Scattering of Single-Particle Excitations by a Vortex in a Clean Type-II Superconductor*

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A study of the scattering of BCS single-particle excitations by an isolated vortex in a clean type-II superconductor is reported. The coupled Bogoliubov equations provide a theoretical framework for calculating quasiparticle wave functions. The vortex introduces superfluid velocity into the metal, depresses the order parameter $\Delta(r)$ at the core, and gives rise to a Bohm-Aharonov effect. The scattering problem has two channels with coupling between particlelike and holelike excitations. The total cross section diverges because of the Bohm-Aharonov effect, but the cross section σ_s for a particle to scatter from one channel to the other remains finite. This quantity σ_s is also equal to one-half the transport cross section. A numerical evaluation of σ_s as a function of the energy and polar angle of the excitations relative to the external field has been made.

I. INTRODUCTION

It has recently been observed that the ultrasonic attenuation[1,2] and thermal conductivity[3] of clean superconducting niobium exhibit a dip near H_{c1}. The magnitude of these transport coefficients suddenly drops with the appearance of flux penetration, proceeds to a minimum, and increases to its normal value as the magnetic field approaches the upper critical field H_{c2}.[3,4]

Forgan and Gough[1] have suggested that the initial decrease may be due to scattering of electronic excitations by the vortices present in the superconductor. The absence of the anomaly in dirty superconductors corroborates their suggestion.[3,5] It is the purpose of this paper to study the scattering effects of a flux line on the BCS [6] quasiparticles and to calculate the cross section of a vortex. In another paper we study the transport properties of a superconductor containing a random array of vortices and employ the results obtained here to compare with experiment.

The spectrum of bound states of a vortex in a clean high-κ superconductor has already been calculated by Caroli, de Gennes, and Matricon.[7,8] We extend their calculational technique to encompass the scattering problem in superconductors with intermediate values of κ ($\kappa \simeq 1/\sqrt{2}$) such as niobium and vanadium.

In Sec. II the problem of a vortex in a clean type-II superconductor is formulated. This is followed by a discussion of the potentials which we shall employ in the Bogoliubov equations. Section III is devoted to the uncoupled problem which represents a normal electron in an inhomogeneous magnetic field. These calculations also correspond to the high-energy limit in the coupled case. A partial-wave analysis of the exact equation is carried out in Sec. IV and we proceed in Sec. V to introduce the envelope approximation. The results of our numerical calculations are also included in this last section.

II. FORMULATION

In the BCS[6] approximation, the electronic excitations of a type-II superconductor containing a vortex are described by the coupled Bogoliubov equations[9]

$$\{i(\partial/\partial t)-(\tau^{(3)}/2m)[(\mathbf{p}-(e/c)\mathbf{A}(r)\tau^{(3)})^2-p_F^2]$$
$$-\Delta(r)\exp(i\phi\tau^{(3)})\tau^{(1)}\}\Psi_{\mathbf{p}}(x)=0. \quad (1)$$

(p_F is the Fermi momentum of the metal, and we take Planck's constant $\hbar$ and Boltzmann's constant k_B equal to 1.) A spinor notation abbreviates the usual form of these equations ($\tau^{(1)}$, $\tau^{(2)}$, and $\tau^{(3)}$ are the Pauli spin matrices) where the wave function is now a two-component vector:

$$\Psi_{\mathbf{p}}(x)=\begin{pmatrix}u_{\mathbf{p}}(x)\\ v_{\mathbf{p}}(x)\end{pmatrix}. \quad (2)$$

We employ the notation $\mathbf{p}$ to denote the momentum operator $-i\partial/\partial\mathbf{x}$ in the curly brackets of Eq. (1) and as a label on the wave function to characterize its incoming properties.

The vortex modifies the magnitude of the order parameter or pair potential function from its constant value of $\boldsymbol{\Delta}$ in the Meissner state and gives it a nonzero coordinate-dependent phase. The introduction of a system of circular cylindrical coordinates (r,ϕ,z) centered

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[8] C. Caroli and J. Matricon, Physik Kondensierten Materie **3**, 380 (1965).

[9] N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau Enterprises, Inc., New York, 1959).

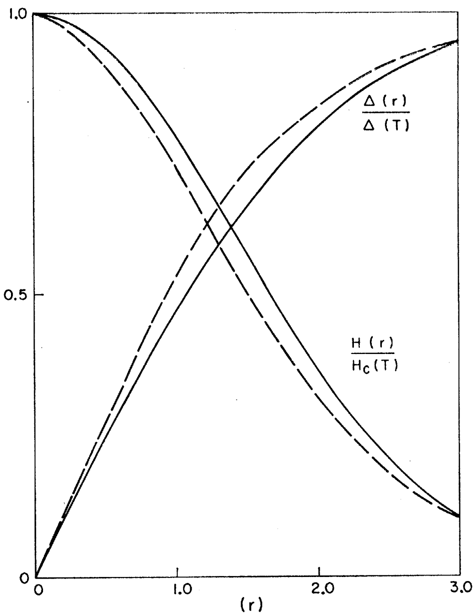

FIG. 1. Approximate potentials employed in Bogoliubov equations: $\Delta(r)$ in units of the order parameter in zero field, $\Delta(T)$ and $H(r)$ in units of the bulk critical field, $H_c(T)$ versus r in units of the penetration depth $\lambda(T)$. The approximate potentials are given by the solid line and numerical solution of the Ginzburg-Landau equations with $\kappa=1/\sqrt{2}$ are given by the dotted line.

along the axis of the vortex is convenient for describing the spatial dependence of this potential and of the magnetic vector potential $\mathbf{A}(r)$.

Equation (1) is translationally invariant in time and in space along the z axis and can be made rotationally invariant about z by introducing a new wave function

$$\Psi_p(x)=\exp(i\tfrac{1}{2}\phi\tau^{(3)})\Phi_p(x).$$

The symmetry of the problem is exploited by extracting a plane wave propagating in the z direction from the new wave function and carrying out a partial-wave expansion in the angular variable ϕ. One has

$$\Phi_p(x)=(2\pi)^{-3/2}\exp[i(kz-Et)]\sum_{\mu=-\infty}^{\infty}e^{i\mu\phi}\Phi_q{}^{\mu}(r), \quad (3)$$

where

$$p^2=q^2+k^2. \quad (4)$$

q and k are the components of momentum perpendicular and parallel to the external field, and μ is restricted to half-integral values in order that the wave function $\Psi_p(x)$ may remain single-valued. The partial-wave component $\Phi_q{}^{\mu}(r)$ satisfies a simplified differential equation which depends only on the coordinate r. The equation is

$$\{E+(2m)^{-1}\tau^{(3)}[d^2/dr^2+r^{-1}d/dr-(\mu/r+mv_s(r)\tau^{(3)})^2 + p_F{}^2-k^2]-\Delta(r)\tau^{(1)}\}\Phi_q{}^{\mu}(r)=0, \quad (5)$$

where the superfluid velocity function is given by

$$v_s(r)=m^{-1}[(2r)^{-1}-(e/c)A(r)]. \quad (6)$$

The potentials which appear in Eq. (5) must be obtained self-consistently from the wave functions in order to arrive at a complete theory of a superconducting vortex.[10] Fortunately we have the Ginzburg-Landau (GL) equations to provide us with the functions that are needed in order to proceed.[11] The solution to these equations gives a self-consistent result in a temperature range restricted by the inequality $(T_c-T)/T_c\ll 1$.[12]

