Emission of Radiation from a System of Many Excited Atoms*

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The Weisskopf-Wigner theory of the natural linewidth of a single isolated atom is extended to a system of $N(\gg 1)$ identical nonoverlapping atoms which are all in the same excited state at time $t=0$. The positions $\mathbf{X}_1 \cdots \mathbf{X}_N$ of the atoms are assumed to fill a volume $\mathcal{V}$ of given shape and size with macroscopically constant density. Emission of radiation from this system takes place only in the form of one narrow, but nonzero-width, bundle of nearly equal photons, which contains all the emitted radiation. If the density of atoms within $\mathcal{V}$ exceeds a certain threshold, the rate of emission of photons has the form of a typical spike. All effects depend sensitively on the shape and size of $\mathcal{V}$ and on the density of atoms within $\mathcal{V}$, and cannot be explained in conventional terms of spontaneous or stimulated emission of radiation.

I. INTRODUCTION AND DISCUSSION OF RESULTS

THE purpose of the present paper is to give a completely quantum-mechanical description of the emission of radiation by an extended system of many atoms. The emission of radiation by a single excited atom is one of the classic problems of quantum electrodynamics, the solution to which is given by Weisskopf and Wigner.[1] Our work is an extension of this method to a system containing many atoms. The system consists of N identical, nonoverlapping two-level atoms located at fixed points $\mathbf{X}_1, \cdots, \mathbf{X}_N$. These points are distributed with uniform macroscopic density over a volume $\mathcal{V}$ of arbitrary shape, but are otherwise uncorrelated. All the atoms of the system interact with the same electromagnetic field. The problem analogous to that treated by Weisskopf and Wigner is to find the time development of the state of this system from an initial state in which all the atoms are excited and no photons are present. The problem involves the correlation between the many photons emitted as well as their rate of emission. The former aspect is absent from the single-atom problem.

Systems of many atoms interacting through a common radiation field have been studied before. Dicke[2] recognized the analogy between a system of two-level atoms and a system of spins, and used it to describe a many-atom system with the techniques developed for magnetic resonance. In this way, he was led to the concept of superradiance and the coherent radiation of an extended system. Schwabl and Thirring[3] discussed a system containing effectively infinitely many atoms, replacing the atoms by a field coupled to the radiation field in such a way as to lead to a soluble problem. Many aspects of their results are mirrored in ours. On a much smaller scale, Stephen[4] investigated two atoms interacting with each other via their common radiation field when one atom is initially excited and the other not. All of these studies use all the degrees of freedom of the radiation field and are not restricted to one or a few modes of that field.

Another line of investigation is inspired by the laser. Most treatments of lasers are based on a single-particle model in which the acts of emission and absorption of radiation by individual atoms are identified, and only one or a few modes of the radiation field are coupled to the atoms. This latter reflects the presence of an optical cavity in a laser. The simplicity gained by these approximations makes possible the introduction of realistic complications occurring in actual lasers, such as pumping mechanisms, nonradiative energy loss mechanisms, Doppler broadening, etc. The theories of Lamb,[5] Scully and Lamb,[6] Lax,[7] and Willis[8] are of this kind. These theories, which are very successful in describing experimental results, are necessarily of a somewhat phenomenological nature. It is one purpose of this paper to see why these models work so well.

The results obtained here are of two kinds, involving the spatial and the temporal behavior of the system. Starting from the initial state with all N atoms excited and no photons present, we find the following general results.

(A) All photons are emitted in a narrow bundle or ray. The direction of emission of the ray depends on the shape of $\mathcal{V}$, being along the axis of a rod, for example. The spread of the bundle is the spread of a plane wave diffracted by an aperture in a screen when the aperture has the form of a section of $\mathcal{V}$ perpendicular to the direction of the ray. For a circular rod, then, the ray has a distribution in angle given by the diffraction pattern for light of the same wavelength passed through

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a circular aperture of diameter equal to that of the rod. The angular correlation within the ray is sharper than the distribution in angle of successive rays. This focusing occurs without the presence of any mirrors or other optical devices.

(B) The rate at which the photons are emitted depends primarily on the density of the atoms. If the N atoms radiated independently in time, the one-atom exponential decay law would also apply to the many-atom system. This is not what is found. Initially the rate of emission is higher than that for independent atoms. Below a critical density the emission rate decreases monotonically with the time, while above this density the emission rate at first increases with the time, reaches a maximum, and then declines rapidly. This maximum can be very high and occurs at a time less than the single-atom mean lifetime. It is physically meaningful to speak of such short time intervals, because the entire system is in a single quantum state. The uncertainty principle relates the uncertainty in the energy of the entire system to the uncertainty in the time, while the single-atom mean lifetime is related to the width of the line emitted by a single atom. Thus for our problem the uncertainty principle requires only that

$$N\delta k\delta t\gtrsim 1$$

if δk is the resolution in energy contemplated in any measurement, or

$$\delta k\delta t\gtrsim 1/N.$$

For large N this puts no significant restriction on the times that may be considered, and therefore the initial condition "all atoms excited and no photons present" is less problematical for large N than for a single atom.

An interesting problem which is not discussed here is the coherence properties of the ray. The photocounting correlations to be obtained in the ray should, in principle, be calculable. The final state contains N photons, but there is a continuum of modes over which they are distributed, so that the state is not at all like that containing N photons in a single mode.

In Sec. II the Weisskopf-Wigner approximation is described and generalized to an N-atom system. Section III is devoted to a system of independently decaying atoms which provides the reference to which the results of later sections are compared. In Sec. IV the two-atom problem is formulated and solved, because here many features of the general system appear in simple form. Section V contains the approximate solution of the N-atom problem. Section VI contains a discussion of the properties of the functions introduced in Sec. V, and Sec. VII is devoted to a discussion of the ray. The time evolution is treated in Sec. VIII, and the role of stimulated emission occupies Sec. IX.

II. WEISSKOPF-WIGNER APPROXIMATION AND ITS APPLICATION TO THE MODEL

As stated in the Introduction, the model consists of a radiation field $A_\mu(\mathbf{x},t)$, subject to box normalization with box volume V and periodic boundary conditions, which interacts with N identical, nonoverlapping atoms with the centers of gravity at the fixed points $\mathbf{X}_1\cdots\mathbf{X}_N$. For convenience, the atom at position $\mathbf{X}_j$ is called A_j. Nonoverlapping of the atoms means that the distance $|\mathbf{X}_i-\mathbf{X}_j|$ between each pair of atoms A_iA_j is so large that the electronic wave functions of A_i and A_j do not overlap and the possible eigenstates and eigenvalues of A_i are not affected by the presence of other atoms A_j, $j\neq i$. Identity of the atoms means that the possible eigenvalues E_n of A_i are also the eigenvalues of A_j, and the possible eigenstates of A_i are identical with those of A_j up to a translation: If $u_n(\mathbf{x}-\mathbf{X}_i)$ is an eigenstate of A_i, then $u_n(\mathbf{x}-\mathbf{X}_j)$ is an eigenstate of A_j, specified by the same quantum numbers E_n. Interaction between the atoms is thus possible only by means of the common radiation field $A_\mu(\mathbf{x},t)$, i.e., by emission and absorption of photons.

An atom A_j will be described in the usual way (Källén,[9] Schweber[10]) by a field $\psi^{(j)}(\mathbf{x},t)$ from which the bilinear atomic current $J_\mu^{(j)}(\mathbf{x},t)$ is constructed, which is independent of the radiation field $A_\mu(\mathbf{x},t)$. The field corresponding to all atoms is the sum of all $\psi^{(j)}(\mathbf{x},t)$ and because of the condition of nonoverlapping of the atoms, the current $J_\mu(\mathbf{x},t)$ corresponding to the whole atomic field is just the sum over all $J_\mu^{(j)}(\mathbf{x},t)$. The interaction Hamiltonian of the system is given by the usual $A_\mu J_\mu$ coupling term

$$H(t)=\int d^3x\, A_\mu(\mathbf{x},t)J_\mu(\mathbf{x},t)$$

$$=\sum_{j=1}^{N}\int d^3x\, A_\mu(\mathbf{x},t)J_\mu^{(j)}(\mathbf{x},t). \tag{2.1}$$

In the Furry interaction picture, the time-development operator $U(t)$ of the system satisfies the equation

$$i\frac{d}{dt}U(t)=H(t)U(t) \tag{2.2}$$

with the initial condition

$$U(0)=1. \tag{2.3}$$

The operator $U(t)$ determines the pure or mixed state of the system for any time $t>0$, if the state of the system is known at $t=0$, regardless of how this state was reached or prepared in the past $t<0$. Condition (2.3) is thus the mathematical description of the principle of "switching on the interaction at $t=0$." A detailed evaluation of (2.3) is given in Sec. VIII.

Let us assume now that a certain pure state $|i\rangle$ of the system is obtained at $t=0$ as a result of some preparation (pumping, etc.) of the system at $t<0$, and that

[9] G. Källén, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1958), Vol. V/1, p. 274.

[10] S. S. Schweber, *An Introduction to Relativistic Field Theory* (Row Peterson and Co., Elmsford, N. Y., 1961), p. 464.

$|i\rangle$ can be described in terms of photons present and the states $u_n(\mathbf{x}-\mathbf{X}_j)$ of the atoms. Taking matrix elements in (2.2), we get

$$i\frac{d}{dt}\langle a|U(t)|i\rangle=\sum_{|z\rangle}\langle a|H(t)|z\rangle\langle z|U(t)|i\rangle, \quad (2.4)$$

where the summation runs over a complete set of intermediate states $|z\rangle$ and $|a\rangle$ is any one of these states. The lowest-order Weisskopf-Wigner approximation consists of the reduction of the intermediate states $|z\rangle$ to an incomplete set of "essential" states only, to be determined later, and an application of the principle that the creation of one photon is always accompanied by the de-excitation of one atom, and the destruction of one photon is always accompanied by the excitation of one atom. We have thus $\langle z|U(t)|i\rangle=0$ and hence $\langle a|U(t)|i\rangle=0$ for all nonessential states $|z\rangle$ or $|a\rangle$. This reduction means also a reduction of the Hilbert space of the original problem to the Hilbert space spanned by the essential states $|z\rangle$ only and the time-development operator $U(t)$ operates in this essential space only. In this space $U(t)$ is strictly unitary.

For the purpose of laser physics it will usually be sufficient to assume that only a few essential atomic states (levels) make up the essential states of the system. For such a situation we can associate to each essential state $|z\rangle$ a number $\mathfrak{N}(|z\rangle)$ composed of the number N_{exc} of possible atomic transitions from the levels in $|z\rangle$ to lower levels, and the number N_{phot} of photons present in $|z\rangle$. The intermediate states in the lowest-order Weisskopf-Wigner approximation are then those states $|z\rangle$, which obey the rule

$$\mathfrak{N}(|z\rangle)=\mathfrak{N}(|i\rangle), \quad \mathfrak{N}(|z\rangle)=(N_{\text{exc}}+N_{\text{phot}}) \text{ in } |z\rangle. \quad (2.5)$$

Higher-order approximations can be obtained by taking into account more intermediate states, for example, the classes of states $|z\rangle$ determined by $\mathfrak{N}(|z\rangle)=\mathfrak{N}(|i\rangle)\pm1$, $\mathfrak{N}(|i\rangle)\pm2$, $\cdots$, until a complete set of states $|z\rangle$ is reached with regard to the essential levels, which corresponds to the exact solution of the problem under the condition that the essential levels are considered complete. If the radiation field has only one mode of oscillation and N two-level atoms are considered which are all in the upper level at $t=0$, then $\mathfrak{N}(|z\rangle)=N$ is a conserved quantity[11–13] and the lowest-order Weisskopf-Wigner approximation is no longer an approximation at all. What is neglected in the lowest-order Weisskopf-Wigner approximation $\mathfrak{N}(|z\rangle)=\mathfrak{N}(|i\rangle)\pm0$ are thus the fluctuations of the quantity $\mathfrak{N}(|z\rangle)$ as defined by (2.5), account of which could be taken in the higher-order approximations $\mathfrak{N}(|z\rangle)=\mathfrak{N}(|i\rangle)\pm1$, $\mathfrak{N}(|i\rangle)\pm2$, $\cdots$, only.

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[12] E. Abate and H. Haken, Z. Naturforsch. **19a**, 857 (1964).
[13] V. F. Chelt'sov, Zh. Eksperim. i Teor. Fiz. **48**, 1139 (1965) [English transl.: Soviet Phys.—JETP **21**, 761 (1965)].

For the present model, we consider essential only two atomic levels corresponding to an excited state $u_e(\mathbf{x}-\mathbf{X}_j)$ and a lower, relatively stable, ground state $u_g(\mathbf{x}-\mathbf{X}_j)$, $j=1, 2, \cdots, N$. The initial state $|i\rangle$ is the state with all atoms in $u_e(\cdots)$ and no photons present. We have thus $N_{\text{phot}}=0$, $N_{\text{exc}}=N$ in $|i\rangle$ and hence

$$\mathfrak{N}(|i\rangle)=N, \quad (2.6)$$

and we shall use the lowest-order Weisskopf-Wigner approximation $\mathfrak{N}(|z\rangle)=\mathfrak{N}(|i\rangle)\pm0$ only. As a consequence of this, the system will be in a state of N photons at $t=\infty$, when all atoms are in their ground level. So, we get automatically a pure N-photon state for the emitted radiation at $t=\infty$, and this is in agreement with the results of Schwabl and Thirring for the model under consideration. From this agreement with these quite independent calculations we can claim some merits for the lowest-order Weisskopf-Wigner approximation. However, physical intuition says also, that the fluctuations of $\mathfrak{N}(|z\rangle)$ should be small in comparison with N, and that the step-by-step introduction $[\mathfrak{N}(|i\rangle)\pm1, \mathfrak{N}(|i\rangle)\pm2, \cdots]$ of such fluctuations should yield a converging approximation scheme. The strongest argument for this hope is the fact that the time-development operator $U(t)$ is kept strictly unitary in any order of approximation, so that all transition probabilities remain properly normalized and, of course, finite.

Formally, as in the discussion of Källén[9] of the theory of Weisskopf and Wigner, only one fixed mode of photon polarization will be considered. However, if that mode is chosen properly, i.e., in accordance with the nature of the atomic transition under consideration and in proper dependence on the wave vector $\mathbf{k}$, no approximation is implied in this procedure and a photon will be specified completely by $\mathbf{k}$ alone. We are not interested here in polarization phenomena and shall not discuss more details concerning polarization.

The special case $N=1$ gives the original Weisskopf-Wigner theory of the natural linewidth. Denoting the state of the atom by e or g only, the absence of any photon by 0, and the presence of a photon of mode $\mathbf{k}$ by $\mathbf{k}$, we have $|i\rangle=|e; 0\rangle$ and the intermediate states are given by

$$|e; 0\rangle, \quad |g; \mathbf{k}\rangle. \quad (2.7)$$

In the case $N=2$, we have $|i\rangle=|e_1,e_2; 0\rangle$ and the intermediate states are

$$|e_1e_2; 0\rangle, \quad |e_1g_2; \mathbf{k}\rangle, \quad |g_1e_2; \mathbf{k}\rangle, \quad |g_1g_2; \mathbf{k}_1\mathbf{k}_2\rangle, \quad (2.8)$$

where the subscripts on e and g refer to the indices of the atoms A_1A_2, and $\mathbf{k}_1\mathbf{k}_2$ indicates the presence of a pair of a photons of modes $\mathbf{k}_1\mathbf{k}_2$, which need not necessarily be different from each other.

For general N, it will be convenient to denote a certain state $|z\rangle$ by the indices $j_1\cdots j_M$ of the atoms $A_{j_1}\cdots A_{j_M}$ which are in the ground level in that state, and by the wave vectors $\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_M$ of the photons in $|z\rangle$. The essential states $|z\rangle$ for the lowest-order Weisskopf-

Wigner approximation for the present system are then given by

$$|j_1\cdots j_M;\mathbf{k}_1\cdots\mathbf{k}_M\rangle,\quad M=0,1,\cdots,N \tag{2.9}$$

where $j_1\cdots j_M$ is any set of M different numbers out of $1, 2, \cdots, N$. The case $M=0$, for which we shall use the notation $|0;0\rangle$, corresponds to the initial state $|i\rangle$ of the system, which would be denoted by $|e_1\cdots e_N;0\rangle$ in the notation of (2.8). $M=N$ corresponds to the state $|g_1\cdots g_N;\mathbf{k}_1\cdots\mathbf{k}_N\rangle$ in the notation of (2.8) and indicates the presence of N photons of modes $\mathbf{k}_1\cdots\mathbf{k}_N$. It is understood that the values of $\mathbf{k}_1\cdots\mathbf{k}_N$ need not be different from each other. If a certain value $\mathbf{k}^{(i)}$ occurs n_i times among $\mathbf{k}_1\cdots\mathbf{k}_N$, the presence of n_i photons of that mode $\mathbf{k}^{(i)}$ is understood.

If we now write down the Eqs. (2.4) with $|a\rangle$ running over the set of states (2.9), we obtain a coupled system of as many linear equations as there are unknown transition amplitudes $\langle a|U(t)|i\rangle$. The establishment of that system of equations and its solution is the problem that remains.

Though straightforward, the act of writing down the equations corresponding to (2.4) contains some interesting points with regard to the role of the "Einstein factors" $(n_i+1)^{1/2}$ about which some remarks should be included here. Let us introduce the notation

$$\alpha_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}(t)=\langle j_1\cdots j_M;\mathbf{k}_1\cdots\mathbf{k}_M|U(t)|0;0\rangle,\quad M=0,1,\cdots,N. \tag{2.10}$$

Taking into account the fact that the Hamiltonian (2.1) has nonvanishing elements $\langle a|H(t)|z\rangle$ only if the number of photons in $|a\rangle$ and $|z\rangle$ differ by ±1, we get for (2.4)

$$i\frac{d}{dt}\alpha_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}(t)=\sum_{(\mathbf{k}')}\sum_{(j')}\langle j_1\cdots j_M;\mathbf{k}_1\cdots\mathbf{k}_M|H(t)|j_1'\cdots j_{M+1}';\mathbf{k}_1'\cdots\mathbf{k}_{M+1}'\rangle\alpha_{\mathbf{k}_1'\cdots\mathbf{k}_{M+1}'}{}^{j_1'\cdots j_{M+1}'}(t)$$
$$+\sum_{(\mathbf{k}')}\sum_{(j')}\langle j_1\cdots j_M;\mathbf{k}_1\cdots\mathbf{k}_M|H(t)|j_1'\cdots j_{M-1}';\mathbf{k}_1'\cdots\mathbf{k}_{M-1}'\rangle\alpha_{\mathbf{k}_1'\cdots\mathbf{k}_{M-1}'}{}^{j_1'\cdots j_{M-1}'}(t),\quad M=0,1,\cdots,N. \tag{2.11}$$

The intermediate states $|j_1'\cdots j_{M\pm1}';\mathbf{k}_1'\cdots\mathbf{k}_{M\pm1}'\rangle$ are chosen in accordance with the Weisskopf-Wigner approximation, as specified by (2.5) or (2.9). The summations in (2.11) have to be carried out in such a way that each different intermediate state appears only once and is counted only once. Hence, Eq. (2.11) must be written down for all possible sets $j_1\cdots j_M$ of M different numbers out of $1, 2, \cdots, N$. We get therefore

$$\binom{N}{M}$$

different α functions of $\mathbf{k}_1\cdots\mathbf{k}_M$ for a fixed value of M, and hence a total number of

$$\sum_{M=0}^{N}\binom{N}{M}=(1+1)^N=2^N \tag{2.12}$$

equations for the same number of different α amplitudes.

