Quantum Theory of an Electron Gas with Anomalous Magnetic Moments in Intense Magnetic Fields

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In this paper we have continued our investigation of the properties of an electron gas in intense magnetic fields. We have obtained expressions for the thermodynamic energy density, particle density, the magnetic moment, and pair density (created in equilibrium with the system) for a gas of electrons with an anomalous magnetic moment, using the exact eigenvalues of the Dirac-Pauli equation for an electron with an anomalous magnetic moment, obtained by Ternov *et al.* According to the solution of Ternov *et al.*, the lowest energy states of the electrons can be zero at certain field strengths. We have shown that pair creation does not occur spontaneously at the expense of the magnetic-field energy, but only at the expense of the thermodynamic energy of other particles of the system. Exact expressions for the pair density are given.

I. INTRODUCTION

In several previous papers[1–4] we have derived thermodynamic properties for an electron gas in a magnetic field, using the exact solution of the Dirac equation for an electron gas in a magnetic field. We have found that the gas becomes very anisotropic. Further, because the lowest energy state of the electron is unaltered, we concluded that spontaneous pair creation will not take place even when the classical spin energy $\mu_\beta \boldsymbol{\sigma}\cdot\mathbf{H}$ (where $\mu_\beta = |e|\hbar/2mc$ is the magnetic moment of the electron and $\mathbf{H}$ is the magnetic field) exceeds mc^2.

However, it is well known[5] that the electron possesses an anomalous magnetic moment amounting to $\mu=(\alpha/2\pi)\mu_\beta$, in addition to its Dirac moment $|e|\hbar/mc$. The question then naturally arises: Will this small amount of additional magnetic moment change our previous conclusions? In this paper we study the effect of the anomalous magnetic moment on the thermodynamic properties of matter. In general, we consider field strengths of the order of $(4\pi/\alpha)m^2c^3/e\hbar \sim 4\times 10^{16}$ G. Admittedly this field strength is rather high and may not be realized in nature. However, our main purpose is to investigate whether spontaneous pair creation will really take place when the effect of the anomalous magnetic moment is included. It is shown that in the presence of an anomalous magnetic moment, in certain cases the separation energy between the positive- and negative-energy states become zero. However, we show that spontaneous pair creation still does not take place at the expense of magnetic-field energy.

II. ANOMALOUS MAGNETIC MOMENT AND DIRAC-PAULI EQUATION

It is well known that the interaction of an electron with an anomalous magnetic moment within any external field A_μ can be described by the Dirac-Pauli equation[5]

$$\left[\gamma_\mu\left(\partial_\mu+\frac{i|e|}{\hbar c}A_\mu\right)-\mu\frac{i}{2\hbar c}F_{\mu\nu}\gamma_\mu\gamma_\nu+\frac{mc}{\hbar}\right]\psi=0\,, \quad (1)$$

where μ is taken to give the correct values of the magnetic moment; it turns out that[6]

$$\mu=\frac{\alpha}{2\pi}\frac{|e|\hbar}{2mc}.$$

The electromagnetic tensor is

$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}.$$

Equation (1) has been solved by Ternov *et al.*,[7] giving the following energy eigenvalues ($x\equiv p_z/mc$, $H_c=m^2c^3/e\hbar$):

$$E(x,n,s)=mc^2\left\{x^2+\left[\left(1+\frac{H}{H_c}(2n+s+1)\right)^{1/2}+s\frac{\alpha}{4\pi}\frac{H}{H_c}\right]^2\right\}^{1/2} \quad (2)$$

$$n=0,1,\cdots,\ s=\pm1.$$

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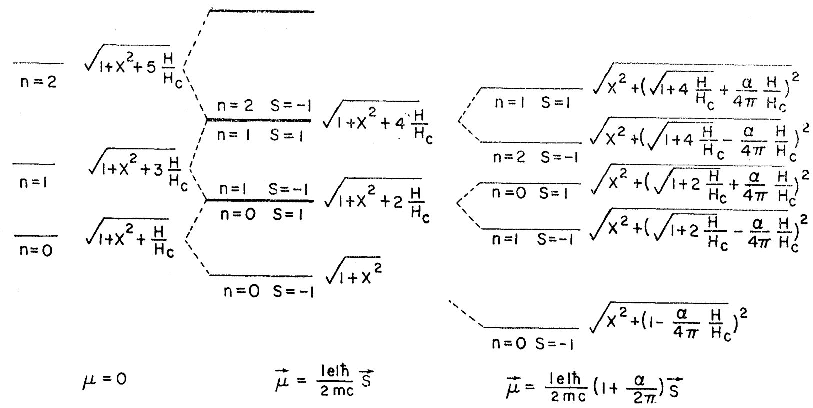

