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Theory of a Single-Mode Gas Laser

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A gas laser oscillating steadily in a single mode is described theoretically using the "generalized Bloch equations" as the point of departure. The nonlinear susceptibility is calculated without utilizing perturbation techniques, so that the theory retains validity for laser fields of arbitrary intensity. Expressions are presented for (1) the "gain profile" which exhibits the phenomenon of dual "hole burning," (2) the power-tuning characteristics, including the conditions for the appearance of the central tuning dip, and (3) the frequency of oscillation, in which are displayed the effects of mode pulling and pushing. Emphasis is on a strongly inhomogeneous line and a high-Q cavity, although some more general results are included as well. For excitations only slightly above threshold, a comparison is made with the Lamb single-mode theory, and in all cases agreement is found.

I. INTRODUCTION

THE behavior of a laser oscillator for which the active material is a gas is characterized in part by a few very simple but important physical features, namely, (a) the cavity supports standing wave fields, (b) atoms are in motion through these fields, and (c) there is a distribution of atomic velocities. Because a standing wave is equivalent to the superposition of two oppositely directed traveling waves, an atom moving through a monochromatic standing wave field sees two Doppler-shifted frequency components in its own rest frame. Consequently, even for a single mode of oscillation the atomic response is to a two-frequency excitation. Many of the unique properties which characterize gas lasers can be traced to this aspect of the interaction between the atom and the electromagnetic field.

In this present paper, this interaction will be handled through the Schrödinger equation, but interactions among atoms (e.g., collisions), the pumping mechanism, and decays will all be treated phenomenologically through the "generalized Bloch equations." Solutions to the Bloch equations for a single-mode laser can be obtained without resort to perturbation theory.

The relevance of the Bloch equations to a description of the laser has been recognized previously by other workers,[1] and in fact some of the NMR terminology is already well established in laser physics. What will be done here will be to incorporate the interaction between an atom and a single-mode standing wave field into the Bloch formalism and to show how this leads to a quantitative description of the laser. As background material we discuss in Secs. II and III the over-all formalism and the equations of motion. Steady-state solutions are derived in Sec. IV.

An important feature of the solutions is the appearance of a saturation factor in which are manifest terms describing the atomic interaction with both Doppler-shifted frequency components. This feature receives further attention in Sec. V, where the gain profile is calculated explicitly and the behavior of the two holes is discussed in detail. A calculation of the nonlinear susceptibility as a function of intensity and frequency is shown to lead to a determination of (a) the power-tuning characteristic, including the conditions for the appearance of the central tuning dip (Sec. VI), and (b) expressions for the actual frequency of oscillation (Sec. VII). Finally, a comparison is made in Sec. VIII with the Lamb theory for the special case of weak fields.

II. FORMALISM

In a gas laser which is operating as an oscillator, an active material with an inverted population density provides gain sufficient to overcome losses while a pair of mirrors at each end of the cavity acts as a feedback mechanism. For a wave which propagates as $\exp(-j\beta z)$, where β is the (complex) propagation constant, the condition which must be satisfied in order that oscillations be sustained is that a wave exactly reproduce itself after one complete pass. Assuming a cavity of length L and end mirrors having reflection coefficients r_1 and r_2, the condition for steady-state oscillation can be expressed as

$$r_1r_2\exp(-2j\beta L)=1. \tag{1}$$

The propagation constant β will be evaluated in terms of the susceptibility $\chi(\omega)$; the latter is defined according to

$$P(\omega)=\chi(\omega)E(\omega), \tag{2}$$

where $E(\omega)$ and $P(\omega)$ are the amplitudes of an applied electric field and an induced atomic polarization, respectively:

$$E(z,t)=\tfrac{1}{2}E(\omega)\exp[j(\omega t-\beta z)]+\text{c.c.},$$

$$P(z,t)=\tfrac{1}{2}P(\omega)\exp[j(\omega t-\beta z)]+\text{c.c.}$$

(we ignore the vector character of the electric field). The susceptibility $\chi(\omega)$ is a complex function, reflecting the fact that $P(z,t)$ and $E(z,t)$ will not be in phase. The plane wave field $E(z,t)$ must satisfy the wave equation

$$\nabla^2E-\epsilon\mu_0(\partial^2E/\partial t^2)=0,$$

in which $\epsilon=\epsilon_0(1+\chi)$ and $\epsilon_0\mu_0=c^{-2}$. As a result we

[1] R. P. Feynman, F. L. Vernon, and R. W. Hellwarth, J. Appl Phys. **28**, 49 (1957); Y. Pao, J. Opt. Soc. Am. **52**, 871 (1962).

find that

$$\beta^2(\omega)=\omega^2[1+\chi(\omega)]c^{-2}.$$

We modify this expression slightly through the addition of a loss term $-j\alpha_0$ to account phenomenologically for losses between the mirrors (e.g., diffraction, scattering); thus we write

$$\beta(\omega)=\beta_0[1+\tfrac{1}{2}\chi(\omega)]-j\alpha_0, \quad (3)$$

where $\beta_0=\omega/c$ is the free space value of β, and we have in addition assumed that $\chi\ll 1$. (Typically χ is of order 10^{-8}. As will be seen shortly, $\chi\sim 1/Q_{\mathrm{cav}}$, and cavity Q's can easily be of order 10^8.)

If we separate χ into its real and imaginary parts according to

$$\chi(\omega)=\chi'(\omega)+j\chi''(\omega),$$

then using the form (3) for $\beta(\omega)$ we require that the condition (1) be satisfied in both magnitude and phase; hence

$$r_1r_2\exp\{[\beta_0\chi''(\omega)-2\alpha_0]L\}=1, \quad (4a)$$

$$\exp\{-2j\beta_0L[1+\tfrac{1}{2}\chi'(\omega)]\}=1. \quad (4b)$$

The condition (4a) can be written equivalently as

$$\chi''(\omega)=1/Q_c,$$

$$[\mathrm{GAIN}]=[\mathrm{LOSS}], \quad (5)$$

where

$$Q_c\equiv(2\pi L/\lambda_0)(2\alpha_0L-\ln r_1r_2)^{-1} \quad (6)$$

and

$$\beta_0=2\pi/\lambda_0,$$

where λ_0 is the wavelength in vacuo. Q_c is termed the "cold cavity Q." In Eq. (5), the physical interpretation is, as indicated, that the gain is equal to the loss when the system is in oscillation. In general the gain $\chi''(\omega)$ is dependent on the field strength (i.e., we have a nonlinear response). If this dependence is known explicitly, Eq. (5) then allows a determination of intensity as a function of oscillation frequency. We note that in a linear theory, for which $\chi''(\omega)$ does not depend upon intensity, no such determination is possible.

The phase condition (4b) leads to an equation which determines the actual frequency of oscillation. If ω_c is the resonant frequency for an empty cavity, satisfying

$$2\omega_cL/c=2\pi q \qquad (q \text{ integer}), \quad (7)$$

then Eq. (4b) can be expressed in the form[2]

$$\omega[1+\tfrac{1}{2}\chi'(\omega)]=\omega_c, \quad (8)$$

where ω is the resonant frequency of mode q in the presence of the gain material. In general the solutions for ω differ from ω_c, the resultant shifts being referred to as "mode pulling and pushing" effects. The derivation of Eq. (8), it should be remembered, is based on the assumption of plane waves for the propagating waveform.

As should be evident now, the procedure will be to calculate the complex susceptibility $\chi(\omega)$ so that its real and imaginary parts can be used in conjunction with Eqs. (5) and (8). This calculation is the subject of the following sections.

III. EQUATIONS OF MOTION

In order to calculate the complex susceptibility $\chi(\omega)$, we assume that an atom is in the presence of a prescribed electric field and ask for the induced atomic polarization. The susceptibility is then identified by comparison with Eq. (2).

We shall represent the atom as an isolated two-level system and postpone temporarily the questions of decays, collisions, and pumping. The state vector $|s\rangle$ for a two-level system can be expanded in the form

$$|s\rangle=\exp(-jW_1t)b_1(t)\,|1\rangle+\exp(-jW_2t)b_2(t)\,|2\rangle, \quad (9)$$

where states $|1\rangle$ and $|2\rangle$ are eigenstates of a Hamiltonian $\hat{H}_0$ with energy eigenvalues W_1 and W_2:

$$\hat{H}_0\,|m\rangle=W_m\,|m\rangle, \qquad m=1,2. \quad (10)$$

In the presence of an electromagnetic field, the amplitude coefficients $b_m(t)$ are governed by a Hamiltonian $\hat{H}$:

$$\hat{H}=\hat{H}_0+\hat{H}_1, \quad (11)$$

where $\hat{H}_1$ represents the interaction between atom and field. It is conventionally taken to be of the form

$$\hat{H}_1=-(e/2m)(\hat{\mathbf{p}}\cdot\mathbf{A}+\mathbf{A}\cdot\hat{\mathbf{p}})+(e^2/2m)\mathbf{A}\cdot\mathbf{A}. \quad (12a)$$

However, it has been shown that in order to treat the atom and the field in a consistent fashion the appropriate interaction Hamiltonian should be[3]

$$\hat{H}_1=-\hat{\boldsymbol{\mu}}\cdot\mathbf{E}, \quad (12b)$$

where $\hat{\boldsymbol{\mu}}$ is the electric dipole operator. Nevertheless, for optical frequencies and within the electric dipole approximation the forms (12a) and (12b) are physically equivalent. To see this, recall that the part of (12a) which is linear in the fields can be transformed to the form (12b), except for a factor ω_0/ω, which is very close to unity (ω_0 is the atomic resonance frequency, and ω is the frequency of the electric field). The part of (12a) which is bilinear in the fields leads (again assuming the electric dipole approximation) merely to an over-all energy shift, which is without physical

[2] If we define the wave number $k(\omega)$ according to $k(\omega)=\beta_0(\omega)[1+\frac{1}{2}\chi'(\omega)]$, then the phase condition (8) is equivalent to the requirement that the cavity length L contain an integral number of half-wavelengths; i.e., that $L=\frac{1}{2}\lambda q$, with $\lambda=2\pi/k$.

