negligible, and we may write the Hamiltonian in the form

$$\mathcal{H}_F \simeq \mathcal{H}_{\mathbf{k}0} + (\beta/m)[\hbar(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{p} + \tfrac{1}{2}(k^2-k_0^2)] + [\hbar/(2mc)^2]\boldsymbol{\sigma}\cdot\nabla V_0 \times \mathbf{p}. \quad (4.10)$$

Equation (4.10) has the form of the working Hamiltonian customarily employed in $\mathbf{k}\cdot\mathbf{p}$ calculations[15,16] but without the appearance of a $\mathbf{k}$-dependent spin-orbit interaction term. It is therefore clear that any linear k terms due to spin-orbit effects may be described by a combination of $\mathbf{k}\cdot\mathbf{p}$ and spin-orbit perturbations and not by means of the $\mathbf{k}$-dependent spin-orbit interaction.[17]

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[17] See E. O. Kane, Ref. 15, Sec. IV-c and reference therein; G. Dresselhaus, Phys. Rev. **100**, 580 (1955).

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Multipole Particles in Equilibrium in General Relativity*

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Axially symmetric static solutions of Einstein's equations with singularities on the z axis are considered. The structure of the singularities is more general than that considered previously by Bergmann, since higher multipole moments are now allowed. The equilibrium conditions for such multipole singularities in an external field are derived. It is shown that solutions having two or more such singularities on the axis may be found in abundance, independently of the signs of the masses of the particles.

1. INTRODUCTION

THE general static axially symmetric vacuum solution of Einstein's field equations is given by the metric of Weyl[1] and of Levi-Civita[2]:

$$ds^2 = e^{2(\nu-\lambda)}(dr^2+dz^2) + r^2e^{-2\lambda}d\phi^2 - e^{2\lambda}dt^2, \quad (1)$$

where $\lambda = \lambda(r,z)$ and $\nu = \nu(r,z)$ satisfy

$$\nabla^2\lambda = \lambda_{rr} + \lambda_{zz} + r^{-1}\lambda_r = 0 \quad (2)$$

and

$$\begin{aligned}\partial\nu/\partial r &= r(\lambda_r^2 - \lambda_z^2),\\ \partial\nu/\partial z &= 2r\lambda_r\lambda_z.\end{aligned} \quad (3)$$

Equation (2) is in fact the integrability condition for Eqs. (3). Hence, given any axially symmetric harmonic function λ, it is always possible, at least locally, to find a function ν that satisfies (3). The global condition for a solution is more severe. It is equivalent to the requirement that, for any closed circuit C in a region where λ is regular,

$$0 = \oint_C \nu_r dr + \nu_z dz = \oint_C r(\lambda_r^2 - \lambda_z^2)dr + 2r\lambda_r\lambda_z dz. \quad (4)$$

Furthermore, it is essential that $\nu = 0$ on the z axis in order to ensure the regularity of the metric there.[3] These conditions have been used in some attempts at proving the impossibility of a static two-body solution in general relativity.

Bergmann[3] has studied the case of two singularities on the z axis, and has shown that the function ν must change its value on going around one of the singularities. Hence the condition $\nu = 0$ cannot be imposed on all regions of the axis which exclude the singularities. This is borne out by direct inspection of the explicit solution of Silberstein representing two such mass singularities.[4,5] The problem with more than two singularities on the axis has been treated by Hoffmann,[6] who finds that if masses of both signs are permitted, static configurations with three or five singularities are always possible.

However, these results deal exclusively with the case of "monopole" singularities, i.e., singularities of the form $\lambda = -M/R$ $(R^2 = r^2+z^2)$. Now the most commonly accepted solution representing a particle, the Schwarzschild solution, is not of this form but appears as a rod singularity in the cylindrical coordinates.[1] Thus even a spherically symmetric particle will have higher multipole moments in Weyl's coordinates. Clearly then it would be desirable to treat the case of particles having

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[1] H. Weyl, Ann. Physik **54**, 117 (1917); **54**, 185 (1919).

[2] T. Levi-Civita, Rend. Accad. Lincei **26**, No. 2, 307 (1917); **27**, No. 1, 3 (1918); **27**, No. 2, 183 (1918); **28**, No. 1, 3, 104 (1919).

[3] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1942), p. 206.

[4] L. Silberstein, Phys. Rev. **49**, 268 (1936).

