Quantum Field Theory of Particles with Both Electric and Magnetic Charges*

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The quantum field theory of particles with both electric and magnetic charges is developed as an obvious extension of Schwinger's quantum field theory of particles with either electric or magnetic charge. Two new results immediately follow. The first is the chiral equivalence theorem which states the unitary equivalence of the Hamiltonians describing the system of particles with electric and magnetic charges e_n, g_n and the system with charges $e_n' = \cos\theta$, $e_n + \sin\theta\, g_n$, $g_n' = -\sin\theta\, e_n + \cos\theta\, g_n$. This result holds in particular in the absence of physical magnetic charges. The second result is that if physical magnetic charges do occur, then, in consequence of chiral equivalence, the charge quantization condition applies, not to the separate products $e_m g_n$, but to the combinations $e_m g_n - g_m e_n$, which must be integral multiples of 4π. The general solution of this condition leads to the introduction of a second elementary quantum of electric charge e_2, the electric charge on the Dirac monopole, besides the first elementary charge e_1, the charge on the electron. There are no other free parameters.

I. INTRODUCTION

THE equations of Maxwell have a natural generalization which allows as sources both the electric current density j_e^μ and a magnetic current density j_g^μ:

$$\dot{\mathbf{E}} = \nabla\times\mathbf{H} - \mathbf{j}_e, \qquad \dot{\mathbf{H}} = -\nabla\times\mathbf{E} - \mathbf{j}_g, \tag{1.1}$$

$$\nabla\cdot\mathbf{E} = j_e^0, \qquad \nabla\cdot\mathbf{H} = j_g^0. \tag{1.2}$$

These equations, together with the Lorentz force

$$\mathbf{F} = j_e^0\mathbf{E} + \mathbf{j}_e\times\mathbf{H} + j_g^0\mathbf{H} - \mathbf{j}_g\times\mathbf{E}, \tag{1.3}$$

are invariant under the substitutions

$$\begin{aligned}\mathbf{E} &\to \cos\theta\,\mathbf{E} - \sin\theta\,\mathbf{H},\\ \mathbf{H} &\to \sin\theta\,\mathbf{E} + \cos\theta\,\mathbf{H},\end{aligned} \tag{1.4}$$

$$\begin{aligned}j_e^\mu &\to \cos\theta\, j_e^\mu - \sin\theta\, j_g^\mu,\\ j_g^\mu &\to \sin\theta\, j_e^\mu + \cos\theta\, j_g^\mu,\end{aligned} \tag{1.5}$$

as is well known. Because of this invariance, it cannot be asserted that the absence of physical magnetic charge (namely, the absence of particles that interact in a particular way with electrons), which nature has heretofore manifested, reflects an asymmetry between electric and magnetic charge. It can only be said that the electric and magnetic current densities observed up to now are proportional.

A quantum field theory of particles with both electric and magnetic charges is easily constructed as an obvious extension of Schwinger's field theory[1] of particles with either electric or magnetic charges.[2,3] Once this is done, the invariance of the Maxwell-Lorentz equations under the substitution (1.4), (1.5) is seen to correspond to an identity satisfied by the Hamiltonian $H(\mathbf{E}^r,\mathbf{H}^r,e_n,g_n)$, regarded as a function of the radiation fields $\mathbf{H}^r$, $\mathbf{E}^r$ and the electric and magnetic charges of the nth charge bearing field:

$$H(\cos\theta\,\mathbf{E}^r - \sin\theta\,\mathbf{H}^r,\ \sin\theta\,\mathbf{E}^r + \cos\theta\,\mathbf{H}^r, e_n, g_n) = H(\mathbf{E}^r,\mathbf{H}^r,\ \cos\theta\, e_n + \sin\theta\, g_n, -\sin\theta\, e_n, +\cos\theta\, g_n). \tag{1.6}$$

Furthermore, one easily constructs the unitary operator $U(\theta)$ which effects the substitution on the left-hand side of this equation, thereby obtaining the result

$$U(\theta)H(\mathbf{E}^r,\mathbf{H}^r,e_n,g_n)U^\dagger(\theta) = H(\mathbf{E}^r,\mathbf{H}^r,\ \cos\theta\, e_n + \sin\theta\, g_n,\ -\sin\theta\, e_n + \cos\theta\, g_n). \tag{1.7}$$

Systems that are described by unitarily equivalent Hamiltonians are physically indistinguishable, so that no experiment can decide between alternative descriptions of the world that differ by a simultaneous rotation of electric and magnetic charges of all particles. In quantum theory, the transformation (1.4) is a chiral transformation on the photon state of momentum $\mathbf{k}$ and helicity λ:

$$|\mathbf{k},\lambda\rangle \to U(\theta)|\mathbf{k},\lambda\rangle = e^{i\lambda\theta}|k,\lambda\rangle, \tag{1.8}$$

so that one may state alternatively that no experiment can measure the relative phase of left and right circularly polarized light, nor, consequently, the absolute plane of polarization of linearly polarized light.

Because of chiral equivalence, it is found that Dirac's charge quantization condition[4] applies, not to the products $e_m g_n/4\pi$, but to the chiral-invariant combinations

$$(1/4\pi)(e_m g_n - g_m e_n) = 0, \pm 1, \pm 2, \cdots. \tag{1.9}$$

Half-integral quantization, implying half-integral angular momentum in the static electromagnetic fields, which is allowed in Dirac's single-particle theory, is forbidden by chiral invariance and locality in the com-

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[2] J. Schwinger, Phys. Rev. **151**, 1048 (1966); **151**, 1055 (1966).
[3] N. Cabbibo and E. Ferrari, Nuovo Cimento **23**, 1147 (1962); T. M. Yan, Phys. Rev. **150**, 1349 (1966); **155**, 1423 (1967).
[4] P. A. M. Dirac, Proc. Roy. Soc. (London) **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948).

mutator of the electric and magnetic vector potentials. This accords with physical intuition for it directly relates the vector nature of the electromagnetic field to the absence of spin-$\frac{1}{2}$ angular momentum. The integral quantization condition was proposed by Schwinger, but with a somewhat different mathematical justification [see Eqs. (3.38) and (3.39)].[5]

The solution to the charge quantization condition is easily expressed by introducing a two-dimensional charge vector $\mathbf{q}_n=(e_n,g_n)$. If $\mathbf{q}_1$ and $\mathbf{q}_2$ are two charge vectors which satisfy $(\mathbf{q}_1\times\mathbf{q}_2)/4\pi=(e_1g_2-g_1e_2)/4\pi=1$, then the most general solution to (1.9) is

$$\mathbf{q}_n=Z_{n1}\mathbf{q}_1+Z_{n2}\mathbf{q}_2\,,\quad Z_{n\alpha}=0,\pm1,\pm2,\cdots. \tag{1.10}$$

If $\mathbf{q}_1$ is the charge vector of the electron, $\mathbf{q}_1{}^2=e^2=4\pi/137$, then Eq. (1.10) introduces a second elementary quantum of electric charge e', the electric charge of the Dirac monopole, defined by $\mathbf{q}_1\cdot\mathbf{q}_2=ee'$. The product ee' is a measure of the strength of the parity and time-reversal violating interaction.

