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Exactly Soluble Nonrelativistic Model of Particles with Both Electric and Magnetic Charges

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We consider the quantum-mechanical problem of the interaction of two particles, each with arbitrary electric and magnetic charges. It is shown that if an additional $1/r^2$ potential, of appropriate strength, acts between the particles, then the resulting Hamiltonian possesses the same higher symmetry as the nonrelativistic Coulomb problem. The bound-state energies and the scattering phase shifts are determined by an algebraic and gauge-independent method. If the electric and magnetic coupling parameters are α and $\mu=0, \pm\frac{1}{2}, \pm 1, \cdots$, then the bound states correspond to the representations $n_1+n_2=|\mu|, |\mu|+1, \cdots$, $n_1-n_2=\mu$ of $SU_2\otimes SU_2\sim O_4$, and the scattering states correspond to the representations of $SL(2,C)\sim O(1,3)$ specified by $\mathbf{J}^2-\mathbf{K}^2=\mu^2-\alpha'^2-1$, $\mathbf{J}\cdot\mathbf{K}=\alpha'\mu$, with $\alpha'=\alpha/v$. Thus, as α and μ are varied, all irreducible representations of O_4 and all irreducible representations in the principal series of $O(1,3)$ occur. The scattering matrix is expressed in closed form, and the differential cross section agrees with its classical value. Some results are obtained which are valid in a relativistic quantum field theory. The S matrix for spinless particles is found to transform under rotations like a $\mu\to-\mu$ helicity-flip amplitude, which contradicts the popular assumption that scattering states transform like the product of free-particle states. It is seen that the Dirac charge quantization condition means that electromagnetic interactions are characterized not by one but by two, and only two, free parameters: the electronic charge $e\approx(137)^{-1/2}$, and the electric charge of the magnetic monopole, whose absolute magnitude is not fixed by the Dirac quantization condition but which defines a second elementary quantum of electric charge.

1. INTRODUCTION

THE interaction of two charged particles with coupling parameter $\alpha=-e_1e_2/4\pi$ ($\alpha>0$ for attraction) is described nonrelativistically by a Hamiltonian which possesses the higher symmetry of $O(4)$ for bound states and $O(1,3)$ for continuum states, as is well known. We will show that if the two particles have, in addition to their electric charges e_1 and e_2, magnetic charges g_1 and g_2 so that the coupling parameters are $\alpha=-(e_1e_2+g_1g_2)/4\pi$, and $\mu=(e_1g_2-g_1e_2)/4\pi c$, then the Hamiltonian possesses the same higher symmetry as the pure Coulomb Hamiltonian, provided that an additional scalar potential $V=\mu^2/2mr^2$ is also present. The nonrelativistic charge-monopole interaction is well explored in the literature,[1–7] but the study of the higher symmetry of the related problem, nor of an additional Coulomb interaction, does not seem to be represented. In the concluding section we discuss those features of the model that have general validity, and make some over-all observations about the magnetic monopole problem. *Inter alia* there is a discussion of the transformation properties of the S matrix, and it is seen that the Dirac quantization condition leads not to one, but to two elementary quanta of electric charge.

The magnetic field $\mathbf{B}(\mathbf{r})$ due to a magnetic monopole of strength g at the origin is

$$\mathbf{B}(\mathbf{r})=(g/4\pi)\hat{r}/r^2 \tag{1.1}$$

and may be obtained as the curl of a vector potential $\mathbf{A}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r})=\nabla\times\mathbf{A}(\mathbf{r}), \tag{1.2}$$

with $\mathbf{A}(\mathbf{r})$ given by

$$\mathbf{A}(\mathbf{r})=\frac{g}{4\pi}\frac{\mathbf{r}\times\hat{n}}{r(r-\mathbf{r}\cdot\hat{n})} \tag{1.3a}$$

or by

$$\mathbf{A}(\mathbf{r})=\frac{g}{4\pi}\frac{\mathbf{r}\times\hat{n}\mathbf{r}\cdot\hat{n}}{r[r^2-(\mathbf{r}\cdot\hat{n})^2]}. \tag{1.3b}$$

For these $\mathbf{A}(\mathbf{r})$, Eqs. (1.1) and (1.2) agree except along the singular line $\mathbf{r}=c\hat{n}$, with $0<c<\infty$ (1.3a) or $-\infty<c<\infty$ (1.3b). Accordingly, the dynamics of a pair of particles, of masses m_1 and m_2, electric charges e_1 and e_2, magnetic charges g_1 and g_2, is specified by the Hamiltonian

$$H=\frac{1}{2m_1}\left(\mathbf{p}_1-\frac{e_1g_2-g_1e_2}{4\pi c}\mathbf{D}(\mathbf{r}_1-\mathbf{r}_2)\right)^2+\frac{1}{2m_2}\left(\mathbf{p}_2-\frac{e_2g_1-g_2e_1}{4\pi c}\mathbf{D}(\mathbf{r}_2-\mathbf{r}_1)\right)^2+\frac{e_1e_2+g_1g_2}{4\pi}\times\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}+V(|\mathbf{r}_1-\mathbf{r}_2|), \tag{1.4}$$

where

$$\mathbf{D}(\mathbf{r})=\frac{\mathbf{r}\times\hat{n}\;\mathbf{r}\cdot\hat{n}}{r[r^2-(\mathbf{r}\cdot\hat{n})^2]}, \tag{1.5}$$

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[2] I. Tamm, Z. Physik **71**, 141 (1931).
[3] M. Fierz, Helv. Phys. Acta **17**, 27 (1944); P. Banderet, *ibid.* **19**, 503 (1946).
[4] A. Goldhaber, Phys. Rev. **140**, B1407 (1965).
[5] B. Zumino, in *Strong and Weak Interactions-Present Problems*, edited by A. Zichichi (Academic Press Inc., New York, 1966).
[6] A. Peres, Phys. Rev. **167**, 1449 (1968).
[7] C. A. Hurst, Center of Theoretical Studies Report, Coral Gables, Fla. (unpublished).

and $V(r)$ is an arbitrary additional potential interaction.