[3] E. A. Power and S. Zienau, Phil. Trans. Roy. Soc. **251**, 427 (1959). In their derivation of the form (12b) the electric dipole approximation has been assumed. This means simply that the spatial variations of the fields are sufficiently slow compared to atomic dimensions that it is a good approximation to treat the fields as constants in the integrals for the matrix elements.

significance, so it can be removed by a simple transformation.

We shall define the following bilinear combinations $b_m(t)b_n^*(t)$:

$$\rho \equiv b_2b_2^* - b_1b_1^*, \tag{13a}$$

$$\rho_{12} \equiv b_1b_2^*. \tag{13b}$$

The quantity ρ denotes the population difference, and ρ_{12} may be related to the atomic polarization P (since $P = \langle s \mid \hat{\mu} \mid s \rangle$) according to

$$P = \mu\rho_{12} \exp(j\omega_0 t) + \text{c.c.}, \tag{14}$$

where

$$\mu \equiv \langle 1 \mid \hat{\mu} \mid 2 \rangle, \tag{15}$$

$$\omega_0 \equiv W_2 - W_1 > 0. \tag{16}$$

Then the Schrödinger equation for this two-level system takes the form

$$\begin{aligned} \dot{\rho} &= 2j\mu\rho_{12} \exp(j\omega_0 t) E(t) + \text{c.c.}, \\ \dot{\rho}_{12} &= j\mu\rho \exp(-j\omega_0 t) E(t). \end{aligned} \tag{17}$$

Here, and throughout, we shall ignore the vector character of the electric field.

The equations of motion (17) will now be modified to include the effects of pumping, decays, and collisions through the addition of phenomenological relaxation terms of the type introduced by Bloch[4] for magnetic-resonance problems. We then have the following "generalized Bloch equations":

$$\dot{\rho} = 2j\mu\rho_{12} \exp(j\omega_0 t) E(t) + \text{c.c.} - (\rho - \rho_0)/T_1, \tag{18a}$$

$$\dot{\rho}_{12} = j\mu\rho \exp(-j\omega_0 t) E(t) - \rho_{12}/T_2. \tag{18b}$$

For a discussion of these equations within the context of magnetic resonance we refer the reader to the literature.[5] We mention in passing that included here among the energy-exchange T_1-type processes are (a) decays, (b) hard collisions, and (c) the pumping mechanism itself. The inclusion of pumping is not conventional. As a result both ρ_0 (the nonthermal equilibrium value of ρ when the system is being pumped but is not oscillating) and T_1 are functions of the pump rate. The T_2 relaxation term describes those processes which destroy the phase correlation among atoms, so that the polarization tends to decay with a time constant T_2.

IV. STEADY-STATE SOLUTIONS AND HOMOGENEOUS RESPONSE

The equations of motion (18) will be solved for an atom which moves with a velocity v through the standing wave field

$$E(z,t) = E \cos\beta z \cos\omega t. \tag{19}$$

When this field is transformed from laboratory coordinates (z, t) to atomic rest frame coordinates (z', t'), it assumes the form

$$\begin{aligned} E(z',t') = \tfrac{1}{4}E\{&\exp[j(\omega_1 t' - \beta_1 z')] \\ &+ \exp[j(\omega_2 t' + \beta_2 z')] + \text{c.c.}\}, \end{aligned} \tag{20}$$

with

$$\begin{aligned} \omega_1 &= \omega(1 - v/c), & \beta_1 &= \beta(1 - v/c), \\ \omega_2 &= \omega(1 + v/c), & \beta_2 &= \beta(1 + v/c). \end{aligned} \tag{21}$$

Thus in its own rest frame the atom sees a different Doppler-shifted frequency associated with each of the two traveling-wave components. The expression (20) is correct to first order in v/c.

With this form for the electric field, the equations of motion (18) become

$$\begin{aligned} \dot{\rho} = \tfrac{1}{2}j\mu E\rho_{12}[&\exp(-j\phi_1) + \exp(-j\phi_2)] \\ &+ \text{c.c.} - (\rho - \rho_0)/T_1, \end{aligned} \tag{22a}$$

$$\dot{\rho}_{12} = \tfrac{1}{4}j\mu E\rho[\exp(j\phi_1) + \exp(j\phi_2)] - \rho_{12}/T_2, \tag{22b}$$

where

$$\begin{aligned} \phi_1 &\equiv (\omega_1 - \omega_0)t' - \beta_1 z', \\ \phi_2 &\equiv (\omega_2 - \omega_0)t' + \beta_2 z'. \end{aligned} \tag{23}$$

We have assumed here that the rapidly oscillating sum frequency components have a negligible influence compared to the slowly varying difference frequency terms (this is the so-called "rotating wave approximation").

To obtain solutions to (22) we shall assume, as a first approximation, that the population difference ρ is nearly constant. The conditions under which this assumption is valid will appear shortly. If $\rho = \rho^{(0)} =$ const, then Eq. (22b) can be integrated directly to yield

$$\rho_{12} = \tfrac{1}{4}(j\mu E)\rho^{(0)} \left(\frac{\exp(j\phi_1)}{T_2^{-1} + j(\omega_1 - \omega_0)} + \frac{\exp(j\phi_2)}{T_2^{-1} + j(\omega_2 - \omega_0)} \right). \tag{24}$$

Insertion of this result into Eq. (22a) leads to the following equation for ρ:

$$\dot{\rho} = -\rho^{(0)}[A + B_1 \cos(\phi_2 - \phi_1) + B_2 \sin(\phi_2 - \phi_1)] + \rho_0/T_1, \tag{25}$$

where

$$AT_1 = 1 + B_1T_1,$$

$$B_1T_1 = s^2 \left(\frac{1}{1 + (\omega_1 - \omega_0)^2 T_2^2} + \frac{1}{1 + (\omega_2 - \omega_0)^2 T_2^2} \right),$$

$$B_2T_1 = s^2 \left(\frac{-(\omega_1 - \omega_0)T_2}{1 + (\omega_1 - \omega_0)^2 T_2^2} + \frac{(\omega_2 - \omega_0)T_2}{1 + (\omega_2 - \omega_0)^2 T_2^2} \right),$$

$$s^2 = \tfrac{1}{4}(\mu^2 E^2 T_1 T_2). \tag{26}$$

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[4] F. Bloch, Phys. Rev. **70**, 460 (1946).

[5] See, e.g., N. Bloembergen, *Nuclear Magnetic Relaxation* (W. A. Benjamin, Inc., New York, 1961); N. Bloembergen, E. M. Purcell, and R. V. Pound, Phys. Rev. **73**, 679 (1948). Regarding the theoretical basis of the Bloch equations, see A. Abragam, *The Principles of Nuclear Magnetism* (Oxford University Press, New York, 1961), especially Chaps. VIII and XII; also, C. P. Slichter, *Principles of Magnetic Resonance* (Harper and Row, New York, 1963); R. K. Wangsness and F. Bloch, Phys. Rev. **89**, 728 (1952); A. G. Redfield, IBM J. Res. Develop. **1**, 19 (1957).

(s^2 is a measure of the intensity in dimensionless units.) In considering the meaning of Eq. (25), it is necessary to recall the conditions under which the relaxation mechanisms can validly be represented in the phenomenological form suggested by Bloch.[6] The condition requires that the physical quantity which is subject to relaxation be described in the sense of a time average taken over an interval Δt which is long compared to the correlation time[7] but short compared to the relaxation time. Thus in Eq. (25) each term must be understood to be averaged over a time interval Δt which is much less than the relaxation time T_1. The sinusoidal terms, which contain the frequency $\omega_2-\omega_1$, will then average to zero provided that $|\omega_2-\omega_1|$ is sufficiently large[8]:

$$|\omega_2-\omega_1|\gg(\Delta t)^{-1}. \tag{27}$$

If condition (27) is satisfied, Eq. (25) then has (physically) only the stationary state solution $\dot{\rho}=0$, whence

$$\rho^{(0)}=\rho_0/AT_1. \tag{28}$$

So we see that the solution (24) is justified provided that the condition (27) is met. Of course there will be some slowly moving atoms for which the condition (27) will not be met. However, we must recall that our ultimate interest is the net response which is made by atoms of all velocities, i.e., an integral over the entire velocity distribution. We shall assume that of the atoms which contribute significantly to this integral (which is treated in detail in succeeding sections) only a small fraction move slowly enough so as not to satisfy (27). Thus if we assume that Eq. (24) describes the response of all atoms, we shall be evaluating the integrand correctly everywhere except in a small range of the integration, so that the error induced in the net integral is a small one.