There are three obvious possible explanations for the failure to observe mangetic monopoles: (1) They do not exist; (2) they have a very large mass; and (3) their large magnetic charge causes them to annihilate or bind into magnetically neutral matter. The third possibility is susceptible to theoretical analysis. A monopole-antimonopole pair interacts like an electron-position pair, but with coupling strength 137 instead of 1/137. It would be valuable to estimate the relative probabilities of scattering (with emission of radiation) or annihilation, if such a pair is present at $t=-\infty$. This branching ratio also controls the competing outgoing channels in any experiment designed to produce magnetic monopole pairs.[6] If it can be shown to be small, then the Dirac quantization condition may not only explain the quantization of electric charge, but also the difficulty of observing magnetic charge.

In Sec. II the chiral equivalence theorem is proven. In Sec. III the quantization condition on $e_mg_n-g_me_n$ is derived from Lorentz invariance. In Sec. IV it is solved, and the symmetries of the theory are presented.

II. CHIRAL EQUIVALENCE THEOREM

Let us begin by considering the free electromagnetic radiation field. It is conventionally described by the Hamiltonian density[7]

$$T^{00}(\mathbf{x})^r=\tfrac{1}{2}[\mathbf{E}^r(\mathbf{x})^2+\mathbf{H}^r(\mathbf{x})^2] \tag{2.1}$$

and carries the momentum density

$$T^{0i}(\mathbf{x})^r=\tfrac{1}{2}[\mathbf{E}^r(\mathbf{x})\times\mathbf{H}^r(\mathbf{x})-\mathbf{H}^r(\mathbf{x})\times\mathbf{E}^r(\mathbf{x})]^i. \tag{2.2}$$

Here $\mathbf{E}^r(\mathbf{x})$ and $\mathbf{H}^r(\mathbf{x})$ are the transverse radiation fields

$$\nabla\cdot\mathbf{E}^r(\mathbf{x})=\nabla\cdot\mathbf{H}^r(\mathbf{x})=0\,, \tag{2.3}$$

which satisfy the canonical equal-time commutation relations

$$\begin{aligned}&[E_i{}^r(\mathbf{x}),E_j{}^r(\mathbf{x}')]=[H_i{}^r(\mathbf{x}),H_j{}^r(\mathbf{x}')]=0\,,\\&[E_i{}^r(\mathbf{x}),H_j{}^r(\mathbf{x}')]=i\epsilon_{ijk}\nabla_k\delta(\mathbf{x}-\mathbf{x}')\,.\end{aligned} \tag{2.4}$$

These fields, and all others appearing below, are evaluated at a common time t, which is suppressed. From the Hamiltonian $H^r=\int T^{00}(\mathbf{x})^r d\mathbf{x}$ and the canonical commutation relations, the Heisenberg equations of motion yield the desired source-free pair of Maxwell equations involving time derivatives,

$$\dot{\mathbf{E}}^r(\mathbf{x})=-i[\mathbf{E}^r(\mathbf{x}),H]=\nabla\times\mathbf{H}^r(\mathbf{x})\,, \tag{2.5a}$$

$$\dot{\mathbf{H}}^r(\mathbf{x})=-i[\mathbf{H}^r(\mathbf{x}),H]=-\nabla\times\mathbf{E}^r(\mathbf{x})\,. \tag{2.5b}$$

It is not really necessary to impose the transversality condition (2.3) on the radiation fields. For the commutation relations (2.4) show that the longitudinal parts of the field commute with everything, including the Hamiltonian, and are thus time-independent c-number functions of position which may be subtracted out.

All relations appearing up to now are invariant under the chiral substitutions

$$\mathbf{E}^r\to\cos\theta\,\mathbf{E}^r-\sin\theta\,\mathbf{H}^r\,, \tag{2.6a}$$

$$\mathbf{H}^r\to\sin\theta\,\mathbf{E}^r+\cos\theta\,\mathbf{H}^r\,. \tag{2.6b}$$

To make this invariance manifest, it is convenient to introduce a two-dimensional real vector space, which we call chiral space, with elements V^α, $\alpha=1, 2$. This space has two tensors that are invariant under proper rotations, the Kronecker symbol $\delta^{\alpha\beta}$, and the antisymmetric symbol $\epsilon^{\alpha\beta}$, with $\epsilon^{12}=1$.

If we introduce the field variable

$$\mathbf{F}^{\alpha r}=(\mathbf{E}^r,\mathbf{H}^r)\,, \tag{2.6c}$$

then the energy and momentum density are

$$T^{00}(\mathbf{x})^r=\tfrac{1}{2}\mathbf{F}^\alpha\cdot\mathbf{F}^\alpha\,, \tag{2.7a}$$

$$T_{0i}(\mathbf{x})^r=-\tfrac{1}{2}\epsilon_{ijk}\epsilon^{\alpha\beta}F_j{}^{\alpha r}(\mathbf{x})F_k{}^{\beta r}(\mathbf{x})\,. \tag{2.7b}$$

The canonical commutation relations take the form

$$[F_i{}^{\alpha r}(\mathbf{x}),F_j{}^{\beta r}(\mathbf{x}')]=-i\epsilon^{\alpha\beta}\epsilon_{ijk}\nabla_k\delta(\mathbf{x}-\mathbf{x}')\,, \tag{2.8}$$

and the equations of motion become

$$\dot{F}_i{}^{\alpha r}=\epsilon^{\alpha\beta}\epsilon_{ijk}\nabla_jF_k{}^{\beta r}\,. \tag{2.9}$$

As a convenience, which partially avoids the appearance of nonlocal interaction in describing the coupling of the radiation field to electrically charged particles, it is conventional to change variables from the magnetic field $\mathbf{H}^r$ to the vector potential $\mathbf{A}^r$ defined

[5] According to the noted added in proof of Ref. 1, special attention should be given to the point $z=z'$ in Eq. (3.27), with the result that the right-hand sides of Eqs. (3.28) and (3.29) should be doubled.

