$$Z_{11}=Y_5G_2+(Y_6+Y_9-Y_{11}+Y_{13})G_{12}+(Y_6+Y_{10})G_{10}+Y_7(G_4+G_6+G_7+G_{13})+2Y_8(G_2+G_{11})+Y_9G_9 +(Y_{11}+Y_{14})G_3+Y_{12}(G_1-G_4+G_5-G_{13}),$$

$$Z_{12}=-Y_5G_3-(Y_6+Y_7+Y_{11}+Y_{12})G_{11}-(Y_6+Y_9-Y_{13})G_{12}-(Y_7+Y_{10}+Y_{12}+Y_{14})G_2-Y_8(G_1+2G_5+G_8),$$

and

$$Z_{13}=Y_5(G_2+G_4)+Y_6(G_7+G_9)+2Y_7(G_{10}+G_{12})+2Y_8(G_2+G_{11}+G_{12})+2Y_9(G_{12}+G_{13})+Y_{10}G_6 +Y_{11}G_5+2Y_{12}G_3+2Y_{13}G_4+Y_{14}G_1.$$

The Green's functions are then given by

$$G_{xx}(1,1;\omega)=G_1(1+Z_4)+G_2Z_1+2G_3Z_5+G_4(Z_2-2Z_6)+G_5Z_8-G_6Z_9 +(G_7+G_9)Z_{13}+2(G_{10}+G_{12})Z_{11}+2(G_{12}+G_{13})Z_{10},$$

$$G_{zz}(1,1;\omega)=G_1+4G_2Z_7+G_3Z_3-4(G_{11}+G_{12})Z_{12},$$

and

$$G_{xy}(1,1;\omega)=G_1Z_4+G_2Z_2+2G_3Z_6+G_4(Z_1-2Z_5)+G_5Z_9-G_6Z_8+(G_7+G_9)Z_{13}+2(G_{10}+G_{12})Z_{10}+2(G_{12}+G_{13})Z_{11}.$$

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Landau Fermi-Liquid Parameters in Na and K

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Theoretical estimates are presented for the Landau Fermi-liquid parameters in Na and K, and a comparison is made with the experimental values. The calculations are presented in two parts. The effects of the Coulomb interaction between the electrons are taken from previous calculations which use the random-phase approximation and include exchange diagrams approximately. The electron-phonon-interaction effects are calculated using the observed phonon spectra and a screened pseudopotential approximation for the electron-ion coupling. The theoretical estimates for Na are found to be in surprisingly good agreement with six independent experimentally determined parameters. In K, the experimental values are less accurate, but a preliminary comparison is encouraging.

I. INTRODUCTION

THE Landau theory of a Fermi liquid[1] as extended by Silin[2] has been very successful in explaining the qualitative nature of many-body effects in metals. In this theory the effects of the interactions are characterized by an effective mass m^* and an interaction function $f(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ which are to be determined from experiment. Until recently, however, the experimental information on the size of f was very limited. The observation of spin waves[3,4] and high-frequency plasmalike waves[5–7] in Na and K has led to the determination of several of the Legendre coefficients of the function f. In this paper we will be concerned with a comparison of the experimental values of these coefficients, the Landau parameters, with theoretical estimates based on microscopic theory.

In metals there are two sources of interactions between electrons, (a) the Coulomb repulsion between two electrons and (b) the attraction caused by the virtual exchange of phonons. There are, in addition, effects due to the periodic potential of the ions. Na and K have Fermi surfaces which deviate from the free-electron sphere by less than 0.2%,[8] so that we will, for the most part, ignore band-structure effects. The contribution to interaction effects from the Coulomb repulsion, which we will refer to as the electron-electron contribution, may then be obtained from calculations for a uniform electron gas.[9] The derivation of the

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Landau theory for an electron gas with purely Coulombic repulsion has been studied by several authors and a detailed treatment may be found in Nozières's book.[10] Prange and Kadanoff[11] have derived a Landau theory for electrons interacting via the exchange of virtual phonons. Recently Prange and Sachs[12] have given a derivation of the Landau theory in the presence of both electron-electron and electron-phonon interactions.

The electronic-density parameter r_s takes the values 3.96 in Na and 4.87 in K. (r_s is defined such that $\frac{4}{3}\pi r_s^3$ is the volume per electron in units of the Bohr radius cubed.) These values are intermediate between the high density (small r_s) and low density (large r_s) regimes. There is no small parameter in the theory which would allow us to calculate f unambiguously. The basic scheme which is used to evaluate f at metallic densities is the random-phase approximation (RPA) along with the approximate inclusion of so-called exchange diagrams. Since these approximations are widely used elsewhere, for example, in pseudopotential theory, etc., it is of real interest to have some direct test of their accuracy. The first calculation of the effective mass and spin susceptibility for the alkalis using such an approximation, suggested by Nozières and Pines,[13] were carried out by Silverstein.[14,15] Subsequently the present author[16,17] calculated the full Landau interaction function $f(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ using the Hubbard approximation[18] scheme which is similar to the Nozières-Pines approximation. The Landau interaction f has also been calculated by Hedin[19] in a somewhat different approximation.

The calculation of the electron-phonon contribution is greatly simplified by using Migdal's[20] result, which shows that only the lowest-order one-phonon exchange graph is important. The higher-order terms are reduced by powers of the ratio of the electronic and ionic masses. The calculation depends on an accurate knowledge of the phonon frequency sepectrum and the electron-ion coupling. The former has been measured directly by neutron scattering in Na[21] and K.[22] The latter we estimate using a screened pseudopotential approximation. It would be of interest if the accuracy of this latter approximation could be tested directly; however, at present, there is no direct way to measure the electron-phonon contribution to the Landau parameters. The electron-phonon contribution to the effective mass was first calculated in this manner by Ashcroft and Wilkins.[23] Since then a number of authors using different pseudopotentials have also estimated the effective mass.[24,25] However, there have been no calculations of the full f function.

In Sec. II we discuss the connection between the microscopic theory and the Landau theory in the presence of both the electron-electron and electron-phonon interactions. We will derive the results of Prange and Sachs[12] from a different point of view.

In Sec. III we discuss the numerical calculations of the electron-electron contributions. We tabulate the Lengendre coefficients of the Landau f function as calculated previously by the present author[16] and also by Hedin.[19] In general, observed quantities depend on both the electron-electron and electron-phonon. There are, however, some quantities which depend only on the electron-electron interaction. For these quantities we compare the experimental values in Na and K with the theoretical estimates. We find surprisingly good agreement between the two sets of values.

In Sec. IV we report on calculations of the electron-phonon contribution to the effective mass and the higher Landau coefficients for Na and K using several different pseudopotentials and the observed phonon spectra. We examine the effects of the band structure by including more than one plane wave. These we find are fairly small. Our results for the mass enhancement are in agreement with previous calculations. We also examine the magnitude of the departures from isotropy due to the large anisotropy in the observed phonon spectra. These we find are too small to be observable at present.