Let us transform now the above specifications for the summation over $\mathbf{k}_1'\cdots\mathbf{k}_{M\pm1}'$ into conventional multiple sums over $\mathbf{k}_1'\cdots\mathbf{k}_{M\pm1}'$. Since an M-fold sum $\sum_{\mathbf{k}_1'}\cdots\sum_{\mathbf{k}_M'}$ counts each fixed state $|\mathbf{k}_1'\cdots\mathbf{k}_M'\rangle$ P_M times, where P_M is number of different permutations of the ordered elements $\mathbf{k}_1'\cdots\mathbf{k}_M'$, we have to compensate for this multiple counting by multiplying each state $|\mathbf{k}_1'\cdots\mathbf{k}_M'\rangle$ by a factor $P_M{}^{-1}$. After this, the sum can be carried out in the conventional sense of a multiple sum. Defining the "Einstein function" $\epsilon(\mathbf{k}_1\cdots\mathbf{k}_M)$ by

$$\epsilon(\mathbf{k}_1\cdots\mathbf{k}_M)=[\prod_i(n_i!)]^{1/2}, \tag{2.13}$$

where n_i is the number of times a special value $\mathbf{k}^{(i)}$ occurs among $\mathbf{k}_1\cdots\mathbf{k}_M$, we get for the P_M of the set $\mathbf{k}_1\cdots\mathbf{k}_M$ the formula

$$P_M=M!\,\epsilon^{-2}(\mathbf{k}_1\cdots\mathbf{k}_M). \tag{2.14}$$

The results of these considerations can be put together in the symbolic equation

$$\sum_{(\mathbf{k}')}=\frac{1}{(M\pm1)!}\sum_{\mathbf{k}_1'}\cdots\sum_{\mathbf{k}_{M\pm1}'}\epsilon^2(\mathbf{k}_1'\cdots\mathbf{k}_{M\pm1}'), \tag{2.15}$$

which specifies uniquely the summations over the $\mathbf{k}'$'s in Eq. (2.11).

After these specifications of the summations in (2.11), we have to evaluate the appropriate elements of the Hamiltonian $H(t)$ in order to carry out as many of these summations as possible. This can be done in a straightforward manner by representing the operators and states in the elements $\langle j_1\cdots j_M;\mathbf{k}_1\cdots\mathbf{k}_M|H(t)|j_1'\cdots j_{M\pm1}';\mathbf{k}_1'\cdots\mathbf{k}_{M\pm1}'\rangle$ by creation and annihilation operators and using the usual commutation relations. Since a normalized state of M photons of modes $\mathbf{k}_1\cdots\mathbf{k}_M$ is given in terms of creation operators $a_{\mathbf{k}}{}^\dagger$ by

$$|\mathbf{k}_1\cdots\mathbf{k}_M\rangle=\epsilon^{-1}(\mathbf{k}_1\cdots\mathbf{k}_M)a_{\mathbf{k}_1}{}^\dagger\cdots a_{\mathbf{k}_M}{}^\dagger|0\rangle, \tag{2.16}$$

two additional Einstein functions

$$\epsilon^{-1}(\mathbf{k}_1\cdots\mathbf{k}_M)\epsilon^{-1}(\mathbf{k}_1'\cdots\mathbf{k}_{M\pm1}')$$

will appear from the elements of the Hamiltonian on the right-hand side of Eq. (2.11). The details of further straightforward calculations are not relevant to the purpose of this paper and will be skipped. We shall state the results only. It turns out that the elements of

$H(t)$ can be expressed in terms of the elements of the partial Hamiltonian $H^{(j)}(t)$ of the single atom A_j at position $\mathbf{X}_j$, i.e., by

$$\langle g_j;\mathbf{k}|H^{(j)}(t)|e_j;0\rangle = C_{\mathbf{k}}{}^{(j)}[e^{-i(\Delta-k)t}/(2Vk)^{1/2}], \quad (2.17)$$

where Δ is the energy difference of the atomic levels, $k=|\mathbf{k}|$, V is the normalization volume of the radiation field $A_\mu(\mathbf{x},t)$, and $C_{\mathbf{k}}{}^{(j)}$ is given by

$$C_{\mathbf{k}}{}^{(j)} = -ie\int d^3x\, \bar{u}_e(\mathbf{x}-\mathbf{X}_j)\gamma_\mu e_\mu u_g(\mathbf{x}-\mathbf{X}_j)e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad (2.18)$$

where e here denotes the electromagnetic coupling constant, $u_{e,g}(\mathbf{x}-\mathbf{X}_j)$ denotes the wave functions of the excited and the ground state of A_j, respectively, γ_μ is the usual Dirac vector, and e_μ denotes the mode of polarization under consideration (cf. Källén[9]). For our theory, it is of vital importance that the dependence of $C_{\mathbf{k}}{}^{(j)}$ on the position $\mathbf{X}_j$ of A_j is included and can be factored out as

$$C_{\mathbf{k}}{}^{(j)} = e^{-i\mathbf{X}_j\cdot\mathbf{k}}C_{\mathbf{k}}, \quad (2.19)$$

where $C_{\mathbf{k}}$ is defined by putting $\mathbf{X}_j=0$ in (2.18). This is of course due to our assumption that all atoms are identical and have thus identical states $u_e(\cdots)$, $u_g(\cdots)$, and that plane waves have been used in the expansion of $A_\mu(\mathbf{x},t)$, but it cannot be ascribed to the special form of $C_{\mathbf{k}}{}^{(j)}$, as expressed in (2.18). Indeed, a factorization (2.19) must be possible for any type of atomic transition, since only (2.19) guarantees that the Weisskopf-Wigner theory ($N=1$) is independent of the position $\mathbf{X}_1$ of the decaying atom, as will be shown later. Since the presence of other atoms A_i, $i\neq 1$, has no influence on such a pure one-atom property, (2.19) must be true in any case. However, if the system is made up of more than one atom, only one phase factor is arbitrary, because of translation invariance, and the other $N-1$ phase factors contribute to the behavior of the system. Translation invariance requires only that observable facts depend on the distances $\mathbf{X}_i-\mathbf{X}_j$, and that will be the case in the present model.

In the course of the calculations, nearly all summations in (2.15) and (2.11) can be carried out by means of Kronecker δ functions, the factors $1/(M\pm1)!$ compensate for the occurrence of $(M\pm1)!$ equal terms, the additional Einstein functions from (2.16) combine with those in (2.15), and what comes out in the end is

$$i\frac{d}{dt}\alpha_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}(t) = \epsilon^{-1}(\mathbf{k}_1\cdots\mathbf{k}_M)\left\{\sum_{\nu=1}^{N-M}\sum_{\mathbf{k}} C_{\mathbf{k}}{}^{(\bar{j}_\nu)*}\frac{e^{i(\Delta-k)t}}{(2Vk)^{1/2}}\epsilon(\mathbf{k}_1\cdots\mathbf{k}_M,\mathbf{k})\alpha_{\mathbf{k}_1\cdots\mathbf{k}_M,\mathbf{k}}{}^{j_1\cdots j_M,\bar{j}_\nu}(t)\right.$$
$$\left.+\sum_{\nu=1}^{M}\sum_{l=1}^{M} C_{\mathbf{k}_l}{}^{(j_\nu)}\frac{e^{-i(\Delta-k)t}}{(2Vk_l)^{1/2}}\epsilon(\mathbf{k}_1\cdots\mathbf{k}_{l-1},\mathbf{k}_{l+1}\cdots\mathbf{k}_M)\alpha_{\mathbf{k}_1\cdots\mathbf{k}_{l-1},\mathbf{k}_{l+1}\cdots\mathbf{k}_M}{}^{j_1\cdots j_{\nu-1},j_{\nu+1}\cdots j_M}(t)\right\}, \quad M=0,1,\cdots,N. \quad (2.20)$$

The summation over (j_ν) runs over the numbers $j_1\cdots j_M$ as specified on the left-hand side of Eq. (2.20), while the summation over $\bar{j}_\nu$ goes over the set $(1,2,\cdots,N$ minus $j_1\cdots j_M)$, i.e., over the excited atoms in the corresponding state. In the special cases $M=N$ $(M=0)$, the first (second) sum of the right-hand side vanishes. The initial values corresponding to (2.3) are

$$\alpha_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}(0) = \delta_{M,0}, \quad (2.21)$$

with $\delta_{M,0}$ denoting the Kronecker δ function.

In (2.20), the Einstein functions $\epsilon^{-1}(\mathbf{k}_1\cdots\mathbf{k}_M)\times\epsilon(\mathbf{k}_1\cdots\mathbf{k}_M,\mathbf{k})$ combine to the usual "Einstein factors" $(n_i+1)^{1/2}$ as $\mathbf{k}$ runs over the values specified on the left-hand side of (2.20). Similar factors appear also in the second term of the right-hand side of (2.20). The above factorization of these factors is very convenient and gives valuable insight into their origin. We note that their appearance has nothing to do with the Weisskopf-Wigner approximation. Indeed, Eq. (2.11) would be exact if the summation over (j') were not restricted to the intermediate states (2.5) of the Weisskopf-Wigner approximation. Equation (2.15) would still hold for the case of an unlimited summation over system states and the appropriate Einstein functions from (2.16) would also appear. It follows from this, that the ϵ's and α's will always appear in the form of the product

$$\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_M}\cdots(t) = \epsilon(\mathbf{k}_1\cdots\mathbf{k}_M)\alpha_{\mathbf{k}_1\cdots\mathbf{k}_M}\cdots(t) \quad (2.22)$$

as they do in our Eqs. (2.20). Upon inserting (2.22) into (2.20), we obtain a set of equations for the $\tilde{\alpha}$'s which does not contain the "discontinuous" Einstein functions ϵ. $\tilde{\alpha}$ will thus be a "smooth" function of its arguments $\mathbf{k}_1\cdots\mathbf{k}_M$.

Since $|\alpha_{\mathbf{k}_1\cdots\mathbf{k}_m}{}^{j_1\cdots j_M}(t)|^2$ is the probability for the presence, at time t, of M photons of given modes $\mathbf{k}_1\cdots\mathbf{k}_M$ under the condition that the atoms $A_{j_1}\cdots A_{j_M}$ are de-excited in that moment, only $\alpha_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2,\cdots,N}(t)$ should have a nonvanishing limit for $t\to\infty$. The unitarity of $U(t)$ requires that the appropriate probabilities $|\alpha_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2,\cdots,N}(\infty)|^2$ be normalized to unity. Using similar arguments as those leading to (2.15), we find the normalization of these probabilities to be given by the expressions

$$1=\frac{1}{N!}\sum_{\mathbf{k}_1}\cdots\sum_{\mathbf{k}_N}\epsilon^2(\mathbf{k}_1\cdots\mathbf{k}_N)|\alpha_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2,\cdots,N}(\infty)|^2$$
$$=\frac{1}{N!}\sum_{\mathbf{k}_1}\cdots\sum_{\mathbf{k}_N}|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2,\cdots,N}(\infty)|^2. \quad (2.23)$$

The normalization is thus accomplished by the smooth functions $\tilde{\alpha}$. We shall use (2.23) to check and improve the approximations used in the solution of (2.20).

At $t=\infty$, the state of the system is given in the form of a product of the state of the atoms, $|g_1\cdots g_N\rangle$, and a state of photons. Since the atomic state is known, we have to look only on the latter to obtain all possible information about the radiation field at $t=\infty$. Defining a photon state $|f\rangle$ by

$$|f\rangle=\sum_{(\mathbf{k})}\alpha_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)\,|\mathbf{k}_1\cdots\mathbf{k}_N\rangle, \tag{2.24}$$

we see, by reinterpreting Eq. (2.23) as the normalization condition for $|f\rangle$, that $|f\rangle$ is normalized to unity. But (2.23) can also be interpreted as

$$|\langle i|U(\infty)|g_1\cdots g_N;\,f\rangle|^2=1, \tag{2.25}$$

which means that $|f\rangle$ is the state of the radiation field at $t=\infty$. Therefore, $|f\rangle$ contains all possible information, including all coherence properties, of the emitted radiation. For example, if we are interested in the directional distribution of the emitted photons, we can calculate the expectation value $n_{\mathbf{k}}$ of the operator $a_{\mathbf{k}}{}^{\dagger}a_{\mathbf{k}}$ in the state $|f\rangle$. By direct calculation, we find

$$\begin{aligned}n_{\mathbf{k}}&=\langle f|a_{\mathbf{k}}{}^{\dagger}a_{\mathbf{k}}|f\rangle\\&=\frac{N}{N!}\sum_{\mathbf{k}_1}\cdots\sum_{\mathbf{k}_{N-1}}|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_{N-1},\mathbf{k}}{}^{1,2,\cdots,N}(\infty)^2|\,.\end{aligned} \tag{2.26}$$

Higher moments (cf. Jordan[14] and Mukunda and Jordan[15]) of $n^{\mathbf{k}}$ reveal the coherence state of $|f\rangle$, but we shall not discuss them here, since, as we shall see, Eq. (2.26) is rather impractical for the essential results contained in $|f\rangle$.

III. RADIATION FROM N INDEPENDENTLY DECAYING ATOMS AND THE ROLE OF THE FACTORS $(n_i+1)^{1/2}$ IN OUR MODEL

To prepare for the solution of (2.20) and its proper interpretation we collect in this chapter some loosely connected considerations and introduce some useful notations.

We shall consider only such a large quantization volume V for the radiation field $A_\mu(\mathbf{x},t)$, that the usual transition

$$\frac{1}{V}\sum_{\mathbf{k}}\rightarrow\frac{1}{(2\pi)^3}\int d^3k \tag{3.1}$$

is justified. This means, in the first instance, that the radiation field of free space is considered, but if the integral (3.1) is understood as an approximation for the sum (3.1), our results will be valid as long as such an approximation is valid.

For later use, we introduce the function

$$C_k(\mathbf{x})=\frac{k}{2(2\pi)^3}\int d\Omega\,|C_{\mathbf{k}}|^2e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3.2}$$

where $C_{\mathbf{k}}$ is given in (2.18) and (2.19), and the integration goes over the angles in the $\mathbf{k}$ space. We shall also use the function

$$\Gamma(\mathbf{x})=\pi C_k(\mathbf{x})\,|_{k=\Delta}. \tag{3.3}$$

Since $C_{\mathbf{k}}=C_{-\mathbf{k}}$ holds, we get $\Gamma(-\mathbf{x})=\Gamma(\mathbf{x})$. The matrix

$$\Gamma_{ij}=\Gamma(\mathbf{X}_i-\mathbf{X}_j) \tag{3.4}$$

is therefore symmetric and has equal diagonal elements

$$\Gamma_0=\Gamma_{ii}=\Gamma(0). \tag{3.5}$$

$2\Gamma_0$ is the natural linewidth of Weisskopf and Wigner.

The matrix Γ_{ij} plays a vital role in the further analysis of this problem. For a dipole radiation process $|C_{\mathbf{k}}|^2$ is proportional to $\sin^2\theta$, where θ is the angle between $\mathbf{k}$ and the dipole axis. Then, with ϑ_0 denoting the angle between $\mathbf{x}$ and the dipole axis, we get

$$\begin{aligned}\Gamma(\mathbf{x})=3\Gamma(0)\{&\cos^2\vartheta_0[\sin\Delta|\mathbf{x}|-\Delta|\mathbf{x}|\cos\Delta|\mathbf{x}|]/(\Delta|\mathbf{x}|)^3\\&+\tfrac{1}{2}\sin^2\vartheta_0[\sin\Delta|\mathbf{x}|/\Delta|\mathbf{x}|\\&-(\sin\Delta|\mathbf{x}|-\Delta|\mathbf{x}|\cos\Delta|\mathbf{x}|)/(\Delta|\mathbf{x}|)^3]\}.\end{aligned} \tag{3.6}$$

It follows from this that Γ_{ij} will approach the limit

$$\Gamma_{ij}\rightarrow\Gamma_0\delta_{ij} \tag{3.7}$$

if the distances $|\mathbf{X}_i-\mathbf{X}_j|$ are large in comparison with the half wavelength $\frac{1}{2}\lambda$, $\lambda=2\pi/\Delta$, of the emitted photons, and

$$\Gamma_{ij}\rightarrow\Gamma_0 \tag{3.8}$$

if all these distances are small in comparison with $\frac{1}{2}\lambda$.

The former case corresponds to the low-density limit of the active substance, while the latter could be described as a "point laser." A real laser will be somewhere between these two limits, but much closer to (3.7) than to (3.8). However, this does not mean that we can go to the limit (3.7) without caution, as follows from the second remark on the properties of Γ_{ij} below:

The invariance of our problem against a renaming of the atoms requires that all physical results depend only on the eigenvalues $\Lambda_n{}^M$ of the matrices Γ_{ij} corresponding to any set of M atoms, with $M=1, 2, \cdots, N$. For large M, we cannot compute these eigenvalues explicitly, and hence we shall have to circumvent this problem somehow. This will be done indeed later, but it will prove helpful to know the eigenvalues of Γ_{ij} in the limits (3.7) and (3.8). We have

$$\Lambda_1{}^M=\Lambda_2{}^M=\cdots=\Lambda_M{}^M=\Gamma_0 \quad\text{for}\quad \Gamma_{ij}=\delta_{ij}\Gamma_0, \tag{3.7'}$$

and

$$\Lambda_1{}^M=\Lambda_2{}^M=\cdots=\Lambda_{M-1}{}^M=0,\ \Lambda_M{}^M=M\Gamma_0 \text{ for } \Gamma_{ij}=\Gamma_0. \tag{3.8'}$$

[14] T. F. Jordan, Helv. Phys. Acta 37, 697 (1964).
[15] N. Mukunda and T. F. Jordan, J. Math. Phys. 7, 849 (1966).

We may expect that the eigenvalues for a real laser are somewhere between these cases, but we cannot expect that Γ_0 will be a good approximation for all eigenvalues, even if the system is close to the limit (3.7′). The reason for this is that we have to expect from the structure of (3.8′), that the small deviations from Γ_0 of many eigenvalues can sum up to a large deviation from Γ_0 of a few eigenvalues. This is the true reason for some troubles which occur later.

We have included the above remarks in order to see how the case $N=1$, which is identical with the original theory of Weisskopf and Wigner, is embedded in the general case $N>1$. Let us treat now the case $N=1$ in more detail.

The system (2.20) reads for $N=1$

$$\begin{aligned} i\frac{d}{dt}\alpha_0{}^0(t)&=\sum_{\mathbf{k}_1} C_{\mathbf{k}_1}{}^{(1)*}\frac{e^{i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\alpha_{\mathbf{k}_1}{}^1(t),\\ i\frac{d}{dt}\alpha_{\mathbf{k}_1}{}^1(t)&=C_{\mathbf{k}_1}{}^{(1)}\frac{e^{-i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\alpha_0{}^0(t), \end{aligned} \tag{3.9}$$

with the initial condition

$$\alpha_0{}^0(0)=1,\quad \alpha_{\mathbf{k}_1}{}^1(0)=0. \tag{3.10}$$

In order to obtain a solution of (3.9), we follow closely the spirit of the original approach of Weisskopf and Wigner.[1] We introduce new quantities $\beta_0(t)$, $\beta_{k_1}(t)$ by putting

$$\alpha_0{}^0(t)=\beta_0(t),\quad \alpha_{\mathbf{k}_1}{}^1(t)=C_{\mathbf{k}_1}{}^{(1)}\frac{e^{-i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\beta_{k_1}(t). \tag{3.11}$$

Inserting (3.11) into (3.9) yields the equations

$$\begin{aligned} i\frac{d}{dt}\beta_0(t)&=\frac{1}{V}\sum_{\mathbf{k}_1}\frac{|C_{\mathbf{k}_1}|^2}{2k_1}\beta_{k_1}(t),\\ \left(\Delta-k_1+i\frac{d}{dt}\right)\beta_{k_1}(t)&=\beta_0(t). \end{aligned} \tag{3.12}$$

The second equation (3.12) shows that $\beta_{k_1}(t)$ does not depend on the direction of $\mathbf{k}_1$, as was quietly assumed already in (3.11). Going to the limit $V\to\infty$ in accordance with (3.1) and using (3.2), we obtain for the right-hand side of the first equation (3.12) the expression

$$\int_0^\infty C_{k_1}(0)\beta_{k_1}(t)dk_1. \tag{3.13}$$

If we assume (a) that $\beta_k(t)$ is appreciably different from zero only for k in a small neighborhood about Δ, and (b) that $C_k(0)$ is slowly varying in that region, we can (a) extend the limits of the integration over k_1 from (0 to ∞) to ($-\infty$ to ∞), and (b) take $C_{k_1}(0)$ out of the integral over k_1 by ascribing to it the value $C_\Delta(0)=\Gamma_0/\pi$. This transforms the integral (3.13) into

$$\frac{\Gamma_0}{\pi}\int_{-\infty}^{\infty}\beta_{k_1}(t)dk_1, \tag{3.14}$$

where the limits $\pm\infty$ are approached symmetrically, and we obtain instead of (3.12) the system of equations

$$\begin{aligned} i\frac{d}{dt}\beta_0(t)&=\frac{\Gamma_0}{\pi}\int_{-\infty}^{\infty}\beta_{k_1}(t)dk_1,\\ \left(\Delta-k_1+i\frac{d}{dt}\right)\beta_{k_1}(t)&=\beta_0(t), \end{aligned} \tag{3.15}$$

which can be solved exactly. The solution with the proper initial values is

$$\begin{aligned} \beta_0(t)&=e^{-\Gamma_0 t},\\ \beta_{k_1}(t)&=(e^{-\Gamma_0 t}-e^{i(\Delta-k_1)t})/(\Delta-k_1-i\Gamma_0). \end{aligned} \tag{3.16}$$

This statement is easily proved by using the formula

$$\int_{-\infty}^{\infty}dx\,\frac{e^{-zt}-e^{ixt}}{x-iz}=-i\pi e^{-zt}, \tag{3.17}$$

which is true for real t and for each finite complex value of z.