[6] I am grateful to Dr. J. Basdevant for a discussion of this point.

[7] For Lorentz tensors we use the metric $g^{\mu\nu}=(1,-1,-1,-1)$, units $\hbar=c=1$, and Dirac matrices $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$.

by

$$\mathbf{A}^r(\mathbf{x})=\nabla\times\int\mathfrak{D}(\mathbf{x}-\mathbf{x}')\mathbf{H}^r(\mathbf{x}')d\mathbf{x}'. \quad (2.10a)$$

To maintain symmetry in the treatment of electric and magnetic fields, one may, following Schwinger,[1] introduce the second vector potential

$$\mathbf{B}^r(\mathbf{x})=-\nabla\times\int\mathfrak{D}(\mathbf{x}-\mathbf{x}')\mathbf{E}^r(\mathbf{x}')d\mathbf{x}', \quad (2.10b)$$

with

$$\mathfrak{D}(\mathbf{x})=1/4\pi|\mathbf{x}|. \quad (2.11)$$

These definitions are, of course, equivalent to

$$\mathbf{H}^r=\nabla\times\mathbf{A}^r, \quad \nabla\cdot\mathbf{A}^r=0, \quad (2.12a)$$

$$\mathbf{E}^r=-\nabla\times\mathbf{B}^r, \quad \nabla\cdot\mathbf{B}^r=0. \quad (2.12b)$$

The pair of vector potentials constitute a vector in chiral space

$$\mathbf{V}^{\alpha r}=(\mathbf{A}^r,\mathbf{B}^r), \quad (2.13)$$

so that these relations may be written

$$V_i^{\alpha r}(\mathbf{x})=\epsilon^{\alpha\beta}\epsilon_{ijk}\nabla_j\int\mathfrak{D}(\mathbf{x}-\mathbf{x}')F_k^{\beta r}(\mathbf{x}')d\mathbf{x}', \quad (2.14a)$$

$$F_i^{\alpha r}=-\epsilon^{\alpha\beta}\epsilon_{ijk}\nabla_j V_k^{\beta r}; \ \nabla\cdot\mathbf{V}^{\alpha r}=0. \quad (2.14b)$$

From the definition (2.14a) and the canonical commutation relation (2.8), one derives the equal-time commutators

$$[F_i^{\alpha r}(\mathbf{x}),V_j^{\beta r}(\mathbf{x}')]=i\delta^{\alpha\beta}[\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')]^{\mathrm{tr}}, \quad (2.15)$$

$$[V_i^{\alpha r}(\mathbf{x}),V_j^{\beta r}(\mathbf{x}')]=-i\epsilon^{\alpha\beta}\epsilon_{ijk}\nabla_k\mathfrak{D}(\mathbf{x}-\mathbf{x}'), \quad (2.16)$$

where

$$[\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')]^{\mathrm{tr}}=\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')-\nabla_i\nabla_j'\mathfrak{D}(\mathbf{x}-\mathbf{x}').$$

The chiral transformation (2.6) may be implemented by the unitary transformation

$$U(\theta)=\exp(i\theta G), \quad (2.17a)$$

$$G=\frac{1}{2}\int\mathbf{V}^{\alpha r}(\mathbf{x})\cdot\epsilon^{\alpha\beta}\mathbf{F}^{\beta r}(\mathbf{x})d\mathbf{x} =\frac{1}{2}\int(\mathbf{A}^r\cdot\mathbf{H}^r-\mathbf{B}^r\cdot\mathbf{E}^r)d\mathbf{x}. \quad (2.17b)$$

The generator G is a Hermitian quantity which is conserved by the free radiation field. It could be called the "total chirality" because it measures the number of right-hand photons less the number of left-handed photons, and may have a finite value even when the number of infrared photons is infinite. Because the $\mathbf{V}^{\alpha r}$ are defined in terms of the $\mathbf{F}^{\alpha r}$ by chiral-invariant equations, U rotates the $\mathbf{V}^{\alpha r}$ also;

$$U(\theta)\mathbf{F}^{\alpha r}U^\dagger(\theta)=\mathbf{F}^{\beta r}d^{\beta\alpha}(\theta), \quad (2.18a)$$

$$U(\theta)\mathbf{V}^{\alpha r}U^\dagger(\theta)=\mathbf{V}^{\alpha r}d^{\beta\alpha}(\theta), \quad (2.18b)$$

where $d^{\beta\alpha}$ is the matrix of the transformation (2.6).

Now let the radiation field be coupled to fields ψ_n bearing electric charge e_n and magnetic charge g_n. For simplicity we take them to be four-component Dirac fields, so that the electric and magnetic current 4-vectors have the form

$$j_\mu^e(\mathbf{x})=\sum_n e_n\bar{\psi}_n(\mathbf{x})\gamma_\mu\psi_n(\mathbf{x}), \quad (2.19a)$$

$$j_\mu^g(\mathbf{x})=\sum_n g_n\bar{\psi}_n(\mathbf{x})\gamma_\mu\psi_n(\mathbf{x}). \quad (2.19b)$$

The total fields and potentials are given by[1]

$$\mathbf{E}(\mathbf{x})=\mathbf{E}^r(\mathbf{x})-\nabla\int\mathfrak{D}(\mathbf{x}-\mathbf{x}')j_e^0(\mathbf{x}')d\mathbf{x}', \quad (2.20a)$$

$$\mathbf{H}(\mathbf{x})=\mathbf{H}^r(\mathbf{x})-\nabla\int\mathfrak{D}(\mathbf{x}-\mathbf{x}')j_g^0(\mathbf{x}')d\mathbf{x}', \quad (2.20b)$$

$$\mathbf{A}(\mathbf{x})=\mathbf{A}^r(\mathbf{x})+\int\mathbf{a}(\mathbf{x}-\mathbf{x}')j_g^0(\mathbf{x}')d\mathbf{x}', \quad (2.21a)$$

$$\mathbf{B}(\mathbf{x})=\mathbf{B}^r(\mathbf{x})-\int\mathbf{a}(\mathbf{x}'-\mathbf{x})j_e^0(\mathbf{x}')d\mathbf{x}', \quad (2.21b)$$

where $\mathbf{a}(\mathbf{x})$ is a numerical vector function defined by

$$\nabla\times\mathbf{a}(\mathbf{x})=\mathbf{h}(\mathbf{x}), \quad \nabla\cdot\mathbf{a}(\mathbf{x})=0. \quad (2.22a)$$