FIG. 1. Behavior of the energy eigenvalues, Eq. (2), in units of mc^2 for the three cases explained in Sec. II. It is clear that the main effect of the anomalous magnetic moment is to remove the degeneracy between the levels with quantum numbers n and $s=1$, and $n+1$ and $s=-1$. The scale is arbitrary.

We have checked the calculations of Ternov *et al.*[7] and have confirmed their results.

III. PROPERTIES OF ENERGY STATES

The behavior of Eq. (2) is schematically shown in Fig. 1 for the three cases (a) when the spin is equal to zero (i.e., no magnetic moment); (b) when the magnetic moment is identical to the Dirac magnetic moment $|e|\hbar/2mc$, and (c) when the anomalous magnetic moment is included, in addition to the Dirac magnetic moment considered in (b). It is seen that the effect of a magnetic field in the z direction is to cause the energy in the x and y directions to become quantized. However, when the magnetic moment of the electron takes the value of the Dirac magnetic moment, there is a degeneracy between the case with the principal quantum number n and the spin parallel to the field $(s=+1)$ and the case with the principal quantum number $n+1$ and the spin antiparallel to the field $(s=-1)$. This degeneracy is, however, removed when the anomalous magnetic moment is included. The two degenerate levels then split, in much the same way as the removal of degeneracy in the Zeeman splitting of atomic spectral lines.

The lowest energy states of the electron in the absence of z momentum $(p_z=x\,mc=0)$ are zero when the field strengths are given by

$$H=H_c\left(\frac{4\pi}{\alpha}\right)^2\left\{\eta+\left[\eta^2+\left(\frac{\alpha}{4\pi}\right)^2\right]^{1/2}\right\}$$

$$\begin{aligned}&\simeq 2\eta H_c(4\pi/\alpha)^2 \quad (\eta\neq 0),\\ &=(4\pi/\alpha)H_c \qquad (\eta=0),\end{aligned} \tag{3}$$

where $\eta=0,1,2,3,\cdots,\infty$. For the case $x=0$, Eq. (2) becomes

$$E(0,n,s)=\pm mc^2\left[\left(1+\frac{H}{H_c}(2n+s+1)\right)^{1/2}+s\frac{\alpha}{4\pi}\frac{H}{H_c}\right]. \tag{4}$$

We have calculated the first five energy states as functions of field strengths as shown in Fig. 2.