Since we shall use the arguments leading from (3.13) to (3.14) throughout this paper, it is worthwhile to compare the results (3.16) with the formally exact solution of (3.9) as given by Källén.[9] We see that the transition from (3.13) to (3.14) involves only a few inaccuracies referring to a small shift of the line peak and to small deviations of the actual line shape from the symmetric, purely Lorentzian line shape obtained in (3.16), and also to a deviation from the purely exponential decay law (3.16) for small t. It is also worthwhile to mention that the normalization condition

$$|\alpha_0{}^0(t)|^2+\sum_{\mathbf{k}_1}|\alpha_{\mathbf{k}_1}{}^1(t)|^2=1, \tag{3.18}$$

which holds exactly for any time t as a consequence of the structure of Eqs. (3.9), still holds if the α's are expressed by the β's as given in (3.16) and the summation over $\mathbf{k}_1$ is approximated in the same way as in (3.13). This means that these arguments do not seriously violate the unitarity of the time-development operator and hence the normalization of the probabilities. We consider these small inaccuracies as nonessential for the problem at hand and conclude that the arguments leading from (3.13) to (3.14) are valid within satisfactory limits. The simplification gained by these arguments is one key to a satisfactory solution of (2.20).

In the solution (3.16) the essential features of the process usually called "spontaneous emission" are still contained: A single atom, somehow excited at $t=0$,

jumps "spontaneously" to its ground state and emits a photon. The lifetime of the excited state is $(2\Gamma_0)^{-1}$, the linewidth of the emitted radiation is $2\Gamma_0$, and the requirements of the uncertainty principle are thus met correctly. It follows from (3.11) that the probability for the photon to have the direction of $\mathbf{k}_1$ is proportional to $|C_{\mathbf{k}_1}^{(1)}|^2=|C_{\mathbf{k}_1}|^2$ and hence independent of the position $\mathbf{X}_1$ of the atom, as it must be.

Tracing back this result we see that it is due to the factorization (2.19). If (2.19) were not true, there would be a dependence on $\mathbf{X}_1$ in Eq. (3.12) and hence in Γ_0. The factorization (2.19) is thus necessary to obtain the translation invariance of the decay of a single atom. This proves the statements made in connection with the discussion of (2.19).

Considered against the case of a large N, the simplicity of the above theory is due to the fact that the N-by-N matrix Γ_{ij} reduces to the single element Γ_0, and that all Einstein factors are 1. Our general model yields a generalization and modification of this process, and not merely a repetition of independent decay processes.

To point out the latter possibility, let us consider a purely fictitious model consisting of N independently radiating atoms, each of which is described by the above theory of Weisskopf and Wigner. This must be thought of as arising from the atoms being in different physical spaces or the emission processes being separated by time intervals $T\gg(2\Gamma_0)^{-1}$. In this model, we can ask for the probability for "A_1 emitting a photon of mode $\mathbf{k}_1$, $A_2\cdots$, and A_N emitting a photon of mode $\mathbf{k}_N$," which is given by the expression

$$|\alpha_{\mathbf{k}_1}{}^1(\infty)|^2\times|\alpha_{\mathbf{k}_2}{}^1(\infty)|^2\times\cdots\times|\alpha_{\mathbf{k}_N}{}^1(\infty)|^2. \quad (3.19)$$

But this probability has a sense only in this fictitious model, where the photons can be associated with the atoms and are thus made distinguishable. If we dispense with this fictitious distinguishability, we can only ask for the probability of finding "N photons of modes $\mathbf{k}_1\cdots\mathbf{k}_N$" in the same sense that this phrase was used previously. The probability that the N photons are in the modes $\mathbf{k}_1\cdots\mathbf{k}_N$ under the condition that each atom decays independently of the other atoms, is given by

$$\frac{N!}{\epsilon^2(\mathbf{k}_1\cdots\mathbf{k}_N)}|\alpha_{\mathbf{k}_1}{}^1(\infty)|^2\times|\alpha_{\mathbf{k}_2}{}^1(\infty)|^2\times\cdots\times|\alpha_{\mathbf{k}_N}{}^1(\infty)|^2. \quad (3.20)$$

We shall use this fictitious nonlasing model as a convenient "background" for our real model and the probability (3.20) as a convenient normalization for the probability $|\alpha_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2,\cdots,N}(\infty)|^2$, that the N photons from our real model are in the modes $\mathbf{k}_1\cdots\mathbf{k}_N$. Both intentions are met by considering the ratio

$$R_{\mathbf{k}_1\cdots\mathbf{k}_N}=|\alpha_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2\cdots N}(\infty)|^2\Big/\frac{N!}{\epsilon^2(\mathbf{k}_1\cdots\mathbf{k}_N)}\prod_{\nu=1}^{N}|\alpha_{\mathbf{k}_\nu}{}^1(\infty)|^2=\frac{|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2\cdots N}(\infty)|^2}{N!\prod_{\nu=1}^{N}|\alpha_{\mathbf{k}_\nu}{}^1(\infty)|^2}, \quad (3.21)$$

where we have used the definition (2.22) of $\tilde{\alpha}$.

If the atoms of the real model could radiate independently of each other, i.e., "spontaneously," we would obtain $R_{\mathbf{k}_1\cdots\mathbf{k}_N}=1$. Deviations from this value are thus due to the mutual influence the atoms have on each other in the real system, and $R_{\mathbf{k}_1\cdots\mathbf{k}_N}$ is a convenient measure for such "self-stimulation effects." We shall see that $R_{\mathbf{k}_1\cdots\mathbf{k}_N}$ assumes values as high as $N!$, and that the occurrence of the photon avalanche in the form of a single ray, the direction of that ray and similar features are most easily understood against the background of the nonlasing model, i.e., in terms of $R_{\mathbf{k}_1\cdots\mathbf{k}_N}$. Note that R depends on $\mathbf{X}_1\cdots\mathbf{X}_N$.

We are now in a position to give a complete discussion of the role of the Einstein factors $(n_1+1)^{1/2}$ in our model. The quantity (3.20) is "discontinuous" in $(\mathbf{k}_1\cdots\mathbf{k}_N)$ due to our dispensing with the distinguishability of photons in that fictitious model. $|\alpha_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2,\cdots N}(\infty)|^2$ contains the same discontinuous factors $\epsilon^{-2}(\mathbf{k}_1\cdots\mathbf{k}_N)$ as (3.20) does, such that these factors cancel in (3.21) and R becomes "smooth." We have to conclude from this that the factors ϵ play the same role in both models, which means that they do nothing else but to take care of the proper normalization requirements of a statistical theory of actually indistinguishable photons. This was indeed the reason why these factors had to be introduced, but it follows from their cancellation in (3.21) that they do not lead to observable effects in the real model: The occurrence and shaping of a ray of photons, our most important result, has nothing to do with these factors. Such effects would also appear in a symmetric quantum theory of distinguished objects, where no Einstein factors appear. We would like to mention also, that the Einstein functions ϵ act on a domain of measure zero in the limit $V\to\infty$, and do not possess any δ-like singularities in that limit. It is thus trivial that they cannot lead to measurable facts in that limit, and yet we obtain very strong effects for $V\to\infty$.

IV. TWO-ATOM PROBLEM

Since a solution of Eq. (2.20) with the same accuracy as for $N=1$ is obtainable for $N=2$, and since, at least in rudimentary forms, nearly all effects of the general case are already present for $N=2$, we feel that the complete treatment of the two-atom problem provides the best key for the understanding of the general case.

For $N=2$, Eqs. (2.20) specialize to

$$\begin{aligned}
i\frac{d}{dt}\alpha_0{}^0(t)&=\sum_{\mathbf{k}_1} C_{\mathbf{k}_1}{}^{(1)*}\frac{e^{i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\alpha_{\mathbf{k}_1}{}^1(t)+\sum_{\mathbf{k}_1} C_{\mathbf{k}_1}{}^{(2)*}\frac{e^{i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\alpha_{\mathbf{k}_1}{}^2(t),\\
i\frac{d}{dt}\alpha_{\mathbf{k}_1}{}^1(t)&=C_{\mathbf{k}_1}{}^{(1)}\frac{e^{-i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\alpha_0{}^0(t)+\sum_{\mathbf{k}_2}\epsilon(\mathbf{k}_1\mathbf{k}_2)C_{\mathbf{k}_2}{}^{(2)*}\frac{e^{i(\Delta-k_2)t}}{(2Vk_2)^{1/2}}\alpha_{\mathbf{k}_1\mathbf{k}_2}{}^{12}(t),\\
i\frac{d}{dt}\alpha_{\mathbf{k}_1}{}^2(t)&=C_{\mathbf{k}_1}{}^{(2)}\frac{e^{-i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\alpha_0{}^0(t)+\sum_{\mathbf{k}_2}\epsilon(\mathbf{k}_1\mathbf{k}_2)C_{\mathbf{k}_2}{}^{(1)*}\frac{e^{i(\Delta-k_2)t}}{(2Vk_2)^{1/2}}\alpha_{\mathbf{k}_1\mathbf{k}_2}{}^{12}(t),\\
i\frac{d}{dt}\alpha_{\mathbf{k}_1\mathbf{k}_2}{}^{12}(t)&=\epsilon^{-1}(\mathbf{k}_1\mathbf{k}_2)\left\{C_{\mathbf{k}_1}{}^{(2)}\frac{e^{-i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\alpha_{\mathbf{k}_2}{}^1(t)+C_{\mathbf{k}_2}{}^{(2)}\frac{e^{-i(\Delta-k_2)t}}{(2Vk_2)^{1/2}}\alpha_{\mathbf{k}_1}{}^1(t)+C_{\mathbf{k}_1}{}^{(1)}\frac{e^{-i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\alpha_{\mathbf{k}_1}{}^2(t)+C_{\mathbf{k}_2}{}^{(1)}\frac{e^{-i(\Delta-k_2)t}}{(2Vk_2)^{1/2}}\alpha_{\mathbf{k}_1}{}^2(t)\right\},
\end{aligned}\tag{4.1}$$

with initial values

$$\alpha_0{}^0(0)=1,\quad \alpha_{\mathbf{k}_1}{}^1(0)=\alpha_{\mathbf{k}_1}{}^2(0)=\alpha_{\mathbf{k}_1\mathbf{k}_2}{}^{12}(0)=0.\tag{4.2}$$

In order to solve this system of equations, we generalize the ansatz (3.11) to

$$\begin{aligned}
\alpha_0{}^0(t)&=\beta_0(t),\\
\alpha_{\mathbf{k}_1}{}^1(t)&=C_{\mathbf{k}_1}{}^{(1)}\frac{e^{-i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}[\beta_{k_1}{}^{(+)}(t)+e^{i\mathbf{k}_1\cdot(\mathbf{X}_1-\mathbf{X}_2)}\beta_{k_1}{}^{(-)}(t)],\\
\alpha_{\mathbf{k}_1}{}^2(t)&=C_{\mathbf{k}_1}{}^{(2)}\frac{e^{-i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}[\beta_{k_1}{}^{(+)}(t)+e^{i\mathbf{k}_1\cdot(\mathbf{X}_2-\mathbf{X}_1)}\beta_{k_1}{}^{(-)}(t)],\\
\alpha_{\mathbf{k}_1\mathbf{k}_2}{}^{12}(t)&=\epsilon^{-1}(\mathbf{k}_1\mathbf{k}_2)(C_{\mathbf{k}_1}{}^{(1)}C_{\mathbf{k}_2}{}^{(2)}\\
&\quad+C_{\mathbf{k}_1}{}^{(2)}C_{\mathbf{k}_2}{}^{(1)})\frac{e^{-i(\Delta-k_1)t}}{(2Vk_1)^{1/2}}\frac{e^{-i(\Delta-k_2)t}}{(2Vk_2)^{1/2}}\\
&\quad\times\{\beta_{k_1k_2}{}^{(+)}(t)+[(e^{i\mathbf{k}_1\cdot(\mathbf{X}_1-\mathbf{X}_2)}+e^{i\mathbf{k}_2\cdot(\mathbf{X}_1-\mathbf{X}_2)})^{-1}\\
&\quad+(e^{i\mathbf{k}_1\cdot(\mathbf{X}_2-\mathbf{X}_1)}+e^{i\mathbf{k}_2\cdot(\mathbf{X}_2-\mathbf{X}_1)})^{-1}]\beta_{k_1k_2}{}^{(-)}(t)\}.
\end{aligned}\tag{4.3}$$

It is assumed that $\beta_{k_1}{}^{(\pm)}(t)$, $\beta_{k_1k_2}{}^{(\pm)}(t)$ are functions of the energies k_1, k_1k_2 only, which are appreciably different from zero only in small neighborhoods of $k_1=\Delta$, $k_1=\Delta$ and $k_2=\Delta$, of dimensions Γ_0 or $\Gamma_0{}^2$, just as in (3.11).

Inserting (4.3) into (4.1), going to the limit $V\to\infty$ as discussed in (3.1), and carrying out the integration over the angles in the integrals over $k_{1,2}$, we introduce the function $C_k(\mathbf{X}_1-\mathbf{X}_2)$, as defined in (3.2), into the remaining integrals of type (3.13). If $C_k(\mathbf{X}_1-\mathbf{X}_2)$ does not vary much in a region around $k=\Delta$ of dimension Γ_0, we can take $C_k(\mathbf{X}_1-\mathbf{X}_2)$ out of the integrals and obtain thus the elements Γ_{ij} as factors in front of the k integrals. Extending also the limits of the k integrations to go from $-\infty$ to ∞, we obtain for the functions $\beta_{k_1}{}^{(\pm)}(t)$, $\beta_{k_1k_2}{}^{(\pm)}(t)$ the coupled set of equations

$$\begin{aligned}
i\frac{d}{dt}\beta_0{}^0(t)&=\frac{2\Gamma_0}{\pi}\int_{-\infty}^{\infty}dk_1\beta_{k_1}{}^{(+)}(t)+\frac{2\Gamma_{12}}{\pi}\int_{-\infty}^{\infty}dk_1\beta_{k_1}{}^{(-)}(t),\\
\left(\Delta-k_1+i\frac{d}{dt}\right)\beta_{k_1}{}^{(+)}(t)&=\beta_0(t)+\frac{\Gamma_0}{\pi}\int_{-\infty}^{\infty}dk_2\beta_{k_1k_2}{}^{(+)}(t)+\frac{\Gamma_{12}}{\pi}\int_{-\infty}^{\infty}dk_2\beta_{k_1k_2}{}^{(-)}(t),\\
\left(\Delta-k_1+i\frac{d}{dt}\right)\beta_{k_1}{}^{(-)}(t)&=\frac{\Gamma_{12}}{\pi}\int_{-\infty}^{\infty}dk_2\beta_{k_1k_2}{}^{(+)}(t)+\frac{\Gamma_0}{\pi}\int_{-\infty}^{\infty}dk_2\beta_{k_1k_2}{}^{(-)}(t),\\
\left(\Delta-k_1+\Delta-k_2+i\frac{d}{dt}\right)\beta_{k_1k_2}{}^{(\pm)}(t)&=\beta_{k_1}{}^{(\pm)}(t)+\beta_{k_2}{}^{(\pm)}(t).
\end{aligned}\tag{4.4}$$

The equations for $\beta_{k_1}{}^{(+)}(t)$ and $\beta_{k_1}{}^{(-)}(t)$ show that a secular problem is involved in the solution of this set of equations, and that the eigenvalues $\Gamma_0\pm\Gamma_{12}$ of Γ_{ij} must appear in the solution of (4.4), which is indeed the case. The exact solution of (4.4) is

$$\begin{aligned}
\beta_0(t)&=e^{-2\Gamma_0t},\\
\beta_{k_1}{}^{(\pm)}&=\frac{1}{2}\left(e^{-(\Gamma_0+\Gamma_{12})t}\frac{e^{-(\Gamma_0-\Gamma_{12})t}-e^{i(\Delta-k_1)t}}{\Delta-k_1-i(\Gamma_0-\Gamma_{12})}\pm e^{-(\Gamma_0-\Gamma_{12})t}\frac{e^{-(\Gamma_0+\Gamma_{12})t}-e^{i(\Delta-k_2)t}}{\Delta-k_2-i(\Gamma_0+\Gamma_{12})}\right),\\
\beta_{k_1k_2}{}^{(\pm)}(t)&=\frac{1}{2}\left[\frac{1}{\Delta-k_1-i(\Gamma_0-\Gamma_{12})}\left(\frac{e^{-2\Gamma_0t}-e^{i(\Delta-k_1+\Delta-k_2)t}}{\Delta-k_1+\Delta-k_2-2i\Gamma_0}-e^{i(\Delta-k_1)t}\frac{e^{-(\Gamma_0-\Gamma_{12})t}-e^{i(\Delta-k_2)t}}{\Delta-k_2-i(\Gamma_0+\Gamma_{12})}\right)+\frac{1}{\Delta-k_2-i(\Gamma_0-\Gamma_{12})}\right.\\
&\quad\left.\times\left(\frac{e^{-2\Gamma_0t}-e^{i(\Delta-k_1+\Delta-k_2)t}}{\Delta-k_1+\Delta-k_2-2i\Gamma_0}-e^{i(\Delta-k_2)t}\frac{e^{-(\Gamma_0+\Gamma_{12})t}-e^{i(\Delta-k_1)t}}{\Delta-k_1-i(\Gamma_0+\Gamma_{12})}\right)\right]\pm\frac{1}{2}\left[\frac{1}{\Delta-k_1-i(\Gamma_0+\Gamma_{12})}\left(\frac{e^{-2\Gamma_0t}-e^{i(\Delta-k_1+\Delta-k_2)t}}{\Delta-k_1+\Delta-k_2-2i\Gamma_0}\right.\right.\\
&\quad\left.\left.-e^{i(\Delta-k_1)t}\frac{e^{-(\Gamma_0-\Gamma_{12})t}-e^{i(\Delta-k_2)t}}{\Delta-k_2-i(\Gamma_0-\Gamma_{12})}\right)+\frac{1}{\Delta-k_2-i(\Gamma_0+\Gamma_{12})}\left(\frac{e^{-2\Gamma_0t}-e^{i(\Delta-k_1+\Delta-k_2)t}}{\Delta-k_1+\Delta-k_2-2i\Gamma_0}-e^{i(\Delta-k_2)t}\frac{e^{-(\Gamma_0-\Gamma_{12})t}-e^{i(\Delta-k_1)t}}{\Delta-k_1-i(\Gamma_0-\Gamma_{12})}\right)\right].
\end{aligned}\tag{4.5}$$

In order to avoid retardation effects and to give thus an unambiguous physical meaning to the initial condition (2.3) and its equivalents, we have to keep the distance $|\mathbf{X}_1-\mathbf{X}_2|$ so small that a light signal can travel several times between A_1 and A_2 during the lifetime of the excited states. This requirement imposes on $|\mathbf{X}_1-\mathbf{X}_2|$ the condition

$$|\mathbf{X}_1-\mathbf{X}_2| \ll L_0=(2\Gamma_0)^{-1}. \tag{4.6}$$

Accidentally, it follows from (3.6) that (4.6) will keep the variation of the function $C_k(\mathbf{X}_1-\mathbf{X}_2)$ small in the essential interval $\Delta-\Gamma_0<k<\Delta+\Gamma_0$ for any case of interest, so that the solution (4.5) should have about the same accuracy within the limits (4.6) as solution (3.16) has for $N=1$. For lifetimes of 3.3×10^{-8} sec, we obtain $L_0=10^3$ cm, and (4.6) is not a very serious physical restriction.