Finally, in Sec. V we combine the two sets of calculations to give theoretical estimates for the Landau parameters in Na and K. A comparison with the observed values in these materials shows good qualitative and even quantitative agreement between the theoretical estimates and the experimental values.

II. REVIEW OF LANDAU THEORY INCLUDING BOTH ELECTRON-ELECTRON AND ELECTRON-PHONON INTERACTIONS

First let us define our terms. We define the electron-phonon contribution to include all graphs which involve one or more phonon lines. The electron-electron contribution includes all graphs with no phonon lines. Thus the electron-phonon contribution includes electron-electron interaction lines to all orders.

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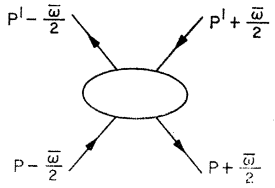

FIG. 1. The proper four-point vertex function ${}^0\tilde{\Gamma}(p,p',\tilde{\omega})$. The proper four-point vertex is defined as including all graphs which cannot be cut into two separate pieces by cutting a single interaction line carrying the four-momentum vector $\tilde{\omega}$.

It is customary in discussing the electron-phonon problem to start from the self-energy. We will take a different approach and begin by noting the Landau interaction function $f(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ is defined in microscopic theory (see Nozières[10]) as a certain limit of the proper four-point vertex function ${}^0\tilde{\Gamma}$, shown in Fig. 1.

$$f(\mathbf{k}\sigma,\mathbf{k}'\sigma')=\lim_{\tilde{\omega}\to 0}\lim_{q/\epsilon\to 0} 2\pi i z_{\mathbf{k}} z_{\mathbf{k}'}\,{}^0\tilde{\Gamma}(\mathbf{k}\sigma,\mathbf{k}'\sigma',\tilde{\omega}) \quad (1)$$

$$=2\pi i z_{\mathbf{k}} z_{\mathbf{k}'}\,{}^0\tilde{\Gamma}^0(\mathbf{k}\sigma,\mathbf{k}'\sigma'). \quad (2)$$

We will use, with certain exceptions, Nozières's notation. $z_{\mathbf{k}}$ is the quasiparticle renormalization factor and the proper four-point vertex is defined as including all graphs which cannot be cut into two separate pieces by cutting a single interaction line carrying the four-momentum vector $\tilde{\omega}=(\mathbf{q},\epsilon)$. The four-momentum vectors p, $=(\mathbf{k},\omega)$, and p' are taken on the energy shell and $|\mathbf{k}|=k_F$. The other limit of the four-point vertex function defines the two-particle scattering function which we denote by $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ [26]:

$$g(\mathbf{k}\sigma,\mathbf{k}'\sigma')=\lim_{\tilde{\omega}\to 0}\lim_{q/\epsilon\to\infty} 2\pi i z_{\mathbf{k}} z_{\mathbf{k}'}\,{}^0\tilde{\Gamma}(\mathbf{k}\sigma,\mathbf{k}'\sigma',\tilde{\omega}) \quad (3)$$

$$=2\pi i z_{\mathbf{k}} z_{\mathbf{k}'}\,{}^0\tilde{\Gamma}^\infty(\mathbf{k}\sigma,\mathbf{k}'\sigma'). \quad (4)$$

Migdal[20] showed some years ago that the calculation of electron-phonon effects was greatly simplified by the existence of a small parameter, i.e., the ratio of electronic and ionic masses (m/M). We may apply his results directly to show that the contribution of graphs of the form shown in Fig. 2 are all of order $(m/M)^{1/2}$. The difference in momenta in all of these graphs is typically of order k_F and it follows at once from Migdal's arguments that they give negligible contributions to f. The graphs of the form shown in Fig. 3 are not small. They may be included by using electron-phonon vertices and phonon propagators which are fully renormalized for Coulomb interactions.

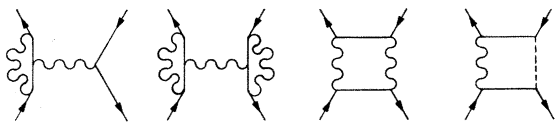

FIG. 2. Examples of phonon renormalization effects which are negligible by Migdal's theorem. A wavy line denotes a phonon propagator and a dashed line a Coulomb interaction line.

[26] We denote the scattering function by $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$, rather than $A(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ as Nozières (Ref. 10) has done, to avoid a conflict with the notation A_n for the Legendre coefficients of the f function.

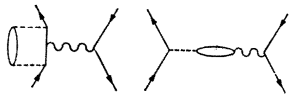

FIG. 3. Examples of Coulomb renormalization effects.

In graphs of the general form shown in Fig. 4 the intermediate electron and hole momenta are constrained to differ by $\tilde{\omega}$. Since we require the limit $\tilde{\omega}\to 0$, we may not simply rule out such graphs. Let us consider a representative graph shown in Fig. 4(b). To avoid unessential complication let us follow Migdal[20] and take a constant electron-phonon coupling and a Debye model. We shall work throughout this paper at $T=0$. Denoting by Γ_2 the contribution of this graph, we see that

$$\Gamma_2(p,p',\tilde{\omega})\sim\lambda^2\int d^4p''\mathfrak{D}(p-p'')\mathfrak{D}(p''-p') \times G_0(p''+\tfrac{1}{2}\tilde{\omega})G_0(p''-\tfrac{1}{2}\tilde{\omega}). \quad (5)$$

We will use atomic units. $\mathfrak{D}(p)$ is the phonon propagator, $G_0(p)$ is the unperturbed one-electron propagator, and λ is the dimensionless electron-phonon coupling constant. It is straightforward to evaluate Γ_2 [20] and, examining the result for both p and p' on the energy shell and on the Fermi surface, we find that

$$\Gamma_2(\mathbf{k},\mathbf{k}',\tilde{\omega})\sim\frac{\lambda^2}{k_F{}^2 q}\left(\ln\left|\frac{k_F q+\epsilon}{k_F q-\epsilon}\right|-\pi i\theta(k_F q-|\epsilon|)\right) \times\int_{-\omega_0/2}^{\omega_0/2} dt(\theta(t+\tfrac{1}{2}\epsilon)-\theta(t-\tfrac{1}{2}\epsilon)), \quad (6)$$

where ω_0 is the Debye frequency. Taking the limits appropriate to the f function, we see that

$$\lim_{\tilde{\omega}\to 0}\lim_{q/\epsilon\to 0}\Gamma_2(\mathbf{k},\mathbf{k}',\tilde{\omega})\sim\frac{\lambda^2}{k_F}, \quad (7)$$

while in the other limit, appropriate to the scattering

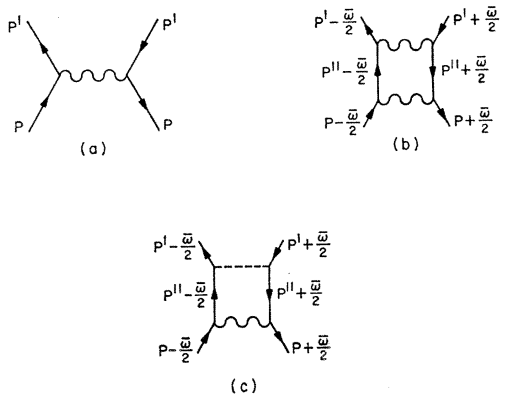

FIG. 4. (a) One-phonon-exchange graph; (b) and (c), graphs which carry intermediate four-momentum $\tilde{\omega}$.

function g,

$$\lim_{\tilde{\omega}\to 0}\lim_{q/\epsilon\to\infty}\Gamma_2(\mathbf{k},\mathbf{k}',\tilde{\omega})\sim\frac{\lambda^2}{k_F}O\left(\frac{m}{M}\right)^{1/2}. \quad (8)$$

This shows that only the lowest-order one-phonon exchange graph contributes to the scattering function $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$. To calculate the electron-phonon contribution to the Landau f function one must include all graphs of the form shown in Fig. 4(b).

