dc Josephson Effect for Strong-Coupling Superconductors

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The maximum zero-voltage (Josephson) current I_s in identical-superconductor tunnel junctions is shown to be related to the response of the bulk superconductor to ac fields. Strong-coupling effects reduce I_s to 78.8% of the weak-coupling prediction for Pb and to 91.1% of that for Sn, corresponding to similar strong-coupling reductions in the bulk supercurrent response of these materials.

In this communication we point out that a close connection exists between the magnitude of the maximum zero-voltage (Josephson) current I_s in a superconductor-insulator-superconductor tunnel junction and the supercurrent response of the bulk superconductors to ac fields. We find that for junctions fabricated from a single superconducting material,

$$I_s=(1/2eR_N)\lim_{\omega\to0^+}\omega\sigma_2(\omega)/\sigma_N, \tag{1}$$

where R_N is the normal-state junction resistance and where $\sigma_2(\omega)/\sigma_N$ is the imaginary part of the normalized conductivity of the bulk superconductor at energy ω in the extreme anomalous limit. Equation (1) is valid in the strong-coupling theory of superconductors at zero and finite temperatures. For BCS or weak-coupling superconductors in which the gap parameter $\Delta(\omega)$ is a real constant Δ, Eq. (1) agrees with the weak-coupling expression for I_s

$$I_{sw}=(\pi\Delta/2eR_N)\tanh(\Delta/2kT), \tag{2}$$

previously derived by Ambegaokar and Baratoff.[1] We have calculated I_s in the strong-coupling theory using values of $\Delta(\omega)$ determined from single-particle tunneling data[2] and find I_s is less than I_{sw} for Pb–Pb and Sn–Sn junctions. These results are consistent with direct measurements of I_s,[3,4] and they are in good agreement through Eq. (1) with recent work on the bulk conductivity.[5–7]

A general expression for I_s has been derived by Ambegaokar and Baratoff.[1] For identical strong-coupling superconductors it can be put in the form[8]

$$I_s=\frac{2}{\pi eR_N}\int_0^\infty d\omega_1 d\omega_2\left\{\frac{f(\omega_1)-f(\omega_2)}{\omega_1-\omega_2}+\frac{1-f(\omega_1)-f(\omega_2)}{\omega_1+\omega_2}\right\}$$
$$\times\mathrm{Re}\left\{\frac{\Delta(\omega_1)}{[\omega_1^2-\Delta^2(\omega_1)]^{1/2}}\right\}\mathrm{Re}\left\{\frac{\Delta(\omega_2)}{[\omega_2^2-\Delta^2(\omega_2)]^{1/2}}\right\}, \tag{3}$$

where $f(\omega)=[\exp(\omega/kT)+1]^{-1}$ is the Fermi function and where here and in all that follows $\Delta(\omega)$ for ω real means $\Delta(\omega+i\epsilon)$, $\epsilon=0^+$. Expressing $f(\omega)$ as a sum over its poles, transforming the integrals to contour integrals in the complex plane, and using standard techniques of many-body theory, we can transform Eq. (3) to

$$I_s=(\pi\Delta_0/2eR_N)[\tanh(\Delta_0/2kT)]/[1-\Delta_1'(\Delta_0)]$$
$$-(eR_N)^{-1}\int_{\Delta_0^+}^\infty d\Omega\tanh\frac{\Omega}{2kT}\mathrm{Im}\left\{\frac{\Delta^2(\Omega)}{\Omega^2-\Delta^2(\Omega)}\right\}, \tag{4}$$

which is useful for deriving (1) and more convenient than (3) for numerical computation. Here $\Delta_1(\Omega)=\mathrm{Re}[\Delta(\Omega)]$, $\Delta_1'(\Omega)=d\Delta_1(\Omega)/d\Omega$, and Δ_0 is that value of Ω for which $\Omega=\Delta_1(\Omega)$. In deriving (4), we used the fact that for physical systems $|\Delta_2(\Delta_0)|\ll\Delta_0$, where $\Delta_2(\Omega)=\mathrm{Im}[\Delta(\Omega)]$. The lower limit Δ_0^+ of the integral in (4) is meant to exclude contributions from the pole at $\Omega=\Delta_0$. In the weak-coupling limit for which $\Delta_1(\Omega)=\Delta$ and $\Delta_2(\Omega)=0$, Eq. (4) correctly reduces to (2).

For the superconductors Pb and Sn at $T=0$°K numerical evaluation of (4) using the $\Delta(\Omega)$ data of McMillan and Rowell[2] gives $I_s/I_{sw}=0.788$ for Pb–Pb junctions and 0.911 for Sn–Sn junctions, where I_{sw} is computed from $\Delta_0=1.40$ and 0.61 meV, respectively.[9] The observed maximum Josephson current is often reduced below the ideal I_s by extraneous factors such

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[9] These gap values satisfy $\Delta_0=\Delta_1(\Delta_0)$, where $\Delta(\omega)$ is adjusted to fit the structure in the single-particle tunneling current above the gap and is accurate to approximately 1%. These values of Δ_0 correspond to the midpoint of the rise in tunneling current, a choice required to provide a good fit to the conductance data for voltages just above the gap, and to provide a reasonable value for the Coulomb pseudopotential. The value $\Delta_0=1.34$ meV sometimes quoted for Pb corresponds to the first rise in current and is definitely too small to meet either of these conditions. Cf. Ref. 2.

as noise and trapped flux; however, the largest reported values of I_s/I_{sw} for Sn–Sn junctions approximate the strong-coupling prediction.[3] The largest reported value for Pb–Pb junctions is 0.6,[4] substantially less than the strong-coupling prediction, but good-quality Pb–Pb junctions are notoriously more difficult to make than Sn–Sn junctions.