The justification for this Hamiltonian is that the Heisenberg equations of motion for $\mathbf{r}_i$ and

$$\boldsymbol{\pi}_i = \mathbf{p}_i - (4\pi c)^{-1}(e_i g_j - g_i e_j)\mathbf{D}(\mathbf{r}_i - \mathbf{r}_j),$$

$i, j = 1, 2$, yield the Newtonian equations of motion

$$\dot{\mathbf{r}}_i = \boldsymbol{\pi}_i/m_i, \tag{1.6a}$$

$$\dot{\boldsymbol{\pi}}_i = \frac{(e_i e_j + g_i g_j)}{4\pi}\frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} + \frac{(e_i g_j - g_i e_j)}{4\pi c} \times \frac{1}{2}\left[\left(\frac{\boldsymbol{\pi}_i}{m_i} - \frac{\boldsymbol{\pi}_j}{m_j}\right) \times \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} - \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|} \times \left(\frac{\boldsymbol{\pi}_i}{m_i} - \frac{\boldsymbol{\pi}_j}{m_j}\right)\right] - \boldsymbol{\nabla}_i V(|\mathbf{r}_i - \mathbf{r}_j|). \tag{1.6b}$$

The first term in Eq. (1.6b) is a Coulomb force, the term proportional to $\mathbf{v}_i = \boldsymbol{\pi}_i/m_i$ is a Lorentz force, and the term proportional to $\mathbf{v}_j = \boldsymbol{\pi}_j/m_j$ represents the force produced by Biot-Savart fields. Only the difference, $\mathbf{v}_i - \mathbf{v}_j$, occurs, in accordance with Galilean invariance. These are the desired classical expressions, symmetrized where necessary so that the quantum-mechanical operators are Hermitian, and are correct to order v/c. The interaction violates parity and time-reversal invariance separately, but is invariant under their product PT.

Upon introducing the usual relative and c.m. coordinates

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2),$$
$$\mathbf{p} = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/(m_1 + m_2), \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2,$$

one obtains[8]

$$H = \frac{1}{2M}\mathbf{P}^2 + \frac{1}{2m}[\mathbf{p} - \mu\mathbf{D}(\mathbf{r})]^2 - \frac{\alpha}{r} + V(r), \tag{1.7}$$

where $M = m_1 + m_2$ and $m = (m_1^{-1} + m_2^{-1})^{-1}$ are the total and reduced masses, respectively, and where

$$\alpha \equiv -(e_1 e_2 + g_1 g_2)/4\pi \tag{1.8}$$

and

$$\mu \equiv (e_1 g_2 - g_1 e_2)/4\pi_c \tag{1.9}$$

are the electric and magnetic coupling parameters.

From now on we set the total momentum $\mathbf{P}$ equal to zero and reexpress the Hamiltonian (1.7) in a form which is manifestly gauge-invariant and rotationally-invariant. In terms of

$$\boldsymbol{\pi} \equiv \mathbf{p} - \mu\mathbf{D}(\mathbf{r}), \tag{1.10}$$

the Hamiltonian takes the form

$$H = \boldsymbol{\pi}^2/2m - \alpha/r + V(r), \tag{1.11}$$

which is associated with the gauge-invariant and rotationally invariant commutation relations

$$[x_i, x_j] = 0, \tag{1.12a}$$

$$[\pi_i, x_j] = -i\delta_{ij}, \tag{1.12b}$$

$$[\pi_i, \pi_j] = i\mu\epsilon_{ijk}x_k/r^3. \tag{1.12c}$$

The commutators (1.12b) and (1.12c) appear to violate the Jacobi identity, since we have, for $i \neq j \neq k$,

$$[\pi_i, [\pi_j, \pi_k]] + [\pi_j, [\pi_k, \pi_i]] + [\pi_k, [\pi_i, \pi_j]] = 4\pi\mu\delta(\mathbf{r}) \tag{1.13}$$

instead of zero. This paradox can be resolved by a study of the domain of definition[9] of the operator π_i, defined by Eq. (1.10), and of the conditions under which it can be extended to a self-adjoint operator. Such a study has been effected by Hurst[7] for the angular-momentum operator $\mathbf{J}$ which will be introduced shortly. The result is that the paradox is removed only for values of μ which satisfy the Dirac quantization condition

$$\mu = 0, \pm\tfrac{1}{2}, \pm 1, \cdots. \tag{1.14}$$

We will obtain this condition by algebraic arguments which also lead to an elucidation of the higher symmetry.

2. ASSOCIATED HAMILTONIAN AND ITS CONSTANTS OF MOTION

The first thing to do in a problem which has spherical symmetry is to identify the angular-momentum operator $\mathbf{J}$. In the classical problem it is well known that $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$ is not a constant of the motion because of the angular momentum $-\mu\hat{r}$ in the crossed static electric and magnetic fields, given by the spatial integral of the moment of the Poynting vector. The quantity $\mathbf{J} = m\mathbf{r} \times \dot{\mathbf{r}} - \mu\hat{r}$ is conserved, however; so it is natural to consider the corresponding quantum-mechanical operator

$$\mathbf{J} = \mathbf{r} \times \boldsymbol{\pi} - \mu\hat{r}. \tag{2.1}$$

One easily verifies from Eqs. (1.12) that $\mathbf{J}$ satisfies

$$[J_i, x_j] = i\epsilon_{ijk}x_k, \quad [J_i, \pi_j] = i\epsilon_{ijk}\pi_k, \tag{2.2}$$

so it is the generator of rotations of position $\mathbf{r}$ and velocity $\boldsymbol{\pi}/m$. Consequently it commutes with any scalar formed out of them, and in particular, the Hamiltonian itself,

$$[\mathbf{J}, H] = 0. \tag{2.3}$$

Likewise, it is the generator of rotations for any vector formed out of $\mathbf{x}$ and $\boldsymbol{\pi}$, and in particular, $\mathbf{J}$ itself,

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{2.4}$$

[8] If in Eq. (1.4), the symmetric form (1.5) for D had not been chosen, then a gauge transformation would be necessary before the separation of the c.m. and relative motion could be effected.

[9] Clearly, the left-hand side of Eq. (1.13) can only be applied to wave functions which vanish at the origin.

In addition to $\mathbf{J}$, it is convenient to introduce the new dynamical variable

$$D=\tfrac{1}{2}(\mathbf{r}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\mathbf{r}). \tag{2.5}$$

The advantage of D over the more customary variable

$$\pi_r=\tfrac{1}{2}(\hat{r}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\hat{r})=r^{-1/2}Dr^{-1/2} \tag{2.6}$$

is that π_r is the generator of translations in the radial direction and thus leads to negative radii. This is meaningless, so that π_r cannot be self-adjoint, whereas D is the generator of radial dilatations, which is all right. Equations (2.1) and (2.5) may be inverted, to express $\boldsymbol{\pi}$ in terms of $\mathbf{J}$ and D:

$$\boldsymbol{\pi}=\hat{r}r^{-1/2}Dr^{-1/2}+\tfrac{1}{2}(\mathbf{J}\times\hat{r}-\hat{r}\times\mathbf{J})/r. \tag{2.7}$$

The commutation relations (1.12) become, in terms of the new variables,

$$[x_i,x_j]=0, \tag{2.8a}$$

$$[J_i,J_j]=i\epsilon_{ijk}J_k,\quad [J_i,D]=0, \tag{2.8b}$$

$$[J_i,x_j]=i\epsilon_{ijk}x_k,\quad [D,x_i]=-ix_i. \tag{2.8c}$$

These commutation relations are formally those of the Lie algebra of the three-dimensional similarity group S_3 which consists of the dilatations plus the three-dimensional Euclidean group E_3 of rotations and translations.[10] The representations of this group are easily found in terms of the known representations of the Euclidean group.[11] It is not our purpose to do this here. However, we note in passing that the irreducible representations of this group are labelled by the invariant $\hat{x}\cdot\mathbf{J}$, a helicitylike quantity which takes on the values

$$\hat{x}\cdot\mathbf{J}=0,\pm\tfrac{1}{2},\pm1,\cdots. \tag{2.9}$$

From Eq. (2.1) we have $\hat{x}\cdot\mathbf{J}=-\mu$; thus we find

$$\mu=0,\pm\tfrac{1}{2},\pm1,\cdots. \tag{2.10}$$

This is an algebraic derivation of the Dirac quantization which is alternative to those of Peres[6] and Hurst.[7] However, we refer the interested reader to the exposition of Hurst for a Hilbert-space analysis of how the quantization condition arises.

