Energy Width of Spin Waves in the Heisenberg Ferromagnet*

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A calculation of the energy width of spin waves whose energy is small compared to kT in a Heisenberg ferromagnet is presented. Full account is taken of all two-particle processes. For $2JS/kT\gg1$ and $2JSa^2\lambda^2/kT\ll1$, the energy width $\Delta\omega_\lambda$ of a spin wave of wave number λ is $\Delta\omega_\lambda\sim\lambda^4T^2[A+B\ln(kT/2JSa^2\lambda^2)+C\ln^2(kT/2JSa^2\lambda^2)]$. The disagreement with some previous treatments is explained physically by their neglect of two many-body effects.

I. INTRODUCTION

BOTH the static and dynamic properties of the Heisenberg ferromagnet at low temperature have been studied by many authors. Although there is general agreement as to the low-temperature thermodynamics, there is presently some controversy as to the dynamical behavior of long-wavelength spin waves at moderately low temperatures. Specifically we refer to the calculation of the width in energy, $\Sigma''(\lambda)$, of spin waves of momentum λ whose energy ϵ_λ is much smaller than kT. Recently Cooke and Gersch[1] and also Marshall and Murray[2] have found the result $\Sigma''(\lambda)\sim\lambda^2T^3$, whereas Kashcheev and Krivoglaz[3] obtained $\Sigma''(\lambda)\sim\lambda^4T^2\ln^2(T/\lambda^2)$, using the first Born approximation. Our work substantiates and simultaneously generalizes this latter result. We find

$$\Sigma''(\lambda)\sim\lambda^4T^2[A+B\ln(T/\lambda^2)+C\ln^2(T/\lambda^2)], \quad (1)$$

correct to all orders in $1/S$ for $2JS/kT\gg1$ and $2JSa^2\lambda^2/kT\ll1$.

Although our calculations are rather formal, we can clearly identify two crucial effects which are essential to a correct treatment of the problem. The first and most obvious of these is the principle of detailed balancing. This principle asserts that in the calculation of the lifetime of a state one should calculate a net transition probability which is the difference between the rate of scattering out of the state minus the rate of scattering into the state. Accordingly in $\Sigma''(\lambda)$ one finds a factor $[1-\exp(-\beta\epsilon_\lambda)]$, where $\beta\equiv1/kT$. In the regime where $\beta\epsilon_\lambda\gg1$ this factor is essentially unity. But in the present case $\beta\epsilon_\lambda\ll1$, so that this factor becomes approximately $2JSa^2\lambda^2/kT$. Inclusion of this factor in Refs. 1 and 2 would result in an energy width $\Sigma''(\lambda)\sim\lambda^4T^2$. The presence of the additional logarithmic factor is due to a typical many-body effect. For the Fermi gas one knows that many-body effects are partially taken into account by modifying matrix elements by the appropriate occupation numbers so that particles scatter only into empty states. For the Bose gas, on the other hand, scattering into an occupied state σ is enhanced by the factor $1+n_\sigma$. In the regime $\beta\epsilon_\lambda\ll1$, the state λ and its neighbors are significantly occupied even though the simultaneous condition $2JS\beta\gg1$ insures that the total number of particles is small. Thus the condition $\beta\epsilon_\lambda\ll1$ means that the state λ must be treated as being in a partially excited Bose gas. Inclusion of the enhancement factors $1+n_\sigma$ leads to the expansion of Eq. (1) in agreement with Ref. 3.

II. ENERGY WIDTH DUE TO TWO-PARTICLE PROCESSES

We use the Dyson-Maleev[4] boson representation for spin operators

$$S^+=(2S)^{1/2}(1-a^\dagger a/2S)a, \quad (2a)$$

$$S^-=(2S)^{1/2}a^\dagger, \quad (2b)$$

$$S^z=S-a^\dagger a, \quad (2c)$$

so that the Heisenberg Hamiltonian becomes

$$\mathcal{H}=E_0+\mathcal{H}_0+V, \quad (3)$$

with

$$E_0=-NzJS^2, \quad (4a)$$

$$H_0=\sum_\lambda \epsilon_\lambda a_\lambda{}^\dagger a_\lambda, \quad (4b)$$

$$V=(2N)^{-1}\sum_{\mathrm{Kpp}'} V_{\mathrm{K}}(\mathbf{p},\mathbf{p}')a_{\mathrm{K}/2+\mathrm{p}}{}^\dagger a_{\mathrm{K}/2-\mathrm{p}}{}^\dagger a_{\mathrm{K}/2+\mathrm{p}'}a_{\mathrm{K}/2-\mathrm{p}'}\,. \quad (4c)$$

Here[5]

$$\epsilon_\lambda=2JzS(1-\gamma_\lambda), \quad (5a)$$

$$V_{\mathrm{K}}(\mathbf{p},\mathbf{p}')=-Jz[\gamma_{\mathrm{p}-\mathrm{p}'}+\gamma_{\mathrm{p}+\mathrm{p}'}-\gamma_{\mathrm{K}/2+\mathrm{p}}-\gamma_{\mathrm{K}/2-\mathrm{p}}], \quad (5b)$$

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[1] J. F. Cooke and H. A. Gersch, Phys. Rev. **153**, 641 (1967).

[2] W. Marshall and G. Murray, J. Appl. Phys. **39**, 380 (1968).

[3] V. N. Kashcheev and M. A. Krivoglaz, Fiz. Tverd. Tela **3**, 1541 (1961) [English transl.: Soviet Phys.—Solid State **3**, 1117 (1961)]; R. A. Tahir-Kheli and D. ter Haar [Phys. Rev. **127**, 95 (1962)] give an expression for $\Sigma''(\epsilon_\lambda)$ which, when evaluated, would give our results.