In Fig. 1 we plot the results of Neumann and Tewordt[13] for the order parameter $\Delta(r)$ and magnetic field $H(r)$ obtained by solving the GL equations for a superconductor with $\kappa=1/\sqrt{2}$. We shall employ the following analytic approximations for these potentials in our calculations:

$$\Delta(r)=\Delta[1-\exp(-r^2/4\lambda^2)]^{1/2} \quad (7)$$

and

$$H(r)=(c/4e\lambda^2)\exp(-r^2/4\lambda^2), \quad (8)$$

where $\lambda(T)$ is the penetration depth at temperature T. The magnetic field is equal to the curl of the magnetic vector potential, which implies that

$$mv_s(r)=(1/2r)\exp(-r^2/4\lambda^2). \quad (9)$$

In units of the bulk critical field $H_c(T)$, the magnetic field of a vortex in our approximation has the form

$$H(r)=H_c(T)\exp(-r^2/4\lambda^2)/\sqrt{2}\kappa. \quad (10)$$

We also plot Eqs. (7) and (8) in Fig. 1 for $\kappa=1/\sqrt{2}$. The approximate functions $H(r)$ and $\Delta-\Delta(r)$ exceed the numerical ones in the range $0\leq r\leq 3\lambda(T)$. For large r they approach their limiting value much more rapidly than the true GL solutions; we will therefore obtain incorrect wave functions for large partial-wave number μ, but such waves make a minor contribution to the cross section. Moreover, this "Gaussian" behavior of the potentials for large r ensures rapid convergence in the integration of Eq. (5).

Just below the critical temperature the GL parameter of clean niobium is 0.76 while that of vanadium is 0.848.[14,15] These two metals constitute the only two clean type-II superconductors presently known and both have relatively small values of κ.

The potentials of Eqs. (7) and (8) can be improved in two ways: (1) Numerical solution of the GL equations may be employed to obtain very accurate results for κ values appropriate for niobium and vanadium. (2) Higher-order terms of the free-energy functional

[10] P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966), Chap. 5.

[11] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950) [English transl.: *Men of Physics*: *L. D. Landau I*, edited by D. ter Haar (Pergamon Press, Inc., Oxford, 1965), Part 2, p. 138].

[12] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **36**, 1918 (1959) [English transl.: Soviet Phys.—JETP **9**, 1364 (1959)].

[13] L. Neumann and L. Tewordt, Z. Physik **189**, 55 (1966).

[14] D. K. Finnemore, T. F. Stromberg, and C. A. Swenson, Phys. Rev. **149**, 231 (1966).

[15] R. Radebaugh and P. H. Keesom, Phys. Rev. **149**, 209 (1966).

may be kept to obtain a correction to the potentials for temperatures below the immediate neighborhood of T_c.[13] This latter correction is important in clean type-II superconductors for small values of κ as we point out in the next paragraph.

As the temperature tends to T_c, where the GL solutions are valid, the lower critical field H_{c1} of the superconductor approaches the upper critical field H_{c2} and it becomes increasingly difficult to study the individual nature of an isolated vortex. At temperatures of $0.9T_c$ the calculations of Neumann and Tewordt indicate that the correction to the GL magnetic field at $r=0$ is equal to $+6\%$ for a clean superconductor with $\kappa=1/\sqrt{2}$. At temperatures much lower than T_c the vortex potentials for such a superconductor are presently unknown. We expect however that the approximate expressions of Eqs. (7) and (8) remain qualitatively correct over the entire temperature range.

III. UNCOUPLED CASE

Before solving Eq. (5) we examine its properties in the absence of the coupling term $\Delta(r)\tau^{(1)}$. This corresponds to the limit $E\gg\Delta$, and the solution of the equation describes the wave function of a normal electron in the presence of the inhomogeneous magnetic field due to a vortex.[16] Since the equations are uncoupled in this case we need only consider a single component $u_q^\mu(r)$ of the wave function. The other component $v_q^\mu(r)$ satisfies the same equation with the modification

$$E\rightarrow -E; \qquad \mu\rightarrow -\mu \tag{11}$$

in the square brackets of Eq. (12). The scalar equation that must be solved for the electron wave function is

$$[2mE+d^2/dr^2+r^{-1}d/dr-(\mu/r+mv_s)^2 +p_F{}^2-k^2]u_q{}^\mu(r)=0 \tag{12}$$

with the boundary condition that $u_q^\mu(r)$ be square integrable at the origin.

If we ignore the superfluid velocity function in Eq. (12), an exact solution is obtained in terms of Bessel functions of half-integral order. The asymptotic value of the exact wave function will therefore behave as follows:

$$u_q{}^\mu(r)\sim J_{|\mu|}(qr)\cos\delta_\mu - Y_{|\mu|}(qr)\sin\delta_\mu, \tag{13}$$

where $Y_{|\mu|}(qr)$ is the Neumann function which diverges at the origin. In order to simplify notation in the remainder of this paper the absolute value brackets about μ in the Bessel functions will be made implicit.

Except for a small group of states with

$$k\simeq(p_F{}^2+2mE)^{1/2}$$

traveling in a narrow cone about the z axis, the wave function varies rapidly over a length interval which is much shorter than the penetration depth. The WKB approximation is well suited for this situation, and we may immediately write down an expression for the phase shifts[17]

$$\delta_\mu=\tfrac{1}{2}|\mu|\pi-qr_0+\int_{r_0}^{\infty}dr\,\{[q^2-(\mu/r+mv_s)^2]^{1/2}-q\}, \tag{14}$$

where the transverse component of momentum q is given by

$$q=(p_F{}^2-k^2+2mE)^{1/2}, \tag{15}$$

and r_0, the classical turning point of the trajectory, is a root of

$$q=|\mu/r_0+mv_s(r_0)|. \tag{16}$$

The integrand in Eq. (14) is expanded in powers of mv_s and only the leading, nonvanishing term is retained, so that

$$\delta_\mu=-\int_{|\mu|/q}^{\infty}\frac{dr\,\mu m v_s(r)}{(q^2r^2-\mu^2)^{1/2}}. \tag{17}$$