Here $\mathbf{h}(\mathbf{x})$ is nonzero along a string, described in Sec. III, and satisfying

$$\nabla\cdot\mathbf{h}(\mathbf{x})=-\delta(\mathbf{x}). \quad (2.22b)$$

The definition of the total fields may be written in chiral-covariant form. The same is true of the potentials only if $\mathbf{a}(\mathbf{x})$ satisfies the important symmetry condition

$$\mathbf{a}(\mathbf{x})=\mathbf{a}(-\mathbf{x}), \quad (2.23)$$

which is assumed to hold for the remainder of this section. If a charge vector

$$q_n^\alpha=(e_n,g_n) \quad (2.24)$$

is introduced for each particle type n, the total fields and potentials may be written

$$\mathbf{F}^\alpha(\mathbf{x})=\mathbf{F}^{\alpha r}(\mathbf{x})+\mathbf{F}^{\alpha s}(\mathbf{x})=\mathbf{F}^{\alpha r}(\mathbf{x}) -\nabla\int\mathfrak{D}(\mathbf{x}-\mathbf{x}')\sum_n q_n^\alpha\bar{\psi}_n(\mathbf{x}')\gamma^0\psi_n(\mathbf{x}')d\mathbf{x}', \quad (2.25)$$

$$\mathbf{V}^\alpha(\mathbf{x})=\mathbf{V}^{\alpha r}(\mathbf{x})+\mathbf{V}^{\alpha s}(\mathbf{x})=\mathbf{V}^{\alpha r}(\mathbf{x}) +\epsilon^{\alpha\beta}\int\mathbf{a}(\mathbf{x}-\mathbf{x}')\sum_n q_n^\beta\bar{\psi}_n(\mathbf{x}')\gamma^0\psi_n(\mathbf{x}')d\mathbf{x}'. \quad (2.26)$$

With these definitions, the total Hamiltonian describing the radiation and particle fields and their interaction is given by

$$H=\int T^{00}(\mathbf{x})d\mathbf{x}\,, \tag{2.27a}$$

$$T^{00}=\tfrac{1}{2}\mathbf{F}^{\alpha}\cdot\mathbf{F}^{\alpha}+\sum_n \bar{\psi}_n[\boldsymbol{\gamma}\cdot(-i\boldsymbol{\nabla}-q_n{}^{\alpha}\mathbf{V}^{\alpha})+m_n]\psi_n\,, \tag{2.27b}$$

where the ψ_n are antisymmetrized with respect to derivative and spinor index. Maxwell's equations (1.1) result from the Heisenberg equations of motion. This Hamiltonian is not invariant under the chiral transformation (2.6) or (2.18). However, the chiral equivalence theorem follows immediately. Because U, Eq. (2.17), effects the substitution (2.18) on $\mathbf{E}^r$ and $\mathbf{H}^r$, it effects the same substitution on any function of them, including T^{00}:

$$U(\theta)T^{00}(\mathbf{E}^r,\mathbf{H}^r)U^{\dagger}(\theta)=T^{00}(\cos\theta\,\mathbf{E}^r-\sin\theta\,\mathbf{H}^r,\ \sin\theta\,\mathbf{E}^r+\cos\theta\,\mathbf{H}^r)\,. \tag{2.28}$$

Furthermore, each $q_n{}^{\alpha}$ in T^{00} is contracted via $\delta^{\alpha\beta}$ or $\epsilon^{\alpha\beta}$ with $\mathbf{F}^{\alpha r}$ or $\mathbf{V}^{\alpha r}$ or another $q_n{}^{\alpha}$. Hence the transformation (2.28) may equivalently be stated

$$U(\theta)T^{00}(e_n,g_n)U^{\dagger}(\theta)=T^{00}(\cos\theta\,e_n+\sin\theta\,g_n,\ -\sin\theta\,e_n+\cos\theta\,g_n)\,. \tag{2.29}$$

Systems that are descirbed by unitarily equivalent Hamiltonians are physically indistinguishable; consequently no experiment can detect a simultaneous rotation of the charge vector of all particles. This result is familiar classically, particularly for $\theta=\frac{1}{2}\pi$, where it is known as the duality principle. Its consequence for the Dirac charge quantization condition is the subject of Sec. III.

III. CHARGE QUANTIZATION CONDITION

In the present section we propose to examine how the condition of Lorentz invariance leads to the quantization of charge. Although the commutation relations (2.15) and (2.16) and the Hamiltonian (2.27) are not manifestly local, one may nevertheless use them directly, with some brute force, to prove that the over-all theory is Lorentz-covariant, provided that the charges satisfy certain conditions. However, it is more convenient to use a formalism in which locality is as manifest as possible, and which provides a clearer insight into the origin of the Dirac strings. For this purpose we adapt Schwinger's device of group parameters[8] to the present problem.

Let $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ be a pair of arbitrary c-number functions, regarded as an infinite set of free parameters, and let the state vectors be functionals of them, as specified below. The fields and potentials are allowed to depend on them and on functional derivatives with respect to them. If one introduces the extended fields

$$\boldsymbol{\mathcal{E}}(\mathbf{x})=\mathbf{E}^r(\mathbf{x})-\boldsymbol{\nabla}\int d\mathbf{x}'\mathfrak{D}(\mathbf{x}-\mathbf{x}')\frac{1}{i}\frac{1}{\delta\lambda(\mathbf{x}')}\,, \tag{3.1a}$$

$$\boldsymbol{\mathcal{H}}(\mathbf{x})=\mathbf{H}^r(\mathbf{x})-\boldsymbol{\nabla}\int d\mathbf{x}'\mathfrak{D}(\mathbf{x}-\mathbf{x}')\frac{1}{2}\frac{\delta}{\delta\mu(\mathbf{x}')}\,, \tag{3.1b}$$