[4] F. J. Dyson, Phys. Rev. **102**, 1217 (1956); S. V. Maleev, Zh. Eksperim. i Teor. Fiz. **33**, 1010 (1957) [English transl.: Soviet Phys.—JETP **6**, 776 (1956)].

[5] Strictly speaking Eq. (5b) is incorrect, because momentum is only conserved to within a reciprocal lattice vector. However, the potential and spin-wave energies and hence the t matrix are periodic functions of all wave vectors, so that we may use the reduced-zone scheme.

in the usual notation.[6] It can be shown[7] that the spin Green's function $\langle\langle S^+ : S^- \rangle\rangle$, i.e., the dynamical susceptibility, can be expressed in terms of boson Green's function. In this way one concludes[8] that near resonance, i.e., for $\omega \sim \epsilon_\lambda$,

$$\langle\langle S^+; S^- \rangle\rangle \approx 2\langle S^z \rangle G(\lambda, \omega), \tag{6}$$

where $G(\lambda, \omega)$ is the usual single-particle boson Green's function with associated self-energy $\Sigma_\lambda(\omega)$ given by

$$G(\lambda, \omega) = [\omega - \epsilon_\lambda - \Sigma_\lambda(\omega)]^{-1}. \tag{7}$$

Physically $\Sigma_\lambda(\omega)$ describes the modification of the energy due to interactions with other particles. The energy shift is given by $\mathrm{Re}\Sigma_\lambda(\epsilon_\lambda)$ and the energy width due to interactions is given by $\mathrm{Im}\Sigma_\lambda(\epsilon_\lambda)$.

We will calculate $\Sigma_\lambda(\omega)$ according to the usual rules of diagrammatic perturbation theory.[9] In this formulation $\Sigma_\lambda(\omega)$ is obtained by summing the contributions of all irreducible graphs with one line entering and one line leaving the diagram. It is desirable to classify diagrams according to the number of particles whose interactions are being described. For each particle interacting with the spin wave of momentum λ, it can be shown that one obtains a factor $\langle n \rangle$, where $\langle n \rangle$ is the density of spin waves; $\langle n \rangle \sim T^{3/2}$. Thus to lowest order in the density of spin waves we must compute $\Sigma_\lambda(\omega)$ from ladder graphs as shown in Fig. 1. The sum over all ladder graphs can be expressed exactly by

$$\Sigma_\lambda(z) = -(2/N\beta) \times \sum_{\mathbf{p}z'} t(z+z'; \lambda+\mathbf{p}; \tfrac{1}{2}(\lambda-\mathbf{p}), \tfrac{1}{2}(\lambda-\mathbf{p}))(z'-\epsilon_{\mathbf{p}})^{-1}, \tag{8}$$

where z and z' are on the set of imaginary frequencies[9] $2l\pi i/\beta$ and the t matrix satisfies

$$t(z; \mathbf{K}; \mathbf{p}, \mathbf{p}') = V_{\mathbf{K}}(\mathbf{p}, \mathbf{p}') - (N\beta)^{-1} \times \sum_{\rho z'} V_{\mathbf{K}}(\mathbf{p}, \boldsymbol{\rho}) \frac{1}{[z' - \epsilon(\frac{1}{2}\mathbf{K}+\boldsymbol{\rho})]} \frac{1}{[z - z' - \epsilon(\frac{1}{2}\mathbf{K}-\boldsymbol{\rho})]} \times t(z; \mathbf{K}; \boldsymbol{\rho}, \mathbf{p}'). \tag{9}$$

The sum over z' may be performed to yield

$$t(z; \mathbf{K}; \mathbf{p}, \mathbf{p}') = V_{\mathbf{K}}(\mathbf{p}, \mathbf{p}') + N^{-1} \sum_{\rho} V_{\mathbf{K}}(\mathbf{p}, \boldsymbol{\rho}) \frac{1 + n(\frac{1}{2}\mathbf{K}+\boldsymbol{\rho}) + n(\frac{1}{2}\mathbf{K}-\boldsymbol{\rho})}{z - \epsilon(\frac{1}{2}\mathbf{K}+\boldsymbol{\rho}) - \epsilon(\frac{1}{2}\mathbf{K}-\boldsymbol{\rho})} \times t(z; \mathbf{K}; \boldsymbol{\rho}, \mathbf{p}'), \tag{10}$$

[6] T. Oguchi, Phys. Rev. **117**, 117 (1960).
[7] S. V. Peletminskii and V. G. Bar'yakhtar, Fiz. Tverd. Tela **6**, 219 (1964) [English transl.: Soviet Phys.—Solid State **6**, 174 (1964)].
[8] R. Silberglitt and A. B. Harris Phys. Rev. (to be published).
[9] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960); see also L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962), Chap. 13.