Substituting the potential of Eq. (7) into Eq. (17) and introducing the new dummy variable of integration

$$\bar{r}=\tfrac{1}{2}r\lambda^{-1}, \tag{18}$$

we obtain the tabulated integral[18]

$$\delta_\mu=-\tfrac{1}{2}\bar{\mu}\int_{|\bar{\mu}|}^{\infty}\frac{d\bar{r}\exp(-\bar{r}^2)}{\bar{r}(\bar{r}^2-\bar{\mu}^2)^{1/2}} =-\mathrm{sgn}(\mu)\tfrac{1}{4}\pi\,\mathrm{erfc}(|\bar{\mu}|), \tag{19}$$

where $\bar{\mu}=\mu/2q\lambda$ and $\mathrm{erfc}(z)$ is the complementary error function

$$\mathrm{erfc}(z)=2\pi^{-1/2}\int_z^{\infty}\exp(-t^2)\,dt. \tag{20}$$

To obtain the scattering amplitude it is necessary to construct incoming plane-wave states from the partial-wave functions of Eq. (12). At distances larger than $|\mu|/q$ the leading term in Hankel's asymptotic expansion for the Bessel functions simplifies Eq. (13) and we may write

$$u_q{}^\mu(r)\sim(2/\pi qr)^{1/2}\cos(qr-\tfrac{1}{2}|\mu|\pi-\tfrac{1}{4}\pi+\delta_\mu) =(2/\pi qr)^{1/2}\cos(qr-\tfrac{1}{2}|l|\pi-\tfrac{1}{4}\pi+\delta_\mu+\tfrac{1}{4}\pi\,\mathrm{sgn}\mu), \tag{21}$$

where the integral wave number l is given by

$$l=\mu+\tfrac{1}{2}. \tag{22}$$

Incoming plane-wave states are constructed by summing an infinite series of partial-wave functions with the appropriate constant coefficients

$$u_{\mathbf{q}}(\mathbf{r})\sim e^{-i\frac{1}{2}\phi}\sum_l i^{|l|}(2\pi)^{-1} \times\exp[i(l\phi+\delta_l+\tfrac{1}{4}\pi\,\mathrm{sgn}l)]u_q{}^l(r), \tag{23}$$

[16] Hide Yoshioka, J. Phys. Soc. Japan **21**, 948 (1966).

[17] M. L. Goldberger and K. M. Watson, *Collison Theory* (John Wiley & Sons, Inc., New York, 1964), Chap. 6, p. 324.

[18] I. S. Gradshteyn and I. M. Ryshik, *Table of Integrals Series and Products* (Academic Press Inc., New York, 1965).

where $u_q^l(r)$, δ_l, and sgn$l$ are equal to $u_q^\mu(r)$, δ_μ, and sgnμ for the value of μ prescribed by Eq. (22).

In the asymptotic region we may employ Eq. (21) to obtain an explicit representation for the wave function

$$u_{\mathbf{q}}(\mathbf{r})\sim e^{-i\frac{1}{2}\phi}(2\pi)^{-1}\times\{\sum_l i^{|l|}e^{il\phi}(2/\pi qr)^{1/2}\cos(qr-\tfrac{1}{2}|l|\pi-\tfrac{1}{4}\pi)+\exp[i(qr-\tfrac{1}{4}\pi)]r^{-1/2}f(\phi)\},\quad(24)$$

where the scattering amplitude is simply

$$f(\phi)=(2\pi q)^{-1/2}\sum_l[\exp[2i(\delta_l+\tfrac{1}{4}\pi\,\mathrm{sgn}l)]-1]e^{il\phi}=(2\pi q)^{-1/2}e^{i\frac{1}{2}\phi}\sum_\mu[\exp[2i(\delta_\mu+\tfrac{1}{4}\pi\,\mathrm{sgn}\mu)]-1]e^{i\mu\phi}.\quad(25)$$

The term $\frac{1}{4}\pi$ sgnμ which is added to the phase shift is identical with that which occurs in a study of the Bohm-Aharonov (BA) effect for a flux line of strength $c\pi/e$.[19] In a study of flux lines with an even number of quanta of circulation this contribution is absent since the wave function $u_{\mathbf{q}}(\mathbf{r})$ remains invariant under 2π rotations about the z axis.

The differential scattering cross section is equal to the absolute magnitude of the square of the scattering amplitude

$$d\sigma/d\phi=|f(\phi)|^2.\quad(26)$$

The total integrated cross section diverges as a result of the BA effect. For very small scattering angles we may ignore the phase shift δ_μ in Eq. (25) and the resulting sum is elementary. The differential scattering cross section for small angles is given below:

$$\lim_{\phi\to0}d\sigma/d\phi\simeq2/\pi q\phi^2.\quad(27)$$

The transport cross section remains finite and is evaluated from Eqs. (25) and (26). With the notation

$$\bar{\delta}_\mu=\delta_\mu+\tfrac{1}{4}\pi\,\mathrm{sgn}\mu=\tfrac{1}{4}\pi\,\mathrm{erf}(\bar{\mu}),\quad(28)$$

we have

$$\sigma_{||}=\int_{-\pi}^{\pi}d\phi\,(d\sigma/d\phi)(1-\cos\phi)=2q^{-1}\sum_\mu\sin^2(\bar{\delta}_{\mu+1}-\bar{\delta}_\mu).\quad(29)$$

The phase shift $\bar{\delta}_\mu$ is a continuous and slowly varying function of the variable μ. To a good approximation we may employ the definition of the derivative to write

$$\sigma_{||}=2q^{-1}\sum_\mu(\bar{\delta}_\mu/d_\mu)^2.\quad(30)$$

From Eq. (28) it is easy to show that

$$d\bar{\delta}_\mu/d\mu=\pi^{1/2}2^{-1}\exp(-\bar{\mu}^2)/(2q\lambda).\quad(31)$$

With this result we obtain the transport cross section

$$\sigma_{||}=\pi q^{-1}(2q\lambda)^{-1}\int_0^\infty\exp(-t^4)\,dt=\pi q^{-1}\Gamma(\tfrac{1}{4})(8q\lambda)^{-1}=1.4/q^2\lambda,\quad(32)$$

where the summation over μ is approximated by integration since the number of partial waves involved is so large. In niobium, for example, $p_F\lambda(T)$ is equal to 10^2 at $T=0$ and increases as T tends to T_c where it diverges.[14] We have the order-of-magnitude result

$$\sigma_{||}\gtrsim10^{-2}p_F^{-1},\quad(33)$$

which is too small to explain the anomaly discussed in the Introduction.

We may also obtain a measure of the asymmetry of the scattering by calculating the integrated cross section averaged over sin ϕ:

$$\sigma_\perp=\int_{-\pi}^{\pi}d\phi\,(d\sigma/d\phi)\sin\phi=q^{-1}\sum_\mu\sin(2\bar{\delta}_\mu-2\bar{\delta}_{\mu+1}).\quad(34)$$

Making the same approximations as in the calculation of the transport cross section we have

$$\sigma_\perp\simeq-2q^{-1}\sum_\mu d\bar{\delta}_\mu/d\mu=-\pi q^{-1}.\quad(35)$$

This cross section is small but finite and of the order πp_F^{-1}, which in niobium and vanadium is 12 and 7 Å, respectively.[14,15] The sign of the cross section indicates that the electrons prefer to rotate against the superfluid motion.

IV. PHASE SHIFTS

The scattering theory of the coupled set of equations constitutes a two-channel problem. In a normal metal it is well known that there are particle excitations above the Fermi surface and holes below which possess the same energy. The superconducting interaction mixes the particle-hole configurations near the Fermi surface and the physical distinction between particles and holes is lost. It remains possible, however, to talk of single-particle excitations that have wave vectors whose magnitude is in excess of or less than p_F. Superfluid motion or a local depression of the order parameter in a superconductor permits particlelike and holelike excitations to couple so that a scattering event may involve a change of channels.