and makes the replacements

$$\mathbf{A}^r(\mathbf{x})\to\mathbf{A}^r(\mathbf{x})+\boldsymbol{\nabla}\lambda(\mathbf{x})\,,$$
$$\mathbf{B}^r(\mathbf{x})\to\mathbf{B}^r(\mathbf{x})+\boldsymbol{\nabla}\mu(\mathbf{x})\,,$$

then the local commutation relations (2.4) between the fields remain unchanged and the nonlocal commutation relations (2.15) between fields and potentials become local, as desired, but the nonlocal commutation relations (2.16) between potentials remain unchanged. If, instead, one substitutes for the potentials

$$\mathfrak{A}(\mathbf{x})=\mathbf{A}^r(\mathbf{x})+\int d\mathbf{x}'\mathbf{a}(\mathbf{x}-\mathbf{x}')\frac{1}{2}\frac{\delta}{\delta\mu(\mathbf{x}')}+\boldsymbol{\nabla}\lambda(\mathbf{x})\,, \tag{3.2a}$$

$$\mathfrak{B}(\mathbf{x})=\mathbf{B}^r(\mathbf{x})+\int dx'\mathbf{b}(\mathbf{x}-\mathbf{x}')\frac{1}{2}\frac{\delta}{\delta\lambda(x')}+\boldsymbol{\nabla}\mu(\mathbf{x})\,, \tag{3.2b}$$

then the commutator of $\mathfrak{A}$ and $\mathfrak{B}$ becomes

$$[\mathfrak{A}_i(\mathbf{x}),\mathfrak{B}_j(\mathbf{x}')]=-i\epsilon_{ijk}\nabla_k\mathfrak{D}(\mathbf{x}-\mathbf{x}')+i\nabla_j a_i(\mathbf{x}-\mathbf{x}')+i\nabla_i b_j(\mathbf{x}'-\mathbf{x})\,,$$

where $\mathbf{a}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ are to be chosen so as to make this expression local, if possible, by cancelling the term in $\mathfrak{D}(\mathbf{x}-\mathbf{x}')$ for $\mathbf{x}\neq\mathbf{x}'$. Unless they satisfy

$$\mathbf{b}(\mathbf{x})=-\mathbf{a}(-\mathbf{x})\,, \tag{3.3}$$

new terms symmetric in i and j are introduced, instead of cancelling the antisymmetric old ones. With this choice one obtains the locality condition

$$\boldsymbol{\nabla}\mathfrak{D}(\mathbf{x})+\boldsymbol{\nabla}\times\mathbf{a}(\mathbf{x})=0\,,\quad \mathbf{x}\neq0\,.$$

It cannot be satisfied for all $\mathbf{x}\neq0$, but must be violated at least on a singular line, called a Dirac string;

$$\boldsymbol{\nabla}\times\mathbf{a}(\mathbf{x})=-\boldsymbol{\nabla}\mathfrak{D}(\mathbf{x})+\mathbf{h}(\mathbf{x})\,, \tag{3.4}$$

where $\mathbf{h}$ is nonzero only along a string to be specified below, and obeys

$$\boldsymbol{\nabla}\cdot\mathbf{h}(\mathbf{x})=-\delta(\mathbf{x})\,. \tag{3.5}$$

To complete the specification of $\mathbf{a}$, one may, without loss of generality, impose

$$\boldsymbol{\nabla}\cdot\mathbf{a}(\mathbf{x})=0\,, \tag{3.6}$$

which can always be achieved by a unitary transformation.

[8] J. Schwinger, Nuovo Cimento **30**, 278 (1963).

The commutator of $\mathfrak{A}$ and $\mathfrak{B}$ now becomes

$$[\mathfrak{A}_i(\mathbf{x}),\mathfrak{B}_j(\mathbf{x}')]=-i\epsilon_{ijk}h_k(\mathbf{x}-\mathbf{x}'). \qquad (3.7)$$

If a rotation θ is performed on the chiral vector $(\mathfrak{A},\mathfrak{B})$, then in the commutator, $\mathbf{h}(\mathbf{x}-\mathbf{x}')$ is replaced by $\cos^2\theta \times\mathbf{h}(\mathbf{x}-\mathbf{x}')-\sin^2\theta\,\mathbf{h}(\mathbf{x}'-\mathbf{x})$. Hence if chiral symmetry is to be maintained in this commutator, $\mathbf{h}(\mathbf{x})$ must be odd:

$$\mathbf{h}(-\mathbf{x})=-\mathbf{h}(\mathbf{x}), \qquad (3.8)$$

so that $\mathbf{a}(\mathbf{x})$ is even:

$$\mathbf{a}(-\mathbf{x})=\mathbf{a}(\mathbf{x}). \qquad (3.9)$$

The string function $\mathbf{h}$ in this case cannot be semi-infinite, but must be the difference of a semi-infinite string function and its image,

$$\mathbf{h}(\mathbf{x})=-\frac{1}{2}\int_0^\infty[\delta(\mathbf{x}-\mathbf{s})-\delta(-\mathbf{x}-\mathbf{s})]d\mathbf{s},$$

where the integral extends along any path from the origin to infinity. If it is chosen along the z axis, one obtains

$$\mathbf{h}_z(\mathbf{x})=-\tfrac{1}{2}\hat{z}\epsilon(z)\delta(x)\delta(y). \qquad (3.10)$$

If one introduces the chiral notation

$$\nu^\alpha(\mathbf{x})=[\lambda(\mathbf{x}),\mu(\mathbf{x})];\quad \frac{\delta}{\delta\nu^\alpha(\mathbf{x})}=\left[\frac{\delta}{\delta\lambda(\mathbf{x})},\frac{\delta}{\delta\mu(\mathbf{x})}\right],$$

then the definitions, (3.1) and (3.2), of the extended fields and potentials may be written

$$\mathfrak{F}^\alpha(\mathbf{x})=\mathbf{F}^{\alpha r}(\mathbf{x})-\nabla\int d\mathbf{x}'\mathfrak{D}(\mathbf{x}-\mathbf{x}')\frac{1}{i}\frac{\delta}{\delta\nu^\alpha(\mathbf{x}')}, \qquad (3.11)$$

$$\mathfrak{V}^\alpha(\mathbf{x})=\mathbf{V}^{\alpha r}(\mathbf{x})+\int d\mathbf{x}'\mathbf{a}(\mathbf{x}-\mathbf{x}')\epsilon^{\alpha\beta} \times\frac{1}{i}\frac{\delta}{\delta\nu^\beta(\mathbf{x}')}+\nabla\nu^\alpha(\mathbf{x}). \qquad (3.12)$$