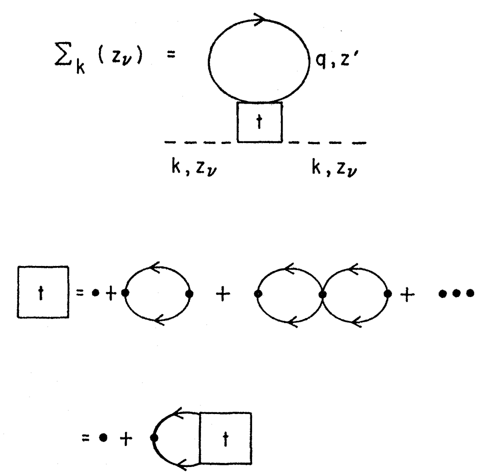

FIG. 1. The diagrammatic series for $\Sigma_\lambda(z)$.

where

$$n(\mathbf{p}) \equiv [\exp(\beta\epsilon_{\mathbf{p}}) - 1]^{-1}, \tag{11a}$$

$$\epsilon(\mathbf{p}) \equiv \epsilon_{\mathbf{p}}. \tag{11b}$$

By the use of the dispersion relation for the t matrix,

$$t(z; \mathbf{K}; \mathbf{p}, \mathbf{p}') = V_{\mathbf{K}}(\mathbf{p}, \mathbf{p}') + \pi^{-1} \int_{-\infty}^{+\infty} \frac{\mathrm{Im}t(\omega - i\epsilon; \mathbf{K}; \mathbf{p}, \mathbf{p}')}{z - \omega} d\omega, \tag{12}$$

one can perform the z' sum in Eq. (8); the result is

$$\Sigma_\lambda(z) = \Sigma_\lambda^{(1)} + \frac{2}{N\pi} \sum_{\mathbf{p}} \int \frac{\mathrm{Im}t(\omega - i\epsilon; \lambda+\mathbf{p}; \frac{1}{2}(\lambda-\mathbf{p}), \frac{1}{2}(\lambda-\mathbf{p}))}{z + \epsilon_{\mathbf{p}} - \omega} \times [f(\epsilon_{\mathbf{p}}) - f(\omega)], \tag{13}$$

where $\Sigma_\lambda^{(1)}$ is the Hartree energy from the first term on the right-hand side of Eq. (12) and

$$f(x) = (e^{\beta x} - 1)^{-1}. \tag{14}$$

As noted above, we seek to calculate the energy width, which is given by $\mathrm{Im}\Sigma_\lambda(\epsilon_\lambda)$:

$$\mathrm{Im}\Sigma_\lambda(\epsilon_\lambda) \equiv \Sigma''(\lambda) = (2/N) \times \sum_{\mathbf{p}} \mathrm{Im}t(\epsilon_\lambda + \epsilon_{\mathbf{p}} - i\epsilon; \lambda+\mathbf{p}; \tfrac{1}{2}(\lambda-\mathbf{p}), \tfrac{1}{2}(\lambda-\mathbf{p})) \times [f(\epsilon_{\mathbf{p}}) - f(\epsilon_{\mathbf{p}} + \epsilon_\lambda)]. \tag{15}$$

Since $\mathrm{Re}t$ is a much more convenient quantity than $\mathrm{Im}t$, we use the optical theorem, discussed in Appendix A:

$$\mathrm{Im}t(\omega - i\epsilon; K; \mathbf{p}, \mathbf{p}') = \frac{\pi}{N} \sum_{\rho} t(\omega - i\epsilon; \mathbf{K}; \mathbf{p}, \boldsymbol{\rho}) \times [1 + n(\tfrac{1}{2}\mathbf{K}+\boldsymbol{\rho}) + n(\tfrac{1}{2}\mathbf{K}-\boldsymbol{\rho})] \times \delta(\omega - \epsilon(\tfrac{1}{2}\mathbf{K}+\boldsymbol{\rho}) - \epsilon(\tfrac{1}{2}\mathbf{K}-\boldsymbol{\rho})) t(\omega + i\epsilon; \mathbf{K}; \boldsymbol{\rho}, \mathbf{p}'). \tag{16}$$

Substituting this into Eq. (15) we find

$$\Sigma''(\lambda)=\frac{2\pi}{N^2}\sum_{\mathbf{p}\rho}f(\epsilon_{\mathbf{p}})[1+f(\epsilon_{\mathbf{p}}+\epsilon_\lambda)][1+f(\epsilon_\lambda)]^{-1}$$
$$\times\,\delta(\epsilon_\lambda+\epsilon_{\mathbf{p}}-\epsilon_\rho-\epsilon_{\lambda+\mathbf{p}-\rho})[1+f(\epsilon_\rho)+f(\epsilon_{\lambda+\mathbf{p}-\rho})]$$
$$\times\, t(\epsilon_\lambda+\epsilon_{\mathbf{p}}-i\delta;\,\boldsymbol{\lambda}+\mathbf{p};\,\tfrac{1}{2}(\boldsymbol{\lambda}-\mathbf{p}),\,\boldsymbol{\rho}-\tfrac{1}{2}(\boldsymbol{\lambda}+\mathbf{p}))$$
$$\times\, t(\epsilon_\lambda+\epsilon_{\mathbf{p}}+i\delta;\,\boldsymbol{\lambda}+\mathbf{p};\,\boldsymbol{\rho}-\tfrac{1}{2}(\boldsymbol{\lambda}+\mathbf{p}),\,\tfrac{1}{2}(\boldsymbol{\lambda}-\mathbf{p})),\quad(17)$$

where we have used the identity

$$f(x)-f(y)=f(x)[1+f(y)]/[1+f(y-x)].\quad(18)$$

In conformity with our previous discussion, note that $\Sigma''(\lambda)$ is proportional to $[1+f(\epsilon_\lambda)]^{-1}=1-\exp(-\beta\epsilon_\lambda)\approx\beta\epsilon_\lambda$. Note also the presence in Eq. (17) of the factor $[1+f(\epsilon_\rho)+f(\epsilon_{\lambda+\mathbf{p}-\rho})]$, which gives the enhancement due to scattering into thermally occupied states. Having obtained an exact but convenient[10] expression for the energy width due to two-particle processes, the aim of the present section has been achieved.