In the uniform case, Eq. (5) reduces to

$$\{E+(2m)^{-1}\tau^{(3)}[d^2/dr^2+r^{-1}d/dr-\mu^2r^{-2}+p_F^2-k^2]-\Delta\tau^{(1)}\}w_q^{(0)\mu}(r)=0,\quad(36)$$

which has the two real solutions

$$w_q^{(0)\mu}(r)=\begin{pmatrix}u(E)\\v(E)\end{pmatrix}J_\mu(qr)$$

[19] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 428 (1959).

and

$$\bar{w}_q^{(0)\mu}(r)=\begin{pmatrix} v(E) \\ u(E) \end{pmatrix} J_\mu(\bar{q}r), \tag{37}$$

where

$$u(E)=\{\tfrac{1}{2}[1+(E^2-\Delta^2)^{1/2}E^{-1}]\}^{1/2}$$

and

$$v(E)=\{\tfrac{1}{2}[1-(E^2-\Delta^2)^{1/2}E^{-1}]\}^{1/2} \tag{38}$$

are the BCS coherence factors.[6] The two momenta q and $\bar{q}$ are given by

$$q^2=p_F^2-k^2+2m(E^2-\Delta^2)^{1/2}$$

and

$$\bar{q}^2=p_F^2-k^2-2m(E^2-\Delta^2)^{1/2}. \tag{39}$$

The potentials of Eqs. (7) and (8) couple the two channels and introduce a phase shift in the asymptotic wave functions. From Fetter's work one can show that the real solutions of Eq. (5) behave as follows in regions remote from the core of the vortex[20,21]:

$$w_q^\mu(r)\sim\cos\chi_\mu\eta[J_\mu(qr)\cos\delta_\mu^{(1)}-Y_\mu(qr)\sin\delta_\mu^{(1)}]+\sin\chi_\mu\bar{\eta}[J_\mu(\bar{q}r)\cos\delta_\mu^{(1)}+Y_\mu(\bar{q}r)\sin\delta_\mu^{(1)}],$$

$$\bar{w}_q^\mu(r)\sim-\sin\chi_\mu\eta[J_\mu(qr)\cos\delta_\mu^{(2)}-Y_\mu(qr)\sin\delta_\mu^{(2)}]+\cos\chi_\mu\bar{\eta}[J_\mu(\bar{q}r)\cos\delta_\mu^{(2)}+Y_\mu(\bar{q}r)\sin\delta_\mu^{(2)}]. \tag{40}$$

The normalization vector η has components $u(E)$ and $v(E)$ and

$$\bar{\eta}=\tau^{(1)}\eta. \tag{41}$$

The coupling angle χ_μ is found to be nonzero for scattering of electronic excitations by a vortex. As in the case of normal electrons we evaluate the Bessel functions for $qr\gg|\mu|$ and introduce the integral wave number l given by Eq. (22):

$$w_q^l(r)\sim(2/\pi qr)^{1/2}\eta\cos\chi_l\times\cos(qr-\tfrac{1}{2}|l|\pi-\tfrac{1}{4}\pi+\delta_l^{(1)}+\tfrac{1}{4}\pi\,\mathrm{sgn}l)+(2/\pi\bar{q}r)^{1/2}\bar{\eta}\sin\chi_l\times\cos(\bar{q}r-\tfrac{1}{2}|l|\pi-\tfrac{1}{4}\pi-\delta_l^{(1)}+\tfrac{1}{4}\pi\,\mathrm{sgn}l)$$

and

$$\bar{w}_q^l(r)\sim-(2/\pi qr)^{1/2}\eta\sin\chi_l\times\cos(qr-\tfrac{1}{2}|l|\pi-\tfrac{1}{4}\pi+\delta_l^{(2)}+\tfrac{1}{4}\pi\,\mathrm{sgn}l)+(2/\pi qr)^{1/2}\bar{\eta}\cos\chi_l\times\cos(\bar{q}r-\tfrac{1}{2}|l|\pi-\tfrac{1}{4}\pi-\delta_l^{(2)}+\tfrac{1}{4}\pi\,\mathrm{sgn}l). \tag{42}$$

Linear combinations of these two functions, $\Phi_q^l(r)$ and $\bar{\Phi}_q^l(r)$, are chosen so as to obtain new functions with incoming wave limited to a single channel. The group velocity for waves of momentum q is equal to

$$v_g=dE/dq=q(E^2-\Delta^2)^{1/2}/mE, \tag{43}$$

which is positive, whereas that for waves of momentum $\bar{q}$ is equal to

$$\bar{v}_g=dE/d\bar{q}=-\bar{q}(E^2-\Delta^2)^{1/2}/mE, \tag{44}$$

which is negative.

It follows that the waves carrying energy inward towards the z axis have the functional form $e^{-iqr}r^{-1/2}$ and $\exp(i\bar{q}r)\,r^{-1/2}$; the corresponding wave functions are given by

$$\Phi_q^l(r)=i^{|l|}\{w_q^l(r)\cos\chi_l\exp[i(\delta_l^{(1)}+\tfrac{1}{4}\pi\,\mathrm{sgn}l)]-\bar{w}_q^l(r)\sin\chi_l\exp[i(\delta_l^{(2)}+\tfrac{1}{4}\pi\,\mathrm{sgn}l)]\}\sim i^{|l|}(2/\pi qr)^{1/2}\eta\cos(qr-\tfrac{1}{2}|l|\pi-\tfrac{1}{4}\pi)-(2/\pi qr)^{1/2}\eta\exp[i(qr-\tfrac{1}{4}\pi)]i\pi T_{11}^l-(2/\pi qr)^{1/2}\bar{\eta}e^{il\pi}\exp[-i(\bar{q}r-\tfrac{1}{4}\pi)]i\pi T_{21}^l \tag{45}$$

and

$$\bar{\Phi}_q^l(r)=i^{-|l|}\{w_q^l(r)\sin\chi_l\exp[i(\delta_l^{(1)}-\tfrac{1}{4}\pi\,\mathrm{sgn}l)]+\bar{w}_q^l(r)\cos\chi_l\exp[i(\delta_l^{(2)}-\tfrac{1}{4}\pi\,\mathrm{sgn}l)]\}\sim i^{-|l|}(2/\pi\bar{q}r)^{1/2}\bar{\eta}\cos(\bar{q}r-\tfrac{1}{2}|l|\pi-\tfrac{1}{4}\pi)-(2/\pi qr)^{1/2}\eta e^{-il\pi}\exp[-i(qr-\tfrac{1}{4}\pi)]i\pi T_{12}^l-(2/\pi\bar{q}r)^{1/2}\bar{\eta}\exp[-i(\bar{q}r-\tfrac{1}{4}\pi)]i\pi T_{22}^l. \tag{46}$$

We have introduced the T matrix in Eqs. (45) and (46) without preceding its introduction by a formal definition. In the simple treatment of this paper it may be regarded simply as a symbol to abbreviate the following expressions:

$$-\pi T_{11}^l=\cos^2\chi_l\exp(i\bar{\delta}_l^{(1)})\sin\bar{\delta}_l^{(1)}+\sin^2\chi_l\exp(i\bar{\delta}_l^{(2)})\sin\bar{\delta}_l^{(2)},$$

$$-\pi T_{22}^l=\sin^2\chi_l\exp(i\delta_l'^{(1)})\sin\delta_l'^{(1)}+\cos^2\chi_l\exp(i\delta_l'^{(2)})\sin\delta_l'^{(2)},$$

and

$$-\pi T_{12}^l=-\pi T_{21}^l=\sin\chi_l\cos\chi_l[\exp(i\delta_l^{(1)})\sin\delta_l^{(1)}-\exp(i\delta_l^{(2)})\sin\delta_l^{(2)}], \tag{47}$$

where the phase shifts $\bar{\delta}_l^{(i)}$ and $\delta_l'^{(i)}$ are given by

$$\bar{\delta}_l^{(i)}=\delta_l^{(i)}+\tfrac{1}{4}\pi\,\mathrm{sgn}l \tag{48}$$

and

$$\delta_l'^{(i)}=\delta_l^{(i)}-\tfrac{1}{4}\pi\,\mathrm{sgn}l \tag{49}$$

with i running from 1 to 2.