It is customary to formulate the Landau theory in terms of the f function. However, it is equally possible to use the g function instead. The g-function formulation is more convenient in discussing anisotropy due to the phonons since, as shown above, the phonons enter in a much simpler way. We will, therefore, in Sec. IV express our results for the electron-phonon system in terms of the g function.

For an isotropic situation the two functions f and g are simply related. They may be expanded in Legendre polynomials in the angle $\theta_{\mathbf{k},\mathbf{k}'}$ and in the absence of spin orbit coupling may be separated into singlet and triplet contributions. Thus

$$f(\mathbf{k}\sigma,\mathbf{k}'\sigma')=\sum_{l=0}^{\infty}(f_l^s+f_l^a\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')P_l(\cos\theta_{\mathbf{k}\cdot\mathbf{k}'}). \quad (9)$$

We define our Fermi-liquid coefficients such that

$$A_l=\frac{m^*k_F}{\pi^2(2l+1)}f_l^s, \quad B_l=\frac{m^*k_F}{\pi^2(2l+1)}f_l^a. \quad (10)$$

We also define an additional set of parameters $g_{s,l}$ and $g_{a,l}$ which are related in the same way to g as A_l and B_l are to f. In an isotropic material these two sets of parameters are connected by a simple relationship:

$$g_{s,l}=A_l/(1+A_l), \quad g_{a,l}=B_l/(1+B_l). \quad (11)$$

In a general case, such as a real metal where there are anisotropies in the shape of the Fermi surface, in the phonon spectrum, etc., the two functions f and g are related by an integral equation.[10]

$$f(\mathbf{k}\sigma,\mathbf{k}'\sigma')=g(\mathbf{k}\sigma,\mathbf{k}'\sigma')+\sum_{\mathbf{k}''\sigma''}f(\mathbf{k}\sigma,\mathbf{k}''\sigma'')\times\delta(\epsilon_{\mathbf{k}''}-\mu)g(\mathbf{k}''\sigma'',\mathbf{k}'\sigma'). \quad (12)$$

The arguments that we have given here to show that only the lowest-order one-phonon exchange contributes to g are quite general and apply to anisotropic situations. The result that we find is that

$$g_{\mathrm{ep}}(\mathbf{k}\sigma,\mathbf{k}'\sigma')=2\pi i z_{\mathbf{k}}z_{\mathbf{k}'}\Gamma_1^{\mathrm{ep}}(\mathbf{k},\mathbf{k}')\delta_{\sigma,\sigma'}, \quad (13)$$

where $\Gamma_1^{\mathrm{ep}}(\mathbf{k},\mathbf{k}')$ is the contribution of the graph shown in Fig. 4(a). The subscript or superscript ep denotes the inclusion of only electron-phonon contributions, and ee will denote only electron-electron contributions. Quantities which include both will be unadorned.

The electron-electron contribution and the electron-phonon contribution to the four-point vertex function simply add, in the forward scattering limit, and we get the result, due to Prange and Sachs,[12]

$$g(\mathbf{k}\sigma,\mathbf{k}'\sigma')=2\pi i z_{\mathbf{k}}z_{\mathbf{k}'}({}^0\tilde{\Gamma}_{\mathrm{ee}}^{\infty}(\mathbf{k}\sigma,\mathbf{k}'\sigma')+\Gamma_1^{\mathrm{ep}}(\mathbf{k},\mathbf{k}')\delta_{\sigma,\sigma'}). \quad (14)$$

The combined f function is then determined through the integral equation (12).

An important result which can be derived using Migdal's arguments[12,27] is that

$$v_{\mathrm{ee}}^{-1}(\mathbf{k})z_{\mathrm{ee}}(\mathbf{k})=v^{-1}(\mathbf{k})z(\mathbf{k}), \quad (15)$$

where v is the magnitude of the quasiparticle velocity. Thus we can rewrite Eq. (14) as

$$g(\mathbf{k}\sigma,\mathbf{k}'\sigma')=2\pi i z_{\mathrm{ee}}(\mathbf{k})z_{\mathrm{ee}}(\mathbf{k}')(v(\mathbf{k})v(\mathbf{k}')/v_{\mathrm{ee}}(\mathbf{k})v_{\mathrm{ee}}(\mathbf{k}'))\times[{}^0\tilde{\Gamma}_{\mathrm{ee}}^{\infty}(\mathbf{k}\sigma,\mathbf{k}'\sigma')+\Gamma_1^{\mathrm{ep}}(\mathbf{k},\mathbf{k}')\delta_{\sigma,\sigma'}]. \quad (16)$$

With these results it is possible to discuss the effects of the electron-electron and electron-phonon interactions on the transport and equilibrium properties of the system. Such a discussion may be found in the articles by Heine, Nozières, and Wilkins[27] and Prange and Sachs[12] and we shall not repeat it here.

III. ELECTRON-ELECTRON INTERACTION

We turn now to the evaluation of the contribution from the electron-electron interaction. The basic approximation used is one in which all diagrams in the RPA are included exactly and the lowest class of exchange diagrams are approximated. Hubbard[18] and Nozières and Pines[13] have shown that for large momentum transfers (or short distance) the exchange graphs will cancel half the direct RPA terms. Various authors have calculated transport coefficients using interpolation schemes which are based on this result. Silverstein[14,15] calculated the effective mass and Pauli spin susceptibility using the numerical interpolation scheme proposed by Nozières and Pines.[13] The present author[16,17] calculated the Landau function $f(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ using an analytic interpolation scheme proposed by Hubbard.[18] Hedin[19] calculated the Landau f function to second order in a staticly screened interaction. Because the interaction is staticly screened, his calculation is not an exact RPA calculation of the direct terms. He does, however, include the second-order exchange graphs explicitly, whereas the author has only included these graphs approximately. We take the view that the size of the differences between these two calculations is indicative of the size of the errors in approximating the exchange graphs. There are, of course, many classes of graphs whose magnitude has not been estimated at all and the more serious error lies in neglecting these graphs. There is no small parameter in the problem, so that

[27] V. Heine, P. Nozières, and J. Wilkins, Phil. Mag. **13**, 741 (1966).

