erez-Rosen coordinates, is the Erez-Rosen monopole solution with an arbitrary monopole moment. Erez and Rosen considered this solution only when the moment was $\frac{1}{2}$, it being then the Schwarzschild solution. When the total mass of this solution is kept constant but the density is increased to infinity, the solution goes in the limit into the Curzon monopole solution. The general Erez-Rosen monopole solution, which we call the augmented Schwarzschild metric, is discussed in Sec. 3.

APPENDIX B

We present here an axially symmetric, static, vacuum solution to the Einstein equations with two parameters m and M, such that $m=0$ gives the Schwarzschild solution with mass M, and $M=0$ gives the Curzon monopole solution with mass m. Asymptotically (i.e., for large r), the solution appears to have mass $M+m$. The motivation for finding this metric arose from the observation that the *negative*-mass Curzon metric could be considered to be due to a *point* source. We thus wished to explore the possibility that a combination of this solution with a positive-mass Schwarzschild solution could lead to a pointlike solution with positive total mass. This combined Schwarzschild-Curzon solution, however, does not turn out to be pointlike.

The combined Schwarzschild-Curzon metric is

$$\begin{aligned} ds^2=&[(1-M/2r)/(1+M/2r)]^2e^{-2m/rG}dt^2\\ &-(1+M/2r)^4[e^{-4mA/MG-m^2B^2/r^2G^4}(dr^2+r^2d\theta^2)\\ &+e^{2m/rG}r^2\sin^2\theta\, d\phi^2], \qquad \text{(B1)}\end{aligned}$$

where

$$\begin{aligned} G&=+[(1-M^2/4r^2)^2+(M^2/r^2)\cos^2\theta]^{1/2},\\ A&=1-G+M^2/4r^2-M/2r,\\ B&=(1-M^2/4r^2)\sin\theta.\end{aligned}$$

It is easily seen that $M=0$ and $m=0$ give, respectively, the Curzon and Schwarzschild metrics. An expansion of g_{00} in powers of $1/r$ shows that the asymptotic total mass is $M+m$.

It is easy to see from the metric (B1) that, for $M=0$ and $m<0$, the solution is pointlike at $r=0$. The equipotential surface at $r=0$ is not an event horizon, however, since g_{00} becomes infinite rather than zero as $r\rightarrow 0$.

If we now take $M>0$ and $m<0$, we find that there is an event horizon at $r=\frac{1}{2}M$, except at the equator, where g_{00} again becomes infinite as $r\rightarrow\frac{1}{2}M$. There is in this case no event horizon at $r=0$, since $g_{00}\rightarrow 1$ as $r\rightarrow 0$. An examination of the remaining part of the metric shows that neither the event horizon nor $r=0$ is pointlike.

Finally, we consider the case of $M<0$ and $m>0$. Now g_{00} vanishes only on the equatorial circle $r=-\frac{1}{2}M$, $\theta=\frac{1}{2}\pi$. Once again we find that the metric does not have the geometry corresponding to a point source.