The matrix elements T_{11}^l and T_{22}^l pertain to scattering in the q and $\bar{q}$ channels, respectively, and are affected by the BA effect. The elements T_{12}^l and T_{21}^l exist because of the coupling between channels and have no term $\tfrac{1}{4}\pi\,\mathrm{sgn}l$ present in them.

[20] A. L. Fetter, Phys. Rev. **140**, A1921 (1965).

[21] Fetter has calculated the reaction matrix for a spherical potential in an infinite superconductor. The modifications required to treat a cylindrical potential are not very great and we omit the details here.

To obtain incoming plane-wave solutions of the problem one must evaluate the right-hand side of Eq. (3), and this gives us

$$\Phi_p(x)\sim(2\pi)^{-3/2}\exp[i(kz-Et)]e^{-i\frac{1}{2}\phi}\{\eta\exp(i\mathbf{q}\cdot\mathbf{r})+\eta\exp[i(qr-\tfrac{1}{4}\pi)]f_1(\phi)r^{-1/2}+\bar{\eta}\exp[-i(\bar{q}r-\tfrac{1}{4}\pi)](q/\bar{q})^{1/2}g(\phi-\pi)r^{-1/2}\} \quad (50)$$

and

$$\bar{\Phi}_p(x)\sim(2\pi)^{-3/2}\exp[i(kz-Et)]e^{-i\frac{1}{2}\phi}\{\bar{\eta}\exp(-i\bar{\mathbf{q}}\cdot\mathbf{r})+\eta\exp[i(qr-\tfrac{1}{4}\pi)]g(\phi-\pi)r^{-1/2}+\bar{\eta}\exp[-i(\bar{q}r-\tfrac{1}{4}\pi)]f_2(\phi)r^{-1/2}\}, \quad (51)$$

where the scattering amplitudes are

$$f_1(\phi)=e^{i\frac{1}{2}\phi}(2/\pi q)^{1/2}\sum_\mu(-i\pi T_{11}{}^\mu)e^{i\mu\phi}, \quad (52)$$

$$f_2(\phi)=e^{i\frac{1}{2}\phi}(2/\pi\bar{q})^{1/2}\sum_\mu(-i\pi T_{22}{}^\mu)e^{i\mu\phi}, \quad (53)$$

and

$$g(\phi)=e^{i\frac{1}{2}\phi}(2/\pi q)^{1/2}\sum_\mu(-i\pi T_{12}{}^\mu)e^{i\mu\phi}. \quad (54)$$

The scattering cross section is equal to the radial flux of outgoing single-particle excitations per unit angle divided by the total incoming flux. The quasiparticle density is equal to the inner product of the wave function with itself:

$$\rho_n=\Psi^\dagger(x)\Psi(x) \quad (55)$$

and satisfies the conservation equation

$$\partial\rho_n/\partial t+\nabla\cdot\mathbf{j}_n=0 \quad (56)$$

with the current density

$$\mathbf{j}_n=(2im)^{-1}\{\Psi^\dagger(x)\tau^{(3)}\nabla\Psi(x)-[\nabla\Psi^\dagger(x)]\tau^{(3)}\Psi(x)-2i(e/c)\mathbf{A}(r)\Psi^\dagger(x)\Psi(x)\}. \quad (57)$$

One may verify Eq. (56) by carrying out the time derivative of ρ_n and employing Eq. (1) where necessary.

We evaluate the scattered current for the wave function of Eq. (50). The magnetic potential term may not be ignored because of the $1/r$ term, but the superfluid velocity function will make no contribution when r is sufficiently large. The result of this calculation is

$$(j_n)_r=q(E^2-\Delta^2)^{1/2}[\,|f_1(\phi)|^2+|g(\phi-\pi)|^2]/mE(2\pi)^3. \quad (58)$$

The incident flux of quasiparticles is equal to the group velocity divided by the volume of the system which in continuum normalization is $(2\pi)^3$. We have already calculated the group velocity in Eq. (43) so that the differential scattering cross section may be immediately written down:

$$d\sigma_n/d\phi=|f_1(\phi)|^2+|g(\phi-\pi)|^2. \quad (59)$$

A similar expression can be derived with the wave function of Eq. (51). However, we will have no need for it in the remainder of this paper, and we do not calculate it here.

On a length scale $p_F{}^{-1}$, the order parameter and superfluid velocity are smooth and slowly varying functions. It follows from conventional scattering theory that the amplitudes $f_1(\phi)$ and $f_2(\phi)$ are narrowly peaked in the forward direction. The same argument applies to the distribution of wave vectors $\mathbf{q}$ and $\bar{\mathbf{q}}$ for particles changing channels in a scattering event. However, the difference in sign of the group velocity for the two channels causes energy and momentum to be carried in the opposite direction from that of the incident beam. This fact is reflected in the amplitude $g(\phi-\pi)$, which is peaked about $\phi=\pi$.

We may introduce another density function ρ_e which is given by

$$\rho_e=\Psi^\dagger(x)\tau^{(3)}\Psi(x). \quad (60)$$

The subscript e is employed to denote this function since in second quantization the off-diagonal components of the corresponding operator are proportional to the electric charge density operator. The current density is

$$\mathbf{j}_e=(2im)^{-1}\{\Psi^\dagger(x)\nabla\Psi(x)-[\nabla\Psi^\dagger(x)]\Psi(x)-2i(e/c)\mathbf{A}(r)\Psi^\dagger(x)\tau^{(3)}\Psi(x)\}; \quad (61)$$

however, a conservation equation similar to Eq. (56) is not satisfied by these two functions except in special cases. In particular, eigenstates of the energy operator will satisfy the conservation equation in regions where the energy gap is a constant and where there is no superfluid motion. Since the density function ρ_e appears in calculations of transport properties it is useful at this time to introduce a cross section $d\sigma_e/d\phi$ which in a scattering event is equal to the radial flux of outgoing current j_e per unit angle divided by the total incoming flux of the same current. We evaluate the radially scattered current for the wave function of Eq. (50) and ignore interference terms between channels. These have an oscillatory dependence on the radial coordinate of the form $\exp[i(q-\bar{q})r]$ which will average out to zero over some finite length except in the low-energy limit when q equals $\bar{q}$. The radial component of the current in the region $r\to\infty$ is