TABLE I. The Landau parameters calculated for an electron gas with purely Coulomb interactions.

	Hubbard approx. from I		Hedin			Screened exchange
	$r_s=2$	$r_s=4$	$r_s=2$	$r_s=4$	$r_s=5$	$r_s=4$
A_0^{ee}	−0.35	−0.69	−0.33	−0.55	−0.64	−0.32
A_1^{ee}	−0.001	+0.06	−0.05	−0.03	−0.015	−0.04
A_2^{ee}	−0.03	−0.05	−0.02	−0.025	−0.92	−0.005
A_3^{ee}	0.004	0.004				
B_0^{ee}	−0.235	−0.28	−0.25	−0.31	−0.32	−0.32
B_1^{ee}	−0.06	−0.06	−0.08	−0.09	−0.09	−0.04
B_2^{ee}	−0.03	−0.02	−0.01	−0.001	0.005	−0.005
B_3^{ee}	0.004	−0.001				
χ/χ_0	1.31	1.47	1.27	1.41	1.45	1.42
m_{ee}^*/m	1.00	1.06	0.95	0.97	0.985	0.96
Sum rule	+0.20	+0.23	+0.10	+0.03	+0.11	+0.26

there is no justification for ignoring such contributions. We shall attempt to assess the reasonableness of our results by comparing the calculations with experiment, wherever possible. We know that there are many cancellations among higher-order graphs, so that great care must be exercised in going beyond the simpler approximations. It is possible that merely including more diagrams will not lead to more accurate results, especially if sum rules, conservation laws, or Ward identities are violated.

In Table I we quote the results found for the electron-electron contributions to the Landau parameters A_n^{ee} and B_n^{ee}. A detailed description of the author's calculation of the Landau f function is given in Ref. 16 (hereafter referred to as I) and we will not repeat it here.[28] The results quoted in Table I were found from numerical integration of the results in I. For comparison we quote the values found by Hedin.[19] These were obtained by numerical integration using the values for the f function quoted in Table V of his paper. We quote only the values that do not include the quasiparticle renormalization factors. As pointed out in I, there are large cancellations between the quasiparticle renormalization factors and vertex corrections. The cancellations are exact only in the limit of zero momentum transfer but calculations, reported in I, of the lowest-order vertex corrections suggest that approximate cancellation will occur at higher momentum transfers. We take the position that, omitting all frequency-dependent vertex corrections and all quasiparticle renormalization factors, is a better approximation than including the latter, but omitting the former.

There is good qualitative agreement between the two calculations. The only major discrepancy is in the sign of A_1^{ee}. We find A_1^{ee} positive for $r_s \gtrsim 2$ corresponding to an enhanced effective mass, whereas Hedin finds A_1^{ee} negative and correspondingly a reduced effective mass. Silverstein[14] using the Nozières-Pines approximation also found an enhanced mass and we shall see below that an enhanced mass is in better agreement with the experimental results for Na and K. There are also significant differences in the magnitudes of some of the other coefficients.

For comparison we show in Table I the results of a calculation keeping only the lowest-order exchange graph in the RPA for $r_s=4$. In this approximation, $A_n \equiv B_n$. The results are much closer to Hedin's but there are significant differences in magnitude.

Brinkman, Platzman, and Rice[29] have derived a sum rule on the coefficients of the f function. They show that due to the exclusion principle the exact f function for an interacting electron gas satisfies the relation

$$1+\frac{B_0^{ee}}{1+B_0^{ee}}+\sum_{l=1}^{\infty}(2l+1)\left(\frac{A_l^{ee}}{1+A_l^{ee}}+\frac{B_l^{ee}}{1+B_l^{ee}}\right)=0. \quad (17)$$

It is interesting to examine how well the various approximations satisfy this sum rule. In the last column of Table I we have tabulated the left-hand side of Eq. (17), neglecting all terms with $l \geqslant 3$.[30] We see that the sum rule is reasonably well approximated by the calculations. Hedin's values give rather better numerical agreement than those calculated with the Hubbard approximation. This is not surprising since in his calculations the second-order exchange and direct graphs are treated on the same footing, whereas in the Hubbard approximation the direct terms are exactly included and the exchange terms are approximated. Thus, if we had extended the sum over l to infinity, the Hubbard approximation would still not give the right answer. On the other hand, Hedin's calculation would not satisfy the sum rule either. This can be seen by examining the origin of the sum rule. The sum rule is really a condition on $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ rather than on $f(\mathbf{k}\sigma,\mathbf{k}'\sigma')$. Thus in using Eq. (17) to give a condition on f we are including some terms of arbitrary order in the screened potential in g, whereas to satisfy the sum rule we should expand g consistently to a given order. In summary, neither approximation satisfies the sum rule exactly, but neither violates it badly.

We now wish to compare the electron gas calculations with experiment. In general, observed quantities depend on both the electron-phonon and electron-electron interactions. There are, however, some quantities which depend only on the electron-electron interaction. These offer a direct check on the accuracy of this part of the calculation. The Pauli spin susceptibility χ is independent of the electron-phonon interaction.[12,31] It can be determined in two ways, by direct measurement of the

[28] Note our present definition of A_n differs from that in Ref. 15 by a factor of $(2n+1)^{-1}$.

[29] W. F. Brinkman, P. M. Platzman, and T. M. Rice, Phys. Rev. (to be published).

[30] In both these approximations $f(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ has a logarithmic infinity as $\mathbf{k}' \to -\mathbf{k}$. Thus the series is only conditionally convergent. The true f function does not have this infinity (Ref. 29). In using just the first few terms of the sum for these approximate f functions, we wish to examine how well the first few coefficients can mimic the behavior of the true f function.

[31] C. Herring, in *Magnetism*, edited by G. T. Rado and H. Suhl (Academic Press Inc., New York, 1966), Vol. IV.

power absorbed in CESR by determining m^* and B_0 and using the relation

$$\frac{\chi}{\chi_0}=\frac{m^*/m}{1+B_0}, \tag{18}$$

where χ_0 is the spin susceptibility in the absence of interactions. The parameters m^* and B_0 are the observed values and contain electron-phonon effects which cancel in the ratio of Eq. (18). Similarly, the combination

$$\frac{\kappa}{\kappa_0}=\frac{m^*/m}{1+A_0} \tag{19}$$

is independent of the electron-phonon interaction. κ may be regarded as the "compressibility" of the electron gas. However, A_0 has not been measured and the observed compressibility of a metal such as Na and K depends on a number of terms besides κ, so that there are no reliable values for this quantity.