They obey the commutation relations

$$[\mathfrak{F}_i{}^\alpha(\mathbf{x}),\mathfrak{F}_j{}^\beta(\mathbf{x}')]=-i\epsilon^{\alpha\beta}\epsilon_{ijk}\nabla_k\delta(\mathbf{x}-\mathbf{x}'), \qquad (3.13)$$

$$[\mathfrak{F}_i{}^\alpha(\mathbf{x}),\mathfrak{V}_j{}^\beta(\mathbf{x}')]=i\delta^{\alpha\beta}\delta_{ij}\delta(\mathbf{x}-\mathbf{x}'), \qquad (3.14)$$

$$[\mathfrak{V}_i{}^\alpha(\mathbf{x}),\mathfrak{V}_j{}^\beta(\mathbf{x}')]=i\epsilon^{\alpha\beta}\epsilon_{ijk}h_k(\mathbf{x}-\mathbf{x}'). \qquad (3.15)$$

The first two are local, but the last has its support on a string. The import of the present discussion is that if vector potentials are introduced for both electric and magnetic fields, then manifest locality cannot be achieved for the commutators, as is already obvious on dimensional grounds. The least possible violation is along the strings introduced by Dirac. Furthermore, if chiral invariance is to be maintained in the commutators, then the string function must be antisymmetric.

Upon comparing the expressions (2.25) and (2.26) for the physical fields and potentials with Eqs. (3.11) and (3.12) for the extended fields and potentials, we see that, except for the gradient term in the potentials, they become identical when applied to states whose functional dependence on the gauge parameters $\nu^\alpha(\mathbf{x})=[\lambda(\mathbf{x}),\mu(\mathbf{x})]$ satisfies

$$\frac{1}{i}\frac{\delta}{\delta\nu^\alpha(\mathbf{x})}|\nu\rangle=j^{0\alpha}(\mathbf{x})|\nu\rangle, \qquad (3.16)$$

$$j^{\mu\alpha}(\mathbf{x})=\sum_n q_n{}^\alpha\bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_n(\mathbf{x}). \qquad (3.17)$$

This dependence takes the explicit form

$$|\nu\rangle=U(\nu)|\quad\rangle, \qquad (3.18)$$

$$U(\nu)=\exp\left(i\int\nu^\alpha(\mathbf{x})j^{0\alpha}(\mathbf{x})d\mathbf{x}\right), \qquad (3.19)$$

where $|\quad\rangle$ is independent of ν^α. [These states would not be normalizable if the inner product included a functional integration over $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$, but it does not. The $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are simply free parameters.[8]]

The motivation for introducing the parametric method is that the Hamiltonian density

$$\theta^{00}=\tfrac{1}{2}\mathfrak{F}^\alpha\cdot\mathfrak{F}^\alpha+\sum\bar{\psi}_n[\boldsymbol{\gamma}\cdot(-i\nabla-q_n\mathfrak{V}^\alpha)+m_n]\psi_n, \qquad (3.20)$$

when applied to states (3.18), satisfies

$$\theta^{00}(\mathbf{x})U(\nu)|\quad\rangle=U(\nu)T^{00}(\mathbf{x})|\quad\rangle, \qquad (3.21)$$

where $T^{00}(\mathbf{x})$ is the physical Hamiltonian density defined by Eq. (2.27). Now $T^{00}(\mathbf{x})$ defines a Lorentz-covariant theory if its satisfies[8,9]

$$[T^{00}(\mathbf{x}),T^{00}(\mathbf{x}')] =-i[T^{0i}(\mathbf{x})+T^{0i}(\mathbf{x}')]\nabla_i\delta(\mathbf{x}-\mathbf{x}'). \qquad (3.22)$$

The verification of this equation is replaced by the verification of

$$[\theta^{00}(\mathbf{x}),\theta^{00}(\mathbf{x}')]=-[\theta^{0i}(\mathbf{x})+\theta^{0i}(\mathbf{x}')]\nabla_i\delta(\mathbf{x}-\mathbf{x}'), \qquad (3.23)$$

which is much simpler because θ^{00} depends on fields and potentials with local, or almost local, commutators.

The calculation of the commutator (3.23) from (3.20) leads to nonlocal contributions on strings, arising from the commutator of the potentials $[\mathfrak{V}_i{}^\alpha(\mathbf{x}),\mathfrak{V}_j{}^\beta(\mathbf{x}')]$. However, the singular operator products appearing in (3.20) should in fact be replaced by a limit of nonsingular products such as[1]

$$\bar{\psi}_n(\mathbf{x})\boldsymbol{\gamma}\cdot[-i\nabla-q_n{}^\alpha\mathfrak{V}^\alpha(\mathbf{x})]\psi_n(\mathbf{x})\rightarrow\bar{\psi}_n(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\epsilon})\frac{i\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}}{\epsilon^2} \times\psi_n(\mathbf{x}-\tfrac{1}{2}\boldsymbol{\epsilon})\exp\left[iq_n{}^\alpha\int_{\mathbf{x}-\epsilon/2}^{\mathbf{x}+\epsilon/2}\mathfrak{V}^\alpha(\mathbf{x}_1)\cdot d\mathbf{x}_1\right], \qquad (3.24)$$

[9] P. A. M. Dirac, Rev. Mod. Phys. **34**, 592 (1962); J. Schwinger, Phys. Rev. **127**, 324 (1962); **130**, 406 (1963); **130**, 800 (1963).