Thus for atoms of velocity v, the solutions we adopt are those given by Eqs. (24) and (28). The polarization P is simply related to ρ_{12} [see Eq. (14)] and it is immediately evident that a polarization is being induced at each of the driving frequencies ω_1 and ω_2. Because we have formulated the oscillation condition

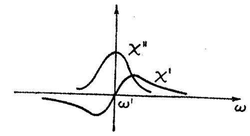

FIG. 1. The real and imaginary parts of the susceptibility $\chi(\omega)=\chi'(\omega)+j\chi''(\omega)$ for an atom having center frequency $\omega'=\omega_0(1+v/c)$.

(1) in terms of traveling waves it is sufficient to know the response to just one of the traveling-wave components which appear in $P(z',t')$. Transforming $P(z',t')$ to a laboratory reference $P(z,t)$ and applying the definition (2) to the ω_1 component we find that (see Fig. 1)

$$\chi(\omega)=\{j\chi_0''/[1+j(\omega_1-\omega_0)T_2]\}\mathcal{S}, \tag{29}$$

where, as before, $\omega_1=\omega(1-v/c)$, and $\mathcal{S}$ is the "saturation factor" which describes how the equilibrium value of ρ is reduced from its zero-field value of ρ_0 to the new value $\rho^{(0)}$ [cf. Eq. (28)]:

$$\rho^{(0)}=\rho_0\mathcal{S},$$
$$\mathcal{S}=1/AT_1 \qquad (0<\mathcal{S}\leq 1). \tag{30}$$

χ_0'' is the "midband unsaturated gain":

$$\chi_0''\equiv\mu^2\rho_0T_2. \tag{31}$$

Although $\chi(\omega)$ is the response to a single traveling wave, the medium is nevertheless still subject to saturation by both traveling-wave components of the standing-wave field, and this is reflected in the saturation factor $\mathcal{S}$ [cf. Eqs. (30) and (26), where we can see clearly the effects of each traveling-wave component].

The response of atoms belonging to a single velocity class is said to be "homogeneous" since the response is shared equally by all atoms and there is nothing to distinguish the behavior of one atom from that of another. If now we consider atoms having a spectrum of velocities, then different velocity groups will be found to be subject to varying degrees of saturation. In particular the saturation may be strong for some atoms and negligible for others. In such a case the response is said to be "strongly inhomogeneous." This situation obtains for some (but not all) gas lasers. The properties of lasers operating on lines which are homogeneously or inhomogeneously broadened may show important qualitative differences. These will be discussed in detail below.

V. GAIN PROFILE

A discussion of the gain profile is really a digression from the main problem of finding the complex susceptibility for an ensemble of atoms subject to saturation and having a distribution of velocities. It is a worthwhile digression, however, because it provides some physical insight into the nature of the effects to be described later (i.e., Lamb notch, mode pulling and pushing).

The "gain profile" is the response to an arbitrarily weak traveling-wave probe of tunable frequency ω

[6] The validity conditions for the Bloch equations, which shall be assumed to prevail here, are discussed in, e.g., Abragam (Ref. 5, Chap. XII, Sec. IA) and Slichter (Ref. 5, Secs. 5.6–5.8).

[7] The correlation time characterizes that part of the Hamiltonian which is randomly fluctuating and which is associated with the relaxation processes. A correlation time may be identified with the mean time between collisions, but it is to be distinguished from the relaxation time T_2, which describes the decay of the polarization.

[8] It is possible to treat Eq. (25) by successive iterations, starting with the solution $\rho=\rho^{(0)}$, before the time averaging is performed. The first iteration yields

$$\rho(t)=\rho^{(0)}[1-(\omega_2-\omega_1)^{-1}(-B_1\sin(\phi_2-\phi_1)+B_2\cos(\phi_2-\phi_1))]$$

and displays the "population pulsation" terms which are seen to be small if $|\omega_2-\omega_1|$ is sufficiently large. However, we must note here that (i) the expansion parameter is proportional to the intensity [cf. Eq. (26) for B_1 and B_2] so that the expansion is not good for high intensities, and (ii) the time averaging is still required in order to make the solutions compatible with the validity conditions for the Bloch equations.

when the system is subject to strong signal saturation by a standing wave of frequency ω_s. We are interested in the imaginary part of the response, $\chi''(\omega, \omega_s)$, as a function of ω, ω_s being held fixed. The response just to the weak probe signal is obtained from Eq. (29) with $\mathcal{S}=1$. To take account of strong signal saturation we include the saturation factor $\mathcal{S}$ evaluated at $\omega=\omega_s$. Finally, we obtain the total response by integrating over all atomic velocities which are weighted with the distribution function $g(v)$. Hence we have

$$\chi''(\omega, \omega_s) = \int_{-\infty}^{\infty} dv g(v) h(\omega, v) \mathcal{S}(\omega_s, v), \qquad (32)$$

where $h(\omega, v)$ is the homogeneous response made by atoms of velocity v to the weak probe of frequency ω:

$$h(\omega, v) = \chi_0''/\{1+[\omega(1-v/c)-\omega_0]^2 T_2^2\}. \qquad (33)$$

The saturation factor $\mathcal{S}(\omega_s, v)$ is given by Eqs. (30) and (26) except that ω_1 and ω_2 are to be evaluated at the frequency ω_s of the standing wave; thus in place of Eq. (21) we have

$$\omega_1=\omega_s(1-v/c),$$
$$\omega_2=\omega_s(1+v/c). \qquad (34)$$

To calculate the integral (32) we make a change of variable from v to ω' according to

$$\omega'=\omega_0(1+v/c). \qquad (35)$$

Physically ω' represents the atomic resonance frequency as seen by a laboratory observer. Then we have, to first order in v/c,

$$\omega_1-\omega_0=\omega_s-\omega',$$
$$\omega_2-\omega_0=\omega'-(2\omega_0-\omega_s),$$
$$\omega(1-v/c)-\omega_0=\omega-\omega'. \qquad (36)$$

We have assumed that all frequency differences are reduced from optical frequencies by a factor of order of magnitude v/c, so that, for example, $(v/c)\omega=(v/c)\omega_0$ to first order in v/c. Typically, v/c is of order 10^{-6}. Then we have

$$\chi''(\omega, \omega_s) = \frac{\chi_0''}{2\pi T_2} \int_{-\infty}^{\infty} g(\omega') \frac{dx}{1+(x-\delta_1)^2} \times \left[1+s^2\left(\frac{1}{1+x^2}+\frac{1}{1+(x-\Delta)^2}\right)\right]^{-1}. \qquad (37)$$

For convenience we have defined the following dimensionless parameters:

$$s^2 \equiv \tfrac{1}{4}\mu^2 E^2 T_1 T_2,$$
$$w \equiv (1+s^2)^{1/2}+1,$$
$$x \equiv T_2(\omega'-\omega_s),$$
$$\delta_1 \equiv T_2(\omega-\omega_s),$$
$$\delta_2 \equiv T_2[\omega-(2\omega_0-\omega_s)],$$
$$\Delta \equiv \delta_1-\delta_2 = 2T_2(\omega_0-\omega_s). \qquad (38)$$

We shall refer to s^2 as the "dimensionless intensity." The distribution $g(\omega')$ has been assumed to be normalized according to

$$\int g(\omega') d\omega'/2\pi = 1. \qquad (39)$$

We shall evaluate the integral (37) under the assumption that the distribution $g(\omega')$ varies slowly over the range for which the other factors are significant. This means that only atoms in a relatively small velocity range contribute to the integral; this will be the case provided the width of the response to the weak probe, $2/T_2$, is small compared to the width of the distribution, $g(\omega')$, i.e., the Doppler width. (As will be seen shortly, this is a necessary but not a sufficient condition that the response be strongly inhomogeneous.) The factor $g(\omega')$ may now be taken outside the integral and replaced by $g(\omega)$ since the homogeneous response function is resonant at $x=\delta_1$ (i.e., $\omega'=\omega$). The evaluation of the resulting integral is discussed in Appendix A. Here we consider only the special case for which $\Delta \gg w$; the physical significance of this condition will appear shortly. In this case we find that

$$\chi''(\omega, \omega_s) = g(\omega)(\chi_0''/2T_2)[G(s^2, \delta_1)+G(s^2, \delta_2)-1] \quad (\Delta \gg w), \qquad (40)$$

where

$$G(s^2, \delta) \equiv 1-[ws^2/(w-1)](\delta^2+w^2)^{-1} \qquad (41)$$

and $w(s^2)$ is defined above, Eq. (38).