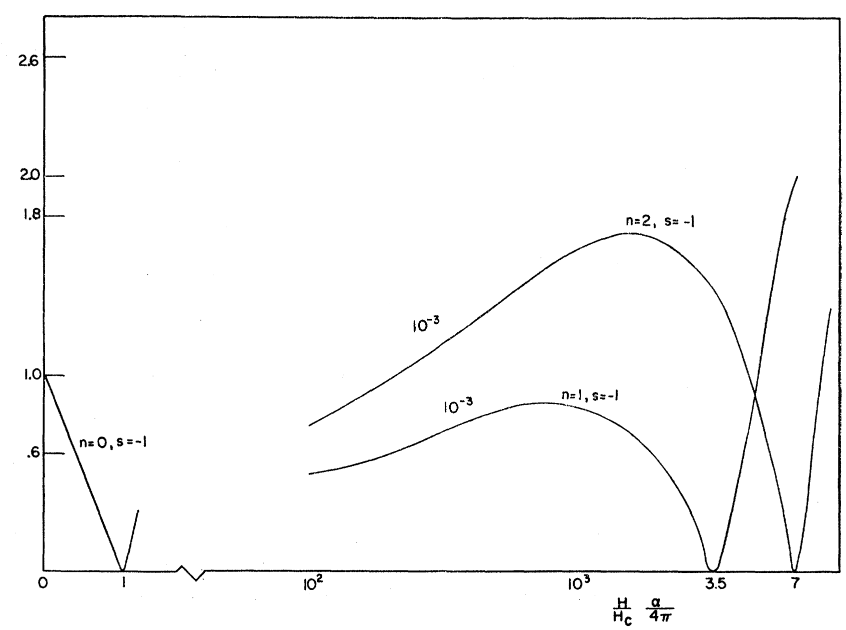

FIG. 2. Energy levels of Eq. (2) (in units of mc^2) as a function of the parameter $(H/H_c)\alpha/4\pi$ for the case $x=0$. The lowest level $n=0$ and $s=-1$ is shown to reduce to zero for $(H/H_c)\alpha/4\pi=1$. The ordinate is the energy of the electrons in mc^2 units. The numbers 10^{-3} indicate that the actual value of the energy has been multiplied by 10^{-3} to fit into this graph.

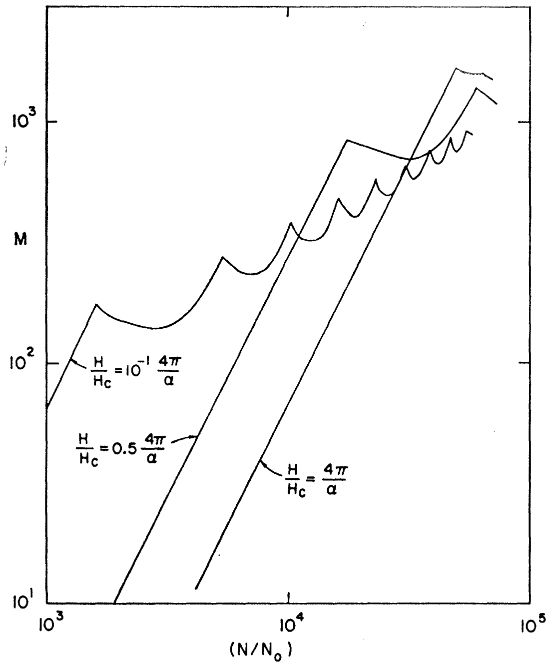

FIG. 3. Relation between the magnetic moment M ($M=\mathfrak{M}/\mathfrak{M}_0$, $\mathfrak{M}_0=2\mu_B\pi^{-2}\lambda_c^{-3}$) and the electron density N/N_0 ($N_0=\pi^{-2}\lambda_c^{-3}$) for the degenerate case. M exhibits the same undulating behavior as in the case without an anomalous magnetic moment (See Ref. 3). The various peaks correspond to the excitation of different magnetic states. At a given density the magnetic moment decreases as the field strength increases, until the first magnetic quantum state is excited.

Consider the case of $n=0$ and $s=-1$. The energy state of an electron is then $mc^2\{x^2+[1-(\alpha/4\pi)/H/H_c]^2\}^{\frac{1}{2}}$. If it were possible for the z momentum to vanish identically, then the energy state of the electron in the state $n=0$ and $s=-1$ would be

$$E(0,1,-1)=+mc^2\left[\left(1-\frac{\alpha}{4\pi}\frac{H}{H_c}\right)^2\right]^{1/2}=mc^2\left(1-\frac{\alpha}{4\pi}\frac{H}{H_c}\right), \tag{5}$$

which changes sign after the magnetic field is increased beyond the critical value $H/H_c=4\pi/\alpha$. Does this mean that the energy of the electrons will become less than that of the positrons at this field strength? If this happens then spontaneous pair creation will be possible. However, this is incorrect because the value of x can never be zero. As long as there is one electron in the universe the Fermi energy of the electron is nonzero, the operation in Eq. (5) is not permissible, and the sign of the energy state is an invariant property of the electron or the positron. This also means that the positive- and negative-energy levels of the electrons will never cross each other (noncrossing property). Naturally, this implies that spontaneous pair creation in a magnetic field without an external energy source is forbidden.