It follows from the first lines of formulas (4.5) and (4.3) that the state $|e_1e_2;\,0\rangle$ [cf. (2.8)] dies out like $e^{-2(2\Gamma_0)t}$, which is exactly the law one expects for a system of two independently radiating atoms, as discussed in the preceding chapter. At first sight this is surprising, since the presence of a nonexcited atom has a marked influence on the lifetime of a single excited atom, as has been shown by Stephen.[4] For the present case, one must expect an even more pronounced mutual influence of the atoms on each other, and the above result can only be explained as being due to an inherent symmetry of the model and the initial condition (2.3). If this is so, we have to expect

$$\alpha_0{}^0(t)=e^{-N\Gamma_0 t} \tag{4.7}$$

for any N, and this is at least not in contradiction to the results of the following sections. More details of (4.7) will be considered in Sec. VIII.

We start with a discussion of the leading features of the solution (4.5). This can be done by observing that the brackets { } in the expressions (4.3) for $\alpha_{\mathbf{k}_1}{}^1$, $\alpha_{\mathbf{k}_1}{}^2$, and $\alpha_{\mathbf{k}_1\mathbf{k}_2}{}^{12}$ can be replaced approximately by

$$\begin{aligned}\beta_{k_1}(t)&=\beta_{k_1}{}^{(+)}(t)+\beta_{k_1}{}^{(-)}(t),\\ \beta_{k_1k_2}(t)&=\beta_{k_1k_2}{}^{(+)}(t)+\beta_{k_1k_2}{}^{(-)}(t).\end{aligned} \tag{4.8}$$

Indeed, if $|\mathbf{X}_1-\mathbf{X}_2|$ is such that $e^{i\mathbf{k}\cdot(\mathbf{X}_1-\mathbf{X}_2)}$ may be approximated by 1, i.e., for $|\mathbf{X}_1-\mathbf{X}_2|<\pi\Delta^{-1}$, we can replace those brackets by (4.8). But if $|\mathbf{X}_1-\mathbf{X}_2|$ is large, $|\mathbf{X}_1-\mathbf{X}_2|>\pi\Delta^{-1}$, we have $\Gamma_{12}\approx0$ and hence $\beta_{k_1}{}^{(-)}(t)\approx0$, $\beta_{k_1k_2}{}^{(-)}(t)\approx0$, which is also compatible with (4.8). Introducing the notation

$$\gamma=\Gamma_{12}/\Gamma_0, \tag{4.9}$$

we obtain the expressions

$$\begin{aligned}\beta_0(t)&=e^{-2\Gamma_0 t},\\ \beta_{k_1}(t)&=e^{-(1+\gamma)\Gamma_0 t}\frac{e^{-(1-\gamma)\Gamma_0 t}-e^{i(\Delta-k_1)t}}{\Delta-k_1-i(1-\gamma)\Gamma_0},\\ \beta_{k_1k_2}(t)&=[\ ]_{(1)}=-ie^{i(\Delta-k_1+\Delta-k_2)t}\\ &\quad\times\int_0^t dt'e^{-i(\Delta-k_1+\Delta-k_2)t'}(\beta_{k_1}(t')+\beta_{k_2}(t')),\end{aligned} \tag{4.10}$$

where $[\ ]_{(1)}$ indicates the contents of the first bracket in the formula (4.5) for $\beta_{k_1k_2}{}^{(\pm)}(t)$. The equivalence of that expression with the integral (4.10) is easily checked.

In the approximation (4.10) it is easily seen how the system develops in time. For very short times, the state $|e_1e_2;\,0\rangle$ is predominant, and $\beta_{k_1}(t)$ and $\beta_{k_1k_2}(t)$ are very small. This state dies out as t obtains the value of about one half of the natural lifetime of the isolated atom, and the states $|g_1e_2;\,\mathbf{k}_1\rangle$ and $|e_1g_2;\,\mathbf{k}_1\rangle$ become dominant around that moment, since $|e_1e_2;\,0\rangle$ is no longer and $|g_1g_2;\,\mathbf{k}_1\mathbf{k}_2\rangle$ is not yet present, as follows from the third line of formula (4.10). As the time proceeds, these intermediate states vanish and only the states $|g_1g_2;\,\mathbf{k}_1\mathbf{k}_2\rangle$ are present. It follows from (4.5), that the limits

$$\tilde\beta_{k_1k_2}{}^{(\pm)}=\lim_{t\to\infty}e^{-i(\Delta-k_1+\Delta-k_2)t}\beta_{k_1k_2}{}^{(\pm)}(t) \tag{4.11}$$

are well defined and exist. Defining $\tilde\beta_{k_1k_2}$ correspondingly, we obtain

$$\tilde\beta_{k_1k_2}=\left(1+\frac{-2i\Gamma_0\gamma}{\Delta-k_1+\Delta-k_2-2i\Gamma_0}\right)\frac{1}{\Delta-k_1-i(1+\gamma)\Gamma_0}\times\frac{1}{\Delta-k_2-i(1+\gamma)\Gamma_0}. \tag{4.12}$$

Since the first factor of (4.12) varies only between the values 1 and $1+\gamma$, it has only a slight influence on the line shape of the emitted photons, which is a broadened Lorentzian line as given by the other two factors. The first factor has only the effect that the probability for both photons to have an energy $\Delta+\delta$ is a little smaller than the probability that one photon has energy $\Delta+\delta$, the other $\Delta-\delta$. Apart from this effect, the first factor of (4.12) could be replaced by $1+\gamma$.

It is interesting to note what has been lost in the approximation (4.10) in comparison with (4.5). For the intermediate states $|g_1e_2;\,\mathbf{k}\rangle$ and $|g_2e_1;\,\mathbf{k}\rangle$, there are some small additional terms with a long lifetime $[2(\Gamma_0-\Gamma_{12})]^{-1}$ which must be considered as traces of the trapping effect discussed by Stephen.[4] For $t\to\infty$ there is a weak dependence of the line shape of the emitted radiation on the direction of emission, since $\tilde\beta_{k_1k_2}{}^{(+)}$ and $\tilde\beta_{k_1k_2}{}^{(-)}$ are coupled with coefficients depending on $\mathbf{X}_1-\mathbf{X}_2$ and the direction of emission. These small effects should be of secondary importance for larger N.

The most important effects contained in the present model are those connected with the directions into which the photons are actually emitted. We can study related questions by considering the ratio $R_{\mathbf{k}_1\mathbf{k}_2}$ as defined by (3.21). We get from (4.3) and (3.11)

$$\begin{aligned}R_{\mathbf{k}_1\mathbf{k}_2}=\tfrac{1}{2}|&(e^{i\mathbf{k}_1\cdot(\mathbf{X}_1-\mathbf{X}_2)}+e^{i\mathbf{k}_2\cdot(\mathbf{X}_1-\mathbf{X}_2)})\tilde\beta_{k_1k_2}{}^{(+)}\\ &+(1+e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{X}_1-\mathbf{X}_2)})\tilde\beta_{k_1k_2}{}^{(-)}|^2\\ &\times|(\Delta-k_1-i\Gamma_0)(\Delta-k_2-i\Gamma_0)|^2.\end{aligned} \tag{4.13}$$

In the limit $\mathbf{X}_1 \to \mathbf{X}_2$, $R_{k_1k_2}$ does not depend sensitively on the direction of $\mathbf{k}_1\mathbf{k}_2$, which shows that the photons are emitted into the same directions as in the case of independently radiating atoms, and the line-shape effects discussed above are the only important results for that case. For $|\mathbf{X}_1-\mathbf{X}_2|>\pi\Delta^{-1}$, we may expand $R_{\mathbf{k}_1\mathbf{k}_2}$ in powers of γ, which becomes small for this case. Retaining only terms linear in γ, we obtain

$$R_{\mathbf{k}_1\mathbf{k}_2}\approx[1+\cos(\mathbf{k}_1-\mathbf{k}_2)\cdot(\mathbf{X}_1-\mathbf{X}_2)]$$
$$-\gamma[\cos\mathbf{k}_1\cdot(\mathbf{X}_1-\mathbf{X}_2)+\cos\mathbf{k}_2\cdot(\mathbf{X}_1-\mathbf{X}_2)]$$
$$\times\mathrm{Re}\left[\frac{-2i\Gamma_0}{\Delta-k_1+\Delta-k_2-2i\Gamma_0}\right.$$
$$\left.\times\left(\frac{\Delta-k_1-i\Gamma_0}{\Delta-k_2-i\Gamma_0}+\frac{\Delta-k_2-i\Gamma_0}{\Delta-k_1-i\Gamma_0}\right)\right]. \quad (4.14)$$

The factor involving γ is bounded for all photon pairs, since $\mathrm{Re}[\cdots]$ assumes values of order of magnitude 2 only, and the above expansion in powers of γ is quite unproblematic. If γ is small enough, we can neglect the second term, but even then $R_{\mathbf{k}_1\mathbf{k}_2}$ is different from unity, which is the most important result of this paper. $R_{\mathbf{k}_1\mathbf{k}_2}\neq 1$ indicates here a mutual influence of the two radiating atoms over large distances, which has the effect that the emission of certain pairs $\mathbf{k}_1\mathbf{k}_2$ of photons is favored ($R_{\mathbf{k}_1\mathbf{k}_2}\approx 2$) over the emission of the same pairs from independently radiating atoms, while other pairs are suppressed ($R_{\mathbf{k}_1\mathbf{k}_2}\approx 0$) and cannot be emitted. The favored pairs $\mathbf{k}_1\mathbf{k}_2$ are the solutions of the equation

$$(\mathbf{k}_1-\mathbf{k}_2)\cdot(\mathbf{X}_1-\mathbf{X}_2)=0 \bmod 2\pi, \quad (4.15)$$

which shows that the effect depends on the geometrical relations between the photon wave vectors $\mathbf{k}_1\mathbf{k}_2$ and the positions of the atoms. The photon pairs with $\mathbf{k}_1=\mathbf{k}_2$ are always favored, but one should not forget that (4.15) is empty unless $\Delta-\Gamma_0<|\mathbf{k}_i|<\Delta+\Gamma_0$, for $i=1, 2$, is true. Dicke[2] found (4.14) with $\gamma=0$ for the directional correlation between the two photons in this case.

This effect can neither be explained in terms of "spontaneous emission," nor in terms of "stimulated emission," but it is a generalization of both of them such that "self-stimulation" might be an adequate notation for its basic features. We could also speak of an interference effect in the phase space of the two photons.

Although self-stimulation is obviously an important laser effect, this model gives only an obscure picture of a laser. One reason for this is that $N=2$ is smaller than the dimension 3 of ordinary space, as will come out clearly from the treatment of the general case. Another reason is the fact that our model has no statistical properties since the positions $\mathbf{X}_1\mathbf{X}_2$ are fixed. If we consider a Gibbs ensemble of two-atom systems, described by a density function $f(\mathbf{X}_1,\mathbf{X}_2)>0$ in the appropriate Γ space of the points $(\mathbf{X}_1,\mathbf{X}_2)$, and if $f(\mathbf{X}_1,\mathbf{X}_2)$ is properly chosen such that the ensemble represents a "laser rod of two atoms," we find indeed that the Gibbs expectation value for the probability of the emission of a pair $\mathbf{k}_1\mathbf{k}_2$ parallel to the axis of the rod is larger than the Gibbs expectation value for a pair not parallel to the axis, and that pairs $\mathbf{k}_1\neq\mathbf{k}_2$ have very small expectation values. Since neither a two-atom laser nor a Gibbs ensemble of such lasers can be realized, we shall be content with these remarks. Similar considerations will turn out much more simple for large N.

V. APPROXIMATE SOLUTION FOR LARGE N

From the experience gained from the case $N=2$ we expect that a solution of Eqs. (2.20) with the same accuracy as in the cases $N=1, 2$ should not be too complicated as long as the necessary linear equations and the secular problems of the type

$$\det(\Gamma_{ij}-\Lambda\delta_{ij})=0 \quad (5.1)$$

can be solved explicitly. Even if such a procedure were possible for large N, say, $N=10^{20}$, we would not expect all the details of the formal solution of (2.20) to be equally important and we might even fear that the complexity of that solution would hide more than reveal the physics we are interested in. This means that some sort of elimination of unimportant details must be carried out and a reasonable approximate solution of (2.20) must be sought for.

To exclude pathological situations, we apply in this chapter the restrictions made in the introduction, that the atoms are distributed randomly over a given volume $\mathcal{V}$, such that their macroscopic density is constant in $\mathcal{V}$. In order that light signals can travel several times between each pair of atoms during the natural lifetime of the atoms, we restrict the linear dimensions L of $\mathcal{V}$ to be such that

$$L\ll L_0 \quad (5.2)$$

holds, where L_0 is the bound given in (4.6). For an actual laser, we have the additional inequality

$$\lambda\ll L, \quad (5.3)$$

where λ is the wavelength of the emitted radiation. Remembering the properties (3.7′) and (3.7″) of the eigenvalues of Γ_{ij}, we shall not go to the formal limit $\Gamma_{ij}\to\Gamma_0\delta_{ij}$ as a consequence of (5.3), although we can solve the Eqs. (2.20) with the accuracy of the cases $N=1, 2$ in that limit. Our approximations will of course be such that this limit is treated correctly.

To obtain an approximate solution of (2.20), for $\alpha_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}(t)$ we make the ansatz

$$\alpha_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}(t)=\epsilon^{-1}(\mathbf{k}_1\cdots\mathbf{k}_M)\left(\prod_{l=1}\frac{e^{-it(\Delta-k_l)}}{(2Vk_l)^{1/2}}C_{\mathbf{k}_l}\right)$$
$$\times p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}\beta_{k_1\cdots k_M}(t), \quad (5.4)$$

where $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ is a given function to be discussed below and $\beta_{k_1\cdots k_M}(t)$ is an unknown symmetric function of the photon energies $k_1\cdots k_M$ and of the time t, whose form depends on the shape and magnitude of $\mathfrak{V}$. If $\beta_{k_1\cdots k_M}(t)$ were replaced by an unknown function $\beta_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}(t)$, we would only have introduced new quantities $\beta_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}(t)$ for $\alpha_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}(t)$ and no approximation would be involved in (5.4). The assumed simple nature of $\beta_{k_1\cdots k_M}(t)$ is thus the point in the ansatz (5.4), the structure of which is designed so that it can lead to a generalization of the approximations (4.10) of the case $N=2$. The crucial function $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ is given by

$$p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}=\mathop{\mathbf{P}}_{\kappa_1\cdots\kappa_M}^{\mathbf{k}_1\cdots\mathbf{k}_M}\exp[i(\mathbf{X}_{j_1}\cdot\boldsymbol{\kappa}_1+\cdots+\mathbf{X}_{j_M}\cdot\boldsymbol{\kappa}_M)]$$

$$=\mathop{\mathbf{P}}_{\nu_1\cdots\nu_M}^{j_1\cdots j_M}\exp[i(\mathbf{k}_1\cdot\mathbf{X}_{\nu_1}+\cdots+\mathbf{k}_M\cdot\mathbf{X}_{\nu_M})], \tag{5.5}$$

where the symbol

$$\mathop{\mathbf{P}}_{x_1\cdots x_M}^{a_1\cdots a_M}F(x_1\cdots x_M) \tag{5.6}$$

indicates the sum over the $M!$ terms $F(a_1,a_2\cdots)+F(a_2,a_1\cdots)+\cdots$ arising from inserting for $x_1\cdots x_M$ all $M!$ permutations of the elements $a_1\cdots a_M$ into F. By convention, the "permutation sum" (5.6) has the values 1 for $M=0$ and $F(a_1)$ for $M=1$.

In order to explore the potentials of (5.4), we calculate the various parts of the system (2.20) with the α's replaced by the expressions (5.4). The left-hand side of (2.20) leads so to an expression $\mathfrak{L}$ given by

$$\mathfrak{L}=\epsilon^{-1}(\mathbf{k}_1\cdots\mathbf{k}_M)\left(\prod_{l'=1}^{M}\frac{e^{-it(\Delta-k_{l'})}}{(2Vk_{l'})^{1/2}}C_{\mathbf{k}_{l'}}\right)p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$$

$$\times\left(\sum_{\nu=1}^{M}(\Delta-k_\nu)+i\frac{d}{dt}\right)\beta_{k_1\cdots k_M}(t). \tag{5.7}$$

Denoting the double sum $\epsilon^{-1}\sum_{\nu=1}{}^{M}\sum_{l=1}{}^{M}\cdots$ on the right-hand side of (2.20) by $\mathfrak{R}_2$, we obtain by the same process the expression

$$\mathfrak{R}_2=\epsilon^{-1}(\mathbf{k}_1\cdots\mathbf{k}_M)\left(\prod_{l'=1}^{M}\frac{e^{-it(\Delta-k_{l'})}}{(2Vk_{l'})^{1/2}}C_{\mathbf{k}_{l'}}\right)$$

$$\times p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}\sum_{l=1}^{M}\beta_{k_1\cdots k_{l-1},k_{l+1}\cdots k_M}(t). \tag{5.8}$$

The appearance in $\mathfrak{R}_2$ of $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$, which has been built up from the p's corresponding to $M-1$ photons by carrying out the sum $(\sum_\nu)$, is a sign that the ansatz (5.4) fits to some degree to the structure of (2.20). $\mathfrak{L}$ and $\mathfrak{R}_2$ could be equated without any trouble. Trouble arises only from the first sum

$$\epsilon^{-1}\sum_{\nu=1}^{N-M}\sum_{\mathbf{k}}$$

on the right-hand side of (2.20), for which the ansatz (5.4) yields an expression $\mathfrak{R}_1$ given by

$$\mathfrak{R}_1=\epsilon^{-1}(\mathbf{k}_1\cdots\mathbf{k}_M)\left(\prod_{l'=1}^{M}\frac{e^{-it(\Delta-k_{l'})}}{(2Vk_{l'})^{1/2}}C_{\mathbf{k}_{l'}}\right)$$

$$\times\frac{\Gamma_0}{\pi}\int_{-\infty}^{\infty}\beta_{k_1\cdots k_M,k}(t)dk$$

$$\times\mathop{\mathbf{P}}_{\kappa_1\cdots\kappa_M}^{\mathbf{k}_1\cdots\mathbf{k}_M}\exp[i(\mathbf{X}_{j_1}\cdot\boldsymbol{\kappa}_1+\cdots+\mathbf{X}_{j_M}\cdot\boldsymbol{\kappa}_M)]$$

$$\times(N-M+S_{\kappa_1\cdots\kappa_M}{}^{j_1\cdots j_M}), \tag{5.9}$$

with

$$S_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}=\frac{1}{\Gamma_0}\sum_{\nu=1}^{N-M}\sum_{\mu=1}^{M}\Gamma_{j_\mu j_\nu}e^{i\mathbf{k}_\mu\cdot(\mathbf{X}_{j_\mu}-\mathbf{X}_{j_\nu})}. \tag{5.10}$$

Here, we have carried out the integration over $\mathbf{k}$ in the same way as in the corresponding transition (3.13) $\rightarrow$ (3.14) for $N=1$ and in the derivation of Eqs. (4.4) for $N=2$, i.e., it was assumed that $\beta_{k_1\cdots k_M}(t)$ is appreciably different from 0 only in a small neighborhood of the point $k_1=\Delta$, $k_2=\Delta$, $\cdots$, $k_M=\Delta$.