III. ASYMPTOTIC EVALUATION OF THE ENERGY WIDTH

So far we have calculated exactly the energy width of spin waves due to two-particle processes, which will give $\Sigma''(\lambda)$ correct to lowest order in the density of spin waves. The present section is devoted to an asymptotic evaluation of Eq. (17) in the regime

$$2JS\beta\gg1,\quad(19a)$$
$$\beta\epsilon_\lambda\ll1.\quad(19b)$$

The first of these conditions is necessary for the rapid convergence of the density expansion which we have already used. The second condition means that we consider spin waves whose energy is less than kT; hence they are in thermally populated states. Generally each f factor in Eq. (17) will contribute a factor of the density, i.e., of $T^{3/2}$. This was the justification for neglecting processes involving more than two particles, which involves at least two f factors. This type of argument is only valid if the f factors refer to *independent* energies. For example, $f^2(\epsilon_\lambda)\sim T^{3/2}$. In Eq. (17) note that ϵ_λ and $\epsilon_\lambda+\epsilon_{\mathbf{p}}$ are not independent because $\beta\epsilon_{\mathbf{p}}\sim1$. Also due to the delta-function condition, $\epsilon_{\mathbf{p}}+\epsilon_{\lambda+\mathbf{p}-\rho}=\epsilon_\lambda+\epsilon_{\mathbf{p}}$, so that when $\epsilon_{\mathbf{p}}$ is small $\epsilon_{\mathbf{p}}+\epsilon_\lambda$, and hence $\epsilon_{\mathbf{p}}+\epsilon_{\lambda+\mathbf{p}-\rho}$, is also small. Thus all the f factors in Eq. (17) must be kept. We may affect a simplification by analyzing Eq. (17) when the Neumann iterative solution to Eq. (10) is used for the t matrix. In this case the f factors associated with the internal lines of the t matrices need not be kept, since they represent independent momenta. Accordingly, we may evaluate Eq. (17) using the zero-temperature t matrix.

[10] The optical theorem may also be used to give a convenient expression for $\Sigma_\lambda'(\omega)$.

Even so, it is not feasible to evaluate Eq. (17) exactly. We can develop an asymptotic expansion in the regime characterized by Eq. (19) by evaluating Eq. (17), keeping only the lowest number of powers of momenta needed for a nonzero value of the t matrix. Additional powers of momenta will give terms smaller in the ratio $(a\lambda)$ or $(kT/2JS)^{1/2}$, both of which are small parameters. Accordingly, we evaluate the t matrix at zero energy (where it is real) and will presently take the limit of zero total momentum. Thus

$$\Sigma''(\lambda)\sim\frac{2\pi\beta\epsilon_\lambda}{N^2}\sum_{\mathbf{p}\rho}f(\epsilon_{\mathbf{p}})[1+f(\epsilon_{\mathbf{p}}+\epsilon_\lambda)]$$
$$\times[1+f(\epsilon_\rho)+f(\epsilon_{\mathbf{p}+\lambda-\rho})]\delta(\epsilon_\lambda+\epsilon_{\mathbf{p}}-\epsilon_\rho-\epsilon_{\lambda+\mathbf{p}-\rho})$$
$$\times\, t(0;\,\boldsymbol{\lambda}+\mathbf{p};\,\tfrac{1}{2}(\boldsymbol{\lambda}-\mathbf{p}),\,\boldsymbol{\rho}-\tfrac{1}{2}(\boldsymbol{\lambda}+\mathbf{p}))$$
$$\times\, t(0;\,\boldsymbol{\lambda}+\mathbf{p};\,\boldsymbol{\rho}-\tfrac{1}{2}(\boldsymbol{\lambda}+\mathbf{p});\,\tfrac{1}{2}(\boldsymbol{\lambda}-\mathbf{p})).\quad(20)$$

For the t matrices we use the explicit expression[11]

$$t(0;\,\boldsymbol{\lambda}+\mathbf{p};\,\boldsymbol{\gamma},\,\boldsymbol{\mu})=4J\sum_{i,j=xyz}[\cos\tfrac{1}{2}(\boldsymbol{\lambda}+\mathbf{p})_i-\cos\mu_i]$$
$$\times[1-(1/2S)B]^{-1}\cos\gamma_j.\quad(21)$$

It is now permissible to evaluate the matrix B for $\boldsymbol{\lambda}+\mathbf{p}=0$;

$$B_{ij}=-N^{-1}\sum_\rho\frac{\cos\rho_i\,(1-\cos\rho_j)}{3-\cos\rho_x-\cos\rho_y-\cos\rho_z}.\quad(22)$$

Note in particular that

$$\sum_j B_{ij}=0,\quad(23)$$

so that also

$$\sum_j[1-(1/2S)B]^{-1}{}_{ij}=1.\quad(24)$$

In this way we obtain an expression for the t matrix for small momenta as

$$t(0;\,\boldsymbol{\lambda}+\mathbf{p};\,\boldsymbol{\gamma},\,\boldsymbol{\mu})$$
$$\approx-2J(\tfrac{1}{2}(\boldsymbol{\lambda}+\mathbf{p})+\boldsymbol{\mu})(\tfrac{1}{2}(\boldsymbol{\lambda}+\mathbf{p})-\boldsymbol{\mu}),\quad(25)$$

where we have taken account of Eq. (24). Note that we have not neglected higher orders in $1/S$; their contribution vanishes due to Eq. (24). Thus Eq. (20) becomes

$$\Sigma''(\lambda)=\frac{8\pi\beta\epsilon_\lambda J^2}{N^2}\sum_{\mathbf{p}\rho}f(\epsilon_{\mathbf{p}})[1+f(\epsilon_{\mathbf{p}}+\epsilon_\lambda)][\boldsymbol{\rho}\cdot(\boldsymbol{\lambda}+\mathbf{p}-\boldsymbol{\rho})]$$
$$\times(\boldsymbol{\lambda}\cdot p)\delta(\epsilon_\lambda+\epsilon_{\mathbf{p}}-\epsilon_\rho-\epsilon_{\lambda+\mathbf{p}-\rho})[1+f(\epsilon_\rho)+f(\epsilon_{\lambda+\mathbf{p}-\rho})].\quad(26)$$

Using the symmetry between $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}+\mathbf{p}-\boldsymbol{\rho}$ we note that $1+f(\epsilon_\rho)+f(\epsilon_{\mathbf{p}+\lambda-\rho})$ may be replaced by $1+2f(\epsilon_\rho)$.