The Doppler curve $g(\omega)$ is of course centered at $\omega=\omega_0$ and is generally taken to be a Gaussian, although in the absence of thermal equilibrium it need not be. The function $G(s^2, \delta)$ is unity except in the vicinity of $\delta=0$, where it is Lorentzian shaped with half-width w. Hence the coefficient in Eq. (40) has two "holes," one at $\delta_1=0$, the other at $\delta_2=0$, these values corresponding in frequency to $\omega=\omega_s$ and $\omega=\omega_0+(\omega_0-\omega_s)$, respectively. The assumption that $\Delta \gg w$ means that $\delta_1-\delta_2 \neq 0$, so that the holes in this instance are distinct. The coefficient $G(s^2, \delta_1)+G(s^2, \delta_2)-1$ then appears as shown in Fig. 2; thus the gain profile $[\chi''(\omega, \omega_s)$ versus $\omega]$ follows the Doppler curve $g(\omega)$ [see Fig. 3(a)] except at $\omega=\omega_s$ and $\omega=\omega_0+(\omega_0-\omega_s)$, where, according to popular terminology, "holes are burned into the gain curve."[9] The holes are symmetric about $\omega=\omega_0$ and are separated by a frequency difference $2|\omega_0-\omega_s|$. The hole centered at the actual oscillation frequency ω_s we call the "primary hole," while that at $\omega=\omega_0+(\omega_0-\omega_s)$ we refer

[9] Hole burning was first described as early as 1947 by N. Bloembergen; see especially Fig. 2.5, p. 47, of the first reference in Ref. 5. The earliest discussion with respect to gas lasers was given by W. R. Bennett, Jr., Phys. Rev. **126**, 580 (1962); Appl. Opt. Suppl. **1**, 24 (1962), especially p. 57. Also see Bennett's article in *Quantum Electronics*, edited by P. Grivet and N. Bloembergen (Columbia University Press, New York, 1964), Vol. 1, especially pp. 450–456. Experimental observation of holes has been reported by D. Launois and A. Kastler, Compt. Rend. **264B**, 868 (1967).

to as the "image hole." Obviously the existence of two holes when there is but a single mode of oscillation reflects the fact that an atom traveling through a standing-wave field interacts with both Doppler-shifted traveling-wave components.

When expressed in the dimensionless units of Eqs. (38), we may describe the holes as being of width $2w$ ($2w/T_2$ in frequency units) and depth $(w-2)/(w-1)$, each having an area $\pi s^2/(1+s^2)^{1/2}$. The holes are separated by an amount $|\Delta|$. The condition $\Delta \gg w$ means simply that the separation of the holes is large compared to their width. Note that the area of the hole is a measure of the intensity of the cavity field, especially for weak fields for which $s^2 \ll 1$.

Having demonstrated the existence of holes in the gain curve, we can now state more precisely the condition, assumed above, that the response be strongly inhomogeneous. It is simply that the hole width be small when compared with the full Doppler width. If this condition is not met, i.e., if the hole width is comparable to the Doppler width ("weakly inhomogeneous" broadening), the signal causes the entire line to saturate as in a homogeneous response, and in this limit the concept of a hole, which implies selective saturation, ceases to be a useful one. We note in this connection that the hole width $2w/T_2$ increases with (a) dephasing frequency $1/T_2$ ("collision broadening") and (b) intensity s^2 ("power broadening").[10] Thus even though a line may suffer considerable Doppler broadening, this in itself is no guarantee that the response will be strongly inhomogeneous. The inhomogeneity can be destroyed by the above effects, whose tendency is to cause all atoms to saturate equally and thereby to make the response more nearly homogeneous.[11]

If the holes are not well separated, the expression for $\chi''(\omega,\omega_s)$ is not so simple, although it can still be handled. There is, however, one other special case for which the gain profile is quite simple, namely, when the oscillation frequency is at line center (i.e., $\Delta=0$). In this case (see Appendix A),

$$\chi''(\omega,\omega_s)\,|_{\omega_s=\omega_0}=[\chi_0''g(\omega)/2T_2]G(2s^2,\delta_0), \quad (42)$$

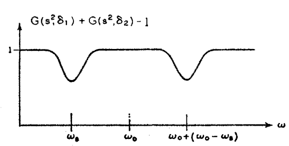

FIG. 2. $\{G(s^2,\delta_1)+G(s^2,\delta_2)-1\}$ versus ω for the case that $\Delta \gg w$.

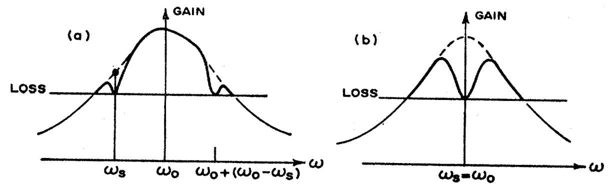

FIG. 3. Gain versus frequency for (a) two distinct holes ($\Delta \gg w$) and (b) one hole at line center ($\Delta=0$).

where $\delta_0=T_2(\omega-\omega_0)$ and $G(s^2,\delta)$ is as given by Eq. (41). Hence when $\omega_s=\omega_0$ there is a single hole located at $\omega=\omega_0$ ($\delta_0=0$) of width $2w_0$, depth $(w_0-2)/(w_0-1)$, and area $2\pi s^2/(1+2s^2)^{1/2}$, where

$$w_0=(1+2s^2)^{1/2}+1. \quad (43)$$

Evidently as the oscillation frequency is brought toward line center the two holes merge into a single hole. Associated with this overlap is a reduction at line center of the output power. The general expressions for power versus tuning will be derived in Sec. VI, but we can give here a simple explanation for the existence of the central tuning dip (the so-called "Lamb notch").[12] The intensity s^2 is determined from the condition (5) that the gain at the oscillation frequency be equal to the loss, i.e., that $\chi''(\omega_s)=Q_c^{-1}$, where $\chi''(\omega_s)$ means the response to the saturating signal itself:

$$\chi''(\omega_s)\equiv\chi''(\omega,\omega_s)\,|_{\omega=\omega_s}, \quad (44)$$

Hence the gain at the frequency of oscillation saturates by an amount which allows the "gain = loss" condition to be satisfied. This is illustrated in Fig. 3 for the two cases that (a) the holes are well separated (intensity s_1^2) and (b) $\omega_s=\omega_0$ (intensity s_0^2). Except for the variation of the envelope $g(\omega)$ the "gain = loss" condition would require the equality of the hole depths in the two cases. As we saw above, the expression for the hole depth is $(w-2)/(w-1)$, where for case (a) w is given in Eq. (38), and in case (b) $w=w_0$ [Eq. (43)]. Hence the equality of hole depths means that

$$s_1^2=2s_0^2,$$

i.e., the intensity at line center is reduced by a factor of 2. Inclusion of the variation of $g(\omega)$ modifies this result and, as we shall see, allows for the possibility of a threshold for the appearance of the Lamb notch.

VI. POWER-TUNING CHARACTERISTIC

We have already indicated in Sec. II that the "gain = loss" equation (5) can be used to determine the power-tuning relationship once the gain $\chi''(\omega_s)$ is known explicitly as a function of intensity E^2 and frequency of oscillation ω_s. We write ω_s to indicate that we are now interested in the response to the strong

[10] Broadening effects have received considerable attention in the literature. See, e.g., W. R. Bennett, Jr., *et al.*, Phys. Rev. Letters **18**, 688 (1967); P. T. Bolwijn and C. Th. J. Alkemade, Phys. Letters **25A**, 632 (1967); A. Szöke and A. Javan, Phys. Rev. **145**, 137 (1966); P. W. Smith, J. Appl. Phys. **37**, 2089 (1966); R. L. Fork and M. A. Pollack, Phys. Rev. **139**, A1408 (1965); B. L. Gyorffy, M. Borenstein, and W. E. Lamb, Jr., *ibid.* **169**, 340 (1968).

[11] A third mechanism which can prevent hole burning is cross relaxation, in which there is an exchange of excitation between atoms of neighboring velocity groups. In this paper we make no attempt to consider such effects.

[12] An explanation for the Lamb notch in terms of the holes of the gain curve was first given by W. R. Bennett, Jr., Appl. Opt. Suppl. **1**, 24 (1962), especially p. 59.

saturating field itself. In calculating $\chi''(\omega_s)$ it is possible to apply Eq. (44) to the results of Appendix A for $\chi''(\omega, \omega_s)$, but it turns out to be simpler from the point of view of interpretation to treat this integral independently. The integral in question is similar to that given above in Eq. (37) except that we now take $\omega=\omega_s$ (i.e., $\delta_1=0$):

$$\chi''(\omega_s)=\frac{\chi_0''}{2\pi T_2}\int_{-\infty}^{\infty} g(\omega')\,\frac{dx}{1+x^2}\times\left[1+s^2\left(\frac{1}{1+x^2}+\frac{1}{1+(x-\Delta)^2}\right)\right]^{-1}. \quad (45)$$

For the Gaussian $g(\omega')$, we write

$$g(\omega')=2T_2^*\exp[-(T_2^*)^2(\omega'-\omega_0)^2/\pi], \quad (46)$$

where $1/T_2^*$ is a measure of the Doppler width, and the amplitude has been chosen to satisfy the normalization requirement (39). Then we have

$$\chi''(\omega_s)=\frac{\chi_0''}{2\pi T_2}\int_{-\infty}^{\infty} du\, g(u)\times\frac{1+u^2+\frac{1}{4}\Delta^2}{(1+u^2+\frac{1}{4}\Delta^2)^2-u^2\Delta^2+2s^2(1+u^2+\frac{1}{4}\Delta^2)} \quad (47)$$

for which the variable of integration is

$$u=x-\tfrac{1}{2}\Delta=T_2(\omega'-\omega_0) \quad (48)$$

and

$$g(u)=2T_2^*\exp(-4\alpha u^2), \quad (49)$$

where $(4\pi\alpha)^{1/2}$ is the ratio of the hole width $1/T_2$ to the Doppler width $1/T_2^*$:

$$T_2^*/T_2\equiv(4\pi\alpha)^{1/2}. \quad (50)$$

The integral (47) is evaluated in Appendix B for arbitrary values of α and also for the limiting cases of $\alpha\gg1$ (the homogeneous limit) and $\alpha\ll1$ (the strongly inhomogeneous limit). It is the latter case we discuss here.