[5] A. Einstein and N. Rosen, Phys. Rev. **49**, 404 (1936).

[6] B. Hoffmann, in *Les Théories Relativistes de la Gravitation* (Centre National de la Recherche Scientifique, Paris, 1962), p. 237.

a general multipole structure

$$\lambda=\sum_{n=0}^{\infty}-M_nR^{-(n+1)}P_n(\cos\theta). \tag{5}$$

That we might expect static situations with two such multipole particles is clear from the result of Hoffman, since we might now coalesce monopoles, while maintaining equilibrium between them, until two particles with higher multipole moments are formed. Hoffmann comes to the conclusion that such structures could not be formed, but his argument is purely heuristic and is apparently wrong in view of the present results. In this paper the general condition of equilibrium between such multipole particles will be derived. It will be shown that static situations can arise even if the masses of the particles (as represented by the monopole moments) are of the same sign. An explicit solution with a monopole in equilibrium with a monopole-dipole is given.

The use of singularities to represent particles may, with good reason, be objected to. It may be argued that only solutions that are regular everywhere, and in which matter is represented by nonvanishing Einstein tensor, are physically acceptable. Under such conditions Bondi[7,8] has shown that if λ is nonsingular and corresponds in the vacuum regions to the potential arising in a Newtonian multibody problem, then the global condition (4) is satisfied if and only if there is no net Newtonian force on any of the bodies. However, it is not easy to see how far this goes towards a solution of the relativistic problem, for, while we have managed to replace our problem with a corresponding Newtonian one in the vacuum regions, the correspondence breaks down completely inside the regions of matter. In these regions it will be necessary to give not just the mass density (as in the Newtonian case) but also an equation of state. In general relativity the static two-body problem should be posed, then, in the following way: Given a static vacuum solution which is the exterior solution to two nonintersecting regions of matter, can the metric (assumed to be C^1 everywhere to satisfy the Lichnerowicz conditions[9]) be extended into these regions in such a way that (a) the mass density (given, for example, by $-G_{44}$) is everywhere ≥ 0 and (b) a reasonable equation of state holds throughout the regions? Unfortunately, there is no universal agreement as to what constitutes a "reasonable" equation of state, and to this extent a more precise formulation of the problem must be left up to individual taste. It is now that we can see the advantages of treating singularities. Since these are limiting cases, one may approach arbitrarily close to the singular points without anywhere being hampered by awkward junction conditions. This turns out to be of considerable help in handling the path integrals that arise in Eq. (4). We can then find very explicit conditions for the existence of solutions with two (or more) singularities. These solutions will be valid everywhere but will be incomplete[10] at a finite number of "points" which it is now required to cover up with regions of matter satisfying conditions (a) and (b) above.

2. COMPLEX-VARIABLE FORMALISM

It will be useful to introduce a pair of conjugate complex variables ζ and $\bar{\zeta}$ in place of r and z

$$\zeta=r+iz,\quad \bar{\zeta}=r-iz.$$

Treating ζ and $\bar{\zeta}$ as independent variables, we have

$$\frac{\partial}{\partial\zeta}=\tfrac{1}{2}\left(\frac{\partial}{\partial r}-\frac{i\partial}{\partial z}\right),\quad \frac{\partial}{\partial\bar{\zeta}}=\tfrac{1}{2}\left(\frac{\partial}{\partial r}+\frac{i\partial}{\partial z}\right).$$

Hence the two equations (3) may be replaced with a single complex equation

$$\partial\nu/\partial\zeta=(\zeta+\bar{\zeta})(\partial\lambda/\partial\zeta)^2. \tag{6}$$

The advantage of this approach is that, if we are given λ as a function of ζ and $\bar{\zeta}$, then ν may be obtained immediately from (6) by a simple integration. As an example, we derive the Weyl metric that corresponds to a dipole potential,

$$\lambda=-DR^{-2}\cos\theta=\tfrac{1}{2}Di(\zeta-\bar{\zeta})(\zeta\bar{\zeta})^{-3/2}.$$

Then

$$\frac{\partial\nu}{\partial\zeta}=-\tfrac{1}{16}D^2\frac{(\zeta+\bar{\zeta})(3\bar{\zeta}-\zeta)^2}{\zeta^5\bar{\zeta}^3}.$$