Note that the condition

$$H/H_c=4\pi/\alpha \tag{6}$$

can also be written as

$$H=4\rho e/r_0^2, \tag{7}$$

where $r_0=e^2/mc^2$ is the classical radius of the electron. Equation (7) is independent of $\hbar$ and is a classical result.[8] Similarly, Eq. (3) can be written as

$$H=\frac{4\pi e}{r_0^2}\frac{4\pi}{\alpha}\left\{\eta+\left[\eta^2+\left(\frac{\alpha}{4\pi}\right)^2\right]^{1/2}\right\}. \tag{8}$$

IV. MAGNETIC MOMENT

In previous papers[1–4] we have shown that spontaneous magnetization cannot take place in a noninteracting electron gas. This conclusion is not altered by the inclusion of the anomalous magnetic moment of the electron, as expected. The magnetic moment of an electron gas with an anomalous magnetic moment is obtained in a similar way as we have done previously; here we will not repeat the intermediate steps but will give the results ($\phi=kT/mc^2$; μ is the chemical potential in mc^2 units).

$$\begin{aligned}\frac{\mathfrak{M}}{\mathfrak{M}_0}=&\tfrac{1}{2}b_0^2C_2\left(\frac{\phi}{b_0},\frac{\mu}{b_0}\right)+\tfrac{1}{2}b_0(1-b_0)C_1\left(\frac{\phi}{b_0},\frac{\mu}{b_0}\right)+\tfrac{1}{2}\sum_{n=1}^{\infty}a_n^2C_2\left(\frac{\phi}{a_n},\frac{\mu}{a_n}\right)\\&+\tfrac{1}{2}\sum_{n=1}^{\infty}b_n^2C_2\left(\frac{\phi}{b_n},\frac{\mu}{b_n}\right)-\frac{H}{H_c}\sum_{n=1}^{\infty}\frac{na_n}{a_n+b_n}C_1\left(\frac{\phi}{a_n},\frac{\mu}{a_n}\right)-\frac{H}{H_c}\sum_{n=1}^{\infty}\frac{nb_n}{a_n+b_n}C_1\left(\frac{\phi}{b_n},\frac{\mu}{b_n}\right)\\&-\tfrac{1}{4}\sum_{n=1}^{\infty}a_n(a_n-b_n)C_1\left(\frac{\phi}{a_n},\frac{\mu}{a_n}\right)+\tfrac{1}{4}\sum_{n=1}^{\infty}b_n(a_n-b_n)C_1\left(\frac{\phi}{b_n},\frac{\mu}{b_n}\right),\end{aligned} \tag{9}$$

where

$$\mathfrak{M}_0=\frac{2}{\pi^2}\frac{\mu_B}{\lambda_c^3},\quad \lambda_c=\frac{\hbar}{mc},\quad a_n=\left(1+2n\frac{H}{H_c}\right)^{1/2}+\frac{\alpha}{4\pi}\frac{H}{H_c},\quad b_n=\left(1+2n\frac{H}{H_c}\right)^{1/2}-\frac{\alpha}{4\pi}\frac{H}{H_c}. \tag{10}$$

[8] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965), Chap. 9.

The functions $C_k(\phi,\mu)$ are defined by the following equations:

$$C_1(\phi,\mu)=\int_0^\infty (1+v^2)^{-1/2}\left[1+\exp\left(\frac{(1+v^2)^{1/2}-\mu}{\phi}\right)\right]^{-1}dv,$$

$$C_2(\phi,\mu)=C_3(\phi,\mu)-C_1(\phi,\mu),$$

$$C_3(\phi,\mu)=\int_0^\infty (1+v^2)^{1/2}\left[1+\exp\left(\frac{(1+v^2)^{1/2}-\mu}{\phi}\right)\right]^{-1}dv,$$

$$C_4(\phi,\mu)=\int_0^\infty \left[1+\exp\left(\frac{(1+v^2)^{1/2}-\mu}{\phi}\right)\right]^{-1}dv. \quad (11)$$

The properties of the C_k functions ($k=1,2,3,4$) have been extensively studied previously and we will not elaborate their properties here.

In Fig. 3 we have plotted the magnetic moment of an electron gas as a function of density for the following values of field strengths: $H/H_c=10^{-1}(4\pi/\alpha)$, $\frac{1}{2}(4\pi/\alpha)$, and $4\pi/\alpha$. Generally the induced magnetization is only 10^{-3} of the impressed field. This conclusion is almost identical to that obtained previously. We conclude, therefore, that the inclusion of the anomalous magnetic moment does not change our main conclusion: Spontaneous magnetization cannot take place in a dense electron gas.