Let us express the Hamiltonian (1.11) in terms of the new variables. One easily finds

$$H=\frac{1}{2m}\pi_r^2+\frac{1}{2mr^2}(\mathbf{J}^2-\mu^2)-\frac{\alpha}{r}+V(r). \tag{2.11}$$

At this point we note that if $V(r)$ takes the particular form

$$V(r)=\mu^2/2mr^2, \tag{2.12}$$

then the net r^{-2} effective potential looks like an ordinary centrifugal barrier, with $\mathbf{J}^2$ replacing $\mathbf{L}^2$, and the resulting H, which we call H^a, the Hamiltonian of the associated problem, is simpler than if $V(r)=0$. One has

$$H^a=\frac{1}{2m}\pi_r^2+\frac{1}{2mr^2}\mathbf{J}^2-\frac{\alpha}{r} \tag{2.13a}$$

or

$$H^a=\frac{1}{2m}\boldsymbol{\pi}^2-\frac{\alpha}{r}+\frac{\mu^2}{2mr^2}. \tag{2.13b}$$

Equation (2.13a) has the same form as the Hamiltonian of the Coulomb problem expressed in radial coordinates,

$$H^c=\frac{1}{2m}p_r^2+\frac{1}{2mr^2}\mathbf{L}^2-\frac{\alpha}{r}, \tag{2.14}$$

and has the same commutation relations. This suggests that the Runge-Lenz vector[12]

$$\mathbf{A}=(1/2m)(\boldsymbol{\pi}\times\mathbf{J}-\mathbf{J}\times\boldsymbol{\pi})-\alpha\hat{r}, \tag{2.15a}$$

whose definition is symmetrical to that of $\mathbf{J}$,

$$\mathbf{J}=\tfrac{1}{2}(\mathbf{r}\times\boldsymbol{\pi}-\boldsymbol{\pi}\times\mathbf{r})-\mu\hat{r}, \tag{2.15b}$$

may be a constant of motion of H^a. This turns out to be the case, and brief manipulative exercises yield

$$[\mathbf{J},H^a]=[\mathbf{A},H^a]=0, \tag{2.16a}$$

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \tag{2.16b}$$

$$[J_i,A_j]=i\epsilon_{ijk}A_k, \tag{2.16c}$$

$$[A_i,A_j]=-i\epsilon_{ijk}J_k(2H^a/m), \tag{2.16d}$$

$$\mathbf{A}\cdot\mathbf{J}=\mathbf{J}\cdot\mathbf{A}=\alpha\mu, \tag{2.16e}$$

$$\mathbf{A}^2-\alpha^2=(2H^a/m)(\mathbf{J}^2-\mu^2+1). \tag{2.16f}$$

Except for the fact that μ has a nonzero value in the last two relations, these equations are identical with the corresponding equations for the Coulomb problem.

3. ENERGY EIGENVALUES AND SCATTERING MATRIX

Having obtained the constants of the motion, one can find the eigenfunctions of H^a, the Hamiltonian of the associated problem, just as for the Coulomb problem, by making a separation of variables in spherical, parabolic, or ellipsoidal coordinates. However, we will forego this here and find instead the energy eigenvalues and scattering matrix by algebraic and gauge-independent methods.

Consider first the possibility of bound states. The method is that introduced by Pauli[13] for the Coulomb

[10] I am grateful to Dr. F. Yndurain for pointing this out to me.

[11] A. S. Wightman, Rev. Mod. Phys. 34, 851 (1962).

[12] There is a rich literature on higher symmetry in the Coulomb problem. References to earlier work may be found in the review articles by M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 330 (1966); 38, 346 (1966).

[13] W. Pauli, Z. Physik 36, 336 (1926).

problem. From Eq. (2.13b) we see that unless $\alpha>0$ the Hamiltonian is positive definite and there are no bound states. Thus we assume $\alpha>0$ and seek any possible bound states. In relations (2.16) we replace H^a by its eigenvalue $E<0$, since it commutes with all quantities appearing there, and set

$$H^a=E=-\tfrac{1}{2}mu^2,\quad u>0. \tag{3.1}$$

The relations (2.16) may be written

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \tag{3.2a}$$

$$[J_i,N_j]=i\epsilon_{ijk}N_k, \tag{3.2b}$$

$$[N_i,N_j]=i\epsilon_{ijk}J_k, \tag{3.2c}$$

$$\mathbf{N}\cdot\mathbf{J}=\alpha'\mu, \tag{3.2d}$$

$$\mathbf{J}^2+\mathbf{N}^2=\alpha'^2+\mu^2-1, \tag{3.2e}$$

where we have introduced

$$\mathbf{N}\equiv\mathbf{A}/u, \tag{3.3}$$

$$\alpha'\equiv\alpha/u. \tag{3.4}$$

Equations (3.2a)–(3.2c) define the Lie algebra of the group $SU_2\otimes SU_2\sim O_4$, as is well known. For, upon introducing

$$\mathbf{J}_\pm\equiv\tfrac{1}{2}(\mathbf{J}\pm\mathbf{N}), \tag{3.5}$$

we have

$$[J_{i\pm},J_{j\pm}]=i\epsilon_{ijk}J_{k\pm}, \tag{3.6a}$$

$$[J_{i+},J_{j-}]=0, \tag{3.6b}$$

$$\mathbf{J}_\pm^2=\tfrac{1}{4}[(\alpha'\pm\mu)^2-1]. \tag{3.6c}$$

The commutators (3.6a) and (3.6b) are the defining relations of $SU_2\otimes SU_2$. It follows that