A straightforward integration now gives

$$\nu=\tfrac{1}{16}D^2\left(\frac{1}{\zeta\bar{\zeta}^3}-\frac{5}{2\zeta^2\bar{\zeta}^2}+\frac{1}{\zeta^3\bar{\zeta}}+\frac{9}{4\zeta^4}\right)+f(\bar{\zeta}).$$

The condition of reality for ν immediately gives

$$f(\bar{\zeta})=\tfrac{1}{16}D^2(9/4\bar{\zeta}^4),$$

and we find that

$$\nu=D^2r^2(r^2-8z^2)/4R^8.$$

Now suppose that ν is given at a point $\zeta_0=r_0+iz_0$. Let C be a curve beginning at ζ_0 and ending at a point ζ_1. Then

$$\nu(\zeta_1)=\nu(\zeta_0)+\int_C(\nu_r dr+\nu_z dz).$$

The integrand on the right may be written

$$2\,\mathrm{Re}[(\partial\nu/\partial\zeta)d\zeta],$$

[7] H. Bondi, Rev. Mod. Phys. **29**, 423 (1957).

[8] H. Bondi, in *Lectures on General Relativity, Brandeis, 1964* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965).

[9] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnétisme* (Masson, Paris, 1955).

[10] See, for example, R. Geroch, Ph.D. thesis, Princeton University, 1967 (unpublished).

and we have

$$\nu(\zeta_1)=\nu(\zeta_0)+2\,\mathrm{Re}\int_C (\zeta+\bar{\zeta})\left(\frac{\partial\lambda}{\partial\zeta}\right)^2 d\zeta. \tag{7}$$

If ζ_0 and ζ_1 are both on the z axis and λ is regular on that portion of the axis lying between ζ_0 and ζ_1, then we see immediately from (7) that $\nu(\zeta_1)=\nu(\zeta_0)$, since $\zeta+\bar{\zeta}=2r=0$ on the z axis. Thus by a trivial addition of a constant it is always possible to make $\nu=0$ on any stretch of the z axis which is free of singularities. Suppose now that λ has a singularity at a point between ζ_0 and ζ_1. We take this point to be $z=0$ for convenience. Then C cannot be chosen to be along the z axis in (7), and we must deform the path about the singular point, for example, by means of a small semicircle of radius a. Introducing "polar" coordinates R, such that

$$r=R\sin\theta,\quad z=R\cos\theta,\quad \zeta=iRe^{-i\theta}, \tag{8}$$

we have that the contribution from the semicircle in (7) is

$$2\,\mathrm{Re}\int_0^\pi a^2 i(e^{-2i\theta}-1)\left(\frac{\partial\lambda}{\partial\zeta}\right)^2 d\theta. \tag{9}$$

This gives the change that ν must experience on crossing a singularity. If λ has the form of Eq. (5), then there is just one singularity, and that at $r=z=0$. We can show then that the integral (9) vanishes. This could be done by direct computation, but there is a simpler way of seeing this result. We may replace the small semicircle with a semicircle of large radius, since the two semicircles may be connected by nonsingular stretches of the z axis on which there is no contribution to the integral. Now $\partial\lambda/\partial\zeta=O(1/R^2)$ as $R\to\infty$; hence the integral (9) vanishes as $a\to\infty$, and therefore as $a\to 0$. Thus a Weyl solution regular everywhere except at the origin will exist for any λ having the form (5). Such a solution will be called, for the purposes of this paper, a "multipole particle." M_0 will be called the monopole moment or mass of the particle, M_1 the dipole moment, M_2 the quadrupole moment, etc. These moments are not to be taken too literally, since they are very much tied down to the Weyl coordinate system and, as mentioned above, the spherically symmetric Schwarzschild solution will have quadrupole and higher moments in these coordinates.

3. EQUILIBRIUM CONDITION FOR A MULTIPOLE PARTICLE

Let λ_n be the potential of a pure 2^n-pole particle, i.e.,

$$\lambda_n=-M_nR^{-(n+1)}P_n(\cos\theta).$$

Let λ' be any other potential which is regular at $r=z=0$. We shall say that λ_n is in equilibrium in λ' if the integral (9) with $\lambda=\lambda_n+\lambda'$ vanishes as $a\to 0$. This means that a 2^n-pole particle could be placed at the origin without affecting the regularity condition $\nu=0$ on the z axis. This equilibrium condition is

$$\lim_{a\to 0} 2\,\mathrm{Re}\int_0^\pi a^2 i(e^{-2i\theta}-1)\times\left[\left(\frac{\partial\lambda_n}{\partial\zeta}\right)^2+2\frac{\partial\lambda_n}{\partial\zeta}\frac{\partial\lambda'}{\partial\zeta}+\left(\frac{\partial\lambda'}{\partial\zeta}\right)^2\right]d\theta=0.$$