$$(j_e)_r=m^{-1}(2\pi)^{-3}q[\,|f_1(\phi)|^2-|g(\phi-\pi)|^2], \quad (62)$$

while the incident flux is $(2\pi)^{-3}q/m$. The differential scattering cross section becomes

$$d\sigma_e/d\phi=|f_1(\phi)|^2-|g(\phi-\pi)|^2. \quad (63)$$

Because of the BA effect the total integrated cross sections σ_n and σ_e diverge. However, their difference remains finite and may be calculated:

$$2\sigma_s=\int_{-\pi}^{\pi}d\phi\,[d\sigma_n/d\phi-d\sigma_e/d\phi]$$

$$=2\int_{-\pi}^{\pi}d\phi\,|\,g(\phi)\,|^2. \qquad (64)$$

The transport cross sections are also finite:

$$\sigma_{n||}=\int_{-\pi}^{\pi}d\phi(d\sigma_n/d\phi)\,(1-\cos\phi)$$

$$=\int_{-\pi}^{\pi}d\phi[\,|\,f_1(\phi)\,|^2-|\,g(\phi)\,|^2](1-\cos\phi)+2\sigma_s \qquad (65)$$

and

$$\sigma_{e||}=\int_{-\pi}^{\pi}d\phi(d\sigma_e/d\phi)\,(1-\cos\phi)$$

$$=\int_{-\pi}^{\pi}d\phi[\,|\,f_1(\phi)\,|^2+|\,g(\phi)\,|^2](1-\cos\phi)-2\sigma_s. \qquad (66)$$

We have shown that in the high-energy limit $E\gg\Delta$, the transport cross section is very small. The first term on the right-hand side of the lower of Eqs. (65) and (66) is of the order p_F^{-1} or less for all E, while the cross section σ_s is very large in the low-energy limit $E\gtrsim\Delta$, as we shall show in the next section. To a good approximation therefore,

$$\sigma_{n||}\simeq-\sigma_{e||}\simeq2\sigma_s. \qquad (67)$$

The appearance of this rather large cross section σ_s for excitations to change channels is strictly a superconducting effect which does not occur when $\Delta=0$ or $E\gg\Delta$.

V. CALCULATIONS

Up to now only the scattering theory of superconductivity has been discussed and the particular differential equation introduced in Sec. II remains to be solved. We begin the solution of Eq. (5) by introducing the change in dependent variable

$$R_q^{\mu}(r)=r^{1/2}\Phi_q^{\mu}(r) \qquad (68)$$

so that the wave equation contains no first derivative in r:

$$\{E-\mu v_s(r)/r-\Delta(r)\tau^{(1)}+(2m)^{-1}\tau^{(3)}[d^2/dr^2+S^2(r)]\}R_q^{\mu}(r)=0, \qquad (69)$$

where the square of the coordinate-dependent wave vector is

$$S^2(r)=p_F^2-k^2-\mu^2r^{-2}+\tfrac{1}{4}r^{-2}-(mv_s)^2. \qquad (70)$$

We assume a real solution of the form

$$R_q^{\mu}(r)=S(r)^{-1/2}[g(r)e^{i\chi}+g^*(r)e^{-i\chi}], \qquad (71)$$

with

$$\chi=\int_{r_0}^{r}dr\,S(r)+\tfrac{1}{4}\pi \qquad (72)$$

and $r>r_0$. In the WKBJ approximation the exponential functions of Eq. (71) are solutions of Eq. (69) in the absence of the first three terms in the curly bracket. To include these we introduce the envelope vector $g(r)$ which we assume is slowly varying compared to χ. With this assumption in mind Eq. (71) is substituted into Eq. (69) and only first derivatives of g are kept so that one arrives at the simplified result[22]

$$[-iS(r)/m]\tau^{(3)}(dg/dr)=(E-\mu v_s/r)g-\Delta(r)\tau^{(1)}g. \qquad (73)$$

If g is a solution of Eq. (73), $\pm\tau^{(1)}g^*$ is still another and we may express the components of g in terms of the complex function $a(r)$

$$g(r)=\begin{pmatrix}a(r)\\ \pm a(r)^*\end{pmatrix}. \qquad (74)$$

[22] See Ref. 8, Eq. (I21).

For the positive sign in Eq. (74) the real and imaginary components of a satisfy the equations

$$-m^{-1}S(r)da_r^{(1)}/dr=[E-\mu v_sr^{-1}+\Delta(r)]a_i^{(1)} \qquad (75)$$

and

$$m^{-1}S(r)da_i^{(1)}/dr=[E-\mu v_sr^{-1}-\Delta(r)]a_r^{(1)}, \qquad (76)$$

where

$$a^{(1)}(r)=a_r^{(1)}(r)+ia_i^{(1)}(r). \qquad (77)$$

We shall use the superscripts (1) and (2) to denote the functions $a(r)$ that acquire a positive or negative sign in Eq. (74). Since the wave function must be finite at the origin one must impose the condition that $a(r)$ be real at r_0 as required by the WKBJ theory of a linear turning point.

To construct $a^{(2)}(r)$ we replace the superscript (1) in Eqs. (75) and (76) with (2) and introduce the new boundary condition

$$a_r^{(2)}(r_0)=0. \qquad (78)$$

We proceed to carry out the sum

$$a^{(2)}(r)=-a_i^{(2)}(r)+ia_r^{(2)}(r) \qquad (79)$$

after solving the equations.

Some exact consequences of Eqs. (75) and (76) are derived by studying the solutions at large r. In the limit $r\rightarrow\infty$, the superfluid velocity function vanishes and the order parameter equals its value in the field-free state. We may easily solve the differential equation

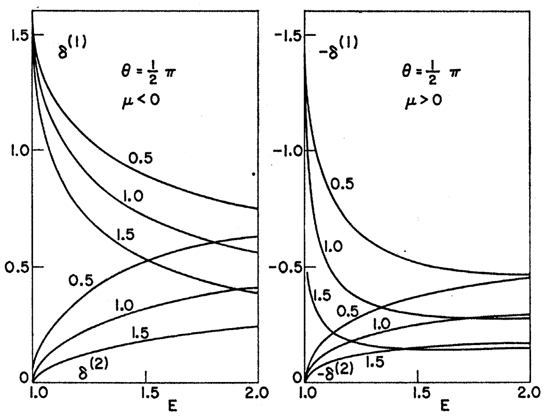

FIG. 2. The phase shifts $\delta_\mu^{(1)}$ and $\delta_\mu^{(2)}$ as a function of energy in units of the order parameter $\Delta(T)$ for $|\mu|/p_F\lambda(T)$ equal to 0.5, 1.0, and 1.5 at $\theta=\frac{1}{2}\pi$, where μ is the partial-wave index.