$$\mathbf{J}_\pm^2=j_\pm(j_\pm+1), \tag{3.7}$$

with

$$j_+,j_-=0,\tfrac{1}{2},1,\cdots,$$

so the eigenvalues are determined by

$$(j_\pm+\tfrac{1}{2})^2=\tfrac{1}{4}(\alpha'\pm\mu)^2. \tag{3.8}$$

If $|\mu|\geq\alpha'$, one has $j_++j_-+1=|\mu|$, which means that $j<|\mu|$ because in the representation (j_+,j_-), the values of j that occur are $|j_+-j_-|\cdots j_++j_-$. This contradicts $j\geq|\mu|$, which follows from $\mathbf{J}\cdot\hat{x}=-\mu$. Consequently we must have

$$\alpha'>|\mu|, \tag{3.9}$$

and hence

$$j_++\tfrac{1}{2}=\tfrac{1}{2}(\alpha'+\mu), \tag{3.10a}$$

$$j_-+\tfrac{1}{2}=\tfrac{1}{2}(\alpha'-\mu) \tag{3.10b}$$

or

$$j_>+j_<+1=\alpha', \tag{3.11a}$$

$$j_>-j_<=|\mu|. \tag{3.11b}$$

Upon introducing the principal quantum number

$$N\equiv j_>+j_<+1=\alpha' \tag{3.12}$$

we have

$$N=|\mu|+1,\ |\mu|+2,\ \cdots, \tag{3.13a}$$

and the energy eigenvalues are determined by $\alpha'=\alpha/u$, so that

$$E=-\tfrac{1}{2}mu^2=-\tfrac{1}{2}m(\alpha^2/N^2). \tag{3.13b}$$

This equation is the familiar Bohr formula, but it has a more general meaning here because the values of N are integral or half-integral, according as $|\mu|$ is, and only assume values greater than $|\mu|$. The angular-momentum values that occur in the degenerate level with principal quantum number N are

$$j=|\mu|,\quad |\mu|+1,\ \cdots,\ N-1. \tag{3.14}$$

This completes our discussion of the bound states and we consider next the continuum scattering problem.

The phase shifts may be obtained algebraically following a method applied previously[14] to the Coulomb problem.[15] We return to Eqs. (2.16) and set

$$H^a=E=\tfrac{1}{2}mv^2,\quad v>0. \tag{3.15}$$

These equations take the form

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \tag{3.16a}$$

$$[J_i,K_j]=i\epsilon_{ijk}K_k, \tag{3.16b}$$

$$[K_i,K_j]=-i\epsilon_{ijk}J_k, \tag{3.16c}$$

$$\mathbf{J}\cdot\mathbf{K}=\alpha'\mu, \tag{3.16d}$$

$$\mathbf{J}^2-\mathbf{K}^2=\mu^2-\alpha'^2-1, \tag{3.16e}$$

with the definitions

$$\mathbf{K}\equiv\mathbf{A}/v, \tag{3.17}$$

$$\alpha'=\alpha/v. \tag{3.18}$$

Equations (3.16a)–(3.16c) define the Lie algebra of the group $SL(2,C)\sim O(1,3)$, whereas Eqs. (3.16d) and (3.16e) specify its irreducible representation labelled by[16]

$$(j_0,c)=(|\mu|,i\alpha'\,\mathrm{sgn}\mu), \tag{3.19}$$

which is a unitary representation in the principal series. The only result[16] which we will need is that in the basis

[14] D. Zwanziger, J. Math. Phys. 8, 1858 (1967). This algebraic method was, in fact, developed for the purpose of making a gauge-invariant calculation of the magnetic-monopole scattering amplitude.

[15] Professor Biedenharn has kindly pointed out to me that L. C. Biedenharn and P. J. Brussaard, *Coulomb Excitation* (Oxford University Press, New York, 1965), contains the suggestion that the Coulomb scattering phase shifts may be obtained by analytic continuation of the Clebsch-Gordan coefficients of O_4.

[16] M. A. Naimark, *Linear Representations of the Lorentz Group* (American Mathematical Society, Providence, R.I., 1957), Chap. 3, Sec. 2, No. 3, Eqs. (51)–(55) and Sec. 5, No. 6, Eqs. (1)–(4).

$|j,m\rangle$,

$$K_3|j,\pm\mu\rangle = i\frac{j^2-\mu^2}{j}\left(\frac{j^2+\alpha'^2}{4j^2-1}\right)^{1/2}|j-1,\pm\mu\rangle \pm\frac{\mu^2\alpha'}{j(j+1)}|j,\pm\mu\rangle - i\frac{(j+1)^2-\mu^2}{(j+1)} \times\left(\frac{(j+1)^2+\alpha'^2}{4(j+1)^2-1}\right)^{1/2}|j+1,\pm\mu\rangle. \quad (3.20)$$

Let us now focus our attention on the in (out) states corresponding to a plane wave of momentum **k** entering (leaving), travelling in the direction $\hat{k}$. Because $\mathbf{J}\cdot\hat{k}$ and $\mathbf{K}\cdot\hat{k}$ are commuting constants of the motion, whatever values they have for appropriate wave packets before (after) scattering, they will have these values at all times, and are suitable labels for the in (out) states. We may suppose that as $t\to-\infty$ $(t\to+\infty)$, the in (out) state travelling in the $\hat{k}$ direction approaches a wave packet which lies in the direction $-\hat{k}$ $(+\hat{k})$, that is, the scattering state approaches an eigenstate of $\hat{x}$. Then from $\mathbf{J}\cdot\hat{x}=-\mu$ we find

$$\mathbf{J}\cdot\hat{k}\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle = \pm\mu\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle. \quad (3.21\text{a})$$

To find the eigenvalue of $\mathbf{K}\cdot\hat{k}$, we observe that

$$\mathbf{A}=(1/2m)(\boldsymbol{\pi}\times\mathbf{J}-\mathbf{J}\times\boldsymbol{\pi})-\alpha\hat{x} =(1/m)(i\boldsymbol{\pi}-\mathbf{J}\times\boldsymbol{\pi})-\alpha\hat{x},$$
$$\mathbf{A}\cdot\hat{k}=(1/m)(i\boldsymbol{\pi}\cdot\hat{k}-\mathbf{J}\cdot\boldsymbol{\pi}\times\hat{k})-\alpha\hat{x}\cdot\hat{k}.$$

As $t\to-\infty$ $(t\to+\infty)$ the scattering states also approach eigenstates of velocity $\boldsymbol{\pi}/m$, with eigenvalue $v\hat{k}$. Hence

$$\mathbf{A}\cdot\hat{k}\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle = (iv\pm\alpha)\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle,$$

or, since $\mathbf{K}=\mathbf{A}/v$,

$$\mathbf{K}\cdot\hat{k}\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle = (i\pm\alpha')\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle. \quad (3.21\text{b})$$