Of the three terms in the square brackets the first gives no contribution, as shown in Sec. 2, while the third gives no contribution, since λ' is regular at the origin. Hence the equilibrium condition reduces to

$$\lim_{a\to 0}\mathrm{Re}\int_0^\pi a^2 i(e^{-2i\theta}-1)\frac{\partial\lambda_n}{\partial\zeta}\frac{\partial\lambda'}{\partial\zeta}d\theta=0. \tag{10}$$

Since λ' is harmonic and regular at $R=0$, we may expand it in a power series in positive powers of R

$$\lambda'=\sum_{m=0}^{\infty}S_mR^mP_m(\cos\theta). \tag{11}$$

We may now apply $\partial/\partial\zeta$ to λ_n and λ' directly, since

$$\frac{\partial}{\partial\zeta}=\tfrac{1}{2}e^{i\theta}\left(-i\frac{\partial}{\partial R}+\frac{1}{R}\frac{\partial}{\partial\theta}\right), \tag{12}$$

and the equilibrium condition (10) becomes, after some computation,

$$0=2M_nS_{n+1}(n+1). \tag{13}$$

For a general multipole particle (5) it is immediately seen that the equilibrium condition is

$$\sum_{n=0}^{\infty}M_n(n+1)S_{n+1}=0. \tag{14}$$

This condition may be expressed directly in terms of derivatives of λ' as follows. From (11) and (12), by induction, we have

$$\frac{\partial^n\lambda'}{\partial\zeta^n}=\sum_{m=0}^{\infty}S_mR^{m-n}Q_m^{(n)}(\theta),$$

where

$$Q_m^{(0)}(\theta)=P_m(\cos\theta)$$

and

$$Q_m^{(n)}(\theta)=\tfrac{1}{2}e^{i\theta}[-i(m-n+1)Q_m^{(n-1)}(\theta)+dQ_m^{(n-1)}/d\theta].$$

Now, since λ' is regular at $\zeta=0$, it must be true that all $Q_m^{(n)}(\theta)$ will vanish for $m<n$ and that $Q_n^{(n)}(\theta)$ is a constant. This may be proved directly by writing the Legendre polynomial $P_m(\cos\theta)$ in the form[11]

$$P_m(\cos\theta)=\sum_{r=0}^{m}A_{m,r}e^{i(-m+2r)\theta},$$

[11] E. Whittaker and G. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1965), p. 303.

where

$$A_{m,r}=A_{m,m-r}=\frac{(2r-1)!!}{(2r)!!}\frac{(2n-2r-1)!!}{(2n-2r)!!},$$

$(2r-1)!!\equiv 1\times 3\times 5\cdots\times(2r-1)$, $2r!!\equiv 2\times 4\cdots\times 2r$.

By induction it may now be shown that

$$\begin{aligned}Q_m{}^{(m)}&=(i/2)^m(-2m)(-2m+2)\cdots(-2)A_{m,0}\\&=(-i/2)^m(2m-1)!!\end{aligned}\tag{15}$$

and

$$Q_m{}^{(m+k)}=0\quad\text{for } k>0.$$

Hence

$$(\partial^n\lambda'/\partial\zeta^n)_{\zeta=0}=Q_n{}^{(n)}S_n\tag{16}$$

and the equilibrium condition (13) can be replaced with the condition

$$(\partial^n\lambda'/\partial\zeta^n)_{\zeta=0}=0.\tag{17}$$

Although this a complex condition that should give rise to two real conditions, it in fact gives rise to only one. This comes about because S_n is real and $Q_n{}^{(n)}$ is real or pure imaginary depending on whether n is even or odd. Hence for n even the imaginary part of $(\partial^n\lambda'/\partial\zeta^n)_{\zeta=0}$ automatically vanishes, while for n odd the real part vanishes. This is equivalent to the statement that, for λ regular on the axis, $(\partial^{2n+1}\lambda/\partial r^{2n+1})_{r=0}$ vanishes. The condition of stability for a monopole is just the familiar one,[3] $\partial\lambda'/\partial z=0$. The conditions for the first few multipoles are

$$\begin{aligned}\text{monopole:}&\quad \lambda'_z=0;\\ \text{dipole:}&\quad \lambda'_{zz}-\lambda'_{rr}=0;\\ \text{quadrupole:}&\quad 3\lambda'_{rrz}-\lambda'_{zzz}=0;\\ \text{octupole:}&\quad \lambda'_{rrrr}+\lambda'_{zzzz}-6\lambda'_{rrzz}=0,\ \text{etc.}\end{aligned}$$