For the case that the Doppler width is large compared with the hole width we find

$$\chi''(\omega_s)=\chi_0''(4\pi\alpha)^{1/2}\exp(-\alpha\Delta^2)\,\mathcal{I}(s^2,\Delta^2), \quad (51)$$

with

$$\mathcal{I}(s^2,\Delta^2)=(1+\zeta/r)[2(\lambda+r)]^{-1/2} \quad (52)$$

and [see Appendix B]

$$\begin{aligned}\Delta&=2T_2(\omega_0-\omega_s),\\ \zeta&=1+\tfrac{1}{4}\Delta^2,\\ r&=[\zeta(\zeta+2s^2)]^{1/2},\\ \lambda&=2+s^2-\zeta.\end{aligned} \quad (53)$$

We call $\mathcal{I}(s^2,\Delta^2)$ the "inhomogeneous saturation factor" since it assumes the value unity for zero fields and is less than unity for nonzero fields. The oscillation condition (5) can then be written

$$R(\Delta)\mathcal{I}(s^2,\Delta^2)=1, \quad (54)$$

where

$$R(\Delta)=R_0\exp(-\alpha\Delta^2),$$
$$R_0\equiv\chi_0''Q_c(T_2^*/T_2)=\mu^2\rho_0T_2Q_c. \quad (55)$$

$R(\Delta)$ is the ratio of the unsaturated gain to the loss, and R_0 is this ratio at line center. Obviously the threshold condition for sustaining oscillations is

$$R_0>1. \quad (56)$$

In Eq. (55), the coefficient T_2^*/T_2 is a measure of the fraction of the homogeneous packets under the Doppler curve which may be saturated. Figure 3(a) illustrates the meaning of Eq. (54), in which $R(\Delta)\mathcal{I}(s^2,\Delta^2)$ signifies the ratio of the saturated gain to the loss. The Gaussian without holes represents the unsaturated gain. At $\omega=\omega_s$, the unsaturated gain is saturated down until the saturated gain is equal to the loss.

From the oscillation condition (54), it is now possible to obtain $s^2(\Delta)$ for a strongly inhomogeneous line as follows:

$$s^2(\Delta)=(r^2-\zeta^2)/2\zeta, \quad (57)$$

where r satisfies

$$(r+\zeta)^2(\zeta R^2-r^2)=4r^2\zeta(1-\zeta) \quad (58a)$$

or, equivalently,

$$\zeta R^2(r+\zeta)^2=r^2(r+3\zeta)(r-\zeta)+4\zeta r^2. \quad (58b)$$

We can verify immediately that oscillations cease when $r=\zeta$, for which Eq. (58b) requires that $R=1$. Physically this means that when the detuning is such that the unsaturated gain is equal to the loss there can be no saturation [i.e., $\mathcal{I}(s^2,\Delta^2)=1$], which in turn implies null intensity fields. Hence the amount of detuning Δ^* which quenches oscillation satisfies

$$R(\Delta^*)=1. \quad (59)$$

Equivalently, this is the condition for threshold with detuning.

Another trivial solution to Eq. (58a) occurs at line center, $\zeta=1$, for which $r=R=R_0$, so that at $\Delta=0$

$$s^2(0)=\tfrac{1}{2}(R_0^2-1). \quad (60)$$

Although the general solution for $s^2(\Delta)$ as given by Eqs. (57) and (58) involves the root of a fourth-degree equation, this complexity is really necessary only to describe the behavior in the region near $\Delta=0$, where the

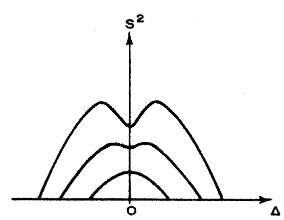

FIG. 4. s^2 versus Δ for increasing excitations R_0, illustrating the appearance of the Lamb notch.

holes of the gain curve partially overlap. In the special cases that (a) the holes are well separated and distinct ($\Delta \gg w$), or (b) the holes overlap completely ($\Delta = 0$), the expression (52) for $\mathcal{J}(s^2, \Delta^2)$ assumes quite simple forms:

$$\text{(a)} \quad \lim_{\Delta \gg w} \mathcal{J}(s^2, \Delta^2) = (1+s^2)^{-1/2}, \tag{61}$$

$$\text{(b)} \quad \lim_{\Delta \to 0} \mathcal{J}(s^2, \Delta^2) = (1+2s^2)^{-1/2}. \tag{62}$$

Hence from Eq. (54) we have

$$s^2(\Delta) = R^2 - 1, \qquad \Delta \gg w \tag{63}$$

and (again)

$$s^2(0) = \tfrac{1}{2}(R_0^2 - 1). \tag{60}$$

We see from these results that if $R(\Delta)$ were flat instead of Gaussian [i.e., if we had $R(\Delta) = R_0$ for all Δ], then the intensity s^2 would drop by a factor of 2 at line center. This effect derives from the fact that the line saturates more strongly at line center than away from line center [as per Eqs. (61) and (62)]. If account is now taken of the variation of R with Δ [Eq. (55)], it is possible to explain the observed behavior of the central tuning dip: For excitations slightly above threshold (i.e., R_0 a little larger than unity) there is no Lamb notch. As the excitation is increased a threshold is reached for the appearance of the Lamb notch (see Fig. 4).

To obtain an analytic description we merely note that a Lamb notch is present or not present according to whether

$$(d^2/d\Delta^2) s^2(\Delta) \,|_{\Delta=0}$$

is positive or negative. From Eqs. (57) and (58), we find that

$$\alpha_c - \alpha > 0 \Leftrightarrow (\text{Lamb notch}),$$

$$\alpha_c - \alpha < 0 \Leftrightarrow (\text{no Lamb notch}), \tag{64}$$

where

$$\alpha_c(R_0) = \frac{(3R_0+1)(R_0-1)}{8R_0^2(R_0+1)^2}. \tag{65}$$

α, defined in Eq. (50), represents the ratio of (unsaturated) hole width to Doppler width. The curvature in $s^2(\Delta)$ at $\Delta = 0$ depends on $\alpha_c - \alpha$ and is determined by two competing effects. Saturation (represented by α_c) and the associated phenomenon of holes which overlap at line center lead to a power dip at $\Delta = 0$, but this must be strong enough to overcome the curvature of the Gaussian (represented by α) which of course is in the opposite sense. The conditions (64) are illustrated in Fig. 5, where it is seen that as the excitation R_0 is increased starting from $R_0 = 1$ we initially have no Lamb notch, but at $R_0 = R_c$ we reach the Lamb-notch threshold. The function $\alpha_c(R_0)$ plotted in Fig. 5 is of course a universal curve and applies to all single-mode laser lines which are inhomogeneously broadened. For a fixed value of α the most pronounced Lamb notch should occur for the excitation $R_0 = R_m$, where R_m

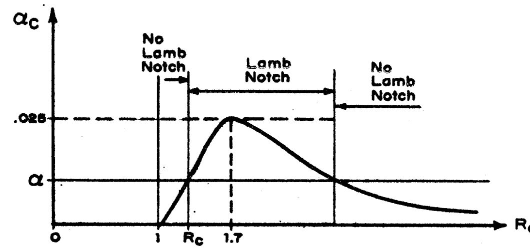

FIG. 5. α_c versus R_0, and the conditions (64) for the Lamb notch. $R_m = 1.7$.

maximizes $\alpha_c(R_0)$ and satisfies

$$3(R_m - 1)^2(R_m + 1) = 4. \tag{66}$$

For R_m we have approximately

$$R_m = 1.7 \tag{67}$$

and

$$\alpha_c(R_m) = 0.025. \tag{68}$$

There are two circumstances identifiable in Fig. 5 for which the Lamb notch is not present: (a) when $\alpha > 0.025$, and (b) assuming $\alpha < 0.025$, if R_0 has been increased beyond R_m until $\alpha_c < \alpha$. In both instances the interpretation is that the hole widths in the gain profile have been increased to the point that the response is essentially homogeneous, in case (a) through collision broadening, and in case (b) through power broadening,[13] as discussed in Sec. V. Thus it is not difficult to understand why gas lasers need not show a central tuning dip in their power-tuning characteristic. In fact the dip has been reported only in a very few cases, of which the He–Ne laser is the most famous. Presumably the Lamb notch can be obtained on other laser lines as well if an attempt is made to avoid excessive collision or power broadening so that the first of conditions (64) can be satisfied.[14]

The variation of the Lamb notch (short of extinction) with the parameters of the system can also be determined with reference to Fig. 5; the discussion of course must relate to the basic parameters R_0, T_2, and T_2^*. (Incidentally, the dependence of these parameters on the system parameters such as pressure, discharge voltage, etc., is not generally a simple one.) For example, it has been reported[15] recently that an increase in pressure is accompanied by a diminution of the central

[13] In a strict sense it is not correct to describe the homogeneous limit of collision broadening or power broadening with results which are based upon the inhomogeneous approximation (51). Nevertheless we can expect Eqs. (64) and (65) and Fig. 5 to provide a valid qualitative description for large values of α and/or R_0. In Appendix B we consider the gain function $\chi''(\omega_s)$ without making the approximation of a strongly inhomogeneous line and thereby are able to verify directly that for (a) $\alpha \gg 1$ or (b) $s^2 \gg 1$ the power-tuning characteristic does not display a Lamb notch.