The energy eigenvalue and the eigenvalues of $\mathbf{J}\cdot\hat{k}$ and $\mathbf{K}\cdot\hat{k}$ characterize the scattering states to within a multiplicative constant. However, it is also necessary to establish conventions for the relative phases of scattering states with different **k**, because Eq. (3.21a) shows that the scattering states are states of helicity $\pm\mu$, and nontrivial phase conventions enter the definition[17] of states of nonzero helicity. We present a fairly detailed discussion because the behavior of the S matrix under rotation has not been given previously. Let us see how Eqs. (3.21) transform under rotations. Because **J** and **K** are vector operators, as seen in Eqs. (3.16a) and (3.16b), we have

$$U(R)\mathbf{J}U^{-1}(R)=R^{-1}\mathbf{J},\quad U(R)\mathbf{K}U^{-1}(R)=R^{-1}\mathbf{K}.$$

Thus from Eq. (3.21a) we find

$$U(R)\mathbf{J}\cdot\hat{k}\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle = \pm\mu U(R)\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle = [U(R)\mathbf{J}\cdot\hat{k}U^{-1}(R)]U(R)\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle = \mathbf{J}\cdot(R\hat{k})U(R)\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle,$$

and similarly for $\mathbf{K}\cdot\hat{k}$. This shows that

$$U(R)\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle = \left|R\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle \exp\left[i\varphi(\mathbf{k},R)\binom{\text{in}}{\text{out}}\right].$$

To each momentum vector **k** let us associate some definite rotation $R_{\mathbf{k}}$ which carries the fixed direction $\hat{z}$ into the direction $\hat{k}$:

$$R_{\mathbf{k}}\hat{z}=\hat{k}. \quad (3.22)$$

The rotation $R_{\mathbf{k}}$ is determined up to an initial rotation through an angle φ about $\hat{z}$, and the phase convention is some definite specification of the angle $\varphi=\varphi(\mathbf{k})$. The relative phases of the states $|\mathbf{k}\text{ in}\rangle$ and $|\mathbf{k}\text{ out}\rangle$ are fixed according to

$$\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle = U(R_{\mathbf{k}})\left|k\hat{z}\binom{\text{in}}{\text{out}}\right\rangle. \quad (3.23)$$

With this convention the scattering states have the definite transformation property

$$U(R)\left|\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle = \left|R\mathbf{k}\binom{\text{in}}{\text{out}}\right\rangle \exp[\pm i\mu\varphi(\mathbf{k},R)] \quad (3.24\text{a})$$

in which the angle $\varphi(\mathbf{k},R)$ is determined from

$$R_{\mathbf{k}'}^{-1}RR_{\mathbf{k}}=R_z[\varphi(\mathbf{k},R)], \quad (3.24\text{b})$$

where $\mathbf{k}'=R\mathbf{k}$ and $R_z(\varphi)$ is a rotation by φ around the z axis. Equations (3.24) are the standard transformation law of states with helicity $\pm\mu$.

Let us now consider the expansion of the scattering states $|\mathbf{k}\text{ in}\rangle$, $|\mathbf{k}\text{ out}\rangle$ in states $|j,m\rangle$ with definite $\mathbf{J}^2$ and J_z (and energy E which is suppressed). From Eq. (3.21a) we find

$$\mathbf{J}\cdot\hat{z}\left|k\hat{z}\binom{\text{in}}{\text{out}}\right\rangle = J_z\left|k\hat{z}\binom{\text{in}}{\text{out}}\right\rangle = \pm\mu\left|k\hat{z}\binom{\text{in}}{\text{out}}\right\rangle, \quad (3.25)$$

[17] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959); G. C. Wick, *ibid.* 18, 65 (1962).

so that the states $|k\hat{z}\,\text{in}\rangle$ have the expansion

$$|k\hat{z}\,\text{in}\rangle=\sum_{j\geq|\mu|}(2j+1)^{1/2}a_j^{\text{in}}|j,\mu\rangle. \tag{3.26}$$

Then, by Eq. (3.23), we may write

$$|\mathbf{k}\,\text{in}\rangle=\sum_{j\geq|\mu|}(2j+1)^{1/2}a_j^{\text{in}}|j,m\rangle\mathfrak{D}_{m,\mu}{}^j(R_{\mathbf{k}}), \tag{3.27a}$$

and similarly

$$|\mathbf{k}\,\text{out}\rangle=\sum_{j\geq|\mu|}(2j+1)^{1/2}a_j^{\text{out}}|j,m\rangle\mathfrak{D}_{m,-\mu}{}^j(R_{\mathbf{k}}), \tag{3.27b}$$

where we have used the known behavior of the states $|j,m\rangle$ under rotation. Equations (3.27) provide the desired expansion of $|\mathbf{k}\,\text{in}\rangle$ and $|\mathbf{k}\,\text{out}\rangle$ in an angular-momentum basis, provided only that the a_j are known. They are found from

$$K_3\left|k\hat{z}\begin{pmatrix}\text{in}\\\text{out}\end{pmatrix}\right\rangle=(i\pm\alpha')\left|k\hat{z}\begin{pmatrix}\text{in}\\\text{out}\end{pmatrix}\right\rangle, \tag{3.28}$$

which is a special case of Eq. (21b), upon substitution of Eqs. (3.20) and (3.26):

$$i\sum_j(2j+1)^{1/2}a_j^{\text{in}}(1-i\alpha')|j,\mu\rangle=i\sum_j(2j+1)^{1/2}a_j^{\text{in}}\times\left[\frac{j^2-\mu^2}{j}\left(\frac{j^2+\alpha'^2}{4j^2-1}\right)|j-1,\mu\rangle-i\frac{\mu^2\alpha'}{j(j+1)}|j,\mu\rangle-\frac{(j+1)^2-\mu^2}{j+1}\left(\frac{(j+1)^2+\alpha'^2}{4(j+1)^2-1}\right)^{1/2}|j+1,\mu\rangle\right]. \tag{3.29}$$

The result for a_j^{out} is obtained by the substitution $\alpha'\to-\alpha'$, $\mu\to-\mu$. Upon equating coefficients of $|j,\mu\rangle$, one obtains the recursion relation

$$\frac{(j+1)^2-\mu^2}{j+1}[(j+1)^2+\alpha'^2]^{1/2}a_{j+1}^{\text{in}}=(2j+1)\left[1-i\alpha'+\frac{\mu^2}{j(j+1)}i\alpha'\right]a_j^{\text{in}}+\frac{j^2-\mu^2}{j}[j^2+\alpha'^2]^{1/2}a_{j-1}^{\text{in}}. \tag{3.30}$$

Upon setting $j=|\mu|$, one obtains

$$a_{|\mu|+1}{}^{\text{in}}=\left(\frac{|\mu|+1-i\alpha'}{|\mu|+1+i\alpha'}\right)^{1/2}a_{|\mu|}{}^{\text{in}}, \tag{3.31}$$

and the general term is easily proven by induction to be

$$a_j^{\text{in}}=\left(\frac{(j-i\alpha')!}{(j+i\alpha')!}\right)^{1/2},\quad j=|\mu|,\ |\mu|+1,\ \cdots \tag{3.32}$$

and by $\alpha'\to-\alpha'$, $\mu\to-\mu$,

$$a_j^{\text{out}}=\left(\frac{(j+i\alpha')!}{(j-i\alpha')!}\right)^{1/2},\quad j=|\mu|,\ |\mu|+1,\ \cdots. \tag{3.33}$$

There is actually an arbitrary coefficient on the right-hand side of these equations, which is independent of j but may depend on k. Its magnitude is fixed at unity by normalization, and its phase to zero by convention.