The crucial point is the appearance of $S_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ in (5.9) which shows that the ansatz (5.4) leads in principle to certain inconsistencies, since in the equations

$$\mathfrak{L}=\mathfrak{R}_1+\mathfrak{R}_2\ (?) \tag{5.11}$$

corresponding to (2.20) the dependence on the directions of the vectors $\mathbf{k}_1\cdots\mathbf{k}_M$ does not cancel out. However, if $S_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ can be replaced with sufficient accuracy by a symmetric function of the photon energies $k_1\cdots k_M$, which may depend on $\mathfrak{V}$, say,

$$S_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}\rightarrow(N-M)M\gamma(k_1\cdots k_M), \tag{5.12}$$

we can take the expression

$$[N-M+S]\rightarrow[(N-M)(1+\gamma M)]$$

out of the operator

$$\mathop{\mathbf{P}}_{\kappa_1\cdots\kappa_M}^{k_1\cdots k_M},$$

which again leads to a factor $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ in $\mathfrak{R}_1$ as in $\mathfrak{L}$ and $\mathfrak{R}_2$. Assuming that (5.12) holds, we obtain for for the β's the system of equations

$$\left(\sum_{\nu=1}^{M}(\Delta-k_\nu)+i\frac{d}{dt}\right)\beta_{k_1\cdots k_M}(t)=\sum_{l=1}^{M}\beta_{k_1\cdots k_{l-1},k_{l+1}\cdots k_M}(t)$$

$$+(N-M)(1+M\gamma(k_1\cdots k_M))\frac{\Gamma_0}{\pi}\int_{-\infty}^{\infty}\beta_{k_1\cdots k_M,k}(t)dk,$$

$$M=0,1,\cdots,N \tag{5.13}$$

with initial conditions

$$\beta_0(0)=1\,,\ \beta_{k_1}(0)=\beta_{k_1k_2}(0)=\cdots=\beta_{k_1\cdots k_N}(0)=0. \quad (5.14)$$

The proper solution of (5.13) depends, of course, on $\gamma(k_1\cdots k_M)$ and hence on $\mathcal{V}$, and meets all the other requirements of ansatz (5.4).

Therefore, in order to use (5.4), we have to justify the replacement (5.12) and this justification will constitute the proof that the ansatz (5.4) leads actually to an approximation of the solution of (2.20). No justification is necessary in the limits $\Gamma_{ij}\to\delta_{ij}\Gamma_0$ and $\Gamma_{ij}\to\Gamma_0$ discussed in (3.7), for which we obtain $\gamma=0$ and $\gamma=1$, respectively. So we are again led to the problem of an approximate interpolation between the limits (3.7), but now we have the proper means to achieve this interpolation by calculating the proper function $\gamma(k_1\cdots k_M)$, which must satisfy $0\leq\gamma\leq1$. The justification of (5.12) and the definition of $\gamma(k_1\cdots k_M)$ will be given in two steps, (A) and (B), to follow below.

(A) First, we make the approximation

$$S_{\kappa_1\cdots\kappa_M}{}^{j_1\cdots j_M}\to\frac{N-M}{\Gamma_0}\frac{1}{\mathcal{V}^*}$$

$$\times\int_{\mathcal{V}}d^3y\{\sum_{\mu=1}^{M}\Gamma(\mathbf{X}_{j_\mu}-\mathbf{y})\exp[i\boldsymbol{\kappa}_\mu\cdot(\mathbf{X}_{j_\mu}-\mathbf{y})]\}\,, \quad (5.15)$$

where $\mathcal{V}^*$ denotes the volume of $\mathcal{V}$. Here, we have approximated the summation over $\mathbf{X}_{j_\nu}$ by a space integral over $\mathcal{V}$, where the factor $(N-M)/\mathcal{V}^*$ takes care of the proper normalization of this process. This is justified as long as the $(N-M)$ points $\mathbf{X}_{j_\nu}$ are distributed randomly over $\mathcal{V}$, and $N-M$ is a large number. We shall take care of exceptional cases below.

Second, in a similar way we replace the sum over $\mathbf{X}_{j_\mu}$ by a space integral over $\mathcal{V}$. To this end, we consider first only the case of large M, and distribute the photons $\mathbf{k}_1\cdots\mathbf{k}_M$ appearing in some permutation $\boldsymbol{\kappa}_1\cdots\boldsymbol{\kappa}_M$ in (5.15) over a number of classes i, which are defined by the condition that

$$\mathbf{k}_\rho\approx\mathbf{k}^{(i)} \quad (5.16)$$

holds with good accuracy for each photon $\mathbf{k}_\rho$ belonging to class i, where $\mathbf{k}^{(i)}$ is a given fixed wave vector characterizing class i. The numbers of such classes can be kept small in comparison with the number M of photons under consideration, such that at least for the leading classes the number m_i of photons in one class i is still a large number. The $\mathbf{X}_{j_\mu}$ corresponding to the $\boldsymbol{\kappa}_\mu$'s belonging to class i are then still distributed randomly over $\mathcal{V}$, and this holds for any permutation $\boldsymbol{\kappa}_1\cdots\boldsymbol{\kappa}_M$ of $\mathbf{k}_1\cdots\mathbf{k}_M$. Hence, we can approximate the partial sum over $\mathbf{X}_{j_\mu}$ corresponding to the $\mathbf{k}_\rho$'s of class i by the expression

$$\frac{m_i}{\mathcal{V}^*}\int_{\mathcal{V}}d^3x\left(\frac{N-M}{\mathcal{V}^*}\int_{\mathcal{V}}d^3y\,\Gamma(\mathbf{x}-\mathbf{y})e^{i\mathbf{k}^{(i)}\cdot(\mathbf{x}-\mathbf{y})}\right) \quad (5.17)$$

and the whole sum over $\mathbf{X}_{j_\mu}$ will be obtained by summing up the contributions of all classes i. This yields

$$S_{\kappa_1\cdots\kappa_M}{}^{j_1\cdots j_M}\to\frac{N-M}{\Gamma_0}\sum_{l=1}^{M}\frac{1}{\mathcal{V}^*}\int_{\mathcal{V}}d^3x\,\frac{1}{\mathcal{V}^*}$$

$$\times\int_{\mathcal{V}}d^3y\,\Gamma(\mathbf{x}-\mathbf{y})e^{i\mathbf{k}_l\cdot(\mathbf{x}-\mathbf{y})}\,, \quad (5.18)$$

where the terms (5.17) appear as m_i single terms in the above sum and the classes have been dissolved formally by the replacement $\mathbf{k}^{(i)}\to\mathbf{k}_l$, which only reverses (5.16).

The accuracy of the combined approximations (5.15), (5.17), and (5.18) is excellent for any reasonable case even in the exceptional cases of small $N-M$ and small M. In the former case, we can carry out the approximations (5.17) and (5.18) first and with high accuracy. After this, the integrand of (5.15) is a smooth function which varies only over distances comparable to the linear dimensions of $\mathcal{V}$ and the y integration will be a good approximation for the summation over $\mathbf{X}_{j_\nu}$ as long as there are at least several such points in $\mathcal{V}$. Only for very small values of $N-M$ is the approximation (5.15) poor, but states with very few excited atoms are governed by the de-excited atoms and an erroneous treatment of the excited atoms can have no influence on the behavior of the system. This is related to such problems as "which atom decays last" and belongs to those inessential details which must and can be eliminated in a reasonable approximation. In the latter case, for small M and hence large $N-M$, we exchange integration and summation in (5.15) such that the integrand of (5.15) becomes smooth over distances comparable to the dimensions of $\mathcal{V}$ and the x integration becomes again a reasonable approximation for the corresponding summation. Besides, the value of S does not count in comparison with $N-M$ for small M, and we need not worry over this limit. The over-all accuracy of the above approximation depends a little on the density of the active atoms and is poorer at low densities, where the average minimal distance between any pair A_iA_j is very large in comparison with λ, but this case is not very interesting in practice. The above approximations are poor only for those states $|j_1\cdots j_M;\mathbf{k}_1\cdots\mathbf{k}_M\rangle$ in which the excited or the de-excited atoms fill out only a macroscopic part of $\mathcal{V}$ with a volume comparable to $\mathcal{V}^*$. But the number of such exceptional states is so small in comparison with the number of "regular" states, where the excited and de-excited atoms are "well mixed," that inaccuracies in the treatment of these exceptional states can also have no influence on the over-all behavior of the system. Therefore, we consider the above approximations as satisfying.

The above step (A) leads to an approximation (5.18) for $S_{\kappa_1\cdots\kappa_M}{}^{j_1\cdots j_M}$, which is already symmetric in $\boldsymbol{\kappa}_1\cdots\boldsymbol{\kappa}_M$ and can therefore be taken out of the permutation sum in (5.9). If $\mathcal{V}$ is a sphere, the expression (5.18) is actually

independent of the directions of the $\mathbf{k}_l$'s, and therefore satisfies all requirements of (5.12). For general $\mathcal{V}$, however, we need a further approximation to follow below.

(B) In the expression (5.18) we average over the directions of the $\mathbf{k}_l$'s, which leads to the definition

$$\gamma(k_1\cdots k_M)=\frac{1}{M}\sum_{l=1}^{M}\gamma(k_l) \tag{5.19}$$

of γ, as introduced in (5.12), with

$$\gamma(k_l)=\frac{1}{\mathcal{V}^*}\int_{\mathcal{V}}d^3x\,\frac{1}{\mathcal{V}^*}\times\int_{\mathcal{V}}d^3y\,\frac{\Gamma(\mathbf{x}-\mathbf{y})}{\Gamma_0}\,\frac{\sin k_l|\mathbf{x}-\mathbf{y}|}{k_l|\mathbf{x}-\mathbf{y}|}. \tag{5.20}$$

In general, a term of the sum in (5.19) differs from a corresponding term of the sum in (5.18) by a $\mathcal{V}$-dependent factor of magnitude 1. This could lead to important variations of $S_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$, as given by (5.18), only if a large number of $\mathbf{k}_l$'s is rotated at the same time. However, these variations of S must be compared to the variations of $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ upon the same rotation of the same $\mathbf{k}_l$'s, and we shall see later that $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ responds in the physically significant region of the phase space of M photons $\mathbf{k}_1\cdots\mathbf{k}_M$ to a significant rotation of one single $\mathbf{k}_l$ by a factor η different from 1 by many orders of magnitude, and by a factor $\sim\eta^n$, if n $\mathbf{k}_l$'s are rotated at the same time. In general, a rotation of one $\mathbf{k}_l$ affects a fraction of $M-1$ in M terms of $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ and only a fraction of 1 in M of the terms in $S_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$. We use this different behavior of S and p for the following justification of step (B).

Assume that the presence of the factor $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ in the solution of (2.20), as introduced in (5.4), is completely false. Then the variations of $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ upon a rotation of $\mathbf{k}_l$'s would have to be compensated by a factor of the β's with variations comparable to those of $(p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M})^{-1}$. The necessity of such a factor would lead to correspondingly striking inconsistencies in Eqs. (4.11), which could arise only from an $S_{\kappa_1\cdots\kappa_M}{}^{j_1\cdots j_M}$ with variations comparable to those of $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$. The actually very weak variations of S can in no way lead to such a compensation, and it is hence a justifiable step of approximation to disregard them completely, as it was done in the transition from (5.18) to (5.20). Therefore, we can consider expression (5.4), with β obeying Eqs. (5.13) and with γ being defined by (5.19), as a reasonable approximation to the actual solution of (2.20). The accuracy of this approximation, as far as steps (A) and (B) are involved, is comparable to the accuracy of thermostatistics and hence beyond reasonable doubt.

The physical significance of the above approximation lies in the separation of the directional distribution of the emitted photons, which is contained in the factors $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$, from their energy distribution, which is influenced also by the β's and from the behavior of the system in time, which is completely contained in the β's. Though many important results can be derived from $p_{\mathbf{k}_1\cdots\mathbf{k}_N}$ alone, it is still interesting to get at least an idea of the explicit form of the functions $\beta_{k_1\cdots k_M}(t)$, $M=0,\ 1,\ \cdots,N$.

To achieve this, we look at the function $\gamma(k_1\cdots k_M)$ which has an important influence on the β's. If $\mathcal{V}$ is a sphere of radius r, the function $\gamma(k)$ can be calculated explicitly for the dipole radiation law (3.6). We obtain

$$\gamma(k)=\tfrac{1}{2}\int_{-1}^{1}d\zeta\{3[\sin ra(\zeta)-ra(\zeta)\cos ra(\zeta)]/[ra(\zeta)]^3\}^2, \tag{5.21}$$

with

$$a(\zeta)=(\Delta^2+k^2+2k\Delta\zeta)^{1/2}.$$

For $k=0$ we have $a(\zeta)=\Delta$ and the integrand does not depend on ζ. Neglecting terms which vary rapidly with $r\Delta\gg1$, we obtain

$$\gamma(0)\approx 9/2(r\Delta)^4 \quad \text{for} \quad r\Delta\gg1. \tag{5.22}$$

For $k=\Delta$ the integral (5.20) can be transformed to

$$\gamma(\Delta)=\int_0^1 d\zeta'\zeta'[3(\sin r\Delta\zeta'-r\Delta\zeta'\cos r\Delta\zeta')/(r\Delta\zeta')^3]^2. \tag{5.21'}$$

The main contributions to this integral come from the region $0\leq\zeta'\lesssim\zeta_0$, where ζ_0 is of the order of magnitude $\sim1/r\Delta$. Therefore we have for $r\Delta\gg1$

$$\gamma(\Delta)\sim 1/(r\Delta)^2\gg\gamma(0). \tag{5.23}$$

We see further that this particular $\gamma(k)$ has a steep maximum at $k=\Delta$ which is due to the long-range term $\sin\Delta|\mathbf{x}|/\Delta|\mathbf{x}|$ in the dipole radiation law (3.6). This maximum will be washed out a little if $\mathcal{V}$ has the shape of a rod, and it vanishes completely, in essence, if $\mathcal{V}$ is a one-dimensional straight line. A similar behavior must be expected for any realistic radiation law. Nevertheless, we put

$$\gamma(k)=\gamma(0)=\gamma=\text{const}, \tag{5.24}$$

and we shall see that this is the only choice for the value of $\gamma(k)$ if we want to neglect its k dependence. In assuming (5.24) we given up the claim to obtain the "fine structure" of the emission line; in fact, we obtain only the "best Lorentzian fit" to the actual solution of Eqs. (5.13). However, we also obtain important results on the time development of the system, which justify our approach even if certain details should have been lost in the approximation (5.24).

Let us look first at the case $N=2$. We see from (5.10), that only $S_{\mathbf{k}_1}{}^{j_1}$ is involved in this problem, and bypassing

step (A), which is not applicable to this simple system, we obtain from step (B)

$$\gamma(k_1)=\frac{\Gamma(\mathbf{X}_1-\mathbf{X}_2)}{\Gamma_0}\frac{\sin k_1|\mathbf{X}_1-\mathbf{X}_2|}{k_1|\mathbf{X}_1-\mathbf{X}_2|}, \tag{5.25}$$

and hence from (5.24)

$$\gamma=\gamma(0)=\Gamma_{12}/\Gamma_0, \tag{5.26}$$

which is in agreement with our former definition (4.9) of γ.

Equations (5.13) and (5.14) become

$$\begin{aligned}
&i\frac{d}{dt}\beta_0(t)=2\frac{\Gamma_0}{\pi}\int_{-\infty}^{\infty}\beta_{k_1}(t)dk_1,\\
&\left(\Delta-k_1+i\frac{d}{dt}\right)\beta_{k_1}(t)=\beta_0(t)+1(+\gamma)\frac{\Gamma_0}{\pi}\int_{-\infty}^{\infty}\beta_{k_1k_2}(t)dk_2,\\
&\left(\Delta-k_1+\Delta-k_2+i\frac{d}{dt}\right)\beta_{k_1k_2}(t)=\beta_{k_1}(t)+\beta_{k_2}(t),
\end{aligned} \tag{5.27}$$

with the initial condition

$$\beta_0(0)=1,\quad \beta_{k_1}(0)=\beta_{k_1k_2}(0)=0. \tag{5.28}$$

Surprisingly enough, the exact solution of Eqs. (5.27) is given by expressions (4.10), which were proven to constitute a very good approximation to the formal solution of (2.20) for $N=2$. This means that the result obtained from (5.4) and the succeeding approximations is much better than could be expected at the outset, and this is so in spite of the fact that the above approximations are far from convincing for $N=2$. The least gain from this result should be an additional confidence in the approximations as developed so far.

Since expressions (4.10) are rather complicated even for the simplest case $N=2$, we expect that the formal solution of (5.13), even with (5.24) taken into account, is still too complicated for practical purposes. However, at least for values of γ and N obeying

$$e^{-1/2}\left(\frac{1+\gamma(N-1)}{1+\gamma(N-2)}\right)^{\frac{1}{2}(1/\gamma-1)}\approx 1, \tag{5.29}$$

the solution of (5.13) is approximated with at least the accuracy of the approximation $N!\approx(N/e)^N$ by the expressions

$$\begin{aligned}
\beta_{k_1\cdots k_M}(t)&=T_M e^{-\Gamma_0 t(N-M)(1+\gamma M)}\\
&\times\prod_{l=1}^{M}\frac{e^{-\Gamma_0 t[1+\gamma(2M-N-1)]}-e^{i(\Delta-k_l)t}}{\Delta-k_l-i\Gamma_0[1+\gamma(2M-N-1)]},\\
&M=0,1,\cdots,N
\end{aligned} \tag{5.30}$$

where $T_M=T_M(\gamma)$ is a normalization factor given by

$$T_M^{-2}(\gamma)=[1+\gamma(M-1)]^{(1/\gamma-1)}e^{-(M-1)},\quad T_0=1. \tag{5.31}$$

If (5.29) is violated, the accuracy of the approximation (5.30) is still better than the accuracy of the formula $N!\approx N^N$, which is usually a sufficient approximation in statistical physics.

To prove the above statements we insert (5.30) into (5.13) and use, at the appropriate places, the approximation (5.29) as well as

$$\begin{aligned}
&e^{\pm\Gamma_0 t\gamma}\approx 1,\\
&1+\gamma(2M-N-1)\approx 1+\gamma(2M-N-1)\pm 2\gamma,
\end{aligned} \tag{5.32}$$

and the assumption that $C_k(\mathbf{X}_i-\mathbf{X}_j)$ can be considered constant in the interval

$$\Delta-\Gamma_0(1+(N-1)\gamma)\lesssim k\lesssim\Delta+\Gamma_0(1+(N-1)\gamma). \tag{5.33}$$

Under these conditions, we obtain from (5.13) an identity, which includes especially the complete proof that (5.30) gives the exact solution of (5.13) for $\gamma=0$. Equation (5.32) does not impose a serious restriction on the validity of (5.30) for realistic values of γ; violations of (5.29) refer only to the normalization of $\beta_{k_1\cdots k_M}$ and lead only to insignificant violations of the normalization condition (2.23). Hence, only (5.33) might be serious, but by a proper redefinition of Γ_0 we would even repair most damage introduced by a violation of (5.33). However, we shall see later that the exact width of $\beta_{k_1\cdots k_M}(t)$ is insignificant once it exceeds a certain value and so we shall not care about the fact that (5.33) might be violated. It is more important to stress the positive features of expressions (5.30), which solve all $N+1$ equations (5.13) with about the same accuracy, assume the correct initial values (5.14) at $t=0$, and yield a nonvanishing limit for $t\to\infty$ only for the correct amplitude $\alpha_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2\cdots N}(t)$. Even more, $\beta_0(t)=\alpha_0{}^0(t)$ assumes the value $e^{-N\Gamma_0 t}$, as it should also in the formal solution of (2.20), as discussed in connection with (4.7).