[11] R. Silberglitt and A. B. Harris, Phys. Rev. Letters **19**, 512 (1967).

Using the quadratic approximation to the energy in the delta functions, we have

$$\Sigma''(\boldsymbol{\lambda})\sim(4\pi\beta\epsilon_\lambda J/S)(2\pi)^{-6}$$
$$\times\int p^2dp\,\sin\theta_p\,d\theta_p\,d\varphi_p\,\rho^2d\rho\,\sin\theta_\rho\,d\theta_\rho$$
$$\times d\varphi_\rho f(\epsilon_p)[1+f(\epsilon_p+\epsilon_\lambda)][1+2f(\epsilon_\rho)]$$
$$\times\delta[\rho^2+\boldsymbol{\lambda}\cdot\mathbf{p}-2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}+\mathbf{p})](\boldsymbol{\lambda}\cdot\mathbf{p})[\boldsymbol{\rho}\cdot\boldsymbol{\lambda}+\mathbf{p}-\boldsymbol{\rho})], \quad (27)$$

where we take the lattice constant to be unity, and extend the momentum integrals to infinity. We can also now redefine

$$f(x)\equiv[\exp(\beta x^2)-1]^{-1} \quad (28)$$

since we wish to obtain $\Sigma''(\boldsymbol{\lambda})$ correct to lowest order in $kT/2JS$. If we write

$$\boldsymbol{\rho}\cdot(\boldsymbol{\lambda}+\mathbf{p})=\rho\,|\,\boldsymbol{\lambda}+\mathbf{p}\,|\cos\theta_\rho, \quad (29a)$$
$$(\lambda+p)^2=\lambda^2+p^2+2\lambda_p\cos\theta_p, \quad (29b)$$
$$\cos\theta_p=x, \quad (29c)$$

the integrations over φ_p, φ_ρ, and θ_ρ can be done and one finds

$$\Sigma''(\boldsymbol{\lambda})=\frac{\beta J\epsilon_\lambda\lambda^2}{8\pi^3S}\int_0^\infty f(\epsilon_p)[1+f(\epsilon_p+\epsilon_\lambda)]p^4dp$$
$$\times\int_{-1}^1\frac{x^2dx}{(\lambda^2+p^2+2\lambda px)^{1/2}}\int_{\rho_<(x)}^{\rho_>(x)}[1+2f(\epsilon_\rho)]\rho d\rho, \quad (30)$$

where

$$\rho_{>,<}^2=\tfrac{1}{2}[\lambda^2+p^2\pm((\lambda^2+p^2)^{1/2}-4\lambda^2p^2x^2)^{1/2}]. \quad (31)$$

Let us evaluate

$$I_1=\int_0^\infty f(\epsilon_p)[1+f(\epsilon_p+\epsilon_\lambda)]p^4dp\int_{-1}^1\tfrac{1}{2}x^2(\rho_>^2-\rho_<^2)$$
$$\times(\lambda^2+p^2+2\lambda px)^{-1/2}dx. \quad (32)$$

We may neglect λ in comparison to p since terms in $(\lambda/p)^2$ will give contributions to I_1 smaller by $\beta\epsilon_\lambda$. Then

$$I_1=\tfrac{1}{3}\int_0^\infty f(\epsilon_p)[1+f(\epsilon_p)]p^5dp \quad (33a)$$
$$=(\pi^2/18)(kT/2JS)^3. \quad (33b)$$

The evaluation of the rest of Eq. (30) is more delicate. We write

$$8\pi^3S\Sigma''(\boldsymbol{\lambda})=\beta J\epsilon_\lambda\lambda^2(I_1+I_2) \quad (34a)$$

and

$$I_2=I_2^{(a)}+I_2^{(b)}, \quad (34b)$$

with

$$I_2^{(a)}=2\int_0^{\xi\lambda}f(\epsilon_p)[1+f(\epsilon_p+\epsilon_\lambda)]p^4dp$$
$$\times\int_{-1}^1\frac{x^2dx}{(\lambda^2+p^2+2\lambda px)^{1/2}}\int_{\rho<(x)}^{\rho>(x)}f(\epsilon_p)\rho d\rho \quad (34c)$$

and $I_2^{(b)}$ similarly an integral over p from $\xi\lambda$ to ∞. We take

$$\xi=(2JS\beta\lambda^2)^{-1/4}, \quad (35a)$$

so that, according to Eq. (19),

$$\xi\gg1, \quad (35b)$$
$$\beta\epsilon_{\xi\lambda}=2JS\beta\xi^2\lambda^2=(2JS\beta\lambda^2)^{1/2}=(\beta\epsilon_\lambda)^{1/2}\ll1. \quad (35c)$$