The scattering matrix is now obtained directly from its definition

$$S(\mathbf{k}',\mathbf{k})=\langle\mathbf{k}\,\text{out}|\mathbf{k}\,\text{in}\rangle, \tag{3.34}$$

and Eq. (3.27):

$$S(\mathbf{k}',\mathbf{k})=\sum_{j\geq|\mu|}(2j+1)a_j^{\text{out}*}a_j^{\text{in}}\times\sum_m\mathfrak{D}_{m,-\mu}{}^j(R_{\mathbf{k}'})\mathfrak{D}_{m,\mu}{}^j(R_{\mathbf{k}}),$$

$$S(\mathbf{k}',\mathbf{k})=\sum_{j\geq|\mu|}(2j+1)\frac{(j-i\alpha')!}{(j+i\alpha')!}\mathfrak{D}_{-\mu,\mu}{}^j(R_{\mathbf{k}'}{}^{-1}R_{\mathbf{k}}). \tag{3.35}$$

The phase shifts

$$e^{2i\delta_j}=\frac{(j-i\alpha')!}{(j+i\alpha')!},\quad j=|\mu|,\ |\mu|+1,\ \cdots \tag{3.36}$$

have the same form as the Coulomb phase shifts, just as the energy eigenvalues (3.13b) look like the Balmer formula, but with the same generalization in meaning. In particular, the angular function which multiplies $e^{2i\delta_j}$ is not a Legendre polynomial. The poles of (3.36) occur at $j-i\alpha mk^{-1}=-n$, $n=1, 2, 3, \cdots$, or

$$k=\frac{i\alpha m}{n+j},\quad E=\frac{1}{2m}k^2=-\frac{1}{2}\frac{\alpha^2}{(n+j)^2}, \tag{3.37}$$

which agrees with (3.13b).

The angular dependence may be made more explicit by using the transformation properties of $S(\mathbf{k}',\mathbf{k})$ under rotation, which are completely standard for helicity amplitudes. We have

$$S(\mathbf{k}',\mathbf{k})=\langle\mathbf{k}'\,\text{out}|\mathbf{k}\,\text{in}\rangle=\langle\mathbf{k}'\,\text{out}|U^\dagger(R)U(R)|\mathbf{k}\,\text{in}\rangle=\langle R\mathbf{k}'\,\text{out}|R\mathbf{k}\,\text{in}\rangle\exp[i\mu\varphi(\mathbf{k}',R)+i\mu\varphi(\mathbf{k},R)],$$

where we have used Eqs. (3.24); therefore,

$$S(R\mathbf{k}',R\mathbf{k})=S(\mathbf{k}',\mathbf{k})\exp[-i\mu\varphi(\mathbf{k}',R)-i\mu\varphi(\mathbf{k},R)]. \tag{3.38}$$

Each term in the expansion (3.35) has this behavior because

$$\begin{aligned}\mathfrak{D}_{-\mu,\mu}{}^j(R_{R\mathbf{k}'}{}^{-1}R_{R\mathbf{k}})&=\mathfrak{D}_{-\mu,\mu}{}^j\{[RR_{\mathbf{k}'}R_z{}^{-1}(\varphi')]^{-1}[RR_{\mathbf{k}}R_z{}^{-1}(\varphi)]\}\\&=\mathfrak{D}_{-\mu,\mu}{}^j[R_z(\varphi')R_{\mathbf{k}'}{}^{-1}R_{\mathbf{k}}R_z{}^{-1}(\varphi)]\\&=e^{-i\mu\varphi'}\mathfrak{D}_{-\mu,\mu}{}^j(R_{\mathbf{k}'}{}^{-1}R_{\mathbf{k}})e^{-i\mu\varphi},\end{aligned}$$

where we have used Eq. (3.24b) and written

$$\varphi \equiv \varphi(\mathbf{k},R), \quad \varphi' \equiv \varphi(\mathbf{k}',R).$$

At this point we interrupt our discussion for a parenthetical observation. The transformation law (3.38) is sufficient to justify a phase-shift expansion even though the S operator, defined in the usual way, does not commute with **J**. For from

$$S(\mathbf{k}',\mathbf{k}) = \langle \mathbf{k}' \text{ out} | \mathbf{k} \text{ in} \rangle = \langle \mathbf{k}' \text{ out} | S | \mathbf{k} \text{ out} \rangle \quad (3.39)$$

we have

$$|\mathbf{k} \text{ in}\rangle = S|\mathbf{k} \text{ out}\rangle, \quad (3.40)$$

so that, by Eq. (3.21a), **J** cannot commute with S. This paradoxical situation arises because the S operator introduced here differs from the S operator of more familiar problems which connects states of the same helicity λ:

$$|\mathbf{k}\lambda \text{ in}\rangle = S|\mathbf{k}\,\lambda \text{ out}\rangle.$$

The differing behavior of the S operator in the two cases reflects the fact that normally the total angular momentum in scattering states is the sum of the angular momenta of the corresponding free-particle states, whereas in the monopole problem there is also an "interaction angular momentum," analogous to an interaction energy, but which remains finite as the interparticle separation becomes infinite. We return now to the angular dependence of $S(\mathbf{k}',\mathbf{k})$.