To prove that (5.30) meets the normalization condition (2.23) with the accuracies stated above, we compute $|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2\cdots N}(\infty)|^2$ as follows from (5.4) and (5.30). We get

$$|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1,2\cdots N}(\infty)|^2=T_N{}^2(\gamma)\sum_{j_1\cdots j_N}^{1\cdots N}\sum_{j_1'\cdots j_N'}^{1\cdots N}\prod_{\nu=1}^{N}\frac{|C_{\mathbf{k}\nu}|^2e^{i\mathbf{k}_\nu(\mathbf{X}_{j\nu}-\mathbf{X}_{j'\nu})}}{2Vk_\nu[(\Delta-k_\nu)^2+\Gamma_0{}^2(1+\gamma(N-1))^2]}. \tag{5.34}$$

Carrying out the summations over $\mathbf{k}_1\cdots\mathbf{k}_N$, as required by the right-hand side $\tilde{R}$ of (2.23), by means of (3.1) and (3.2) and the usual extension of the limits of integration over $\mathbf{k}_\nu$, we obtain for $\tilde{R}$ the expression

$$\tilde{R}=\frac{T_N{}^2}{N![\Gamma_0(1+\gamma(N-1))]^N}\sum_{j_1\cdots j_N}^{1\cdots N}\sum_{j_1'\cdots j_N'}^{1\cdots N}\prod_{\nu=1}^{N}\Gamma(\mathbf{X}_{j_\nu}-\mathbf{X}_{j_\nu'}). \tag{5.35}$$

In the limit $\Gamma_{ij}\to\delta_{ij}\Gamma_0$, $\gamma\to 0$, we obtain $\tilde{R}=1$, as expected, but we obtain $\tilde{R}=1$ even for $\gamma\to 1$, $\Gamma_{ij}\to\Gamma_0$ with the accuracy of Stirling's formula $N!=(N/e)^N$, since the double permutation sum assumes the value $\Gamma_0{}^N(N!)^2$ in that limit. To prove $\tilde{R}=1$ for any value of γ, we observe that operator equation

$$\mathop{\mathsf{P}}_{j_1\cdots j_N}^{1\cdots N}\mathop{\mathsf{P}}_{j_1'\cdots j_{N'}}^{1\cdots N}(\ )=\sum_{j_1=1}^{N}\sum_{j_1'=1}^{N}\{\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N}\sum_{\substack{j_2'=1\\ j_2'\neq j_1'}}^{N}\{\cdots\{\sum_{\substack{j_N=1\\ j_N\neq j_1\\ \neq j_2\\ \vdots\\ \neq j_{N-1}}}^{N}\sum_{\substack{j_{N'}=1\\ j_{N'}\neq j_1'\\ \neq j_2'\\ \vdots\\ \neq j_{N-1}'}}^{N}(\ )\}\cdots\}\} \tag{5.36}$$

holds exactly. Since the summations go over the positions of the atoms, it should be feasible to approximate the Mth of the above double sums by a double integral

$$M^2\frac{1}{\mathfrak{V}^*}\int_{\mathfrak{V}}d^3x_M\frac{1}{\mathfrak{V}^*}\int_{\mathfrak{V}}d^3y_M(\ ), \tag{5.37}$$

with a slight modification due to the nature of the diagonal elements of Γ_{ij} to be discussed below. This approximation is based on the same principles as the former step (A) and takes into account correctly that the Mth double sum has M^2 terms. It makes the Mth double sum independent of $j_1\cdots j_{M-1}$, $j_1'\cdots j_{M-1}'$ and yields thus automatically a factorization of the double permutation sum in the form

$$\mathop{\mathsf{P}}_{j_1\cdots j_N}^{1\cdots N}\mathop{\mathsf{P}}_{j_1'\cdots j_{N'}}^{1\cdots N}\prod_{\nu=1}^{N}\frac{\Gamma(\mathbf{X}_{j_\nu}-\mathbf{X}_{j_\nu'})}{\Gamma_0}=N!\prod_{M=1}^{N}[1+\gamma(M-1)]$$

$$=N![1+\gamma(N-1)]^N T_N{}^{-2}(\gamma), \tag{5.38}$$

with

$$\gamma=\frac{1}{\mathfrak{V}^*}\int_{\mathfrak{V}}d^3x\frac{1}{\mathfrak{V}^*}\int_{\mathfrak{V}}d^3x\frac{\Gamma(\mathbf{x}-\mathbf{y})}{\Gamma_0}. \tag{5.39}$$

We have written the factor occurring from (5.37) in the form $M[1+(M-1)\gamma]$ in order to take care of the fact that the diagonal elements of any $M\times M$ matrix Γ_{ij} give the contribution $M\Gamma_0$ to a double sum over all its elements, while in the integral (5.39), the points $\mathbf{x}=\mathbf{y}$ have measure zero and give no contribution to γ. The second equation (5.38) is obtained by approximating in the usual way the sum

$$\sum_{M=1}^{N}\ln[1+(M-1)\gamma]$$

by the integral

$$\int_0^{N-1}dx\ln(1+x\gamma).$$

This yields still Stirling's formula $N!\approx N^N e^{-N}$ in the worst case $\gamma=1$.

Inserting (5.38) into (5.35), we obtain $\tilde{R}=1$, which proves the statements made about the accuracy of (5.30) under various conditions. If we had replaced T_M by 1 in (5.30), the condition (5.29) would not have arisen, but the normalization of $\beta_{k_1\cdots k_N}(\infty)$ would have been poorer, yielding $\tilde{R}\approx 1$ only in the approximation $N!\approx N^N$, which is of course still quite satisfactory. The inclusion of $T_M(\gamma)$ in (5.30) improves $\beta_{k_1\cdots k_M}(t)$ slightly, as is seen from the case $N=2$, with $T_2{}^2=(1+\gamma)\times e(1+\gamma)^{-1/\gamma}\approx 1+\gamma$, replacing satisfactorily the first factor of formula (4.12). This remark shows even what has been lost in the approximation (5.30) in comparison with the exact solution of (5.13) with (5.24) taken into account: It yields equal probabilities for the events "all N photons have energy $\Delta+\delta$" and "some photons have energy $\Delta+\delta$, the rest having $\Delta-\delta$." The exact solution should slightly favor the second type of event, as is the case for $N=2$. Hence, we expect that approximation (5.30) will "conserve the energy" a little worse than the formal solution of (5.13).

It was also proved that γ must have the value (5.39), which is equivalent to the restriction (5.24) of $\gamma(k_1\cdots k_M)$. For shapes of $\mathfrak{V}$ other than a long rod, the expressions (5.30) can be considered at least as the properly normalized best Lorentzian fit to the actual solution of the unrestricted equations (5.13). This follows from the fact that (5.38) and $\tilde{R}=1$ will hold in any case, such that the normalization of $|\beta_{k_1\cdots k_N}(\infty)|^2$ must in any case cancel the expression (5.38). On purely physical reasons, $|\beta_{k_1\cdots k_N}(\infty)|^2$ must in any case have a maximum at $k_1=\cdots=k_N=\Delta$. Hence, especially in view of the fact that the physical meaning of $\beta_{k_1\cdots k_N}(\infty)$ becomes rather insignificant as its width exceeds a certain value to be discussed later, the above Lorentzian fit should be quite satisfactory. What will be needed in any case is only the fact that $|\beta|^2$ has a maximum at $k_1=\cdots=k_N=\Delta$, even if that maximum is very flat, as in the case of a very broad $|\beta_{k_1\cdots k_N}(\infty)|^2$.

VI. PROPERTIES OF THE FUNCTION $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ FOR LARGE M

The appearance of the factor $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ in the approximate solution of Eqs. (2.20) is the most striking and the most important result obtained in the preceding chapter. Since $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ governs the act of focusing a ray in the emission of radiation from the system of excited atoms almost completely, it seems desirable to collect its most important properties in this separate section.

We shall see that $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ appears in all results in the combination

$$P_{\mathbf{k}_1\cdots\mathbf{k}_M}=\frac{1}{M!}|p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}|^2, \tag{6.1}$$

where the superscripts $j_1\cdots j_M$ have been omitted for simplicity. Hence, we can restrict our discussion to that function. From the definition (5.5) of $p_{\kappa_1\cdots\kappa_M}{}^{j_1\cdots j_M}$, we obtain the formulas

$$\begin{aligned}P_{\mathbf{k}_1\cdots\mathbf{k}_M}&=\frac{1}{M!}\sum_{\kappa_1\cdots\kappa_M}^{\mathbf{k}_1\cdots\mathbf{k}_M}\sum_{\kappa_1'\cdots\kappa_M'}^{\mathbf{k}_1\cdots\mathbf{k}_M}\prod_{\rho=1}^{M}\exp[i\mathbf{X}_{j_\rho}\cdot(\boldsymbol{\kappa}_\rho-\boldsymbol{\kappa}_\rho')]\\&=\frac{1}{M!}\sum_{\kappa_1\cdots\kappa_M}^{\mathbf{k}_1\cdots\mathbf{k}_M}\sum_{\kappa_1'\cdots\kappa_M'}^{\mathbf{k}_1\cdots\mathbf{k}_M}\cos[\mathbf{X}_{j_1}\cdot(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_1')\\&\qquad+\cdots+\mathbf{X}_{j_M}(\boldsymbol{\kappa}_M-\boldsymbol{\kappa}_M')]\\&=\frac{1}{M!}\sum_{\nu_1\cdots\nu_M}^{j_1\cdots j_M}\sum_{\nu_1'\cdots\nu_M'}^{j_1\cdots j_M}\prod_{\rho=1}^{M}\exp[i\mathbf{k}_\rho\cdot(\mathbf{X}_{\nu_\rho}-\mathbf{X}_{\nu_\rho'})]\\&=\frac{1}{M!}\sum_{\nu_1\cdots\nu_M}^{j_1\cdots j_M}\sum_{\nu_1'\cdots\nu_M'}^{j_1\cdots j_M}\cos[\mathbf{k}_1\cdot(\mathbf{X}_{\nu_1}-\mathbf{X}_{\nu_1'})\\&\qquad+\cdots+\mathbf{k}_M\cdot(\mathbf{X}_{\nu_M}-\mathbf{X}_{\nu_M'})].\end{aligned} \tag{6.2}$$

These formulas reveal immediately that $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ is invariant under the translations

$$\mathbf{X}_{j_\nu}\to\mathbf{X}_{j_\nu}+\mathbf{X},\quad \nu=1\cdots M \tag{6.3a}$$

and

$$\mathbf{k}_l\to\mathbf{k}_l+\mathbf{k},\quad l=1\cdots M. \tag{6.3b}$$

The invariance under the translation (6.3a) is obviously a consequence of the invariance of the system under translations. Hence, we note that the phase of $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ can have no physical significance, and no physics has been lost in the transition from $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$ to $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$. The invariance under the transformation (6.3b) is obviously a consequence of (6.3a) and the symmetric structure of the formulas (6.2). It implies physically, that the energies of the emitted photons, or better, the emission line under consideration, are not determined by $p_{\mathbf{k}_1\cdots\mathbf{k}_M}{}^{j_1\cdots j_M}$. This line is contained in the factors $\beta_{k_1\cdots k_M}(t)$, as we shall see later.

The most important property of $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ is its capability of assuming very high maxima for certain points $(\mathbf{k}_1{}^0\cdots\mathbf{k}_M{}^0)$ of the $3M$-dimensional phase space $\mathfrak{K}^M$ of M photons. Indeed, the double permutation sums (6.2) contain $(M!)^2$ terms with maximal value 1 each. The necessary and sufficient condition for $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ to assume its maximal value $M!$ in a point $(\mathbf{k}_1{}^0\cdots\mathbf{k}_M{}^0)$ is

$$(\mathbf{X}_{j_\mu}-\mathbf{X}_{j_\nu})\cdot(\mathbf{k}_\mu{}^0-\mathbf{k}_\nu{}^0)=0 \text{ for any } \mu,\ \nu=1,\ 2\cdots M, \text{ and any } j_\mu,\ j_\nu. \tag{6.4}$$

The necessity of (6.4) is proved as follows: If $(\nu_1\cdots\nu_M)$ is a given arbitrary permutation of the numbers $j_1\cdots j_M$, and if $(\nu_1'\cdots\nu_M')$ is the permutation obtained from $(\nu_1\cdots\nu_M)$ by the exchange of one arbitrary pair of numbers, say, $\nu_\rho\rightleftarrows\nu_\sigma$, the contribution of the pair of permutations $\{(\nu_1\cdots\nu_M)$ and $(\nu_1'\cdots\nu_M')\}$ to the double sum (6.2) is $\cos[\mathbf{k}_\rho(\mathbf{X}_{\nu_\rho}-\mathbf{X}_{\nu_\sigma})+\mathbf{k}_\sigma(\mathbf{X}_{\nu_\sigma}-\mathbf{X}_{\nu_\rho})]$. This contribution is maximal only if (6.4) holds at least "modulo 2π," but "modulo 2π" allows for no additional solutions of (6.3), if $M>2$ and if the points $\mathbf{X}_{j_1}\cdots\mathbf{X}_{j_M}$ are not contained in a regular lattice, which we assume true in all our applications. The sufficiency of (6.4) is proved by showing that the solutions of (6.4) lead indeed to maxima. This, in turn, can be seen "by inspection" from the explicit solutions of (6.4) to be discussed below:

If $\mathfrak{V}$ reduces to a "point" in comparison with the wavelength λ, which corresponds to our former limit $\Gamma_{ij}=\Gamma_0$ and $\gamma=1$, we have $P_{\mathbf{k}_1\cdots\mathbf{k}_M}=M!$ for all points of $\mathfrak{K}^M$. In this case, $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ acts only as a normalization constant in the solution of (2.20).

If $\mathfrak{V}$ is a straight line g ("laser line") of any length $2a$, the manifold of points $(\mathbf{k}_1{}^0\cdots\mathbf{k}_M{}^0)$ solving (6.4) is given by

$$\mathbf{k}_l{}^0=\mathbf{k}+a_l{}^1\mathbf{n}_1+a_l{}^2\mathbf{n}_2,\quad l=1\cdots M \tag{6.5}$$

where $\mathbf{k}$ is an arbitrary but fixed wave vector, $a_l{}^{1,2}$ are arbitrary energies, and $\mathbf{n}_{1,2}$ are unit normal vectors of g, which are orthogonal to each other. The points $\mathbf{k}_l{}^0$ lie in that normal plane $h_g{}^0$ of g, which contains the point $\mathbf{k}$.

If $\mathfrak{V}$ is contained in a plane h, the solutions of (6.4) are

$$\mathbf{k}_l{}^0=\mathbf{k}+a_l\mathbf{n},\quad l=1\cdots M \tag{6.6}$$

where $\mathbf{k}$ and a_l are again arbitrary, and $\mathbf{n}$ is the unit normal vector to the laser plane h. The points $\mathbf{k}_l$ lie on the straight line $g_h{}^0$ normal to h which contains the point $\mathbf{k}$.

If $\mathfrak{V}$ is an arbitrary three-dimensional volume, the only solution of (6.4) is

$$\mathbf{k}_l{}^0=\mathbf{k},\quad l=1\cdots M \tag{6.7}$$

with an arbitrary wave vector $\mathbf{k}$, which yields only one point $\mathbf{k}$ for any $\mathbf{k}_l{}^0$.

It is easily seen that $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ assumes actually its maximal value $M!$ in the points $(\mathbf{k}_1{}^0\cdots\mathbf{k}_M{}^0)$ defined in (6.5)–(6.7). However, we need more details about the structure of the function $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ around these points, and to obtain these details, we consider first the variation of $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ under a change of arguments from $(\mathbf{k}_1{}^0\cdots\mathbf{k}_M{}^0)=(\mathbf{k}\cdots\mathbf{k})$ to $(\mathbf{k}_1{}^0\cdots\mathbf{k}_M{}^0)=(\mathbf{k}\cdots\mathbf{k},\ \mathbf{k}+\mathbf{q})$. The value of $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ changes then from $M!$ to $M!\eta_M{}^2(\mathbf{q})$, with

$$\eta_M{}^2(\mathbf{q})=\frac{1}{M^2}\sum_{\nu=1}^{M}\sum_{\nu'=1}^{M}\cos\mathbf{q}\cdot(\mathbf{X}_{j_\nu}-\mathbf{X}_{j_{\nu'}}). \tag{6.8}$$

Assuming that the points $\mathbf{X}_{j_1}\cdots\mathbf{X}_{j_M}$ are distributed at random over $\mathcal{V}$, this double sum can be approximated by a double integral over $\mathcal{V}$ in the same way as (5.37) approximates the corresponding double sum in (5.36). This yields

$$\eta_M{}^2(\mathbf{q})=\frac{1}{M}+\frac{M-1}{M}\eta^2(\mathbf{q}) \tag{6.9}$$

with

$$\eta^2(\mathbf{q})=\frac{1}{\mathcal{V}^*}\int_{\mathcal{V}}d^3x\,\frac{1}{\mathcal{V}^*}\int_{\mathcal{V}}d^3y\,\cos\mathbf{q}\cdot(\mathbf{x}-\mathbf{y}). \tag{6.10}$$

Here again we have taken care of the nonrandom diagonal elements 1 of the matrix $\cos\mathbf{q}\cdot(\mathbf{X}_i-\mathbf{X}_j)$. It follows from the symmetry of $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ with respect to its arguments $\mathbf{k}_1\cdots\mathbf{k}_M$, that a factorization of $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$ of the form

$$P_{\mathbf{k}_1\cdots\mathbf{k}_M}\approx M!\prod_{\nu=1}^{M}\eta_M{}^2(\mathbf{k}_\nu-\mathbf{k}_\nu{}^0) \tag{6.11}$$

must hold at least around any point $(\mathbf{k}_1{}^0\cdots\mathbf{k}_M{}^0)$ at which there is a maximum of $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$. However, if we apply the operator equation (5.36) to the third form (6.2) of $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$, and approximate the double sums by the double integrals of type (5.37), we again obtain (6.11), which is thus not restricted to a too narrow vicinity of a maximum. If in (6.11) we replace the point $(\mathbf{k}_1{}^0\cdots\mathbf{k}_M{}^0)$ by $(\mathbf{k}_{\text{c.m.}}\cdots\mathbf{k}_{\text{c.m.}})$, where $\mathbf{k}_{\text{c.m.}}$ is the center of the gravity of the vectors $\mathbf{k}_\nu$, i.e.,

$$\mathbf{k}_{\text{c.m.}}=\frac{1}{M}\sum_{\nu=1}^{M}\mathbf{k}_\nu, \tag{6.12}$$

the formula (6.11) yields the same maxima as the original function $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$, independently of the shape of $\mathcal{V}$, and it also has the invariances (6.3a) and (6.3b). It follows from this, that the approximation (6.11), (6.12) has satisfactory accuracy even far away from the maxima of $P_{\mathbf{k}_1\cdots\mathbf{k}_M}$.

The integral (6.10) can be calculated explicitly for at least two specific shapes of $\mathcal{V}$ of great physical interest. If $\mathcal{V}$ is an ellipsoid with semi-axes a, b, c, we get for all values of a, b, c the expression

$$\eta_{ell}(\mathbf{q})=\frac{3}{\xi^2}\left(\frac{\sin\xi}{\xi}-\cos\xi\right)\approx 1-\frac{\xi^2}{10}+\frac{\xi^4}{280}-O(\xi^6), \tag{6.13}$$

with

$$\xi^2=(aq_a)^2+(bq_b)^2+(cq_c)^2, \tag{6.14}$$

where $q_aq_bq_c$ are the components of $\mathbf{q}$ in the direction of the axes a, b, c. The analytic expression (6.13) for ξ can be expressed in terms of the spherical Bessel function of the first kind $j_1(\xi)$ by $3j_1(\xi)/\xi$. If $\mathcal{V}$ is a rectangular parallelepiped with semi-axes a, b, c, we get

$$\eta_{par}{}^2(\mathbf{q})=\frac{\sin^2aq_a}{(aq_a)^2}\,\frac{\sin^2bq_b}{(bq_b)^2}\,\frac{\sin^2cq_c}{(cq_c)^2}, \tag{6.15}$$

with an equivalent meaning of q_a, q_b, and q_c. The physical implications of Eqs. (6.9) to (6.15) will be discussed in Sec. IX. Here we note only that the difference between $\eta_M{}^2(\mathbf{q})$ and $\eta^2(\mathbf{q})$ becomes negligible for small $\mathbf{q}$, and only these refer to physically observable quantities. The following definitions are therefore made with $\eta^2(\mathbf{q})$ instead of $\eta_M{}^2(\mathbf{q})$.