The Pauli spin susceptibility has been measured directly in Na.[32] Schultz and Dunifer[4] have measured B_0 by studying the spin-wave spectrum in Na and K in the presence of a magnetic field. We combine their result with the value the effective mass found by Grimes and Kip[33] to estimate the enhancement of the susceptibility from Eq. (18). In Table II we tabulate these results and the theoretical estimates. The spin-wave value is some 10% lower than the latest directly measured value. In view of the difficulties in making an absolute power absorption measurement, necessary to obtain the Pauli spin susceptibility directly, the agreement appears reasonable. The spin-wave value is in better agreement with the theoretical values from the Hubbard approximation and from Hedin's results. In K there are no direct spin susceptibility measurements and the spin-wave data are much less accurate at present. Kaeck[34] has estimated the spin susceptibility of K by studying the Knight shift in a series of Na-K alloys. He takes the directly measured value in Na as a reference point and extrapolates to pure K. In Table II we show this value and the theoretical estimates for K. The calculated value for K in the Hubbard approximation has been found by extrapolation from the results quoted in I. The comparison of the experimental and theoretical results is encouraging. The theory gives slightly lower answers for the enhancement of the susceptibility than observed but in view of the crudity of the approximation it is perhaps surprising that they are so close.

TABLE II. Experimental and theoretical estimates for the enhancement factor of the spin susceptibility in Na and K. Expt. (1) Na, Schumacher and Vehse,[a] K, Kaeck.[b] Expt. (2) From the spin-wave data[c] and effective mass.[d] Theory (1) using the Hubbard approximation from I. Theory (2) using Hedin's calculation.

χ/χ_0	Na	K
Expt. (1)	1.74±0.1	1.58±0.1
(2)	1.51±0.06	1.68±0.25
Theory (1)	1.47	1.55
(2)	1.41	1.45

[a] Reference 32.
[b] Reference 34.
[c] References 4 and 42.
[d] Reference 33.

TABLE III. Experimental values[a] and theoretical estimate for $g_{s,-l}$ in Na. Theory (1), Hubbard's approximation calculated by present author in I. Theory (2), Hedin's calculations.

	$g_{s,0}-g_{a,1}$	$g_{s,1}-g_{a,1}$	$g_{s,2}-g_{a,2}$
Theory (1)	−1.82	+0.12	−0.03
(2)	−0.74	+0.07	−0.025
Expt.			−0.05±0.06

[a] References 4, 7, and 42.

As we discussed above, Migdal's result[20] implies that only the lowest-order one-phonon exchange diagram enters the scattering function $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$. The contributions of the electron-phonon interaction to the singlet and triplet parts of g are the same and the combination $g_{s,l}-g_{a,l}$ will depend only on the electron-electron contributions. Note that because of the non-linear relation between the interaction function f and the scattering function g, Eq. (12), this is not true of A_l-B_l. Similarly, all electron-electron exchange graphs, i.e., graphs whose spin dependence has the form $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')=g(\mathbf{k},\mathbf{k}')\delta_{\sigma,\sigma'}$, will cancel from the combinations $g_{s,l}-g_{a,l}$. Thus a comparison of the measured and calculated values of $g_{s,l}-g_{a,l}$ directly tests the spin-independent electron-electron part of the scattering function g. The high-frequency waves found by Walsh and Platzman[5-7] measure A_l for $l\geq 2$ and therefore neglecting anisotropies $g_{s,l}$ for $l\geq 2$. The spin-wave spectrum, in principle, can give B_l for all l, so that an experimental determination of $g_{s,l}-g_{a,l}$ for $l\geq 2$ is possible. At present there are preliminary measurements of A_2 and B_2 in Na. In Table III we quote the results and compare them with the calculated values. We also tabulate the calculated values of $g_{s,0}-g_{a,0}$ and $g_{s,1}-g_{a,1}$, though at present there are no experimental values for $g_{s,0}$ and $g_{s,1}$. The theoretical estimates agree with the measured values within experimental values. The large uncertainty in B_2 at present reduces the significance of this test.

In summary, we find, where we have been able to make comparison, a surprisingly good agreement between the experimental values and the available theoretical estimates. In Sec. V we compare the combined electron-electron and electron-phonon results with experiments.

[32] R. T. Schumacher and W. E. Vehse, J. Phys. Chem. Solids 24, 297 (1963).
[33] C. C. Grimes and A. F. Kip, Phys. Rev. 132, 1991 (1963).
[34] J. A. Kaeck, this issue, Phys. Rev. 175, 897 (1968).

IV. ELECTRON-PHONON INTERACTION

As we have remarked previously, the calculations of the electron-phonon contribution is greatly simplified by Migdal's[20] result. In contrast to the electron-electron part we need only calculate one graph, that shown in Fig. 4(a). Thus we would expect our electron-phonon calculations to be more accurate than the electron-electron calculations. We need to know, however, the effective electron-ion coupling which, in fact, involves electron-electron interactions to all orders. We shall use a linearly screened pseudopotential approximation to describe the electron-ion coupling.

The contribution of the graph in Fig. 4(a) may be written immediately:

$$2\pi i z_{\mathbf{k}e e} z_{\mathbf{k}'e e} \Gamma_1^{ep}(\mathbf{k}\sigma,\mathbf{k}'\sigma') = \frac{1}{2NM}\sum_\lambda \left(\frac{\varepsilon_\lambda\cdot(\mathbf{k}-\mathbf{k}')}{\omega_\lambda(\mathbf{k}-\mathbf{k}')}\right)^2 V_{ps}(\mathbf{k}-\mathbf{k}')^2[1+\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'], \quad (20)$$

where N is the number of ions, M is the ionic mass, ε_λ and ω_λ are the polarization vector and frequency of the phonon with wave vector $\mathbf{k}-\mathbf{k}'$ and index λ. $V_{ps}(\mathbf{q})$ is the screened pseudopotential. We will always choose $V_{ps}(\mathbf{q})$ such that it has the correct long-wavelength limit and passes through the values determined by fitting the Fermi surface,[8] at reciprocal lattice vectors.

The frequency spectrum of the phonons in a number of symmetry directions has been determined by neutron scattering in both Na and K. We follow the method suggested by Darby and March[35] and expand the spectrum in cubic harmonics, fit the coefficients to the symmetry directions, and interpolate in this way for arbitrary directions in reciprocal space. We assume that the polarization vectors are purely longitudinal or transverse, so that the transverse phonons only enter via umklapp processes. Grimvall[25] has investigated this approximation for Na. He finds that it tends to underestimate the enhancement of the effective mass by about 10%. This is less than the differences in g which arise from the use of different pseudopotentials.

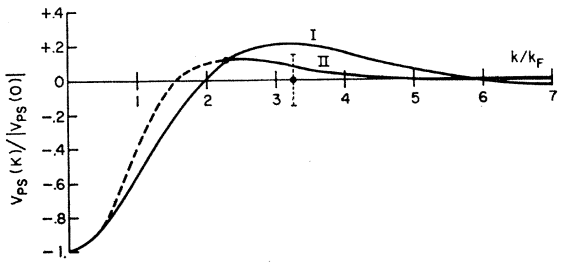

FIG. 5. The pseudopotentials used for Na. Solid curve I, Ashcroft's pseudopotential (Ref. 36) undamped at large K. Solid curve II, Ashcroft's pseudopotential (Ref. 36) with damping factor. Dashed curve, the low node pseudopotential. Solid points denote values at reciprocal-lattice vectors determined from de Haas–Van Alphen data by Lee (Ref. 8).