V. THERMODYNAMIC PROPERTIES

The energy density U and particle density N of an electron gas can be obtained directly from the energy eigenvalues obtained earlier. In analogy to calculations presented in previous papers, we have

$$U=\tfrac{1}{2}U_0\theta(U_1+U_2), \quad (12)$$

$$N=\tfrac{1}{2}N_0\theta(N_1+N_2), \quad (13)$$

where

$$U_0=\frac{1}{\pi^2}\frac{mc^2}{\lambda_c^3},\quad N_0=\frac{1}{\pi^2}\frac{1}{\lambda_c^3},\quad \theta=\frac{H}{H_c}, \quad (14)$$

and

$$U_1=\sum_{n=1}^\infty\int_0^\infty F_1(x,n)E_1(x,n)dx, \quad (15)$$

$$U_2=\sum_{n=0}^\infty\int_0^\infty F_2(x,n)E_2(x,n)dx, \quad (16)$$

$$N_1=\sum_{n=1}^\infty\int_0^\infty F_1(x,n)dx, \quad (17)$$

$$N_2=\sum_{n=0}^\infty\int_0^\infty F_2(x,n)dx, \quad (18)$$

$$F_r(x,n)=\left[1+\exp\left(\frac{E_r(x,n)-\mu}{\phi}\right)\right]^{-1}, \quad (19)$$

$$E_1(x,n)=[x^2+((1+2n\theta)^{1/2}+\theta\alpha/4\pi)^2]^{1/2}, \quad (20)$$

$$E_2(x,n)=[x^2+((1+2n\theta)^{1/2}-\theta\alpha/4\pi)^2]^{1/2}. \quad (21)$$

Upon introducing the C_k functions, Eqs. (15)–(18) can be reduced to convenient form with the transformation in the integration variables

$$v=x/a_n \quad\text{or}\quad v=x/b_n. \quad (22)$$

The results are (with $b_n\neq 0$)

$$U_1=\sum_{n=1}^\infty a_n^2C_3\left(\frac{\phi}{a_n},\frac{\mu}{a_n}\right), \quad (23)$$

$$U_2=\sum_{n=0}^\infty b_n^2C_3\left(\frac{\phi}{b_n},\frac{\mu}{b_n}\right), \quad (24)$$

$$N_1=\sum_{n=1}^\infty a_nC_4\left(\frac{\phi}{a_n},\frac{\mu}{a_n}\right), \quad (25)$$

$$N_2=\sum_{n=0}^\infty b_nC_4\left(\frac{\phi}{b_n},\frac{\mu}{b_n}\right), \quad (26)$$

b_n vanishes when the lowest energy state of the electron is zero. When this happens for some values of $n=m$, we have

$$U_2=\sum_{n=0;\,n\neq m}^\infty b_n^2C_3\left(\frac{\phi}{b_n},\frac{\mu}{b_n}\right)+\mathfrak{F}_1(\phi,\mu), \quad (27)$$

$$N_2=\sum_{n=0;\,n\neq m}^\infty b_nC_4\left(\frac{\phi}{b_n},\frac{\mu}{b_n}\right)+\mathfrak{F}_0(\phi,\mu), \quad (28)$$

where

$$\mathfrak{F}_0(\phi,\mu)=\int_0^\infty\left[1+\exp\left(\frac{x-\mu}{\phi}\right)\right]^{-1}dx=\phi\ln(1+e^{\mu/\phi}), \quad (29)$$

$$\mathfrak{F}_1(\phi,\mu)=\int_0^\infty\left[1+\exp\left(\frac{x-\mu}{\phi}\right)\right]^{-1}xdx. \quad (30)$$

VI. PAIR CREATION

We shall now study the most important case of interest, namely the pair-creation phenomena when the field strength is close to $(4\pi/\alpha)H_c$ or $2\eta H_c(4\pi/\alpha)^2$, $\eta=1,2,3,\cdots,\infty$. At these field strengths the lowest energy state of the electron is zero. It will be of great interest to see if pairs can be created spontaneously.