In order to comply with later definitions, we call a bundle of $\mathbf{k}_l$'s with values close to a given $\mathbf{k}^0$ a "ray" $\mathbf{k}^0$. The "geometrical shape" of the ray $\mathbf{k}^0$ is defined as the three-dimensional region $u\{\mathbf{k}-\mathbf{k}^0\}$ around the point $\mathbf{k}^0$, which contains only points $\mathbf{k}$ such that

$$\eta^2(\mathbf{k}-\mathbf{k}^0)\geq\tfrac{1}{2} \tag{6.16}$$

holds for all points $\mathbf{k}$ of $u\{\mathbf{k}-\mathbf{k}^0\}$. The geometrical shape of ray $\mathbf{k}^0$ corresponding to the above ellipsoid (a,b,c) for $\mathcal{V}$ is the ellipsoid

$$\frac{(k_a-k_a{}^0)^2}{(\xi_0/a)^2}+\frac{(k_b-k_b{}^0)^2}{(\xi_0/b)^2}+\frac{(k_c-k_c{}^0)^2}{(\xi_0/c)^2}=1, \tag{6.17}$$

with ξ_0 being the solution of the equation

$$\frac{3}{\xi^2}\left(\frac{\sin\xi}{\xi}-\cos\xi\right)=\tfrac{1}{2}\sqrt{2}. \tag{6.18}$$

Using the approximation (6.13) of the above function of ξ, we obtain for ξ_0 the approximate value

$$\xi_0\approx[10(1-\tfrac{1}{2}\sqrt{2})]^{1/2}\approx 1.7. \tag{6.19}$$

If $\mathcal{V}$ has a shape different from an ellipsoid, $u\{\mathbf{k}-\mathbf{k}^0\}$ can still be constructed in the following way: We construct, first, all $u\{\mathbf{k}-\mathbf{k}^0\}=u_g\{\mathbf{k}-\mathbf{k}^0\}$ corresponding to the intersections of all straight lines g with $\mathcal{V}$. $u_g\{\mathbf{k}-\mathbf{k}^0\}$ is therefore the space between the parallel planes of h_g [cf. (6.5)] at distances $\pm\xi_0/a_g'$, where $2a_g'$ is the length of the intersection of g with $\mathcal{V}$. The $u(\mathbf{k}-\mathbf{k}^0\}$ for any simple shape of $\mathcal{V}$ is then

$$u\{\mathbf{k}-\mathbf{k}^0\}=\bigcap_g u_g\{\mathbf{k}-\mathbf{k}^0\}. \tag{6.20}$$

The sufficiency of (6.18) is trivial, but its necessity is also easily proved: If $\mathbf{k}'$ is a point outside $u\{\mathbf{k}-\mathbf{k}^0\}$, we can always find a g such that $\mathbf{k}'$ is also outside $u_g(\mathbf{k}-\mathbf{k}^0\}$, and this violates the condition (6.16). If convenient, as in the case of a laser disk, one can construct first $u_h\{k-k^0\}$ corresponding to the intersections of planes h with $\mathcal{V}$ and take later the intersectiono f all $u_h\{k-k^0\}$. If $\mathcal{V}$ is an ellipsoid, Eq. (6.20) yields the ellipsoid (6.17) as it must.

A remark should be included now on the statistical properties of the system. So far, it is still assumed that we know the positions $\mathbf{X}_1\cdots\mathbf{X}_N$ of our atoms. Actually, however, this information has been washed out in several approximations, the latest being Eq. (6.11). Therefore, if we consider a Gibbs ensemble of systems of N atoms at positions $\mathbf{X}_1\cdots\mathbf{X}_N$, described by a

properly chosen density function $f(\mathbf{X}_1\cdots\mathbf{X}_N)\geq 0$ in the "parameter Γ space" of the ensemble, and if we average over the members of the ensemble, we get no visible changes in the results to be discussed in the following sections. The reason for this is seen easily from (6.20), which can be used in two ways: First, if g is allowed to run continuously over all straight lines g having an intersection with $\mathcal{V}$, we get a smooth $u\{\mathbf{k}-\mathbf{k}^0\}$ of the type of the ellipsoid (6.15). However, if we restrict g to all actual "line lasers" contained in the system, i.e., to all actual diagonals of the points $\mathbf{X}_1\cdots\mathbf{X}_N$, that smooth $u\{\mathbf{k}-\mathbf{k}^0\}$ is replaced by a polyhedron with a very large number of corners, which encloses the smooth $u\{\mathbf{k}-\mathbf{k}^0\}$. A change of the microscopic structure $\mathbf{X}_1\cdots\mathbf{X}_N$ affects only the microscopic structure of that polyhedron, but the difference between that polyhedron and a smooth $u\{\mathbf{k}-\mathbf{k}^0\}$ cannot be observed on a macroscopic scale. For the present case, then, we have obtained a proof of Gibbs's hypothesis which states in general that the value of an observable (here the shape of the ray as described by a polyhedron) of a large system is macroscopically indistinguishable from the average value (here the smooth $u\{\mathbf{k}-\mathbf{k}_0\}$) of that observable over an ensemble of such systems.

VII. EXISTENCE AND STATIC PROPERTIES OF A RAY

We can now discuss the physical content of the approximate solution (5.4) and (5.30) of Eq. (2.20) without further mathematical digressions. In this section we look at the system at time $t=\infty$, while its evolution in time will be considered in Sec. VIII. Here, we shall prove first the existence of the many photon collective phenomenon, which has been called a "ray" in Sec. I. Afterwards we shall discuss in detail its most important properties.

Let us consider the ratio $R_{\mathbf{k}_1\cdots\mathbf{k}_N}$, as defined in Eq. (3.21). Combining the results of Eqs. (3.11) and (3.16) with those of (5.4) and (5.30), and using the definition (6.1) of $P_{\mathbf{k}_1\cdots\mathbf{k}_N}$, we obtain

$$R_{\mathbf{k}_1\cdots\mathbf{k}_N}=P_{\mathbf{k}_1\cdots\mathbf{k}_N}\Phi_{k_1\cdots k_N}(\gamma) \tag{7.1}$$

with

$$\Phi_{k_1\cdots k_N}(\gamma)=T_N(\gamma)\prod_{\nu=1}^{N}\frac{(\Delta-k_\nu)^2+\Gamma_0^2}{(\Delta-k_\nu)^2+\Gamma_0^2(1+s)^2}, \tag{7.2}$$

where $T_N(\gamma)$ is given in (5.31) and s is defined by

$$s=(N-1)\gamma=N\gamma=\frac{N}{\mathcal{V}^*}\times\frac{1}{\mathcal{V}^*}\int_{\mathcal{V}}d^3x\times\int_{\mathcal{V}}d^3y\,\frac{\Gamma(\mathbf{x}-\mathbf{y})}{\Gamma_0}. \tag{7.3}$$

As we should, we have $R_{\mathbf{k}_1}=1$ for any value of $\mathbf{k}_1$ and $\Phi_{k_1\cdots k_N}(0)=1$ for any N.

Since $\Gamma(\mathbf{x}-\mathbf{y})$ decreases rapidly with increasing $(\mathbf{x}-\mathbf{y})$, the integral over $\mathbf{y}$ will not depend too sensitively on $\mathcal{V}$ and $\mathbf{x}$. Therefore the integral over $\mathbf{x}$ will be roughly proportional to $\mathcal{V}^*$, and the whole second factor of (7.3) will depend rather insensitively on $\mathcal{V}$. s is therefore mainly a function of the density $N/\mathcal{V}^*$ of atoms and does not depend sensitively on the total number N of atoms in $\mathcal{V}^*$. The N single factors ϕ of $\Phi_{k_1\cdots k_N}(\gamma)$ have thus lower bounds and are actually independent of N. As a consequence of the capability of $P_{\mathbf{k}_1\cdots\mathbf{k}_N}$ to assume maxima of the height $N!\approx(N/e)^N$, $R_{\mathbf{k}_1\cdots\mathbf{k}_N}$ can assume maxima as high as $\sim(\phi N)^N$ for certain sets $(\mathbf{k}_1\cdots\mathbf{k}_N)$ of N photons, with $\phi\to e^{-1}$ for $s\to 0$. This must be ascribed physically to a very strong influence the atoms have on each other under the conditions of the present model, or to very strong effects of "self stimulation": By definition, $R_{\mathbf{k}_1\cdots\mathbf{k}_N}$ would have the value 1, if no such influence were present and the atoms could decay independently of each other.

To see immediate consequences of these effects, we compare the value of $R_{\mathbf{k}_1\cdots\mathbf{k}_N}$ for different sets of photons $(\mathbf{k}_1\cdots\mathbf{k}_N)$ of a given energy $k_1=\cdots=k_N=k^0$, such that $\phi_{k_1\cdots k_N}(\gamma)=\Phi_{k^0\cdots k^0}(\gamma)$ has the same value for all competing sets $(\mathbf{k}_1\cdots\mathbf{k}_N)$. We see from Sec. VI that $R_{\mathbf{k}_1\cdots\mathbf{k}_N}$ assumes its maximal value if and only if the $\mathbf{k}_l$'s are parallel. This means that the relative probabilities for the emission of N photons of above modes $(\mathbf{k}_1\cdots\mathbf{k}_N)$ have very high relative maxima if all photons are in the same mode. For example, it follows from (6.8) that the probability for the emission of N equal photons $\mathbf{k}^0$ is by a factor $\eta_N^{-2}(\mathbf{q})$ larger than the probability for the emission of N photons of modes $(\mathbf{k}^0\cdots\mathbf{k}^0,\mathbf{k}^0+\mathbf{q})$. For realistic situations the value of η_N^{-2} is different from unity by many orders of magnitude, if the angle between $\mathbf{k}^0$ and $\mathbf{k}^0+\mathbf{q}$ is substantial, e.g., if $|\mathbf{q}|\approx|\mathbf{k}^0|\approx|\mathbf{k}^0+\mathbf{q}|$ holds. These relations are only slightly altered by the function $\Phi_{k_1\cdots k_N}(\gamma)$ if the competing sets $(\mathbf{k}_1\cdots\mathbf{k}_N)$ are not restricted to equal energies. We have to conclude from this that the N photons are actually emitted in the form of a bundle of nearly equal photons, which we call a "ray." The emission of N substantially different photons is so much less probable, that such an event is practically impossible.

The existence of the many-photon collective "ray" is independent of the specific atomic transition under consideration. Indeed, the dependence on the one-atom radiation law $|C_{\mathbf{k}}|^2$, as given by (2.18), has dropped out of $R_{\mathbf{k}_1\cdots\mathbf{k}_N}$, leaving only a weak dependency on the far less characteristic one-atom constant Γ_0. The results derived from (7.1) are thus valid for any type of transition in any type of atoms, for which the initial condition (2.3) can be established. The $\mathcal{V}$ dependence of the effects must be attributed to a very long range of the effect of self-stimulation [cf. (4.6) and the discussion following (4.14)], and there is no reason why such effects should not occur for radiation of wavelength shorter than visible light, provided that an effective pumping mechanism can be found for a suitable atomic transition.

We note that the above conclusions are subject only to the approximations (A) and (B) of Sec. V, since the actual form of the function $\beta_{k_1\cdots k_N}(\infty)$, as defined by Eq. (5.13), has no important influence on the above considerations. From now on, however, we shall use all approximations of Sec. V in considering in more detail the stationary properties of a ray.

Since $P_{\mathbf{k}_1\cdots\mathbf{k}_N}$ has maxima of equal height for all diagonal points $(\mathbf{k}_1\cdots\mathbf{k}_N)=(\mathbf{k}^0\cdots\mathbf{k}^0)$ of the phase space $\mathfrak{K}^N$ of N photons of modes $(\mathbf{k}_1\cdots\mathbf{k}_N)$, the value $\mathbf{k}^0$ of the "center of a ray" is not determined by $P_{\mathbf{k}_1\cdots\mathbf{k}_N}$. $\mathbf{k}^0$ follows, however, from the properties of the function $|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2$, as defined in (2.22), because the latter can be considered, in analogy with (3.19) and (3.20), as a probability distribution in $\mathfrak{K}^N$. Factorizing $P_{\mathbf{k}_1\cdots\mathbf{k}_N}$ in accordance with (6.11), we obtain the formula

$$|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2=N!\prod_{\nu=1}^{N}\eta_N{}^2(\mathbf{k}_\nu-\mathbf{k}_{\mathrm{c.m.}})\times\frac{|C_{\mathbf{k}_\nu}|^2}{2Vk_\nu[(\Delta-k_\nu)^2+\Gamma_0{}^2(1+s)^2]},\quad(7.4)$$

where $\mathbf{k}_{\mathrm{c.m.}}$ is the mean value (6.12) of $\mathbf{k}_1\cdots\mathbf{k}_N$. Considered as an "event" in $\mathfrak{K}^N$, we observe the "emission of a bundle of N photons with center $\mathbf{k}^0$" any time N photons of modes $(\mathbf{k}_1\cdots\mathbf{k}_N)$ with $\mathbf{k}_{\mathrm{c.m.}}$ close to $\mathbf{k}^0$ are emitted. Since $\eta_N{}^2(\mathbf{k}_\nu-\mathbf{k}_{\mathrm{c.m.}})$ has a narrow and relatively very high maximum for $\mathbf{k}_\nu=\mathbf{k}_{\mathrm{c.m.}}$, only a narrow bundle, or ray, with

$$\mathbf{k}_1\approx\mathbf{k}_2\approx\cdots\approx\mathbf{k}_N\approx\mathbf{k}_{\mathrm{c.m.}}\quad(7.5)$$

will be emitted, as we know already. Therefore, the emission of a ray with center $\mathbf{k}^0$ will be the more probable the more points $(\mathbf{k}_1\cdots\mathbf{k}_N)$ there are around the point $(\mathbf{k}^0\cdots\mathbf{k}^0)$, such that $|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2$ is still comparable with its absolute maximal value. Consequently, the total probability for the emission of a ray with center $\mathbf{k}^0$ will be proportional to the volume of a $3N$-dimensional neighborhood $\mathfrak{d}^0$ of the point $(\mathbf{k}^0\cdots\mathbf{k}^0)$, which contains only points $(\mathbf{k}_1\cdots\mathbf{k}_N)$ of $\mathfrak{K}^N$ such that $|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2$ has values of the order of magnitude of the maximal value of $|\tilde{\alpha}_{\mathbf{k}^0\cdots\mathbf{k}^0}{}^{1\cdots N}(\infty)|^2$. The latter is reached for

$$|\mathbf{k}^0|=\Delta,\quad(7.6)$$

as is easily seen from (7.4), and therefore the center energy of any ray will be given by (7.6). As a consequence of the factorization (7.4) of $|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2$, an approximate measure for the volume of $\mathfrak{d}^0$ is given by $|C_{\mathbf{k}^0}|^{2N}$ times the Nth power of the volume of the intersection of the geometrical shape of a ray $\mathbf{k}^0$, $u\{\mathbf{k}-\mathbf{k}^0\}$, as defined in Sec. VI, with the spherical shell $\sigma_\tau\{\mathbf{k}-\mathbf{k}^0\}$, of mean radius $|\mathbf{k}^0|=\Delta$ and arbitrary, sufficiently small shell thickness 2τ. The direction of $\mathbf{k}^0$ is therefore determined by the condition

$$|C_{\mathbf{k}^0}|^2\times\mathrm{vol}(u\{\mathbf{k}-\mathbf{k}^0\}\cap\sigma_\tau\{\mathbf{k}-\mathbf{k}^0\})=\max.\quad(7.7)$$

Since the influence of $|C_{\mathbf{k}^0}|^2$ on the solution of (7.7) is small under realistic conditions, and since $|C_{\mathbf{k}^0}|^2$ is due to the initial, rather unrealistic condition that all atoms have *identical states up to a translation*, the direction of $\mathbf{k}^0$ will usually be determined by the second factor of (7.7). For characteristic shapes of $\mathfrak{V}$, this leads to the following results:

If $\mathfrak{V}$ is a sphere, we have a constant volume of the intersection $u\cap\sigma$ and the direction of $\mathbf{k}^0$ remains indetermined. All directions of $\mathbf{k}^0$ will therefore be equally probable for the emitted ray.

If $\mathfrak{V}$ is approximately a disk, then u is approximately a rod normal to the plane of the disk. vol $(u\cap\sigma)$ will be maximal if and only if the axis of the rod u is a tangent of $\sigma_\tau\{\mathbf{k}-\mathbf{k}^0\}$. This means that $\mathbf{k}^0$ will be in the plane of the disk $\mathfrak{V}$, and all directions within that plane are equally probable, if $\mathfrak{V}$ is exactly circular.

If $\mathfrak{V}$ is approximately a rod, which includes nearly all cases of physical interest, then u is a disk in a plane normal to the axis of $\mathfrak{V}$, and vol $(u\cap\sigma)$ will be maximal if and only if the plane of the disk u is a tangent plane of $\sigma_\tau\{k-k^0\}$. $\mathbf{k}^0$ must therefore be parallel to the main axis of $\mathfrak{V}$, but the two possible directions of $\mathbf{k}^0$ remain equally probable. This is true even if $\mathfrak{V}$ is a long cone, as one can easily deduce from (6.20). The relative probability of $\mathbf{k}^0$ being in a direction other than the most probable direction is I^{2N}, where I^2 (≤ 1) is the fractional reduction in the volume of the above intersection for this $\mathbf{k}^0$ relative to that for the most probable $\mathbf{k}^0$. This is a much more slowly varying function that $\eta^2(\mathbf{q})$.

So far, the ray is still defined as the most probable event in $\mathfrak{K}^N$. For a description of what can be observed of the ray in an ordinary three-dimensional $\mathbf{k}$ space, we have to translate the results from $\mathfrak{K}^N$ into the language of a $\mathfrak{K}^1$. This can be done in principle by formulas of the type (2.26), but this would prove unpractical and even misleading physically, since such a procedure includes a weighted averaging process over all possible rays and would thus smear out the characteristic correlations between the emitted photons which lead to the "one-ray" aspects as discussed above. For example, if $\mathfrak{V}$ is a sphere, the number of photons of mode $\mathbf{k}$, as defined by (2.26), would turn out independent of the direction of $\mathbf{k}$. However, we would no longer know that all N photons have been emitted into a neighborhood of the mode $\mathbf{k}$, if a substantial number of photons of mode $\mathbf{k}$ have actually been measured.

Consequently, we can only ask for the most probable distribution of photons around $\mathbf{k}^0$ provided that $\mathbf{k}^0$ is the center of an observed ray. The possibility for such considerations arises from the fact that the maxima of $P_{\mathbf{k}_1\cdots\mathbf{k}_N}$ lie on the diagonal points $(\mathbf{k}^0\cdots\mathbf{k}^0)$ of $\mathfrak{K}^N$, such that an approximate factorization of $\mathfrak{d}^0$ in the form

$$\mathfrak{d}^0\approx\mathfrak{z}\{\mathbf{k}_1-\mathbf{k}^0\}\otimes\cdots\otimes\mathfrak{z}\{\mathbf{k}_N-\mathbf{k}^0\}\quad(7.8)$$

can be obtained around every point $(\mathbf{k}^0\cdots\mathbf{k}^0)$, where $\mathfrak{z}\{\mathbf{k}-\mathbf{k}^0\}$ is a quantitatively well-defined three-dimensional volume around the point $\mathbf{k}^0$. To obtain such a $\mathfrak{z}\{\mathbf{k}-\mathbf{k}^0\}$, we proceed as follows: Consider a system of N atoms which radiate independently of each other by a "distorted" one-atom radiation law

$$W_{\mathbf{k}}^{\text{dist}}\sim\eta^2(\mathbf{k}-\mathbf{k}^0)\frac{|C_{\mathbf{k}}|^2}{2Vk[(\Delta-k)^2+\Gamma_0^2(1+s)^2]} \quad (7.9)$$

with a given fixed $\mathbf{k}^0$. Constructing for this system the analog of expression (3.19), we obtain, up to an irrelevant normalization factor, the product (7.4) for $|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2$, for $\mathbf{k}_{\text{c.m.}}$ equal to that given $\mathbf{k}^0$. This means that the distribution in angle and in frequency of an observed ray $\mathbf{k}^0$ from the real system can be looked at, to the accuracy of the factorization (6.11) for given $\mathbf{k}^0$, *as if* the ray were produced by independently radiating atoms with one-atom radiation law (7.9) instead of the interacting atoms radiating by the real one-atom radiation law $|\alpha_{\mathbf{k}}{}^1(\infty)|^2$, as obtained from the case $N=1$. Looking at (7.9), we see that the "geometrical shape" of ray $\mathbf{k}^0$ is on the same footing with the usual definition of the half-width of a radiation line. Hence, if we define the "physical shape" $\mathfrak{z}\{k-k^0\}$ of a ray by the condition

$$W_{\mathbf{k}}^{\text{dist}}\geq\tfrac{1}{2}W_{\mathbf{k}^0}{}^{\text{dist}} \text{ for all } \mathbf{k}\text{'s of } \mathfrak{z}\{\mathbf{k}-\mathbf{k}^0\}, \quad (7.10)$$

we see that $\mathfrak{z}^0$, as defined by (7.8), contains only points $(\mathbf{k}_1\cdots\mathbf{k}_N)$ such that

$$|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2\geq\left(\frac{1}{2^N}\right)\times\max|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2 \quad (7.11)$$

holds. On the other side, if we consider a point $(\mathbf{k}_1\cdots\mathbf{k}_N)$ with an appreciable number n of $\mathbf{k}$'s taken not from $\mathfrak{z}\{\mathbf{k}-\mathbf{k}^0\}$, then $|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2$ drops by a factor η^{2n} which soon becomes so small that $|\tilde{\alpha}_{\mathbf{k}_1\cdots\mathbf{k}_N}{}^{1\cdots N}(\infty)|^2$ cannot be compared reasonably with its maximal value.