[14] However, in some instances the mechanism responsible for producing a homogeneous response may be cross relaxation; see Ref. 11.

[15] M. A. Pollack, T. J. Bridges, and A. R. Strnad, Appl. Phys. Letters **10**, 182 (1967).

tuning dip. To explain this behavior we note that higher pressures result in higher mean collision frequencies, so that the response becomes more nearly homogeneous.

If a Lamb notch is observable at excitation $R_0=R_m$, further increase in R_0 should cause the dip to diminish and eventually to disappear. This kind of behavior has apparently not been reported in the literature; it would be an interesting way to observe the effects of power broadening.

In the limit of $\alpha\to 0$ the response is strongly inhomogeneous and the Lamb notch is more readily discernible. In fact the case of $\alpha=0$ corresponds to a flat distribution $R(\Delta)$, and in this limit, as we saw above [Eqs. (63) and (60)], the intensity at line center is always reduced by a factor of 2.

For excitations not far above threshold $(R-1\ll 1)$ it is possible to develop the solution for $s^2(\Delta)$ as a power series in $R-1$. We obtain, to second order in $R-1$,

$$s^2(\Delta)=\frac{2\zeta}{\zeta+1}(R-1)\left(1+\frac{\zeta^2-\zeta+4}{2(\zeta+1)^2}(R-1)+\cdots\right). \tag{69}$$

This expression is appropriate to a strongly inhomogeneous line since it is based on the solutions (57) and (58). We may note that the two special cases considered earlier in Eqs. (63) and (60) do not involve $R-1$ to higher than second order even if $R-1$ is not a small quantity. We conclude then that the contributions to (69) beyond second order are either negligible or else they vanish if either (a) $\Delta\gg w(\zeta\gg 1)$ or (b) $\Delta=0(\zeta=1)$. The result (69) becomes in these two cases:

$$s^2(\Delta)=2(R-1)+(R-1)^2, \qquad \Delta\gg w \tag{69'}$$

$$s^2(0)=(R_0-1)+\tfrac{1}{2}(R_0-1)^2. \tag{69''}$$

As would be anticipated, these forms are equivalent to those of Eqs. (63) and (60), respectively. For small excitations the Lamb-notch conditions (64) remain the same except that α_c is given approximately by

$$\alpha_c\approx\tfrac{1}{8}(R_0-1). \tag{70}$$

If it were desired to have the power output function without making the approximation of a strongly inhomogeneous line [which leads to Eq. (51)], it would be necessary to use the exact expression for $\chi''(\omega_s)$ [Eq. (B9)] in Eq. (5). This then yields an implicit transcendental equation for $s^2(\Delta)$.

As was noted earlier, the solutions (57) and (58) assume the much simpler form (63) if ω_s is not in the region where the two holes overlap appreciably. The result (63) can also be obtained by constructing a simpler theory which ignores the standing-wave nature of the oscillations and assumes instead a single traveling wave.[16] Such a theory in effect takes account of the primary hole but not the image hole [it is constructed by dropping the second term of B_1, Eq. (26)]. Although it leads to the correct power output for $\Delta\gg w$, a single-hole theory cannot predict the central tuning dip. However, it does illustrate the fact that the image hole has a negligible influence on the gain at the oscillation frequency ω_s, and hence on the intensity s^2, except near line center, where it overlaps the primary hole (as shown in Fig. 3). On the other hand, as will be seen in Sec. VII, the image hole has important effects when accounting for phase shifts, whereas the primary hole is of no consequence in this regard. This difference in behavior derives from the fact that for packets near resonance the net phase-shift contribution is negligible because the homogeneous dispersion function $\chi'(\omega)$ [Eq. (29)] is odd about the center frequency. Thus the important contributions to the phase shift come from packets well removed from resonance where, due to the long-range character of $\chi'(\omega)$, a homogeneous packet may contribute phase shift but no gain.

VII. FREQUENCY OF OSCILLATION

So far we have used the magnitude part of the oscillation condition to determine the intensity as a function of tuning. As we saw earlier, there is also a phase condition associated with Eq. (1); this was given in Eq. (8) and determines the actual frequency of oscillation.

Application of the phase equation requires a knowledge of the real part of the susceptibility $\chi'(\omega_s)$. It is not possible to calculate χ' from χ'' using the Kramers-Kronig relations because these apply only to a linear response. The Kramers-Kronig equations do have a nonlinear analog,[17] but they are not too useful and shall not be employed here. Instead we calculate $\chi'(\omega_s)$ directly. The expression for $\chi'(\omega_s)$ is obtained using the homogeneous response (29) to the strong saturating signal of frequency ω_s. Thus ω_1 and ω_2 are again given by Eqs. (34). The integral in question then is

$$\chi'(\omega_s)=-\frac{\chi_0''}{2\pi T_2}\int_{-\infty}^{\infty}g(\omega')\,\frac{x\,dx}{1+x^2}\times\left[1+s^2\left(\frac{1}{1+x^2}+\frac{1}{1+(x-\Delta)^2}\right)\right]^{-1}, \tag{71}$$

where x and Δ are as defined in Eqs. (38) and $g(\omega')$ is given in Eq. (46). This expression is to be compared to the one we had earlier [Eq. (45)] for $\chi''(\omega_s)$. The only difference is the extra factor of $-x$ in the numerator. As shown in Appendix C, the exact evaluation of (71) involves expressions similar to those found in the exact evaluation of $\chi''(\omega_s)$. However, in the inhomogeneous limit, $\chi''(\omega_s)$ can be evaluated by an approximation [namely, the factoring of $g(\omega')$ out of the integral, as

[16] P. W. Smith, IEEE J. Quantum Electron. **2**, 62 (1966). Equation (3) of this paper is tantamount to the result (63).

[17] N. Bloembergen, *Non-Linear Optics* (W. A. Benjamin, Inc., New York, 1965), p. 45.

discussed in Appendix B] which is not applicable to the integral (71) because of the difference in behavior between the functions $x(1+x^2)^{-1}$ and $(1+x^2)^{-1}$. As a result, even in the inhomogeneous limit the expression for $\chi'(\omega_s)$ is more involved than that for $\chi''(\omega_s)$.

We find (see Appendix C for details) that

$$\chi'(\omega_s)=\chi_0''(\Delta/\sigma)\exp[-(\Delta/\sigma)^2]\mathcal{G}(s^2,\Delta^2)M(s^2,\Delta^2), \tag{72}$$

with

$$M(s^2,\Delta^2)=A(s^2,\Delta^2)-B(s^2,\Delta^2), \tag{73}$$

where, to first order in $1/\sigma$ $(1/\sigma=\alpha^{1/2})$,

$$A(s^2,\Delta^2)=\frac{r-\zeta}{r+\zeta}\left(\pi^{1/2}-\frac{4x}{\sigma}\right),\qquad \Delta^2\leq\Delta_0^2$$

$$A(s^2,\Delta^2)=\frac{r-\zeta}{r+\zeta}\left[\pi^{1/2}-\frac{4x}{\sigma}+\frac{8x}{\sigma}\int_0^{2y/\sigma}d\eta\, e^{\eta^2}\left(\frac{2y}{\sigma}-\eta\right)\right],\qquad \Delta^2\geq\Delta_0^2 \tag{74}$$

$$B(s^2,\Delta^2)=4x/\sigma,\qquad \Delta^2\leq\Delta_0^2$$

$$B(s^2,\Delta^2)=\frac{2x}{y}\int_0^{2y/\sigma}d\eta\, e^{\eta^2},\qquad \Delta^2\geq\Delta_0^2 \tag{75}$$

and

$$\Delta_0^2=s^4(1+s^2)^{-1},$$
$$x=[\tfrac{1}{2}(r+\lambda)]^{1/2},$$
$$y=[\tfrac{1}{2}(r-\lambda)]^{1/2},$$
$$\sigma^2=1/\alpha=4\pi(T_2/T_2^*)^2. \tag{76}$$

All other quantities were defined earlier [Eqs. (52) and (53)]. For a strongly inhomogeneous response $\sigma\gg 1$.

The gross behavior of the dispersion function $\chi'(\omega_s)$ is dominated by the factor $(\Delta/\sigma)\exp[-(\Delta/\sigma)^2]$, the remaining factors being slowly varying by comparison. Hence $\chi'(\omega_s)$ qualitatively resembles the homogeneous dispersion function $\chi'(x)=x(1+x^2)^{-1}$.