Then in $I_2^{(a)}$ we may approximate $f(x)$ as $1/x$ and in $I_2^{(b)}$ we will be able to neglect λ in comparison to p by Eq. (35b). Then writing $p=q\lambda$ we have

$$I_2^{(a)}=\left(\frac{kT}{2JS}\right)^3\int_0^\xi\frac{q^2dq}{1+q^2}\int_{-1}^1dx\,x^2(1+2qx+q^2)^{-1/2}$$
$$\times\ln\frac{1+q^2+[(1+q^2)^2-4q^2x^2]^{1/2}}{1+q^2-[(1+q^2)^2-4q^2x^2]^{1/2}}, \quad (36)$$

so that

$$I_2^{(a)}=(kT/2JS)^3[\tfrac{2}{3}\ln^2\xi+\tfrac{4}{9}\ln\xi+K], \quad (37)$$

neglecting terms which vanish as $\xi\to\infty$. Naturally in $I_2^{(b)}$ we must find terms to eliminate the ξ dependence. Here

$$K=\lim_{\xi\to\infty}\left[\left\{\int_0^\xi\frac{q^2dq}{1+q^2}\int_{-1}^1dx(1+2qx+q^2)^{-1/2}\right.\right.$$
$$\left.\left.\times\ln\frac{1+q^2+[(1+q^2)^2-4q^2x^2]^{1/2}}{1+q^2-[(1+q^2)^2-4q^2x^2]^{1/2}}\right\}-\tfrac{2}{3}\ln^2\xi-\tfrac{4}{9}\ln\xi\right] \quad (38)$$

and is evaluated in Appendix B.

As we have mentioned, in $I_{(2)}^{(b)}$ we neglect λ in comparison to p because of Eq. (35b). Then

$$I_2^{(b)}\sim4\int_{\xi\lambda}^\infty f(\epsilon_p)[1+f(\epsilon_p)]p^3\int_0^1x^2dx\int_{\rho<(x)}^{\rho>(x)}f(\epsilon_\rho)\rho d\rho. \quad (39)$$

Doing the x integral by parts, one obtains

$$I_2^{(b)}=\tfrac{4}{3}\int_{\xi\lambda}^\infty f(\epsilon_p)[1+f(\epsilon_p)]p^3\left[\int_\lambda^p f(\epsilon_\rho)\rho d\rho\right.$$
$$+\int_p^{(p^2+\lambda^2)^{1/2}}f(\epsilon_\rho)\rho\left(\frac{\rho}{\lambda p}\right)^3(\lambda^2+p^2-\rho^2)^{3/2}d\rho$$
$$\left.+\int_0^\lambda f(\epsilon_\rho)\rho\left(\frac{\rho}{\lambda p}\right)^3(\lambda^2+p^2-\rho^2)^{3/2}d\rho\right]. \quad (40)$$

It is clear now that the second integral in the square brackets is negligible in comparison to the first integral. Furthermore the third integral can be done by expanding $f(\epsilon_\rho)$ in powers of $\beta\epsilon_\rho$ and $(p^2+\lambda^2-\rho^2)^{3/2}$ in powers of $(\lambda^2-\rho^2)/p^2$ since $\rho\leq\lambda\ll p$. Keeping only the leading terms we have

$$\int_0^\lambda f(\epsilon_\rho)\rho\left(\frac{\rho}{\lambda p}\right)^3(\lambda^2+p^2-\rho^2)^{3/2}d\rho\approx\tfrac{1}{3}(kT/2JS). \quad (41)$$

Then

$$I_2^{(b)}=\tfrac{4}{3}\int_{\xi\lambda}^{\infty} f(\epsilon_p)[1+f(\epsilon_p)]p^3\left[\int_{\xi\lambda}^{p} f(\epsilon_\rho)\rho d\rho + \int_{\lambda}^{\xi\lambda} f(\epsilon_\rho)\rho d\rho+\tfrac{1}{3}\left(\frac{kT}{2JS}\right)\right]. \quad (42)$$

Because of Eq. (35c) we can approximate $f(\epsilon_\rho)$ by $(\beta\epsilon_\rho)^{-1}$ in the second ρ integral, so that

$$I_2^{(b)}=\tfrac{4}{3}\int_{\xi\lambda}^{\infty} f(\epsilon_p)[1+f(\epsilon_p)]p^3dp\int_{\xi\lambda}^{p} f(\epsilon_\rho)\rho d\rho +\tfrac{4}{3}\left(\frac{kT}{2JS}\right)(\ln\xi+\tfrac{1}{3})\int_{\xi\lambda}^{\infty} f(\epsilon_p)[1+f(\epsilon_p)]p^3dp. \quad (43)$$

Now one has

$$\int_{\xi\lambda}^{\infty} f(\epsilon_p)[1+f(\epsilon_p)]p^3dp\sim\tfrac{1}{2}(kT/2JS)^2 \times[1+\ln(kT/2JS\xi^2\lambda^2)], \quad (44)$$

and by integration by parts one evaluates

$$\int_{\xi\lambda}^{\infty} f(\epsilon_p)[1+f(\epsilon_p)]p^3dp\int_{\xi\lambda}^{p} f(\epsilon_\rho)\rho d\rho =(4\beta JS)^{-1}\int_{\xi\lambda}^{\infty} f(\epsilon_p)pdp\left\{p^2f(\epsilon_p)+2\int_{\xi\lambda}^{p} f(\epsilon_\rho)\rho d\rho\right\}. \quad (45)$$

Furthermore,

$$\int_{\xi\lambda}^{\infty} f(\epsilon_p)pdp\int_{\xi\lambda}^{p} f(\epsilon_\rho)\rho d\rho=\tfrac{1}{2}\left[\int_{\xi\lambda}^{\infty} f(\epsilon_p)pdp\right]^2 \quad (46a)$$

$$\sim\tfrac{1}{8}(kT/2JS)^2\ln^2(kT/2JS\xi^2\lambda^2). \quad (46b)$$

Also,

$$\int_{\xi\lambda}^{\infty} f^2(\epsilon_p)p^3dp\sim\frac{1}{2}\left(\frac{kT}{2JS}\right)\left(1+\ln\frac{kT}{2JS\xi^2\lambda^2}-\frac{\pi^2}{6}\right). \quad (47)$$