In the limited sense of the above approximations, we can say that an observed ray $\mathbf{k}^0$ is described by (7.9) or by its physical shape $\mathfrak{z}\{\mathbf{k}-\mathbf{k}^0\}$ as defined in (7.10). If $\mathfrak{V}$ is a rod which can be approximately described by a long ellipsoid, we can define as the "line shape" of a ray the intersection of $\mathfrak{z}\{\mathbf{k}-\mathbf{k}^0\}$ with a plane which contains $\mathbf{k}^0$. The line shape $\mathfrak{L}(\Delta-k)$ of such a ray is therefore given by

$$\mathfrak{L}(\Delta-k)\sim\frac{1}{a^4(\Delta-k)^4}\left(\frac{\sin(\Delta-k)a}{(\Delta-k)a}-\cos(\Delta-k)a\right)^2\times\frac{1}{(\Delta-k)^2+\Gamma_0^2(1+s)^2}, \quad (7.12)$$

where $2a$ is the length of the rod. The half-width Γ' of this line is given by the equation $\mathfrak{L}(\Gamma')=\frac{1}{2}\mathfrak{L}(0)$. Expressing this linewidth in terms of fractions $\delta\lambda/\lambda$ of the wavelength $\lambda=2\pi/\Delta$, we have the inequalities

$$\delta\lambda/\lambda\leq 2\xi_0/a\Delta=(\lambda/2a)2\xi_0/\pi \quad (7.13)$$

and

$$\delta\lambda/\lambda\leq(\lambda/L_0)(1+s)(1/\pi). \quad (7.14)$$

L_0 is the distance used in (4.6). With the exception of a transition region where the above bounds are of the same magnitude and (7.12) must be used, the line shape is either given by the first or the second factor of (7.12), depending on whether $L_0\xi_0\ll a(1+s)$ or $L_0\xi_0\gg a(1+s)$ holds.

Comparing the proof of existence of a ray with the above detailed examination of its properties, we note a remarkable difference in the "strength of the forces" leading to the respective results: The existence of a ray is due to an interference—or "self-stimulation"—effect in $\mathfrak{K}^N$, leading to relative probability ratios of order of magnitude $1:\eta_N{}^{2N}$ for the alternatives "ray" or "no ray." The relative probability of two directions of that ray, and to a degree even its mean energy, is governed by relative probability ratios I^{2N}, where I^2 (≤ 1) is of the order of magnitude 1. We can thus state that the ray *is formed* by "forces of strength" $\eta_N{}^{-2}$, but it is *driven into its final direction* by much weaker forces of strength I^{-2} only. Actually, for the case of a spherical $\mathfrak{V}$ we have $I^2=1$ and hence no driving force. This provides the basis for the following hypothesis concerning the forming of a ray in the presence of laser cavities.

Considering the interference nature of the function $P_{\mathbf{k}_1\cdots\mathbf{k}_N}$, which governs the existence of the ray, and the strength of the "ray-forming forces," it is hardly imaginable that the presence of a cavity has any influence on the existence of a ray, since the grid of the points $\mathbf{X}_1\cdots\mathbf{X}_N$ is present in any case. However, it is very well possible and even necessary that the cavity interacts with the above "driving forces I^2," such that the ray is directed into one or another "cavity mode." The existence of a ray would thus be to a high degree analogous to the existence of many molecule collectives, like sound waves in acoustics, or water waves, and the presence of a cavity would thus produce resonance phenomena (modes) analogous to those produced in acoustics by the presence of resonant bodies. As the acoustical resonance phenomena cannot be understood on the basis of molecules *independently* interacting with a resonant body, and the macroscopic phenomenon "sound wave" is needed for a satisfying explanation, so, our hypothesis states, the resonance phenomena of laser physics cannot be understood on the basis of atoms interacting *independently* (i.e., "without seeing each other *at their respective places*") with a macroscopic cavity but some collective, macroscopic phenomenon of the nature of a ray would give an easy understanding of these phenomena.

Since laser activity without mirrors,[16,17] and hence

[16] A. Lempicki and A. Heller, Appl. Phys. Letters **9**, 108 (1966).
[17] W. R. Bennett, J. W. Knutson, G. N. Mercer, and J. L. Detch, Appl. Phys. Letters **4**, 180 (1964).

without distinct cavity modes, has been obtained recently, an explanation of "mirrorless" lasers is needed in any case. The above hypothesis explains easily the practical importance of cavities by putting them into analogy with the resonant bodies of acoustics.

Apart from the triviality that N photons are emitted from N excited atoms, no essential N dependency has been obtained in the above results. Instead of this we had to emphasize the dependency of self-stimulation effects on the shape and size of $\mathcal{V}$ and on the density of excited atoms, as contained, e.g., in the physical line shape (7.12) of the ray. On the basis of this situation we put forward a second hypothesis related to a system of active atoms between mirrors:

Until now, mirrors have been considered exclusively (cf. Lamb[5]) as creating, in the sense of Fox and Li,[18] certain equivalents of cavity modes, which in turn are considered responsible for laser effects. We question the dominance and necessity of this view by hypothetically stating that at least some aspects of laser activity can be understood in terms of the classical "mirror principle," which implies that the actual volume $\mathcal{V}$ is multiplied and/or modified in a certain way and can be replaced by an "effective" $\mathcal{V}_{\text{eff}}$. This is possible only because the number N of actual atoms plays no vital role in all of the above results, such that we need not worry about the fact that the images of excited atoms add nothing to the total energy output $N\Delta$, which remains of course unchanged. Hence, we expect, e.g., that the line-shape properties from a system with mirrors can be obtained to some degree by the simple replacement

$$\mathcal{V} \rightarrow \mathcal{V}_{\text{eff}}. \tag{7.15}$$

For example, looking on Eq. (7.12) for the line shape of a rod, we expect that the actual length $2a$ of the rod can be replaced by an effective length $2a_{\text{eff}} > 2a$, while s should remain more or less constant. However, we do not claim that the mirror principle can replace completely the concept of Fox-Li modes, which is included in our first hypothesis on the interaction of rays with resonant cavities.

VIII. TIME EVOLUTION

In this section, we look at the evolution in time of the ray, starting from the initial state at $t=0$ with N atoms excited and no photons present, and ending with a ray of N photons and no excited atoms. We are describing a single pulse from an ideally pumped system and not the time dependence in the presence of pumps or other external influences. The nearest realistic system is probably the Q-switched laser, and we believe the connection between our model and such a laser is close. This connection is being investigated. It must be emphasized that this time development is that of a pure quantum state involving all the atoms and the radiation field. This type of time development must be distinguished from that in which the radiation and absorption of individual atoms are identified, even when these are treated quantum mechanically by means of a density matrix involving an atom interacting with the radiation field as is done, for example, by Scully and Lamb.[6] It is not comparable to a rate-equation treatment[7,19] for the above reason and because a continuum of modes of the radiation field is considered. A rate-equation formulation with all modes admitted on an equal footing does not lead to the formation of a ray because in such a formulation the necessary correlations between atomic states are not included.

Because of the large number of atoms present and because the system is in a pure quantum state, the Heisenberg uncertainty principle involves the energy of the entire system and not the energies of the atoms separately. If δk is the minimal uncertainty in the energy of an individual atom or photon, then the uncertainty principle requires that

$$N\delta k\ \delta t \gtrsim 1. \tag{8.1}$$

If the photon energies are not to be resolved more closely than the single-atom natural linewidth, then this places no practically effective bound on δt. Times much shorter than the mean lifetime of a single atom can be considered in a meaningful way, and the initial condition used is less problematical for many atoms than for a single atom. A more stringent restriction on meaningful time intervals is imposed by the requirement that retardation can be neglected. Time intervals shorter than L/c, where L is a typical dimension of the system, are not to be considered.

The initial state decays exponentially. According to (5.30),

$$|\alpha_0{}^0(t)|^2 = |\beta_0(t)|^2 = \exp(-2\Gamma_0 N t), \tag{8.2}$$

with $2\Gamma_0$ the single-atom width. The initial state is destroyed by the decay of any one atom. No photons are initially present, and there are no unexcited atoms present to give rise to radiation trapping, so this decay occurs as if the atoms were independent.

The probability of finding the system at time t in the state with M photons of momenta $\mathbf{k}_1 \cdots \mathbf{k}_M$ present and the M atoms $A_{j_1} \cdots A_{j_M}$ de-excited is given by

$$|\alpha_{\mathbf{k}_1 \cdots \mathbf{k}_M}{}^{j_1 \cdots j_M}(t)|^2,$$

but for the present it is enough to consider the corresponding smooth function

$$|\tilde{\alpha}_{\mathbf{k}_1 \cdots \mathbf{k}_M}{}^{j_1 \cdots j_M}(t)|^2 = P_{\mathbf{k}_1 \cdots \mathbf{k}_M} |\beta_{k_1 \cdots k_M}(t)|^2 \tag{8.3}$$

as in (5.4) together with (2.22). Even for M much less than N, $P_{\mathbf{k}_1 \cdots \mathbf{k}_M}$ has a sharp maximum of value $M!$ for parallel photons, so that an intermediate ray of M photons is found early in the decay process. The direction of this intermediate ray is determined by the set of

[18] A. G. Fox and Tingye Li, Bell System Tech. J. **40**, 453 (1961).

[19] D. A. Kleinman, Bell System Tech. J. **43**, 1505 (1964).

M atoms $A_{j_1}\cdots A_{j_M}$ which are de-excited. The state of the system when M photons are present is linear combination of the

$$\binom{N}{M}$$

states with M de-excited atoms. The overwhelming majority of these states correspond to a macroscopically uniform distribution of the de-excited atoms over the volume $\mathcal{V}$ of the system, so that the direction and shape of the intermediate ray are those of the final ray. A determination of which atoms have decayed at time t would destroy the state and eliminate the ray.

The probability $P_s(M,t)$ of finding M photons present is zero for both $t=0$ and $t=\infty$ when $0<M<N$. It reaches a maximum at time $t=t_s(M)$ which depends on the parameter $s=\gamma\cdot N$ of (7.3) and on M. Inverting this equation gives a measure of the number of photons present at time t, which we write as $M_s(t)$. The rate of photon emission $r_s(t)$ is then taken to be

$$r_s(t)=\partial M_s(t)/\partial t. \tag{8.4}$$

From (8.3) it is seen that the only time dependence comes from the $\beta(t)$. Then according to (5.30)

$$\begin{aligned}P_s(M,t)&\propto\int_{-\infty}^{\infty}dk_1\cdots\int_{-\infty}^{\infty}dk_M\,|\beta_{k_1\cdots k_M}(t)|^2\\&\propto\left(\int_{-\infty}^{\infty}dk\frac{e^{-2\Gamma t}+1-2\cos(\Delta-k)te^{-\Gamma t}}{(\Delta-k)^2+\Gamma^2}\right)^M\\&=\left(\pi\frac{1-e^{-2\Gamma t}}{\Gamma}\right)^M,\end{aligned} \tag{8.5}$$

with

$$\Gamma=\Gamma_0[1+\gamma(2M-N-1)]. \tag{8.6}$$

This has a maximum when

$$\begin{aligned}2\Gamma_0 t_s(M)&=\frac{1}{1+\gamma(2M-N-1)}\ln\frac{N+\gamma M(M-1)}{(N+M)(1+\gamma M)}\\&\approx\frac{1}{1+s(2M/N-1)}\ln\frac{1+sM^2/N^2}{(1-M/N)(1+sM/N)}.\end{aligned} \tag{8.7}$$

The inversion of this gives $M_s(t)$. The result is shown in Fig. 1, along with its derivative $r_s(t)$, for several values of s.

For $s=0$, the low-density limit, we have

$$\begin{aligned}M_0(t)&=N(1-e^{-2\Gamma_0 t}),\\r_0(t)&=2\Gamma_0 Ne^{-2\Gamma_0 t},\end{aligned} \tag{8.8}$$

which is just the rate for the decay of independent atoms. It is noteworthy that the ray is still formed at low density, even though the rate of emission does not show a cooperative effect. For all $s<1$ the rate of emission is monotomically decreasing with time, but for $s>1$ it has a maximum which appears when M is about $\frac{1}{2}N$ and whose height increases rapidly with increasing s. For large s the entire process of light emission takes place in a time short compared to the single-atom mean lifetime. As was mentioned above, this is reminiscent of the behavior of a Q-switched laser.

The fact that the maximum of the emission rate occurs at the time when about half of the atoms are de-excited suggests a connection with Dicke's superradiant states. Dicke[2] showed that a system of atoms excited by an appropriate pulse of light so that half the atoms are excited radiates very rapidly with the emission of a ray in the direction of the exciting pulse. We can think of the early part of the radiation process described in the present paper as playing a role analogous to that of Dicke's exciting pulse, leaving the system in a state to emit the rest of the radiation rapidly in the same direction.

IX. MEANING OF "STIMULATED EMISSION"

The laser is supposed to operate on the principle of "stimulated emission," the very name incorporating this supposition. In this section, we try to find out what role stimulated emission plays in the model treated here. There are several features which make this a nontrivial matter: (i) There is a continuum of modes present which makes the concept of occupation number of a mode distinct from that when discrete modes are involved. (ii) The focusing of the ray depends primarily on the geometry of the system and is not enhanced when the number of atoms $\mathfrak{N}$ is increased beyond a certain lower limit. (iii) The rate of emission does increase with increasing number of atoms for a fixed geometry.

Point (i) is closely related to the discussion of Einstein factors given in Sec. III. When a continuum of modes is present, it can be represented by discrete modes with arbitrary accuracy by taking enough discrete modes appropriately distributed in $\mathbf{k}$ space. The occupation number of such a discrete mode is directly proportional to the range of $\mathbf{k}$ per mode, but this is irrelevant for physical results as long as the division is fine enough to describe the experimental resolution contemplated. Thus, there can be no essential dependence on the occupation numbers of the separate modes.

Point (ii) is seen explicitly in (6.9), which we may write as

$$\eta_M{}^2(\mathbf{q})=1/M+\eta^2(\mathbf{q}). \tag{9.1}$$

The quantity $\eta^2(\mathbf{q})$ defined in (6.10) describes the probability for finding a photon with momentum $\mathbf{k}_0+\mathbf{q}$ if $\mathbf{k}_0$ is the central wave vector of the ray, and $\eta_M{}^2(\mathbf{q})$ is the corresponding intermediate quantity when only M ($\gg 1$) photons are present. Since for small $\mathbf{q}$, $\eta^2(\mathbf{q})$ has values greatly exceeding unity, the term $1/M$ in (9.1) plays only a negligible part in determining the focusing of the ray. For very small M it merely makes

the visibility of the interference pattern given by $\eta^2(\mathbf{q})$ slightly less than unity.

Point (iii) does show some effect which can be interpreted as stimulated emission. The rate of emission is determined by (8.7). A rough estimate of the maximum rate for large s can be obtained by putting $M=\frac{1}{2}N$ and treating $1/s$ as a small quantity. Then

$$\Gamma_0 t_s(\tfrac{1}{2}N)\approx 1/s, \quad s\gg 1 \tag{9.2}$$

so that the time required for the de-excitation of half the atoms is inversely proportional to the density for large density. If the radiation emitted were confined in an optical cavity for a time long compared with $1/s\Gamma_0$, the density of photons would be proportional to s and then (9.2) would indeed correspond to stimulated emission, provided, of course, that under these circumstances the relation (9.2) still holds. The results achieved here may, then, be regarded as a generalization of the usual concept of stimulated emission as formulated by Dirac[20] and Heitler,[21] being relevant when the number of photons associated with any one plane-wave mode is not a relevant quantity. This generalized concept of stimulated emission has the usual one as a limiting case when an optical cavity defines a discrete set of modes, as will be shown in a future publication.

It appears, then, that the formation of the ray in space is not a consequence of stimulated emission in the usual sense but the time dependence of the emission at high atomic density is describable in terms of stimulated emission. If, as appears in a preliminary investigation, the effect of an optical cavity is to increase s greatly, the large-s approximation is good. This will be treated in another paper.

Note added in proof. It has been suggested that a more detailed comparison be made of the present calculations with earlier work on laser "spiking." It appears to us that most of the latter [with the exception of Dicke's results on "optical bombing" (Ref. 2)] is based on rate equations of the type first used by H. Statz and G. deMars [e.g., in *Quantum Electronics*, edited by C. H. Townes (Columbia University Press, New York, 1960)]. The application of rate equations *implies* the assumption that one deals with *statistically independent processes* (in the present case, creation processes of single light quanta usually of only one mode) which are *only* subject to some macroscopic conservation law (energy). The latter might be sufficient for such problems as the power supply of the laser, but it is not *sufficient* to determine the inherent properties of the emitted light: If laser spikes were governed *only* by the requirements

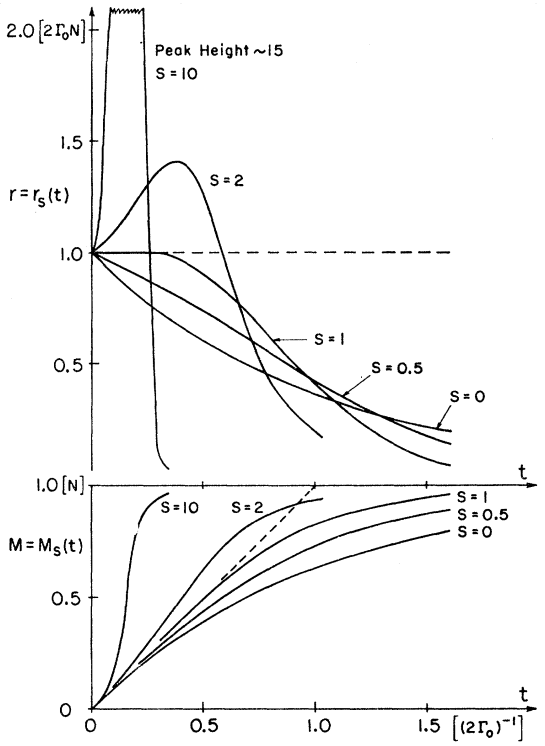

FIG. 1. Plots of the rate of emission, $r_s(t)$, and the number $M_s(t)$ of photons present at time t, as functions of time for several values of the density parameter s defined by (7.3).

of energy conservation, it would be impossible, by the entropy law, to obtain interference effects between two *spikes* of two different lasers [R. L. Magyar and L. Mandel, Nature **198**, 255 (1963); R. L. Pfleegor and L. Mandel, Phys. Rev. **159**, 1084 (1967)]. Furthermore, the many forms in which spiking occurs (there are damped and undamped regular spikes, and, mostly, spikes irregular in time and peak intensity) suggest that steady-state laser operation is rather an exception than the rule, *but solutions of realistic rate equations always approach a steady state.* Therefore we see no common base for a reasonable comparison of results from rate equations with those obtained here by the consideration of *one* quantum-mechanical creation process for *many* photons. The present theory adds an additional *dynamical element* (not a complete explanation) to the fundamental instability at least of solid-state lasers, which is usually ascribed solely to the interaction with one or more cavity modes. Because of our initial condition and other inherent limitations (retardation) the present theory cannot be applied to the recently observed *picosecond substructure* of certain giant pulses, though a solution to that problem, too, most probably will be found on the base of many-photon coherence.

[20] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, New York, 1958), 4th ed., Chap. X.
[21] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed., Chap. V.