[35] J. K. Darby and N. H. March, Proc. Phys. Soc. (London) 84, 591 (1964).

We will discuss the results for Na first. Ashcroft[36] has proposed a simple form for the pseudopotential of the alkali metals,

$$V_{ps}^A(\mathbf{q}) = -\lambda^2(\cos qR_c)/q^2\epsilon(q,0), \quad (21)$$

where $\epsilon(q)$ is the Lindhard dielectric function, and $\lambda^2=(\pi a_0 k_F)^{-1}$ and $R_c=0.88$ Å for Na. In Fig. 5 we plot Aschroft's pseudopotential for Na. The points on Fig. 5 are taken from Lee's fit to the Fermi surface of Na.[8] In the one-plane-wave approximation, which we are using, only momentum transfers $q\leqslant 2k_F$ are relevant. We see that Ashcroft's form fits the de Haas–van Alphen data well in Na. Fong and Cohen[37] have fitted the optical spectrum of NaCl and obtain pseudopotentials for both Na and Cl atoms. Their pseudopotential also has a note near $2k_F$. It is interesting to vary the position of the node and see what effect this has on the parameters. We have arbitrarily constructed a potential with a node near $1.5k_F$, shown as the dashed curve in Fig. 5, which we shall refer to as the low node potential.

We have investigated convergence of the one-plane-wave approximation for Na. In general, the matrix element, in the nearly-free-electron approximation for scattering from a state $|\mathbf{k}\rangle$ to a state $|\mathbf{k}'\rangle$, has the form

$$M_{\mathbf{k},\mathbf{k}'} \sim \sum_{\mathbf{G},\mathbf{G}'} C_{\mathbf{k}',\mathbf{G}'} C_{\mathbf{k},\mathbf{G}}\, \varepsilon_{\mathbf{q}}\cdot(\mathbf{k}'+\mathbf{G}'-\mathbf{k}-\mathbf{G}) \times V_{ps}(\mathbf{k}'+\mathbf{G}'-\mathbf{k}-\mathbf{G}), \quad (22)$$

where $\mathbf{q}$ is the momentum transfer reduced to the first zone and

$$C_{\mathbf{k},0}=C_0, \qquad C_{\mathbf{k},\mathbf{G}}=V_{ps}(\mathbf{G})/(k^2/2m-(\mathbf{k}+\mathbf{G})^2/2m), \quad G\neq 0. \quad (23)$$

C_0 is a normalization constant and the vectors $\{\mathbf{G}\}$ are the reciprocal lattice vectors of the crystal. We have included the twelve $\langle 110\rangle$ vectors. The momentum transfers in the pseudopotential may now be larger than $2k_F$. The Ashcroft pseudopotential (curve I in Fig. 5) is too large at large k. We have arbitrarily reduced it in magnitude to curve II in Fig. 5 by multiplying by the factor $(k/2.3k_F)^{-3}$ for $k\gtrsim 2.3k_F$.

In Table IV we list the results of our numerical calculations. It is convenient to tabulate the moments of the scattering function $g_{ep}(\mathbf{k}\sigma,\mathbf{k}'\sigma')$. We define

$$g_l^{ep} = \frac{m_{ep}^* k_F}{2\pi^2}\int \frac{d\Omega_{\mathbf{k}}}{4\pi}\frac{d\Omega_{\mathbf{k}'}}{4\pi} g_{ep}(\mathbf{k}\sigma,\mathbf{k}'\sigma) P_l(\cos\theta_{\mathbf{k},\mathbf{k}'}), \quad (24)$$

where we have dropped the spin suffix since the singlet and triplet amplitudes are equal. The average effective mass including only electron-phonon interactions is given by

$$m/m_{ep}^* = 1-g_0^{ep}. \quad (25)$$

[36] N. Ashcroft, Phys. Letters 23, 48 (1966).

[37] C. Y. Fong and M. L. Cohen, Phys. Rev. Letters 21, 22 (1968).

TABLE IV. The moments of the scattering $g_{ep}(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ due to electron-phonon interaction in Na evaluated using (1) one-plane-wave and Ashcroft pseudopotential,[a] (2) one-plane-wave and the low node pseudopotential,[a] and (3) multiplane wave and Ashcroft pseudopotential.[a]

	g_0^{ep}	g_1^{ep}	g_2^{ep}	g_3^{ep}
(1)	0.13	0.05	−0.001	0.002
(2)	0.08	0.04	0.02	
(3)	0.17	0.06	0.01	0.001

[a] Reference 36.

The results for g_0^{ep} show a considerable variation. The lowest value is obtained using the low node potential. Previous calculations of the effective mass give values for g_0^{ep} which range from 0.13 (Animalu *et al.*[24]) to 0.16 (Grimvall[25]), while Ashcroft and Wilkins[23] quote 0.15. These authors use pseudopotentials which are similar to Ashcroft's with a node near $2k_F$. We conclude that g_0^{ep} is quite sensitive to the position of the node. The resistivity is extremely sensitive to the position of the node and we expect that the low node potential would not give good estimates of the resistivity. The results for the higher moments are much less sensitive to the pseudopotential form.

A comparison of rows 1 and 3 shows that g_0^{ep} is enhanced by some 30% by going beyond the one-plane-wave approximation. This enhancement is clearly strongly dependent on the behavior of the pseudopotential at large k and is, for example, much larger again with the undamped Ashcroft potential. Unfortunately, there is no good criterion for deciding the form at large k of the pseudopotential and there is no direct experimental test of the size of the electron-phonon contribution in the alkalis. The higher moments g_l^{ep} are not too sensitive to the different approximations.

In K, Lee and Falicov[8] have fitted their de Haas–van Alphen data with a nonlocal pseudopotential. Their pseudopotential has the form

$$V_{ps}^{LF}(q)=V_L(q)+V_{NL}(q), \tag{26}$$

where $V_L(q)$ is a local pseudopotential and V_{NL} is the nonlocal part which, however, for scattering on the Fermi surface, depends only on q. The Fermi-surface data only determine the pseudopotentials at reciprocal lattice points. We need, in the one-plane-wave approximation, a pseudopotential for all values of momentum transfer $q\leq 2k_F$. We interpolate the local part from the known $V_L(110)=0.35$ eV by using an Ashcroft form. For $V_{NL}(q)$ we write

$$V_{NL}(q)=\frac{\epsilon(110)}{\epsilon(q)}F(q), \tag{27}$$

where $\epsilon(q)$ is the Lindhard dielectric function and $F(q)$ is the Fourier transform of a potential of the Heine-Abarenkov form,

$$F_{NL}(\mathbf{r})=\sum_{l=0}^{2} U_l(\mathbf{r})P_l(L)P_l(R), \tag{28}$$

where $P_l(L)$ and $P_l(R)$ project the l component of the wave function on the left and right, respectively. $U_l(\mathbf{r})$ has the form

$$\begin{aligned} U_l(\mathbf{r})&=U_l, \quad r<R_M \\ &=0, \quad r>R_M \end{aligned} \tag{29}$$

and the following values are chosen: $R_M=1.33\times10^{-8}$ cm, $U_0=-1.5$ eV, $U_1=3.0$ eV, and $U_2=-33.5$ eV.