Substituting Eq. (44) through (47) into Eq. (43) one obtains

$$I_2^{(b)}=\{\tfrac{2}{3}(\ln\xi+\tfrac{1}{3})[1+\ln(kT/2JS\lambda^2)-2\ln\xi] +\tfrac{1}{6}[\ln(kT/2JS\lambda^2)-2\ln\xi]^2+\tfrac{1}{3} +\tfrac{1}{3}\ln(kT/2JS\lambda^2)-\tfrac{2}{3}\ln\xi-\pi^2/18\}(kT/2JS)^3 \quad (48a)$$

$$=[-\tfrac{2}{3}\ln^2\xi-\tfrac{4}{9}\ln\xi-\pi^2/18+\tfrac{5}{9}+\tfrac{5}{9}\ln(kT/2JS\lambda^2) +\tfrac{1}{6}\ln^2(kT/2JS\lambda^2)](kT/2JS)^3. \quad (48b)$$

Using this equation together with Eqs. (37) and (33) in Eq. (34) we obtain finally

$$\Sigma''(\lambda)=\frac{J\lambda^4}{8\pi^3S}\left(\frac{kT}{2JS}\right)^2 \times\left(\tfrac{1}{6}\ln^2\frac{kT}{2JS\lambda^2}+\tfrac{5}{9}\ln\frac{kT}{2JS\lambda^2}-0.50\right), \quad (49)$$

where we have used the value $K=-1.06$ from Appendix B. The leading term in Eq. (45) agrees with the results quoted by Kashcheev and Krivoglaz,[3] who, however, did not give the remaining terms. Since the condition $\ln(kT/2JS\lambda^2)\gg 1$ is rather stringent, it is well to have the remaining terms. Further terms in the series will be smaller by at least a factor $(kT/2JS)^{1/2}$ and perhaps even by $kT/2JS$, although the analysis seems to be too complicated to bother with. As a final comment we note that $\Sigma''(\lambda)/\epsilon_\lambda\to 0$ rapidly as $\lambda\to 0$. Thus we conclude that spin waves are perfect modes in this limit.[12] In contrast the formula in Refs. 1 and 2 predicts that $\Sigma''(\lambda)/\epsilon_\lambda$ remains finite in the limit $\lambda\to 0$.

APPENDIX A: OPTICAL THEOREM FOR THE t MATRIX

The t matrix satisfies Eq. (10);

$$t(z;\mathbf{K};\boldsymbol{\lambda},\boldsymbol{\mu})=V_{\mathbf{K}}(\boldsymbol{\lambda},\boldsymbol{\mu}) +N^{-1}\sum_{\rho}V_{\mathbf{K}}(\boldsymbol{\lambda},\boldsymbol{\rho})\frac{1+n(\tfrac{1}{2}\mathbf{K}+\boldsymbol{\rho})+n(\tfrac{1}{2}\mathbf{K}-\boldsymbol{\rho})}{z-\epsilon(\tfrac{1}{2}\mathbf{K}+\boldsymbol{\rho})-\epsilon(\tfrac{1}{2}\mathbf{K}-\boldsymbol{\rho})} \times t(z;\mathbf{K};\boldsymbol{\rho},\boldsymbol{\mu}). \quad (A1)$$

Since z and $\mathbf{K}$ are merely parameters, we will introduce a more convenient matrix notation:

$$t_{\lambda\mu}\equiv t(z;\mathbf{K};\boldsymbol{\lambda},\boldsymbol{\mu}), \quad (A2a)$$

$$V_{\lambda\mu}\equiv V_{\mathbf{K}}(\boldsymbol{\lambda},\boldsymbol{\mu}), \quad (A2b)$$

$$G_{\lambda\mu}\equiv\delta_{\lambda,\mu}\frac{1+n(\tfrac{1}{2}\mathbf{K}+\boldsymbol{\lambda})+n(\tfrac{1}{2}\mathbf{K}-\boldsymbol{\lambda})}{z-\epsilon(\tfrac{1}{2}\mathbf{K}+\boldsymbol{\lambda})-(\tfrac{1}{2}\mathbf{K}-\boldsymbol{\lambda})}, \quad (A2c)$$

so that Eq. (A1) becomes, in matrix notation,

$$t=V+VGt. \quad (A3)$$

We will now derive the optical theorem, assuming V to be real but not Hermitian, as is the case for the Dyson-Maleev Hamiltonian of Eq. (3a). We separate G into its real and imaginary parts (z is complex):

$$G=G'+iG''. \quad (A4)$$

We introduce t_0, which is defined as the solution to

$$t_0=V+VG't_0, \quad (A5)$$

so that

$$t_0=(1-VG')^{-1}V. \quad (A6)$$

We now define

$$s\equiv t_0(1+G''t_0G''t_0)^{-1}(1+iG''t_0), \quad (A7)$$

[12] We are assuming that many spin-wave processes which are of higher order in kT/JS do not give contributions of lower order in λ than $\lambda^4\ln^2\lambda$. If this assumption is incorrect, then our results will be limited to $\lambda\gg\lambda_0(T)$, where $\lambda_0(T)$ is the momentum at which the terms of higher order in kT/JS but lower order in λ become comparable with those retained here.