The equation which governs pair-creation equilibrium is

$$\mu_-+\mu_+=0 \quad (31)$$

or

$$\mu_-=-\mu_+\equiv\mu, \quad (32)$$

where μ_- and μ_+ are the chemical potentials of the electron and positron, respectively. The number density of positrons is given by

$$n_+=N_0[N_1(-\mu,\phi)+N_2(-\mu,\phi)]. \quad (33)$$

The charge-neutrality condition requires that

$$n_--n_+=n_0, \quad (34)$$

where n_0 is some constant. Consider the case of the vacuum; then $n_0=0$ and $\mu=0$. In most cases of interest, all terms in n_- and n_+ are negligible except those with $b_m=0$. We then find from Eq. (28) that

$$n_-=n_+=N_0\phi\ln(1+e^{\mu/\phi})$$
$$=N_0\phi\ln 2, \qquad (35)$$

which is proportional to the temperature. In the presence of matter μ is positive, and pair creation is suppressed. Let us consider the case $(\mu-1)/\phi\gg 1$, with a field strength such that $b_m\approx 0$. The positron density n_+, to the order $\exp(-\mu/\phi)$, is given by

$$n_+=N_0\phi\ln(1+e^{-\mu/\phi}). \qquad (36)$$

The electron density is n_+ plus the number of electrons originally present without pair creation. Let the number density of electrons originally present be n_0. If $n_0\gg n_+$, then n_0 approximately determines the chemical potential through Eq. (13), i.e.,

$$n_0=N_0[N_1(\mu,\phi)+N_2(\mu,\phi)];$$

hence

$$n_-=n_0+n_+$$
$$=N_0[N_1(\mu,\phi)+N_2(\mu,\phi)+\phi\ln(1+e^{-\mu/\phi})]. \qquad (37)$$

Thus it is seen that the number of pairs vanishes at zero temperature, as expected. This means that the energy of pairs created comes from thermodynamic energy from other particles and not from the energy of the magnetic field.

However, the number of pairs created is directly proportional to the temperature in the nondegenerate limit ($\mu-1\ll\phi$) and is proportional to $\phi e^{-\mu/\phi}$ in the degenerate limit. Additional processes like the pair-annihilation process[9] $e^-+e^+\rightarrow\nu+\bar{\nu}$ can take place more favorably when a strong magnetic field is present.[10] This will aid energy dissipation. However, this will not occur until the field strength exceeds $H=(4\pi/\alpha)H_c\simeq 10^{16}$ G. It is not known whether such a strong field may exist in nature.

VII. CONCLUSION

We have discussed the thermodynamic properties of an electron gas in a magnetic field with an anomalous magnetic moment. According to the solutions of the Dirac equation with an anomalous magnetic moment, the lowest energy states of an electron can become zero when the field strengths exceed 10^{16} G. We have shown that spontaneous pair creation cannot take place at the expense of the field energy. We have also derived expressions for the magnetic moment, the energy density, the number density, and the pair density as functions of temperature and the chemical potential.

We now compare the normal and anomalous cases. In the normal case the minimum energy of the electrons is always mc^2 and is independent of the field strength; the magnetic field only quantizes the kinetic energy perpendicular to the field and does not change the rest mass of the electron. In the anomalous case the rest energy of the electrons changes with field strength, first decreasing to zero at $H\sim 10^{16}$ G and then increasing to a value of $\sim 2\times 10^3\ mc^2$ at a field strength of $\sim 10^9$ G, and then decreases to zero again, and so on (see Fig. 2). Since the energy eigenvalues determine the thermodynamic properties of a system, naturally the normal case will differ from the anomalous case.

At a field of $\sim 10^{19}$ G, the rest mass of an electron in the proper quantum state is of the order of twice the mass of the protons. According to our theory the rest mass of a muon in a magnetic field is not altered until the field strength approaches $H_c(\alpha/4\pi)(m_\mu/m)^2\approx 10^{22}$ G. Thus it is energetically possible for an electron in the proper magnetic state to decay into a muon at a field strength of $\sim 10^{19}$ G. In other words, at a field strength of $\sim 10^{19}$ G matter should consist of electrons, muons, and baryons. This result may be of relevance in the theory of neutron stars.

Finally we should point out that the solution of Ternov *et al.* is based on the concept of a point anomalous magnetic moment. It is known from quantum electrodynamics that the anomalous magnetic moment possesses a form factor of electrodynamical nature. It is not known how a finite distribution of the anomalous magnetic moment will alter the solution of Ternov *et al.* and hence ours. This may require further investigation.

Note added in proof. After our paper was prepared for publication, we received a preprint by R. O'Connell on a similar subject.[11] However, O'Connell's paper only discussed the possibility of pair creation while we have given expressions for the pair densities and discussed other relevant facts, among which the most important one is the instability of an electron to decay into a muon in fields as high as 10^{19} G.

ACKNOWLEDGMENTS

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