In Table V we quote the results for the parameters g_l^{ep} in K using the pseudopotential of Lee and Falicov,[8] and also the earlier Ashcroft potential,[36] in the one-plane-wave approximation. The phonon spectrum was treated as in Na, by using a cubic harmonic expansion to interpolate from the measurements of Cowley, Woods, and Dolling.[22] The differences between the two sets of results are small, which is not surprising since the two potentials in the region $q\leqslant 2k_F$ do not differ by much and in particular the position of the node does not vary greatly.

Finally, we turn to the question of anisotropy in the Landau function. The phonon spectrum is quite anisotropic in the alkali metals and in view of the almost spherical Fermi surfaces one expects that this will be a major, if not the greatest, source of anisotropy. In discussing anisotropy due to the phonon spectrum it is much simpler to use the scattering function $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$. However, the simplest expansion is not valid. One may *not* expand in the form

$$g^{ep}(\mathbf{k}\sigma,\mathbf{k}'\sigma')=\sum_{n,m} g_{2n2m}\delta_{\sigma,\sigma'}K_{2n}^{\alpha}(\hat{\mathbf{k}})K_{2m}^{\alpha}(\hat{\mathbf{k}}'), \tag{30}$$

where K_n^{α} is the nth cubic harmonic of type α in the notation of Von der Lage and Bethe,[38] normalized to 4π. This can be seen by considering the limit as $\mathbf{k}'-\mathbf{k}\to 0$. The phonon spectrum depends strongly on the direction of $\mathbf{k}-\mathbf{k}'$ relative to the crystal axes even in the zero-momentum limit. Thus from Eq. (20) $\lim_{\mathbf{k}'-\mathbf{k}\to 0}g^{ep}$ depends on the direction in the limit. A form such as Eq. (30) clearly cannot reproduce such behavior. We must, instead, expand directly in terms of the vector $\mathbf{k}-\mathbf{k}'$

TABLE V. The coefficients g_l^{ep} for K calculated using (1) the Lee-Falicov pseudopotential[a] and (2) the Ashcroft pseudopotential[b] in the one-plane-wave approximation.

	g_0^{ep}	g_1^{ep}	g_2^{ep}
(1)	0.10	0.04	0.001
(2)	0.11	0.05	−0.001

[a] Reference 8.
[b] Reference 36.

[38] F. C. Von der Lage and H. A. Bethe, Phys. Rev. **71**, 612 (1947).

and write

$$g^{\rm ep}(\mathbf{k}\sigma,\mathbf{k}'\sigma')=\sum_{n=0}^{\infty} g_{2n}{}^{\rm ep}(|\mathbf{k}-\mathbf{k}'|,|\varphi_{\mathbf{k},\mathbf{k}'}|) \times K_{2n}{}^{\alpha}\left(\frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|}\right)(\tfrac{1}{2}(1+\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')), \quad (31)$$

where $\varphi_{\mathbf{k},\mathbf{k}'}$ is the remaining independent variable. If we include only the anisotropy due to the phonon spectrum and neglect all anisotropy due to band structure, then it is clear that g_{2n} will depend only on $|\mathbf{k}-\mathbf{k}'|$. We may now expand $g_{2n}(|\mathbf{k}-\mathbf{k}'|)$ as a series of Legendre polynomials in $\theta_{\mathbf{k},\mathbf{k}'}$, and we define for each of the functions $g_{2n}{}^{\rm ep}(|\mathbf{k}-\mathbf{k}'|)$ a set of parameters $g_{2n,l}{}^{\rm ep}$ corresponding to the $g_{s,l}$ defined in Sec. II. The parameters $g_{0,l}{}^{\rm ep}$ are identical to the $g_l{}^{\rm ep}$ which we have evaluated previously. We have also evaluated the lowest anisotropic coefficients $g_{4,l}{}^{\rm ep}$ and $g_{6,l}{}^{\rm ep}$, where $l=0,1,2$, and in Table VI we list the results for these parameters. The calculations were carried out for Na only, using the one-plane-wave approximation, Ashcroft's pseudopotential, and the observed phonon spectrum. We see that the anisotropic coefficients are small, $\lesssim 0.02$. Therefore we expect deviations in any quantity which, in the uniform system, depends on the combinations $1+A_l$ or $1+B_l$ to be of order a few percent or less. The coefficient $g_{4,0}{}^{\rm ep}$ is very small because of a cancellation between large and small q in the function $g_4{}^{\rm ep}(q)$. This is consistent with the measurements of the effective mass,[33] where the anisotropy was found to be less than 2%. The spin-wave spectrum depends sensitively at long wavelengths[3] on the difference B_0-B_1. It can be shown,[29] however, that the form of the spectrum is unchanged at long wavelengths and all corrections involve the squares of the anisotropic coefficients, $g_{4,0}{}^2$, etc. Such corrections we would expect to be very small, and it appears that the predicted size of anisotropic effects even in the spin-wave spectrum is too small to be measured at present.

V. CALCULATED AND EXPERIMENTAL LANDAU COEFFICIENTS

In preceding sections we have discussed the calculation from microscopic theory of the electron-electron and the electron-phonon contributions to the scattering and interaction functions. It is customary to express the experimental results in terms of the moments of the Landau interaction function $f(\mathbf{k}\sigma,\mathbf{k}'\sigma')$. In Sec. IV we expressed our results for the electron-phonon contribution in terms of the scattering function $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$. Strictly speaking, for an anisotropic situation we would solve the integral equation (12) to obtain f from g. However, the anisotropy effects are quite small and less than the uncertainties due to other sources in our calculations. We will therefore ignore the anisotropy effects and use Eq. (11) to relate the moments of f and g.

We will consider Na first. The experimental values for Na are considerably more accurate, at present, than those for K. The paramagnetic spin-wave spectrum of Na has been studied by Schultz and Dunifer.[4] They fitted their transmission spectrum to the theory of Platzman and Wolff[3] and Fredkin and Wilson[39] to obtain B_0, B_1, and B_2. The spin-wave spectrum determines only the triplet amplitude f^a. The high-frequency plasma like waves lead to a determination of A_n, where $n\gtrsim 2$.[5–7] So far A_2 and A_3 have been measured in this way in Na. The effective mass m^* is an independent parameter which has been measured by Azbel-Kaner cyclotron resonance[33] and by specific heat.[40,41] In the presence of both electron-phonon and electron-electron interactions the effective mass is not simply related to the Landau parameters. This gives us six independent quantities to compare with the microscopic estimates.