The phase condition (8) is most conveniently handled when expressed in terms of the cavity $Q(Q_c)$ and the atomic line $Q(Q_a)$, the latter being defined through

$$Q_a=\omega_0/(\Delta\omega)_D=\tfrac{1}{2}(\omega_0T_2^*), \tag{77}$$

since the Doppler width $(\Delta\omega)_D$ may be defined as $2/T_2^*$. Combining Eqs. (72), (51), and (5) the phase Eq. (8) may be written

$$\omega_s-\omega_c=(\sigma/2\pi)(Q_a/Q_c)(\omega_s/\omega_0)(\omega_s-\omega_0)M(s^2,\Delta^2). \tag{78}$$

So far the only approximation that we have made is that the response be strongly inhomogeneous. However the solutions to Eq. (78) of greatest interest are those for which the cavity Q is sufficiently large that the actual oscillation frequency ω_s is not far removed from

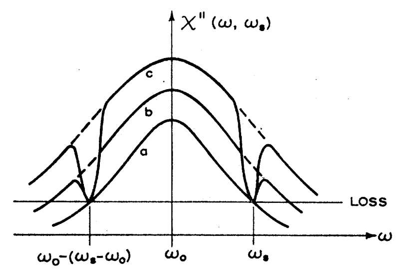

FIG. 6. The gain function $\chi''(\omega,\omega_s)$ versus ω. Each curve represents a different value of unsaturated gain χ_0''. In (a) the gain is at threshold for oscillations at ω_s. In (b) and (c) χ_0'' has been raised above threshold.

the cavity resonant frequency ω_c. This will be the case provided that

$$(\sigma/2\pi)(Q_a/Q_c)M(s^2,\Delta^2)\ll 1.$$

Then, to first order in this small quantity, we have

$$\omega_s=\omega_c+(\sigma/2\pi)(Q_a/Q_c)M(s^2,\Delta^2)\,|_{\omega_s=\omega_c}(\omega_c-\omega_0). \tag{79}$$

To lowest order $\omega_s=\omega_c$. The correction term alters ω_s either away from or toward ω_0 according as $M(s^2,\Delta^2)$ is positive or negative; these solutions correspond, respectively, to mode pushing $(M>0)$ or mode pulling $(M<0)$. From the form (73) we see that the terms $A(s^2,\Delta^2)$ are pushing terms while $B(s^2,\Delta^2)$ represents pulling (both A and B are themselves always positive, $1/\sigma$ being assumed small).

In analyzing the behavior of the mode-pulling and -pushing solutions, it is possible to identify the influence of the primary and image holes which appear in the gain profile. If in calculating $\chi''(\omega,\omega_s)$ we had used a saturation factor $\mathcal{S}$ which took account of interactions with only one traveling-wave component instead of both [in which case B_1, Eq. (26), would contain one term and not two], then the gain profile would show a primary hole but no image hole. In such a truncated theory the evaluation of $\chi'(\omega_s)$ would be as in Eq. (72) except taken in the limit of[18] $s^2\to 0$:

$$\lim_{s^2\to 0}M(s^2,\Delta^2)=-\frac{4}{\Delta}\int_0^{\Delta/\sigma}d\eta\, e^{\eta^2}. \tag{80}$$

Hence if only the primary hole were present we should have power-independent mode pulling and no mode pushing. This power-independent pulling arises from the phase shift which is to be associated with the unsaturated gain (curve a of Fig. 6). (Note that in this linear regime the real and imaginary parts of the susceptibility satisfy the Kramers-Kronig relations.)

[18] An expression for pulling appropriate to a strongly inhomogeneous Doppler broadened line was given earlier by A. Javan, E. A. Ballik, and W. L. Bond [J. Opt. Soc. Am. 52, 96 (1962), in their footnote 9] and is equivalent to Eq. (79) when $M(s^2,\Delta^2)$ is taken to zero order in s^2, as in Eq. (80).

In this single-hole theory the effect of increasing (through pumping) the midband unsaturated gain χ_0'' is to cause a hole to be burned in the gain curve at $\omega=\omega_s$ (curves b and c of Fig. 6, but temporarily ignoring the image hole); nevertheless the total amount of phase shift at $\omega=\omega_s$ is not influenced by the size of the hole, i.e., it is independent of intensity s^2. We can see in a general way how this behavior comes about. The total phase shift at $\omega=\omega_s$ is the sum of contributions from the various homogeneous packets whose center frequencies are distributed under the Doppler curve and whose responses $\chi'(\omega)$ are of the form shown in Eq. (29) and in Fig. 1. Note that for each homogeneous group of center frequency $\omega'=\omega_0(1+v/c)$ the strength of the response is the same for $\chi'(\omega)$ as for $\chi''(\omega)$, in each case being proportional to $\chi_0''\,\mathcal{S}$ and to the weighting factor $g(\omega')$ as well. Because of the odd symmetry of $\chi'(\omega)$ the net phase shift at $\omega=\omega_s$ is positive (negative) if ω_s is greater (less) than ω_0. If the gain χ_0'' is increased, two things happen: (a) Packets which are well removed from the hole contribute net additional phase shift; (b) on the other hand, because of the even symmetry of the hole itself, those packets close to $\omega=\omega_s$ which formerly contributed some net phase shift there no longer do so since the contributions from one side of the hole cancel those from the other side. These two effects oppose each other, and apparently they cancel exactly in the single-hole theory.

If now we take account of the saturation which occurs at the image hole, then obviously those atoms which in the absence of saturation were able to supply additional phase shift as χ_0'' was increased can no longer do so. Thus the image hole causes a reduction in the amount of added phase shift, and hence a reduction in the amount of pulling; i.e., the presence of the image hole leads to mode pushing. This is the origin of the statement that "holes repel each other." These arguments are identical to those given by Bennett (in Sec. 9 of the first of Bennett's papers cited in Ref. 9) except that the discussion there refers more generally to any second hole, not just the image hole, as here.

In summary then (a) there is a residual amount of pulling associated with the unsaturated gain, (b) the primary hole by itself does not alter this pulling, and (c) the presence of the image hole modifies the pulling by introducing a power dependence [Eq. (80) being replaced by (75)], but generally the more important effect is the introduction of the power-dependent pushing terms (74). Whereas for the gain the primary hole is a major influence and the image hole a negligible one (except near line center), for the phase shift the situation is reversed. Also, it is to be noted from Eqs. (74) and (75), which are expansions in powers of $1/\sigma$, that the leading term in $A(s^2,\Delta^2)$ is zero order in $1/\sigma$, whereas $B(s^2,\Delta^2)$ appears at least to first order. Thus for a strongly inhomogeneous line ($\sigma\gg1$) the pushing terms may even dominate.

For weak fields the coefficient $M(s^2,\Delta^2)$ can be expanded to first order in s^2 by using the result

$$\lim_{s^2\ll1}(r-\zeta)/(r+\zeta)=s^2/2\zeta. \tag{81}$$

Then we find that

$$\lim_{s^2\ll1}M(s^2,\Delta^2)=-\frac{4}{\Delta}\int_0^{\Delta/\sigma}d\eta\, e^{\eta^2}+s^2\left[\frac{\pi^{1/2}-4/\sigma}{2\zeta}+\frac{4/\sigma}{\zeta}\int_0^{\Delta/\sigma}d\eta\, e^{\eta^2}\left(\frac{\Delta}{\sigma}-\eta\right)-\frac{2}{\Delta}\int_0^{\Delta/\sigma}d\eta\, e^{\eta^2}\right]. \tag{82}$$

As expected, the lowest-order contribution to pushing is first order in s^2, whereas for pulling there is a zero-order contribution. Hence for weak fields we can expect the pulling terms to dominate. In the approximation that the power-dependent terms are negligible we may write

$$\lim_{s^2\to0}M(s^2,\Delta^2)=-\frac{4}{\Delta}\int_0^{\Delta/\sigma}d\eta\, e^{\eta^2}\approx-\frac{4}{\sigma} \tag{80$'$}$$

if $\Delta/\sigma\ll1$ (i.e., if ω_s is not in the wings of the Gaussian). Then the frequency condition (79) becomes

$$\omega_s=\omega_c+(2/\pi)(Q_a/Q_c)(\omega_0-\omega_c). \tag{83a}$$

This result is formally similar to the mode-pulling equation which is appropriate to a homogeneous line of width $2/T_2$:

$$\omega_s=\omega_c+(Q_h/Q_c)(\omega_0-\omega_c), \tag{83b}$$

where $Q_h=\frac{1}{2}\omega_0T_2$ is the "homogeneous line Q." In (83a) Q_a of course pertains to the full Doppler line [cf. Eq. (77)].

VIII. DISCUSSION

Lamb[19] has earlier given a complete theory for a gas laser which is based on a perturbation expansion for the atomic polarization and is thus valid for weak fields ($s^2\ll1$). Whereas we consider pumping, collisions, and decays all phenomenologically (through the formalism of the Bloch equations), Lamb considers pumping directly, neglects collisions, and considers decays phenomenologically. All of Lamb's results for a single-mode laser can be recovered if we take the present results for a strongly inhomogeneous line and consider excitations only slightly above threshold. For purposes of comparison it is necessary to make the following correspondences[20,21]:

$$R_0\leftrightarrow\mathfrak{N},\qquad T_1\leftrightarrow\gamma_{ab}/\gamma_a\gamma_b,\qquad T_2\leftrightarrow1/\gamma_{ab}. \tag{84}$$

[19] W. E. Lamb, Jr., Phys. Rev. **134**, A1429 (1964).

[20] In Lamb's paper, Ref. 19, $\mathfrak{N}$ is the "relative excitation," γ_a and γ_b are the decay constants of the two states, and $\gamma_{ab}=\frac{1}{2}(\gamma_a+\gamma_b)$. In terms of the lifetimes $\tau(=1/\gamma)$ we can also write $T_1=\frac{1}{2}(\tau_a+\tau_b)$, $T_2=2\tau_a\tau_b/(\tau_a+\tau_b)$.