The result (1) follows directly from a comparison of Eq. (4) with an expression derived by Nam for the imaginary part of the normalized bulk conductivity of a strong-coupling superconductor in the extreme anomalous limit (coherence length $\xi_0 \gg$ wavelength λ).[8] In our notation and with our assumption $|\Delta_2(\Delta_0)| \ll \Delta_0$ this can be written for small positive ω as

$$\begin{aligned}\omega\sigma_2(\omega)/\sigma_N &= \int_{\Delta_0-\omega}^{\Delta_0} d\Omega \left\{ \frac{\Delta_1(\omega+\Omega)\Delta_1(\Omega)+(\omega+\Omega)\Omega}{[(\omega+\Omega)^2-\Delta_1^2(\omega+\Omega)]^{1/2}[\Delta_1^2(\Omega)-\Omega^2]^{1/2}} \right\} \\ &\qquad \times \tanh[(\Omega+\omega)/2kT] \\ &\quad -\int_{\Delta_0^+}^{\infty} d\Omega \{ [n_1(\omega+\Omega)n_2(\Omega)+p_1(\omega+\Omega)p_2(\Omega)] \\ &\qquad \times \tanh[(\Omega+\omega)/2kT] \\ &\quad +[n_1(\Omega)n_2(\omega+\Omega)+p_1(\Omega)p_2(\omega+\Omega)] \\ &\qquad \times \tanh[\Omega/2kT] \}, \end{aligned} \tag{5}$$

where

$$n_1(\Omega)+in_2(\Omega)=\Omega/[\Omega^2-\Delta^2(\Omega)]^{1/2},$$

$$p_1(\Omega)+ip_2(\Omega)=\Delta(\Omega)/[\Omega^2-\Delta^2(\Omega)]^{1/2}.$$

The integrand of the second term reduces for $\omega \to 0$ to twice the integrand of the integral in (4). In the same limit the first term of (5) becomes

$$\frac{\Delta_0 \tanh(\Delta_0/2kT)}{[1-\Delta_1'(\Delta_0)]} \int_{\Delta_0-\omega}^{\Delta_0} \frac{d\Omega}{[(\omega+\Omega-\Delta_0)(\Delta_0-\Omega)]^{1/2}} = \frac{\pi\Delta_0 \tanh(\Delta_0/2kT)}{1-\Delta_1'(\Delta_0)},$$

which is the result required to establish Eq. (1).[10] This simple connection between the tunneling and bulk supercurrents has apparently not previously been recognized, although the weak-coupling expressions (2) and $\sigma_2(\omega)/\sigma_N = (\pi\Delta/\omega)\tanh(\Delta/2kT)$ are well known.[1,11]

It follows from (1) and our calculation of I_s/I_{sw} that the low-frequency bulk value of $\sigma_2(\omega)/\sigma_N$ at $T=0°\mathrm{K}$ should be $0.788\pi\Delta_0/\omega$ for Pb and $0.911\pi\Delta_0/\omega$ for Sn. Such a reduction below the weak-coupling value $\pi\Delta_0/\omega$ has been observed by Palmer and Tinkham for Pb in far-infrared transmission experiments.[5] At low frequencies they find a 25% reduction, which they attribute to strong-coupling effects.

Using an approximate $\Delta(\omega)$ derived by Scalapino, Schrieffer, and Wilkins,[12] Nam has calculated $\omega\sigma_2(\omega)/\sigma_N$ directly and finds a 26% reduction.[1] More recently, Shaw and Swihart have evaluated the real part $\sigma_1(\omega)/\sigma_N$ of the normalized bulk conductivity for Pb and Sn using values for the electron-phonon interaction function $\alpha^2F(\omega)$ derived from tunneling measurements.[7,9,13] From the Ferrell-Glover-Tinkham sum rule[14]

$$\int_0^{\infty} d\Omega [1-\sigma_1(\Omega)/\sigma_N] = \lim_{\omega\to 0} \tfrac{1}{2}\pi\omega\sigma_2(\omega)/\sigma_N, \tag{6}$$

they estimate that $\omega\sigma_2(\omega)/\sigma_N$ is reduced to 79.7% and 91.6% of the weak-coupling values for Pb and Sn, respectively. These theoretical and experimental results are all in satisfactory agreement.

We wish to thank J. M. Rowell for supplying us with accurate values for $\Delta(\omega)$ and for insight into acceptable choices for Δ_0.

[10] The assumption $|\Delta_2(\Delta_0)| \ll \Delta_0$ is not essential to Eq. (1) but is used to avoid mathematical obscurity and to obtain a form (4) useful for numerical computation. Equation (1) can be established for general $\Delta_2(\omega)$ by expressing both I_s and $\sigma_2(\omega)/\sigma_N$ in terms of contour integrals in the complex plane.

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