for the real and imaginary parts of $a^{(1)}(r)$ to obtain

$$a_i^{(1)}=\bar{a}^{(1)}\sin[\zeta(r)+\delta_\mu^{(1)}]$$

and

$$a_r^{(1)}=\bar{a}^{(1)}[(E+\Delta)/(E-\Delta)]^{1/2}\cos[\zeta(r)+\delta_\mu^{(1)}], \quad (80)$$

where

$$\zeta(r)=m\int_{r_0}^{r}(E^2-\Delta^2)^{1/2}S(r)^{-1}dr \quad (81)$$

and $\bar{a}^{(1)}$ is a real constant. In the Meissner state the phase shift $\delta_\mu^{(1)}$ must vanish and Eqs. (80) and (81) become exact for all $r>r_0$. In general $\delta_\mu^{(1)}$ will not be zero and as E tends to Δ the phase shift $\delta_\mu^{(1)}$ must approach $\pm\frac{1}{2}\pi$ for all μ in order that Eq. (81) may remain finite. The complex function $a^{(1)}(r)$ is calculated with the help of Eqs. (77), (80), and (81)

$$a^{(1)}(r)=\bar{a}^{(1)}[E/(E-\Delta)]^{1/2}\{u(E)\exp[i(\zeta(r)+\delta_\mu^{(1)})] + v(E)\exp[-i(\zeta(r)+\delta_\mu^{(1)})]\}. \quad (82)$$

Combining Eqs. (71) and (82) and comparing with Eq. (40) we see that χ_μ is equal to $\frac{1}{4}\pi$. In the high-energy limit Eqs. (80) and (81) are valid for all r and one may rederive the WKBJ results of Sec. II.

In calculating $a^{(2)}(r)$ for large r proper consideration must be paid to Eqs. (78) and (79).

$$a_r^{(2)}(r)=\bar{a}^{(2)}[(E+\Delta)/(E-\Delta)]^{1/2}\sin[\zeta(r)+\delta_\mu^{(2)}], \quad (83)$$

and

$$a_i^{(2)}(r)=-\bar{a}^{(2)}\cos[\zeta(r)+\delta_\mu^{(2)}], \quad (84)$$

where $\bar{a}^{(2)}$ is a real constant. As E tends to Δ the phase shift $\delta_\mu^{(2)}$ must vanish in order that $a_r^{(2)}(r)$ may remain finite. The complex function $a^{(2)}(r)$ is constructed and exhibited on the following line:

$$a^{(2)}(r)=-\bar{a}^{(2)}[E/(E-\Delta)]^{1/2} \times\{-u(E)\exp[i(\zeta(r)+\delta_\mu^{(2)})] + v(E)\exp[-i(\zeta(r)+\delta_\mu^{(2)})]\}. \quad (85)$$

With $a^{(2)}(r)$ we may construct the second wave function of Eq. (40) with the coupling angle $\chi=\frac{1}{4}\pi$.

Unlike the coupling angle χ_μ the phase shifts $\delta_\mu^{(1)}$ and $\delta_\mu^{(2)}$ are rapidly varying functions of the energy E and the partial wave number μ in the range $E\gtrsim\Delta$. The quasiparticle momentum, however, may be restricted to the value p_F, and we write

$$k_F=p_F\sin\theta, \quad (86)$$

where θ is the polar angle of the trajectory of the particles with respect to the external field.

We have solved Eqs. (75) and (76) on the IBM 360/67 digital computer at the Stanford Computation Center. The goal of our first computational effort was to plot the phase shifts $\delta_\mu^{(1)}$ and $\delta_\mu^{(2)}$ as a function of energy for typical values of μ and θ. Some results are plotted in Figs. 2 and 3 where energy is measured in units of $\Delta(T)$. We employ[23]

$$\Delta(T)=3.2T_c(1-T/T_c)^{1/2} \quad (87)$$

and

$$\lambda(T)=0.52\xi_0(1-T/T_c)^{-1/2}, \quad (88)$$

where $\xi_0=v_F/\pi\Delta$ is the coherence length.

A peculiar resonance is observed in all of the phase shifts in the low-energy limit prescribed by $E=\Delta$. In particular the phase shift $\delta_\mu^{(1)}$ is equal to $-\frac{1}{2}\pi\,\mathrm{sgn}\mu$ for all partial-wave numbers μ and polar angles θ, while the other $\delta_\mu^{(2)}$ is identically zero. In Schrödinger theory, a resonant phase shift must be attributed to the characteristics of the scattering potential. In the Bogoliubov problem the resonance previously described exists for a whole class of potentials. The strength and spatial extent of the potentials does not determine the position of the resonance in the energy spectrum but only its width. The group velocity of the waves vanishes identically in the low-energy limit and the medium itself contributes to the resonance.

In the high-energy limit the two phase shifts merge

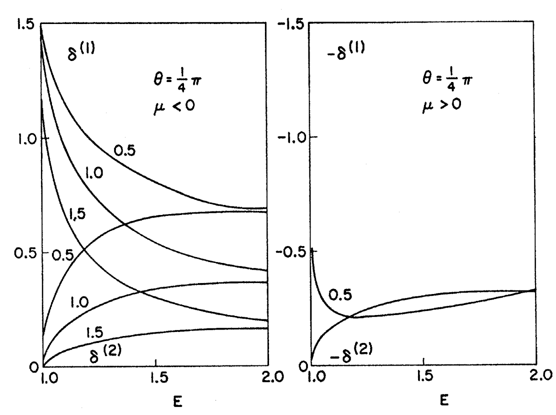

FIG. 3. The phase shifts $\delta_\mu^{(1)}$ and $\delta_\mu^{(2)}$ as a function of energy in units of the order parameter $\Delta(T)$ for $|\mu|/p_F\lambda(T)$ equal to 0.5, 1.0, and 1.5 at $\theta=\frac{1}{2}\pi$, where μ is the partial-wave index.

[23] See Ref. 10, Chaps. 4 and 6.

and the scattering reduces to the one-channel problem examined in Sec. III. One may also observe an asymmetry in the phase shift for positive and negative values of μ. It is known that a low-lying branch in the bound-state spectrum exists only for negative values of μ.[8] Here, the width of the resonance is larger for $\mu<0$.[24]

A better understanding of the results illustrated in Figs. 2 and 3 may be obtained by solving Eqs. (75) and (76) in the WKBJ approximation where we have

$$a_i^{(1)}\simeq\bar{a}^{(1)}\sin\left\{m\int_{r0}^{r}[(E-\mu v_s r^{-1})^2-\Delta(r)^2]^{1/2}S(r)^{-1}dr\right\} \tag{89}$$

and

$$a_r^{(1)}\simeq\bar{a}^{(1)}[(E-\mu v_s r^{-1}+\Delta(r))/(E-\mu v_s r^{-1}-\Delta(r))]^{1/2}\cos\left\{m\int_{r0}^{r}[(E-\mu v_s r^{-1})^2-\Delta(r)^2]^{1/2}S(r)^{-1}dr\right\}, \tag{90}$$

with

$$\bar{a}^{(1)}=[(E-\mu v_s r^{-1})^2-\Delta(r)^2]^{-1/4}. \tag{91}$$

For negative values of μ the square root in the integrand of Eqs. (89) and (90) is real and slowly varying for all r in the range r_0 to ∞. When E approaches Δ the approximation breaks down because of the turning point which exists at large r for $E=\Delta$. With positive values of μ the square-root function is imaginary between the points

$$E-\mu v_s r^{-1}=\pm\Delta(r), \tag{92}$$

and the wave function must be modified over this region. The effect only exists for nonzero $\Delta(r)$ and results in a large modification of the phase shifts for $E\simeq\Delta$. Since the quasiparticles must tunnel through the region specified by Eq. (92) its amplitude and phase at infinity are sensitive to the parameters which characterize the barrier. In particular the dependence of $\delta_\mu^{(1)}-\delta_\mu^{(2)}$ on θ is quite rapid, as can be seen in Figs. 2 and 3.