where it is understood that an average of the direction of $\boldsymbol{\epsilon}$ is performed before letting $\boldsymbol{\epsilon}\to 0$. The nonlocal contribution to the commutator then depends on

$$E=\exp\left[i\int_{\mathbf{x}-\epsilon/2}^{\mathbf{x}+\epsilon/2}[e_m\mathfrak{A}(\mathbf{x}_1)+g_m\mathfrak{B}(\mathbf{x}_1)]\cdot d\mathbf{x}_1,\; i\int_{\mathbf{x}'-\epsilon'/2}^{\mathbf{x}'+\epsilon'/2}[e_n\mathfrak{A}(\mathbf{x}_1')+g_n\mathfrak{B}(\mathbf{x}_1')]\cdot d\mathbf{x}_1'\right],$$

or, by (3.7),

$$E=\exp\left(i\int_{\mathbf{x}-\epsilon/2}^{\mathbf{x}+\epsilon/2}d\mathbf{x}_1\times\int_{\mathbf{x}'-\epsilon'/2}^{\mathbf{x}'+\epsilon'/2}d\mathbf{x}_1'\cdot[e_mg_n\mathbf{h}(\mathbf{x}_1-\mathbf{x}_1')+g_me_n\mathbf{h}(\mathbf{x}_1'-\mathbf{x}_1)]\right). \tag{3.25}$$

We have replaced $\mathfrak{B}^\alpha=(\mathfrak{A},\mathfrak{B})$ and $q_n{}^\alpha=(e_n,g_n)$ by their definitions so as not to assume chiral invariance of the commutator of the extended potentials, guaranteed by $\mathbf{h}(-\mathbf{x})=-\mathbf{h}(\mathbf{x})$, but rather to see to what extent this is required by Lorentz invariance. It is sufficient to consider the alternative of semi-infinite or symmetric infinite strings lying along the z axis:

$$\mathbf{h}(\mathbf{x})=\mathbf{h}_1(\mathbf{x})=-\hat{z}\theta(z)\delta(z)\delta(x)\delta(y) \tag{3.26a}$$

or

$$\mathbf{h}(\mathbf{x})=\mathbf{h}_2(\mathbf{x})=-\tfrac{1}{2}\hat{z}\epsilon(z)\delta(x)\delta(y). \tag{3.26b}$$

The integrand in (3.25) is zero unless the projections on the x-y plane of $\boldsymbol{\epsilon}$ centered at $\mathbf{x}$ and of $\boldsymbol{\epsilon}'$ centered at $\mathbf{x}'$ intersect, in which case one finds in the corresponding alternative cases

$$E_1=\exp\{-i[e_mg_n\theta(z-z')+g_me_n\theta(z'-z)]\times\mathrm{sgn}(\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\cdot\hat{z})\}, \tag{3.27a}$$

$$E_2=\exp\{-\tfrac{1}{2}i(e_mg_n-g_me_n)\times\mathrm{sgn}(z-z')\,\mathrm{sgn}(\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\cdot\hat{z})\}. \tag{3.27b}$$

The nonlocal contributions vanish when $E=1$, that is, when the exponent is an integral multiple of $2\pi i$. If the semi-infinite string function is chosen, one obtains the quantization condition on the products e_mg_n,

$$e_mg_n/4\pi=0,\pm\tfrac{1}{2},\pm1,\cdots, \tag{3.28}$$

as discussed by Dirac[6] and by Schwinger.[1,5] On the other hand, if the infinite antisymmetric string function is used, the quantization condition is much weaker and applies instead to the chiral-invariant combinations

$$(e_mg_n-g_me_n)/4\pi=(q_m{}^\alpha\epsilon^{\alpha\beta}q_n{}^\beta)/4\pi=0,\pm1,\pm2,\cdots. \tag{3.29}$$

Conversely, if it is assumed that the products e_mg_n of electric and magnetic charges are not separately quantized, then Lorentz invariance leads to the weaker chiral-invariant quantization condition (3.29), and to the antisymmetric string function $\mathbf{h}$ which assures chiral invariance of the commutator (3.7) or the potentials (2.21).

When the charge quantization condition holds, the calculation of the commutator of the θ's is straightforward, yielding the right-hand side of (3.23) with extended momentum density

$$\theta^{0i}=\boldsymbol{\theta}^0=\tfrac{1}{2}\mathfrak{F}^\alpha\times\epsilon^{\alpha\beta}\mathfrak{F}^\beta+\sum_n\bar{\psi}_n\gamma^0(-i\nabla-q_n{}^\alpha\mathfrak{B}^\alpha)\psi_n+\tfrac{1}{4}\nabla\times(\bar{\psi}_n\gamma^0\boldsymbol{\sigma}\psi_n).$$

This operator satisfies

$$\boldsymbol{\theta}^0U(\nu)|\;\rangle=U(\nu)\mathbf{T}^0|\;\rangle,$$

with physical momentum density

$$\mathbf{T}^0=\tfrac{1}{2}\mathbf{F}^\alpha\times\epsilon^{\alpha\beta}\mathbf{F}^\beta+\sum_n\bar{\psi}_n(-i\nabla-q_n{}^\alpha\mathbf{V}^\alpha)\psi_n+\tfrac{1}{4}\nabla\times(\bar{\psi}_n\gamma^0\boldsymbol{\sigma}\psi_n). \tag{3.30}$$

It is understood that this expression is properly antisymmetrized, and regularized by a limit such as (3.24). After rearrangement it yields a total momentum

$$\mathbf{P}=\int\mathbf{T}^0d\mathbf{x}=\int(E_i{}^r\nabla A_i{}^r+\sum_n\bar{\psi}_n\gamma^0(-i\nabla)\psi_n)d\mathbf{x} \tag{3.31}$$

and a total angular momentum

$$J=\int\mathbf{x}\times\mathbf{T}^0d\mathbf{x}=\int(E_i{}^r\mathbf{x}\times\nabla A_i{}^r+\mathbf{E}^r\times\mathbf{A}^r+\sum_n\psi_n{}^\dagger(-i\mathbf{x}\times\nabla+\tfrac{1}{2}\boldsymbol{\sigma})\psi_n)d\mathbf{x}-\tfrac{1}{2}\sum_{n,m}q_m{}^\alpha\epsilon^{\alpha\beta}q_n\int d\mathbf{x}d\mathbf{x}'\times\left[\psi_m{}^\dagger(\mathbf{x})\psi_m(\mathbf{x})\left((\mathbf{x}-\mathbf{x}')\times\mathbf{a}(\mathbf{x}-\mathbf{x}')+\frac{1}{4\pi}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\right)\times\psi_n{}^\dagger(\mathbf{x}')\psi_n(\mathbf{x}')\right]. \tag{3.32}$$

IV. SOLUTION OF CHARGE QUANTIZATION CONDITION

Let us write the charge quantization condition (3.29) for a set of charge vectors $q_n{}^\alpha$ ($\alpha=1,2$) as

$$\mathbf{q}_m\times\mathbf{q}_n/4\pi=0,\pm1,\pm2,\cdots. \tag{4.1}$$

This equation is obviously satisfied with 0 on the right for all m and n if all charge vectors are parallel. A chiral transformation may then align all these vectors along the electric axis corresponding to a situation where all particles bear arbitrary electric charge. This agrees with observation, since no magnetically charged particles are known, but fails to account for the quantization of electric charge which was Dirac's original motivation in introducing magnetic charges.