At present there is no way of determining A_0 or A_1. These two would be particularly interesting to measure. A knowledge of A_0 would give the electron-gas compressibility which in turn is related to the second derivative of the ground state energy with respect to density. Thus a knowledge of A_0 would provide a direct test of the various theoretical estimates of the correlation energy of an electron gas. A measurement of A_1 com-

TABLE VI. The anisotropy coefficients $g_{n,l}{}^{\rm ep}$ defined in (30) for Na. These were calculated in the one-plane-wave approximation, with the Ashcroft form of the pseudopotential.[a]

	$g_{0,l}{}^{\rm ep}$	$g_{4,l}{}^{\rm ep}$	$g_{6,l}{}^{\rm ep}$
$l=0$	0.13	−0.002	0.005
$l=1$	0.05	0.008	−0.003
$l=2$	−0.001	0.012	−0.004

[a] Reference 36.

TABLE VII. The experimental values[a] and theoretical estimates for the Landau parameters in Na. Theoretical approximations: (I) Electron-electron calculated by author in I, electron-phonon in multiplane-wave and Ashcroft pseudopotential.[b] (II) Electron-electron from I, electron-phonon in one-plane-wave and Ashcroft pseudopotential.[b] (III) Electron-electron from I, electron-phonon in one-plane-wave and the low node pseudopotential. (IV) Electron-electron from Hedin[c]; electron-phonon as in approx. (I). (V) Electron-electron, screened-exchange approximation, electron-phonon as in approx. (I).

	Expt. (Na)	Theory (I)	(II)	(III)	(IV)	(V)
A_0		−0.62	−0.64	−0.66	−0.45	−0.17
A_1		+0.12	+0.11	+0.10	+0.04	+0.03
A_2	−0.05±0.01	−0.03	−0.04	−0.03	−0.01	+0.006
A_3	0.0 ±0.005	+0.004	+0.005			
B_0	−0.18±0.03	−0.14	−0.17	−0.22	−0.17	−0.17
B_1	+0.05±0.04	+0.01	−0.005	−0.02	−0.02	+0.03
B_2	0.0 ±0.05	−0.01	−0.02	0.00	+0.01	+0.006
B_3		+0.000	+0.001			
m^*/m	1.24±0.02	1.26	1.21	1.15	1.19	1.17

[a] Reference 42.
[b] Reference 36.
[c] Reference 19.

[39] D. R. Fredkin and A. R. Wilson (to be published).
[40] D. L. Martin, Phys. Rev. 124, 438 (1961).
[41] W. H. Lien and N. E. Phillips, Phys. Rev. 133, A1370 (1964).

bined with the known effective mass m^* would enable a determination of the cyclotron mass in a uniform field which, in the absence of band effects, is determined solely by electron-phonon interaction.[12] This would directly test the accuracy of our electron-phonon calculations.

In Table VII we list the experimental values[42] and theoretical estimates for the Landau parameters in Na. We include a variety of different theoretical approximations for comparison purposes. Approximation (I) corresponds to using the Hubbard approximation as calculated by this author in I for the electron-electron part and calculating the electron-phonon part with many plane waves and Ashcroft's pseudopotential. In approximation (II) we use just one plane wave in the electron-phonon part, while in (III) we use the low node pseudopotential and one plane wave. In (IV) and (V) we vary the electron-electron part, using Hedin's[19] results (IV) and a simple screened-exchange approximation (V). These are combined with the electron-phonon parts as in approximation (I).

The over-all agreement between the experimental values and the theoretical estimates is remarkably good. There are some discrepancies between the different estimates but these are unfortunately largest in A_0 and A_1. Comparing first the different electron-electron calculations (I), (IV), and (V), the best agreement is with the Hubbard approximation, approximation (I). B_0 is somewhat off in (I), while (IV) and (V) have m^*/m too low and A_2 considerably off. The best over-all agreement is with approximation (II). Approximation (III) is particularly bad on the mass value. However, it is clear that it would be premature at present to claim that one approximation was decisively better than the others. This is perhaps not too surprising since the difference between approximations (I), (II), and (IV) is not very great.

The theoretical estimates are qualitatively correct for all the six independent parameters and even quantitatively correct for most of them. This must be regarded as a major success for the approximations. As we have stressed before, there is no *a priori* reason to assume that the interaction effects are weak. Thus one might expect that A_l, $B_l \sim 1$ for small values of l and indeed such values are found in He^3. The theoretical estimates based on the simple approximations predict much smaller values in good agreement with experiment. It clearly shows that there must be considerable cancellations among the higher-order graphs for the Coulomb interaction, which we have omitted. It will be interesting to see if the calculations can predict the correct trends for the lower density alkalis. This question will be left to a later publication.

Leggett[43] has recently derived some inequalities for the Landau parameters in metals. He shows that

$$\frac{m^*}{1+A_1} \gtrless m, \quad \frac{m^*}{1+B_1} \gtrless m. \tag{32}$$

It is clear from the results in Table VII that both the experimental values and theoretical estimates satisfy the relations (32).

In K the experimental values are much less well known at present for the B_n. Our earlier electron-gas calculations in I were restricted to the densities $r_s=2$ and 4, so that we cannot use them for K, where $r_s=4.87$. More extensive calculations were done on the effective mass and spin susceptibility, so that we can quote results for m^* and B_0. Hedin[19] calculated at both $r_s=4$ and $r_s=5$. We will linearly interpolate between these two values to obtain estimates for K. In Table VIII we quote the results. An accurate comparison is really only possible for m^*/m and A_2 at present. Hedin's results are combined with phonon calculations from Sec. IV and shown as approximation (II). There are considerable deviations in both m^*/m and A_2 in this approximation. The mass value calculated with the Hubbard approximation is better. More accurate measurements are clearly needed on K to draw a firm conclusion.

In summary, we find very good agreement between the calculated and measured values of the Landau parameters for Na. In view of the obvious crudity of the electron-electron part, it is indeed surprising that the estimates are so close. The best agreement with experiment is obtained by using the Hubbard approximation, as in I, for the electron-electron and the Ashcroft pseudopotential in the electron-phonon calculations.

TABLE VIII. The experimental values[a] and theoretical estimates for the Landau parameters in K. Theory (I) Electron-electron calculated by author in I and electron-phonon using Lee-Falicov potential.[b] (II) Electron-electron from Hedin[c] and same electron-phonon.

		Theory	
	Expt.	(I)	(II)
A_0			−0.58
A_1			+0.04
A_2	−0.030±0.005		−0.02
B_0	−0.28 ±0.1	−0.21	−0.24
B_1	+0.06 ±0.15		−0.04
B_2			+0.003
m^*/m	1.21 ±0.01	1.23	1.11

[a] References 4, 7, 33, and 42.
[b] Reference 8.
[c] Reference 19.

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[42] The experimental values are the best available at present and were communicated to the author by W. M. Walsh, Jr., and by S. Schultz and G. Dunifer. The B_n differ a little from those quoted in Ref. 29.

[43] A. J. Leggett, Ann. Phys. (N. Y.) 46, 76 (1968).