4. CONDITIONS OF EQUILIBRIUM FOR TWO OR MORE MULTIPOLE PARTICLES

Consider a pure 2^m-pole situated at the origin,

$$\lambda_m=-M_mR^{-(m+1)}P_m(\cos\theta).\tag{18}$$

If we expand λ_m about the point $r=0$, $z=a$, then, since it is regular at this point, we have

$$\lambda_m=\sum_{n=0}^{\infty}S_n{}^{(m)}R_a{}^nP_n(\cos\theta_a),\tag{19}$$

where

$$R_a=[r^2+(z-a)^2]^{1/2},\quad \cos\theta_a=(z-a)/R_a.$$

To find the coefficients $S_n{}^{(m)}$ it is only necessary to put $\theta=0$, $R>a$. Then $\theta_a=0$, $R=R_a+a$, and Eqs. (18) and (19) give

$$-M_m(R_a+a)^{-(m+1)}=\sum_n S_n{}^{(m)}R_a{}^n,$$

i.e.,

$$S_n{}^{(m)}=\frac{M_m}{a^{m+n+1}}(-1)^{n+1}\binom{m+n}{n}.\tag{20}$$

Thus from (15) and (16) we have

$$\left(\frac{\partial^n\lambda_m}{\partial\zeta^n}\right)_{\zeta=ia}=-\frac{M_m}{a^{m+n+1}}i^n\frac{2n-1!!}{2^n}\binom{m+n}{n}.\tag{21}$$

Similarly, replacing a with $-a$ and putting $\theta=\pi$, $\theta_{-a}=\pi$, $R=R_{-a}+a$, we have

$$S_n{}^{(m)}(-a)=(-1)^{m+n}S_n{}^{(m)}(a),$$
$$(\partial^n\lambda_m/\partial\zeta^n)_{\zeta=ia}=(-1)^{m+n}(\partial^n\lambda_m/\partial\zeta^n)_{\zeta=-ia}.$$

Consider two multipole particles at $z=\pm a$,

$$\lambda^{(1)}=\sum_{m=0}^{\infty}M_m{}^{(1)}R_a{}^{-(m+1)}P_m(\cos\theta_a)=\sum_{m=0}^{\infty}\lambda_m{}^{(1)},$$

$$\lambda^{(2)}=\sum_{m=0}^{\infty}M_m{}^{(2)}R_{-a}{}^{-(m+1)}P_m(\cos\theta_{-a})=\sum_{m=0}^{\infty}\lambda_m{}^{(2)}.$$

The equilibrium condition (14) at $z=a$ reads

$$\begin{aligned}0&=\sum_{n=0}^{\infty}M_n{}^{(1)}\frac{n+1}{Q_{n+1}{}^{(n+1)}}\sum_{m=0}^{\infty}\left(\frac{\partial^{n+1}\lambda_m{}^{(1)}}{\partial\zeta^{n+1}}\right)_{\zeta=ia}\\&=\sum_{n,m}M_n{}^{(1)}M_m{}^{(2)}|2a|^{-(m+n+2)}(-1)^n\binom{m+n}{n}\\&\qquad\times(m+n+1).\end{aligned}\tag{22}$$

The equilibrium condition at $z=-a$ is identical to this (the law of action and reaction); hence (22) is the necessary and sufficient condition for two multipole particles to be in equilibrium. For example, if at $z=-a$ we have a pure monopole, $M_n{}^{(2)}=0$ for $n>0$, and at $z=a$ we have a monopole-dipole, $M_n{}^{(1)}=0$ for $n>1$, then the equilibrium condition (22) reduces to

$$M_1{}^{(1)}=aM_0{}^{(1)}.$$

By the method described in Sec. 2 an exact solution may now be written that corresponds to this situation. The result is, putting $M_0{}^{(2)}=M_2$, $M_0{}^{(1)}=M_1$, $M_1{}^{(1)}=D$,