and will show that s satisfies Eq. (A3), so that $t=s$:

$$V+VGs=V+VG's+iVG''s \tag{A8a}$$

$$=V+VG't_0(1+G''t_0G''t_0)^{-1}(1+iG''t_0)$$
$$+iVG''t_0(1+G''t_0G''t_0)^{-1}(1+iG''t_0) \tag{A8b}$$

$$=V+VG't_0(1+G''t_0G''t_0)^{-1}(1+iG''t_0)$$
$$+V(1-1+iG''t_0)(1+iG''t_0)$$
$$\times(1+G''t_0G''t_0)^{-1}, \tag{A8c}$$

because $1+iG''t_0$ commutes with $(1+G''t_0G''t_0)^{-1}$. Thus

$$V+VGs=V+VG't_0(1+G''t_0G''t_0)^{-1}(1+iG''t_0)$$
$$+V(1+G''t_0G''t_0)^{-1}(1+iG''t_0)-V \tag{A9a}$$

$$=t_0(1+G''t_0G''t_0)^{-1}(1+iG''t_0) \tag{A9b}$$

$$=s. \qquad \text{Q.E.D.} \tag{A9c}$$

Here we have used

$$(1-iG''t_0)(1+iG''t_0)=(1+G''t_0G''t_0)$$

and also Eq. (A5). Furthermore, since G'' and t_0 are real,

$$t^*=t_0(1+G''t_0G''t_0)^{-1}(1-iG''t_0) \tag{A10a}$$

$$=t_0(1-iG''t_0)(1+G''t_0G''t_0)^{-1} \tag{A10b}$$

$$=(1-it_0G'')t_0(1+G''t_0G''t_0)^{-1}. \tag{A10c}$$

Using Eq. (A7) for t and Eq. (A10c) for t^* we have

$$tG''t^*=t_0(1+G''t_0G''t_0)^{-1}(1+iG''t_0)G''$$
$$\times(1-it_0G'')t_0(1+G''t_0G''t_0)^{-1} \tag{A11a}$$

$$=t_0(1+G''t_0G''t_0)^{-1}(1+G''t_0G''t_0)$$
$$\times G''t_0(1+G''t_0G''t_0)^{-1} \tag{A11b}$$

$$=t_0G''t_0(1+G''t_0G''t_0) \tag{A11c}$$

$$=t'', \tag{A11d}$$

which is the optical theorem. We have thus generalized the optical theorem to the case of a non-Hermitian but real potential.

APPENDIX B: EVALUATION OF K

From Eq. (38) we have that

$$K=\lim_{\xi\to\infty}[I(\xi)-\tfrac{2}{3}\ln^2\xi-\tfrac{4}{9}\ln\xi], \tag{B1a}$$

where

$$I(\xi)=\int_0^\xi \frac{q^2dq}{1+q^2}\int_{-1}^{1}\frac{x^2dx}{(1+2qx+q^2)^{1/2}}$$
$$\times\ln\frac{1+q^2+[(1+q^2)^2-4q^2x^2]^{1/2}}{1+q^2-[(1+q^2)^2-4q^2x^2]^{1/2}}. \tag{B1b}$$

We break the q integral into two ranges about $q=1$. For $q>1$ we write $q=1/t$. Also in the x integral we set $x=(1+q^2)y/2q$. Then for large ξ we find

$$I(\xi)=\tfrac{1}{8}\int_{1/\xi}^{1} q^{-4}(1+q^3)(1+q^2)^{3/2}$$
$$\times\left[J\left(\frac{2q}{1+q^2}\right)-J\left(\frac{-2q}{1+q^2}\right)\right], \tag{B2}$$

where

$$J(y)=(1+y)^{1/2}[\tfrac{2}{5}y^2-\tfrac{8}{15}y+\tfrac{16}{15}]\ln\frac{1+(1-y^2)^{1/2}}{1-(1-y^2)^{1/2}}$$
$$+\tfrac{8}{15}(1-y)^{1/2}(2-y)+\tfrac{32}{15}\ln\left|\frac{1-(1-y)^{1/2}}{1+(1-y)^{1/2}}\right|, \tag{B3}$$

which leads to

$$I(\xi)=\tfrac{4}{15}(I_1+I_2+I_3), \tag{B4}$$

with

$$I_1=-\int_{1/\xi}^{1}\frac{1}{q}\frac{1+q^3}{1+q^2}\ln q[5+2q^2]dq, \tag{B5a}$$

$$I_2=-\int_{1/\xi}^{1} q^{-3}(1+q^3)(2+q^2)dq, \tag{B5b}$$

$$I_3=\int_{1/\xi}^{1} 2q^{-4}(1+q^3)(1+q^2)^{3/2}\ln[(1+q^2)^{1/2}+q]. \tag{B5c}$$

We find that

$$I_1=\tfrac{29}{9}+\tfrac{5}{2}\ln^2\xi-\tfrac{1}{16}\pi^2-\tfrac{3}{2}\int_0^\infty\frac{xdx}{\cosh x}, \tag{B6a}$$

$$I_2=-\xi^2-\ln\xi-\tfrac{4}{3}, \tag{B6b}$$

$$I_3=\ln^2(1+\sqrt{2})+\xi^2+\tfrac{8}{3}\ln\xi-\tfrac{1}{3}+4\int_1^{1+\sqrt{2}}\frac{\ln y dy}{y^2-1}, \tag{B6c}$$

so that

$$K=\tfrac{56}{135}-\tfrac{1}{60}\pi^2+\tfrac{4}{15}\ln^2(1+\sqrt{2})$$
$$-\tfrac{3}{2}\int_0^\infty\frac{xdx}{\cosh x}+4\int_1^{1+\sqrt{2}}\frac{\ln y dy}{y^2-1}. \tag{B7}$$

Numerically we obtain

$$K=-1.06. \tag{B8}$$