We have calculated $\sigma_s(E,\theta)$ by evaluating the phase shifts and performing the necessary summation over partial-wave index μ. The results are plotted in Fig. 4 with σ_s in units of $\lambda(T)\sin\theta$ versus energy in units of $\Delta(T)$ for a range of $\sin\theta$ between 1 and 0.2. The estimated error of the program was 1% although the actual error was possibly 5%. The cross section is a rapidly varying function of E and θ, decreasing to zero for large E and small θ; it diverges when $E=\Delta$ since all the phase shifts are equal to $\pm\frac{1}{2}\pi$. From our numerical calculations we conclude however that this divergence is only logarithmic. A divergence of this kind has been encountered by Fetter[20] in a study of scattering by a spherical δ-shell potential.

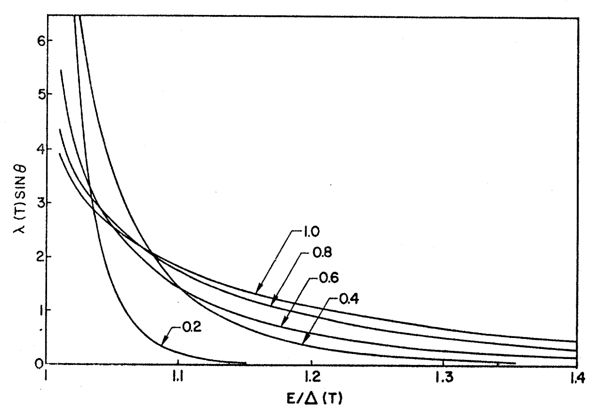

FIG. 4. The cross section $\sigma_s(E,\theta)$ in units of $\lambda(T)\sin\theta$ for scattering particles to change channel as a function of energy in units of the order parameter $\Delta(T)$ for $\sin\theta$ ranging from 1.0 to 0.2 in steps of 0.2.

The cross section σ_s is very large in the range of energies $E\simeq\Delta(T)$. As T tends to T_c this range shrinks to zero, but σ_s diverges like $\lambda(T)$. As a result the thermally averaged cross section will remain finite and nonzero even at $T=T_c$. At temperatures below T_c increased weighting of the low-energy part of the curve will cause the cross section to increase.

It follows from Eqs. (47), (54), and (64) that $\sigma_s(E,\theta)$ is the total scattering cross section for an incoming particle to change channels. The coupling mechanism between particlelike and holelike states of the same energy appears only when Δ is finite and will vanish for normal electrons. It is not possible, therefore, to calculate such a cross section by the analysis of a simple Schrödinger equation of the type studied in Sec. III.

Cross sections like $\sigma_s(E,\theta)$ are not new to superconductivity theory but have been encountered previously in the study of paramagnetic alloys[25] and of current carrying films containing spinless impurities.[26] In all these cases there is the introduction of an interaction into the Hamiltonian which violates the time-reversal invariance of the electrons. As a result, superconducting pairs are broken by the scattering potential which in the case of a vortex results in a region of gapless superconductivity of diameter $\xi(T)$ near the core.

[24] Caroli and Matricon point out that the sense of rotation of the superfluid motion of a vortex fascilitates the formation of bound states for excitations rotating in the opposite direction. In a similar vein, we expect the excitations rotating with the superfluid to experience greater scattering because of the extra energy needed for rotation in that direction.

[25] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **39**, 1781 (1960) [English transl.: Soviet Phys.—JETP **12**, 1243 (1961)].

[26] K. Maki, Progr. Theoret. Phys. (Kyoto) **29**, 603 (1963).

The potentials of a vortex are large and we have not employed the Born approximation as is usually done for scattering by atomic potentials. We have also examined a model for a vortex in which the depression in $\Delta(r)$ has a square-well appearance and the superfluid velocity is limited to the confines of the well. Such a model can be solved analytically and the results may be compared with the more realistic calculations presented here. The bound-state spectrum agrees qualitatively with that of Caroli, de Gennes, and Matricon.[7,8]

Forgan and Gough have estimated a vortex cross section of 40 Å from their measurements of the anomaly in the ultrasonic attenuation near H_{c1}.[1] More recently, Sinclair and Leibowitz have observed a much larger effect in vanadium which suggests a cross section of 200 Å near T_c for the attenuation of longitudinal sound waves propogating in the direction of the external field.[27] In order to compare calculations with experiment it is necessary to obtain a transport theory of superconductors in the presence of vortices. We are carrying out such a program and hope to publish the results soon.

ACKNOWLEDGMENTS

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[27] A. C. E. Sinclair and J. Leibowitz Phys. Rev. **175**, 596 (1968).

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Ultrasonic Investigation of the Isolated-Vortex State near H_{c1} in Vanadium*

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Ultrasonic attenuation in the mixed state of vanadium has been investigated near H_{c1}, where the vortex density is low. The expected rise toward the normal-state value of attenuation at H_{c2} is preceded by an initial absorption decrease as the vortices penetrate. The unexpected dip is interpreted in terms of a reduction in the quasiparticle mean free path, a result of the scattering of quasiparticles by vortices. The scattering width of a vortex was found to be $(2.4\pm0.4)\times10^{-6}$ cm for temperatures near T_c, in good agreement with the theoretical determination 2.1×10^{-6} cm. The apparent width decreases at lower temperatures. Similar measurements have been made in niobium. The scattering width is found to be considerably greater in vanadium, and its temperature dependence is in the opposite direction. A further feature of the present results observed even at rather low fields was the faster-than-linear dependence of the reciprocal attenuation on the internal field.

THE attenuation of ultrasonic waves in a pure type-II superconductor near H_{c1} shows an unexpected decrease as the field initially penetrates. This was first demonstrated in experiments in niobium.[1,2] We believe that the attenuation provides a sensitive test for the vortex properties, and present here results of a detailed investigation of the field dependence of the attenuation. This behavior shows a systematic trend with temperature.

The vanadium sample was the same one as that used by Radebaugh and Keesom[3] for their demonstration of the intrinsic type-II behavior of vanadium. Longitudinal 150-Mc/sec sound waves were propagated axially through the central core (diam 0.1 cm) of the cylindrical sample (length 1.9 cm, diam 0.7 cm). A magnetic field was applied along the sample axis and swept linearly in time. The voltage appearing across a short coil (length 0.3 cm), wound centrally on the sample, was integrated to give the total flux within the coil. With a correction for the flux within the air gap between coil and sample, the average magnetic field within the central region of the sample was determined from measurements in the normal and mixed states.

The magnetization through the sample was not expected to be constant since the demagnetizing coefficient for a cylindrical sample is not uniform. This consideration led us to propagate the sound wave only through the central core of the sample, so that the attenuation level was determined only by the field within this core. Thus, the effect of the nonuniformity in

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[3] R. Radebaugh and P. H. Keesom, Phys. Rev. **149**, 209 (1966); **149**, 217 (1966). The reader is referred to these papers for a complete description of the superconducting properties of vanadium, and of this particular sample.