On the other hand, suppose (4.1) is satisfied for two vectors, call them $\mathbf{q}_1$ and $\mathbf{q}_2$, with

$$\mathbf{q}_1 \times \mathbf{q}_2 / 4\pi = 1. \tag{4.2}$$

Then these two vectors are linearly independent and any vector in the two-dimensional charge space may be written

$$\mathbf{q}_n = c_{n1}\mathbf{q}_1 + c_{n2}\mathbf{q}_2.$$

On forming cross products of $\mathbf{q}_n$ with $\mathbf{q}_1$ and $\mathbf{q}_2$ and using (4.1) and (4.2), one finds that c_{n1} and c_{n2} must be integers, Z_{n1}, Z_{n2},

$$\mathbf{q}_n = Z_{n1}\mathbf{q}_1 + Z_{n2}\mathbf{q}_2, \quad Z_{ni} = 0, \pm 1, \pm 2, \cdots. \tag{4.3}$$

The cross product of any pair of vectors is then

$$\mathbf{q}_m \times \mathbf{q}_n = (Z_{m1}Z_{n2} - Z_{m2}Z_{n1})4\pi,$$

so that (4.1) is satisfied for all m and n. Consequently (4.3) is the most general solution to the charge quantization condition provided that there exists a pair of vectors whose cross product has the minimum nonzero value.

A chiral transformation may be used to align $\mathbf{q}_1$ along the electric axis, so that

$$\mathbf{q}_1 = (e_1, 0), \tag{4.4}$$

with presumably $e_1^2/4\pi = 1/137$. In this case we have

$$\mathbf{q}_2 = (e_2, 4\pi e_1^{-1}). \tag{4.5}$$

There is no further restriction on e_2, the electric charge on the Dirac magnetic monopole, and it defines a second elementary quantum of electric charge.

Let us discuss the discrete symmetries of the Hamiltonian (2.27). For this purpose it is convenient to use the Majorana representation. The Hamiltonian is, of course, invariant under the CPT transformation Θ,

$$\Theta: \quad \mathbf{E}(t,\mathbf{x}),\mathbf{H}(t,\mathbf{x})\psi(t,\mathbf{x}) \rightarrow \mathbf{E}(-t,-\mathbf{x}),\mathbf{H}(-t,-\mathbf{x}),\; \gamma^5\psi^\dagger(-t,-\mathbf{x}), \tag{4.6}$$

and it is also invariant under charge conjugation,

$$C: \quad \mathbf{E},\mathbf{H},\psi \rightarrow -\mathbf{E},-\mathbf{H},\psi^\dagger. \tag{4.7}$$

To discuss parity and time reversal, assume that (4.3) holds and that the electric axis is aligned along $\mathbf{q}_1$. Otherwise a chiral transformation would be included in the definition of P and T. Two cases are to be distinguished, depending on whether or not the magnetically charged particles appearing in the Hamiltonian (2.27) also bear electric charges. If they do not, let ψ_m correspond to the purely electric particles and let ψ_n correspond to the purely magnetic particles. Then conserved-parity and time-reversal operators are defined by

$$P: \quad \mathbf{E}(\mathbf{x}),\mathbf{H}(\mathbf{x}),\psi_m(\mathbf{x}),\psi_n(\mathbf{x}) \rightarrow -\mathbf{E}(-\mathbf{x}),\mathbf{H}(-\mathbf{x}),\; \gamma^0\psi_m(-\mathbf{x}),\gamma^0\psi_n^\dagger(-\mathbf{x}), \tag{4.8}$$

$$T: \quad \mathbf{E}(t),\mathbf{H}(t),\psi_m(t),\psi_n(t) \rightarrow \mathbf{E}(-t),-\mathbf{H}(-t),\; \gamma^0\gamma^5\psi_m(-t),\gamma^0\gamma^5\psi_n^\dagger(-t). \tag{4.9}$$

This is the situation discussed by Ramsey.[10] On the other hand, if a particle, say, $\mathbf{q}_2$, has both electric and magnetic charges, then no conserved-parity or time-reversal operator exists. A convenient chiral-invariant measure of the strength of the parity and time-reversal breaking interaction is

$$\frac{1}{(4\pi)^2}\mathbf{q}_1\cdot\mathbf{q}_2\mathbf{q}_1\times\mathbf{q}_2 = \frac{1}{4\pi}\mathbf{q}_1\cdot\mathbf{q}_2 = \frac{1}{4\pi}e_1e_2, \tag{4.10}$$

the product of the two elementary quanta of electric charge. This could be a fairly large coupling and yet not show up very much in present experiments because magnetically charged particles are bound into magnetically neutral systems via the parity-conserving superstrong coupling constant

$$\mathbf{q}_2^2/4\pi = e_2^2/4\pi + 4\pi/e_1^2 = e_2^2/4\pi + 137.$$

Besides these discrete symmetries, each type of particle in the Hamiltonian (2.27) has its own conserved current $\bar{\psi}_n\gamma^\mu\psi_n$. Finally, there are the gauge transformations involving two gauge functions $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$:

$$\mathbf{A},\mathbf{B},\psi_n \rightarrow \mathbf{A}+\nabla\lambda,\mathbf{B}+\nabla\mu,\psi_n\exp(ie_n\lambda+ig_n\mu), \tag{4.11}$$

but they require a larger framework than the present one to be studied adequately.

Because of the superstrong coupling of the present theory, and because no Lorentz-covariant perturbative expansion exists due to the nonlocal commutator of the vector potentials, new calculational techniques are required to extract physical predictions. However, the nonrelativistic form may be studied unambiguously.[11]

Note added in proof. The quantization proposed here has also been proposed recently by J. Schwinger, Phys. Rev. **173**, 1536 (1968), Eq. (80).

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[10] N. F. Ramsey, Phys. Rev. **109**, 225 (1958).
[11] D. Zwanziger, preceding paper, Phys. Rev. **176**, 1480 (1968).