To be completely definite, let us make a particular choice of $R_{\mathbf{k}}$. For $\hat{k} = (\theta,\varphi)$, with $\varphi = 0$ for $\theta = 0$, set

$$R_{\mathbf{k}} \equiv R_z(\varphi) R_y(\theta). \quad (3.41)$$

Let us also call the "standard orientation" for scattering angle θ the orientation $\hat{k}' = \hat{z}$ and $\hat{k} = \cos\theta\,\hat{z} + \sin\theta\,\hat{x}$, so that in the standard configuration

$$\mathfrak{D}_{-\mu,\mu}{}^{j}(R_{\mathbf{k}'}{}^{-1}R_{\mathbf{k}}) = \mathfrak{D}_{-\mu,\mu}{}^{j}[R_y(\theta)] = d_{-\mu,\mu}{}^{j}(\theta). \quad (3.42)$$

Any other orientation with scattering angle θ is obtainable from the standard orientation by a rotation $R(\mathbf{k}',\mathbf{k})$ which is unique except when $\mathbf{k}$ and $\mathbf{k}'$ are collinear. Hence $S(\mathbf{k}',\mathbf{k})$ may be found from the S for the standard orientation by Eq. (3.38) and we have

$$S(\mathbf{k}',\mathbf{k}) = e^{-i\mu\varphi(\mathbf{k}',\mathbf{k})} \sum_{j \geq |\mu|} (2j+1) \times \frac{(j-i\alpha')!}{(j+i\alpha')!} d_{-\mu,\mu}{}^{j}(\theta), \quad (3.43)$$

where $\cos\theta = \hat{k}\cdot\hat{k}'$,

$$\varphi(\mathbf{k}',\mathbf{k}) \equiv \varphi(\hat{z},R) + \varphi(\cos\theta\,\hat{z} + \sin\theta\,\hat{x},R),$$

and

$$R \equiv R(\mathbf{k}',\mathbf{k}).$$

Equation (3.43) is familiar if $\mu = 0$, or if[3,4] $\alpha' = \alpha/v = 0$, but the scattering amplitude for arbitrary α and μ, and its behavior under rotation, Eq. (3.38), do not seem to have been stated before.[18]

The sum (3.43) yields for the scattering amplitude

$$f(\mathbf{k}',\mathbf{k}) = S(\mathbf{k}',\mathbf{k})/2ik, \quad \theta \neq 0$$

$$f(\mathbf{k}',\mathbf{k}) = \frac{|\mu| + i\alpha'}{ik(1-\cos\theta)} e^{-i\mu\varphi(k',k)}(-1)^{|\mu|+\mu} \times \exp\left[i\alpha' \ln\left(\frac{1-\cos\theta}{2}\right)\right] \frac{(|\mu| - i\alpha')!}{(|\mu| + i\alpha')!}, \quad (3.44)$$

as may be verified using Rodrigues's formula[19] for the $d_{-\mu,\mu}{}^{j}$. Here $\alpha' = \alpha/v$. The differential cross section follows immediately:

$$\frac{d\sigma}{d\omega} = |f(\mathbf{k}',\mathbf{k})|^2 = \frac{\mu^2 + \alpha'^2}{k^2(1-\cos)^2}, \qquad \frac{d\sigma}{d\omega} = \frac{m^2(\mu^2 v^2 + \alpha^2)}{k^4(1-\cos\theta)^2}, \quad (3.45)$$

and has a Rutherfordian angular dependence. The classical trajectory corresponding to the force (1.6b) is easily obtained when $V = \mu^2(2mr^2)^{-1}$, and yields the same differential cross section. It is the sum of the separate cross sections for purely electric and magnetic interaction. However, the phase of the amplitude (3.44), which shows up in interference phenomena, has a more complicated dependence.

4. CONCLUSION

This section consists of a few observations which are unrelated to each other, but mostly have to do with those features of the model that are of general validity.

(i) The associated Hamiltonian with the higher symmetry was obtained by the *ad hoc* introduction of the potential

$$V(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \frac{\mu^2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}. \quad (4.1)$$

This potential leads to a static classical force, Eq. (1.6b), for which there is no evidence or justification, although it is possible that the classical limit of a theory with superstrong and quantized interactions may contain such an unexpected term. However, if the potential (4.1) is present, then two striking features of the nonrelativistic Coulomb interaction of two particles are preserved when the two particles are allowed to bear both magnetic and electric charges: (1) The same differential cross section is obtained both classically and

[18] If $V = 0$, one sees by inspection of Eq. (2.11) that the energy levels and phase shifts may be obtained from present formulas by $j(j+1) \to j(j+1) - \mu^2$ or $j \to [(j+\frac{1}{2})^2 - \mu^2]^{1/2} - \frac{1}{2}$.

[19] A. E. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princton, N. J., 1957), Eqs. (4.1.18)–(4.1.23) and (4.2.6).

quantum-mechanically, and (2) the Hamiltonian exhibits the higher symmetry of O_4 for bound states and $SL(2,C)$ for the continuum.[20] It is quite unexpected that these seemingly exceptional properties of the Coulomb force can, in fact, be maintained in the presence of velocity-dependent forces. The associated Hamiltonian thus seems to define the natural generalization of the Coulomb problem.

(ii) We have seen, Eq. (3.38), that the scattering matrix transforms under rotation like a helicity-flip amplitude, with helicity changing from μ to $-\mu$, even though only spinless particles are present. This result remains true also in the absence of the higher symmetry. Thus the S matrix and also the scattering states do not transform under rotations like the product of free-particle states, contrary to standard assumptions.[21] In relativistic theories of magnetic monopoles also the S matrix and the scattering states cannot transform like the product of free-particle states, for if they did this property would carry over to the Galilean limit. Thus some traditional concepts must be abandoned for a consistent theory of magnetic monopoles.

It is tempting to conjecture how the relativistic S matrix must transform. In general, the scattering amplitude transforms, in spin space, according to an irreducible representation of the little group of a momentum 4-vector, the little group being the subgroup of the homogeneous Lorentz group which leaves the 4-vector invariant. It is interesting to observe that a pair of 4-vectors also has a little group—the subgroup of the homogeneous Lorentz group which leaves both vectors invariant. Let us find the little group of two linearly independent timelike future 4-vectors, p_1 and p_2; $p_i^2=m_i^2>0$, $E_i\geq m_i>0$. Choose the time axis along p_1+p_2 so that one is in the c.m. frame;

$$p_1=((m_1^2+p^2)^{1/2},\mathbf{p}),\quad p_2=((m_2^2+p_2^2)^{1/2},-\mathbf{p}).$$

If the z axis is chosen along $\mathbf{p}$, one sees that the set of Lorentz transformation which leave both 4-vectors invariant is the set of rotations about the z axis. This is the familiar 1-paramenter (call it φ) Abelian Lie group with irreducible representations

$$d_\mu(\varphi)=\exp i\mu\varphi,\quad 0\leq\varphi<4\pi,\quad \mu=0,\pm\tfrac{1}{2},\pm1,\cdots. \tag{4.2}$$