$$\lambda=-\frac{M_1}{R_a}-\frac{D\cos\theta}{R_a{}^2}-\frac{M_2}{R_{-a}},$$

$$\begin{aligned}\nu=&-M_1{}^2r^2/2R_a{}^4+D^2r^2[r^2-8(z-a)^2]/4R_a{}^8\\&-M_2{}^2r^2/2R_{-a}{}^4-2M_1Dr^2(z-a)/R_a{}^6\\&+M_1M_2(R^2-a^2)/2a^2R_aR_{-a}+M_2D(a^4-2za^3-2r^2a^2\\&+2zR^2a-R^4)/2a^3R_a{}^3R_{-a}.\end{aligned}$$

On the z axis we find $\nu=\pm M_2(M_1a-D)/2a^3$, depending on whether $|z|>a$ or $<a$. Thus it is only possible to

make $\nu=0$ uniformly on the z axis if $M_1a=D$, which is just the condition found above.

Finally, the conditions of equilibrium for more than two particles on the z axis may be obtained. Let there be N particles with z coordinates z_i arranged in descending order, $z_1>z_2>\cdots>z_N$. If the ith particle has multipole moments $M_n^{(i)}$, put

$$M_{ij}=\sum_{m,n} M_m^{(i)}M_n^{(j)}|z_i-z_j|^{-(m+n+2)}\binom{m+n}{n}\times(m+n+1)(-1)^n, \quad \text{for } i>j$$
$$=-M_{ji}, \quad \text{for } j>i.$$

Then the equilibrium conditions are

$$\sum_{i=1}^{N} M_{ij}=0.$$

These N conditions are not independent, since

$$\sum_{ij=1}^{N} M_{ij}=-\sum_{ij} M_{ji}=0.$$

Hence only $N-1$ of these conditions are independent. We can specify $\frac{1}{2}(N-1)(N-2)$ of the M_{ij} arbitrarily and solve for the remaining $N-1$. For example, $M_{\alpha\beta}$ $(\alpha,\beta=1,\cdots,N-1)$ could be specified arbitrarily and we may solve explicitly for $M_{N\beta}$:

$$M_{N\beta}=-\sum_{\alpha=1}^{N-1} M_{\alpha\beta}.$$

Finally, if we can find $M_m^{(i)}$, z_i to generate these M_{ij}, we shall have a static N-body situation, provided that $M_m^{(i)}$ do not all vanish for each $i=1,\cdots,N$.

5. CONCLUSION

We have shown that it is possible to find axially symmetric vacuum solutions of Einstein's field equations which have two or more singularities on the z axis. The singularities have the appearance of point particles endowed with a multipole mass structure.[12] The question that must be answered is whether these singular points can or cannot be covered up with a realistic mass distribution.

Suppose, for example, that we wish to fill in the solution representing a monopole-quadrupole at $z=a$ balancing a monopole at $z=-a$. The condition, by (22), is that

$$M_2^{(1)}=-\tfrac{4}{3}a^2M_0^{(1)}.$$

Now the largest negative quadrupole moment that can be achieved is that of a uniform ring of matter. The quadrupole moment for a ring of radius b is $M_2=-\frac{1}{2}b^2M_0$. Thus to achieve the desired quadrupole moment it seems that the matter must be spread out at least to a radius $b=a\sqrt{(8/3)}$. In any case, since this discussion is purely Newtonian in character, we can apply Bondi's theorem to tell us that no Newtonian mass distribution with everywhere-positive density can be found that will fill in the singularities of this solution. There are, of course, static two-body situations in Newtonian mechanics, but they are all of a rather artificial kind (such as the case of a ball in equilibrium at the center of a ring). The equilibrium in these cases is obtained by virtue of some additional symmetry (such as a reflection about the plane $z=0$). One would expect that for physically reasonable mass distributions only the corresponding special static solutions could occur in general relativity.

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[12] The straightforward "appearance" of these singularities is rather misleading. More detailed investigations of the Curzon metric (i.e., the monopole $\lambda=-M/R$) reveal some rather pathological behavior as $R\to 0$ [R. Gautreau and J. Anderson, Phys. Letters **27A**, 60 (1968)]. In fact, it is probably best to regard $R=0$ as representing a different "singular point" for each direction of approach [L. Mysak and G. Szekeres, Can. J. Phys. **44**, 617 (1966)].