Corresponding to any Lorentz transformation Λ and a pair of 4-vectors p_1 and p_2, a unique element of the little group $\varphi(p_1,p_2,\Lambda)$ is determined as follows. Associate with each pair of 4-vectors p_1 and p_2 a Lorentz transformation Λ_{p_1,p_2} such that

$$p_1=\Lambda_{p_1,p_2}p_1^0, \tag{4.3a}$$

$$p_2=\Lambda_{p_1,p_2}p_2^0, \tag{4.3b}$$

with

$$p_1^0=((m_1^2+p^2)^{1/2},0,0,p), \tag{4.4a}$$

$$p_2^0=((m_2^2+p^2)^{1/2},0,0,-p), \tag{4.4b}$$

$$p=\left(\frac{(p_1\cdot p_2)^2-m_1^2m_2^2}{(p_1+p_2)^2}\right)^{1/2},$$

and let $p_1'\equiv\Lambda p_1$, $p_2'\equiv\Lambda p_2$. Then $\varphi(p_1,p_2,\Lambda)$ is defined by

$$R_z[\varphi(p_1,p_2,\Lambda)]=\Lambda_{p_1',p_2'}{}^{-1}\Lambda\Lambda_{p_1,p_2}, \tag{4.5}$$

where $R_z(\varphi)$ is a rotation through an angle φ about the z axis. We conjecture that under the Lorentz transformation Λ, the S matrix $S(p_1,p_2)$ (other variables are suppressed) transforms according to

$$S'(p_1',p_2')=\exp[\pm i\mu\varphi(p_1,p_2,\Lambda)]S(p_1,p_2), \tag{4.6}$$

with

$$\mu=e_1g_2-g_1e_2. \tag{4.7}$$

The upper or lower sign holds according as both particles are in the initial or final state. For rotations in the center-of-mass frame Eqs. (3.38) and (4.7) agree. (This discussion neglects the standard infrared difficulty in the definition of scattering states of charged particles.)

We note with relief that the little group of three or more 4-vectors is the identity, so n-body forces ($n>2$) cannot change the transformation law of the S matrix in this way.

(iii) In a previous work[22] it was argued from the appearance of unwanted analytical singularities in the scattering amplitude that magnetic monopoles cannot exist. That work explicitly assumed [assumption (a)], in accordance with standard[21] practice, that the amplitude transforms like the product of free-particle states. This assumption we have just seen to be incorrect because the amplitude for spinless particles transforms like a helicity-flip amplitude. This allows us to understand the origin of the "unwanted" singularities: They are the kinematical singularities of a helicity-flip amplitude. As such, the analytical behavior of (3.44) is quite conventional. Apart from the kinematical singularities, its only other singularities are poles at the bound-state energies, a normal threshold cut from $E=0$ to $E=\infty$, and a cut in $t=-2k^2(1-\cos\theta)$ from $t=0$ to $t=-\infty$. It has, of course, particularities due to infinite-range forces, namely, lack of separation between scattered and unscattered wave (no clustering), and a cut in t starting at $t=0$ instead of an isolated pole there.

(iv) Accelerator experiments to produce magnetic

[20] There is, in fact, a third feature of the Coulomb interaction which is preserved here. If there are three nonrelativistic particles with electric and magnetic charges and the pairwise interaction (4.1), then if two of the particles are held fixed, the Hamiltonian for the third particle is separable in ellipsoidal coordinates.

[21] R. Streater and A. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964), pp. 24–27. These authors do express reservations about massless particles, however.

[22] D. Zwanziger, Phys. Rev. 137, B647 (1965).

monopole pairs[23] have always yielded negative results. However, we should like to emphasize that they do not lead to unambiguous lower limits for the mass of the monopole because of superstrong attractive Coulombic final-state interactions, characterized by the coupling strength $\alpha = 137/4, 137, \cdots$. These could suppress the production amplitude by causing the oppositely charged magnetic particles to rejoin, radiating strongly, and thus annihilate, or bind, into magnetically neutral particles. It may be relevant in this connection that the solutions to the Dirac equation are unstable for $(j+\frac{1}{2}) < \alpha = 137/4, 137, \cdots$.

One of the most striking experimental characteristics of a magnetic monopole is its property of producing constant ionization along its track as it comes to rest.[1,23] This property is lost if the magnetically charged particle also bears an electric charge. Such a possibility is the subject of our concluding remark. It would be realized in a trivial way if an electrically neutral magnetic monopole were bound (by nuclear forces, for example) to an ordinary nucleus.

(v) We may regard the electric and magnetic charges of a particle as a vector $\mathbf{q} = (e,g)$ in a real two-dimensional linear vector space. The electric and magnetic coupling parameters between two particles i and j are given, according to Eqs. (1.8) and (1.9) and after changing to unrationalized natural units, by the inner and cross products of the charge vectors $\mathbf{q}_i$ and $\mathbf{q}_j$:

$$\alpha_{ij} = e_i e_j + g_i g_j = \mathbf{q}_i \cdot \mathbf{q}_j , \tag{4.8}$$

$$\mu_{ij} = e_i g_j - g_i e_j = \mathbf{q}_i \times \mathbf{q}_j . \tag{4.9}$$

The two-dimensional cross product is a pseudoscalar. These forms are, of course, invariant under proper rotations in the two-dimensional space. The dependence of the coupling parameters on the invariant forms (4.8) and (4.9) is the same in the present nonrelativistic model as in a quantum field theory of particles with both electric and magnetic charges, and the consequences are the same.[24] The Dirac quantization condition applies to the cross products (4.9):

$$\mathbf{q}_i \times \mathbf{q}_j = 0, \pm\tfrac{1}{2}, \pm 1, \cdots . \tag{4.10}$$

As shown in Ref. 24, if there exist two charge vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ satisfying

$$\mathbf{q}_1 \times \mathbf{q}_2 = \tfrac{1}{2} , \tag{4.11}$$

then the most general solution to Eqs. (4.10) is

$$\mathbf{q}_i = Z_{i1}\mathbf{q}_1 + Z_{i2}\mathbf{q}_2 , \tag{4.12}$$

where Z_{i1} and Z_{i2} are integers. If one sets $\mathbf{q}_1 = (e_1,0)$, by appropriate choice of axis, then $\mathbf{q}_2 = (e_2, \frac{1}{2}e_1^{-1})$, by Eq. (4.11). Here, presumably $e_1 = (137)^{-1/2}$, and e_2, the second elementary quantum of electric charge, is the electric charge of the Dirac monopole. Because one may replace $\mathbf{q}_2$ by $\mathbf{q}_2 + Z\mathbf{q}_1$ in (4.11), e_2 is defined modulo e_1.

Note added in proof. Some readers find it difficult to understand why the conserved angular momentum **J** does not commute with the S operator, defined by Eq. (3.40), as may be verified by multiplying Eq. (3.40) by $\mathbf{J}\cdot\hat{k}$ and using Eq. (3.21a). A conserved quantity, by definition, is one which commutes with the total Hamiltonian H. In a scattering theory which allows the decomposition $H = H_0 + V$, a conserved quantity generally does not commute with S unless it also commutes with H_0 and V, separately. In the present case there is no such decomposition of H, but **J** depends explicitly on the coupling parameter $\mu = eg$ describing the strength of the interaction and therefore it should not be expected to commute with S.

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[23] E. Amaldi *et al.*, CERN Report No. 63-13, 1963 (unpublished). This work gives a general survey and bibliography of the subject.

[24] D. Zwanziger, following paper, Phys. Rev. **176**, 1489 (1968).