[21] A similar interpretation of T_1 and T_2 was made by W. E. Lamb, Jr. [in *Lectures in Theoretical Physics*, edited by W. E. Brittin and B. W. Downs (Interscience Publishers, Inc., New York, 1960), Vol. II; see especially pp. 472–477], where the homogeneous response is derived for a traveling-wave excitation.

The correspondence of the mean decay rate γ_{ab} with the dephasing frequency T_2^{-1} is not surprising, since it has in fact been suggested by several authors[22] that the Lamb theory could be extended so as to take account of collisions by just such a generalization.

Assuming the relations (84) we find that the power-tuning characteristic as given by Lamb[23] in Eq. (L96) is equivalent to the above result (69) taking only the term linear[24] in $R-1$; i.e.,

$$s^2(\Delta)=[2\zeta/(\zeta+1)](R-1). \tag{85}$$

Similarly, the condition for threshold[25] with detuning, Eq. (L97), has its counterpart here in Eq. (59), which says merely that at threshold the unsaturated gain and the loss have a unity ratio.

The conditions (64) describing the threshold for the appearance of the central tuning dip are found to reduce to those given in Eqs. (L99) and (L100) provided that α_c is taken to lowest order in R_0-1, as in Eq. (70).

The frequency conditions given in Lamb's paper in Eqs. (L89)–(L93) correspond to our results (79) and (82) except that Lamb does not include power-dependent terms which go as $1/\sigma$; so for comparison we take (82) in abbreviated form:

$$\lim_{s^2\ll 1} M(s^2,\Delta^2)=-\frac{4}{\Delta}\int_0^{\Delta/\sigma} d\eta\, e^{\eta^2}+s^2\frac{\pi^{1/2}}{2\zeta}. \tag{86}$$

Hence Lamb's power-dependent term describes mode pushing only, while just the leading power-independent term is included for mode pulling. Departures from Lamb's results can be expected whenever we fail to satisfy $s^2\ll 1$ and $\sigma\gg 1$. As we have already seen, when s^2 is not small the mode-pushing effect can be significant and possibly dominant.

An important difference between the present theory and that of Lamb is that the use of a phenomenological relaxation time T_1 to include the effects of pumping allows for a very simple treatment of excitation. It is of course also a grosser description, no attempt having been made here to determine the dependence of T_1 on the excitation R_0.

Another difference from the Lamb theory concerns the equations which determine the amplitude and frequency of oscillation. In the Lamb paper, these equations are arrived at by imposing a self-consistency requirement on the electric fields, considering that they induce an atomic polarization which in turn acts as a source term for the fields. An alternate approach is to calculate the explicit dependence of the nonlinear susceptibility upon both the intensity and frequency of oscillation; then the oscillation condition (1) provides the desired equations. This latter method is the one used here. It was also used earlier by Lamb[21] in discussing the equations of motion appropriate to an atom at rest in a monochromatic field (or, what is mathematically equivalent, an atom in motion which interacts with but one traveling-wave component of the standing-wave field).

Although the Lamb theory is based on a third-order perturbation expansion, Lamb's paper[19] does also contain a discussion (see Secs. 16–19) of the case of strong fields. In particular the expression[23] (L183) is analogous to our result (30); in both cases these equations show the effects of saturation by each of the two traveling-wave components. In order to obtain Eq. (L183) it was necessary to neglect the population pulsations and to make a plausible assumption for the rate constant. In our treatment the time-dependent terms of the population difference were averaged to zero (except for slowly moving atoms) because, having chosen to represent the relaxation processes in the phenomenological manner of Bloch, we were then forbidden, as a matter of self-consistency, to describe events which took place over time intervals less than Δt (see discussion of Sec. IV). Also, the fact that we have included pumping in the T_1 relaxation term provides an additional simplification. For the inhomogeneous response Lamb's Eqs. (L185) and (L186) illustrate how the saturation effects are different when the holes are well separated and when they overlap completely. The difference, of course, is a factor of 2, as we have also shown in Eqs. (61) and (62).

Finally, it should be pointed out that the question of the magnitude of the dimensionless intensity s^2 can be phrased in very simple and practical terms. Let us say that a characteristic value of s^2 is its value when the cavity is tuned to the center of the Doppler line. Here the inhomogeneous saturation factor is $(1+2s^2)^{-1/2}$, from Eq. (62). This number may be evaluated if it is known how far the cavity must be detuned to reach threshold, since the ratio of unsaturated gain at the detuning threshold to its value at line center is also equal to the saturation factor for central tuning. The frequency dependence of the unsaturated gain is of course a Gaussian [see Eq. (51)]. Thus, for example, if it is possible to detune from line center by an amount of the order of the Doppler half-width, so that the unsaturated gain falls by a factor of, say, 2, then we have $s^2=1.5$ at line center. The condition, therefore, that a characteristic value of s^2 be much less than unity means simply that the detuning threshold must be small compared to the Doppler half-width.

ACKNOWLEDGMENTS

Many of the ideas presented in this paper were stimulated by a course in quantum electronics which

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[22] Including Lamb, Ref. 19, Sec. 21. For further discussion of collision effects see Ref. 10.

[23] Equations referring to Ref. 19 will be indicated by a prefix L, as in (L96).

[24] A calculation extending Lamb's theory to second order in $R-1$ (fifth order in Schrödinger perturbation theory) was performed by K. Uehara and K. Shimoda, Japan. J. Appl. Phys. **4**, 921 (1965). Their result does not agree with the second-order terms of our expression (69). A higher-order calculation was also made by W. Culshaw, Phys. Rev. **164**, 329 (1967).

[25] Threshold conditions were first discussed by A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1940 (1958).

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APPENDIX A: EVALUATION OF GAIN PROFILE

The integral for the gain profile $\chi''(\omega, \omega_s)$ was given in Eq. (37). In this Appendix, we shall discuss its evaluation for a strongly inhomogeneous line, for which $g(\omega')$ varies slowly in the region where the other factors are significant so that it can be removed from the integrand as $g(\omega)$. Then the integral in question is

$$\chi''(\omega, \omega_s) = (\chi_0''/2\pi T_2) g(\omega) K(s^2, \Delta, \delta_1), \quad \text{(A1)}$$

with

$$K(s^2, \Delta, \delta_1) = \int_{-\infty}^{\infty} \frac{dx}{1+(x-\delta_1)^2} \times \left[1+s^2\left(\frac{1}{1+x^2} + \frac{1}{1+(x-\Delta)^2}\right)\right]^{-1}. \quad \text{(A2)}$$

[See Eq. (38) for definitions of x, Δ, and δ_1.] By making the change of variable $x \to u$, with

$$u = x - \tfrac{1}{2}\Delta,$$

the integral for K can be written in the form

$$K(s^2, \Delta, \delta_1) = \pi - 2s^2 \int_{-\infty}^{\infty} \frac{du}{1+(u+\frac{1}{2}\Delta-\delta_1)^2} \times \frac{1+\frac{1}{4}\Delta^2+u^2}{(1+\frac{1}{4}\Delta^2+u^2)^2 - u^2\Delta^2 + 2s^2(1+\frac{1}{4}\Delta^2+u^2)}. \quad \text{(A3)}$$

Evaluation of this integral is most readily performed with the technique of contour integration, the result being

$$K(s^2, \Delta, \delta_1) = \pi + 2s^2(K_+ - K_-),$$

$$K_\pm = \frac{\pi}{h_+^2 - h_-^2} \frac{(\zeta - h_\pm^2)(1+h_\pm)}{h_\pm[(\delta_1 - \frac{1}{2}\Delta)^2 + (1+h_\pm)^2]}. \quad \text{(A4)}$$

The second denominator of (A3) has four simple poles, two of which have residues which contribute to the integral; these are located at $u = jh_\pm$, where

$$h_\pm^2 = \lambda \pm (\lambda^2 - r^2)^{1/2},$$
$$h_\pm \equiv [\tfrac{1}{2}(\lambda+r)]^{1/2} \pm [\tfrac{1}{2}(\lambda-r)]^{1/2},$$
$$\lambda \equiv 2 + s^2 - \zeta,$$
$$r \equiv (\zeta^2 + 2s^2\zeta)^{1/2} = h_+ h_-,$$
$$\zeta \equiv 1 + \tfrac{1}{4}\Delta^2. \quad \text{(A5)}$$

It is a simple matter to show that $(\lambda + r) > 0$, so that if $h_\pm$ is complex $(\lambda < r)$, then $\mathrm{Re}\, h_\pm > 0$. If $h_\pm$ is real $(\lambda > r)$, it is also positive.

Although the expression (A4) is correct, it is not a result which can be readily interpreted. However there are two special cases for which the integral (A2) can be evaluated in a straightforward fashion. These are (i) $\Delta \gg w$, and (ii) $\Delta = 0$. [See Eq. (38) for definitions of Δ, w, etc., and the accompanying text for the physical interpretation.]

$\Delta \gg w$

The saturation factor $\mathcal{S}(x)$ of Eq. (A2) can be written

$$\mathcal{S}(x) = \{1 + s^2[v_1(x) + v_2(x)]\}^{-1}, \quad \text{(A6)}$$

where

$$v_1(x) = (1+x^2)^{-1},$$
$$v_2(x) = [1-(x-\Delta)^2]^{-1}. \quad \text{(A7)}$$

If Δ is a large quantity, the two Lorentz-type functions, which are peaked at $x=0$ and $x=\Delta$ for $v_1(x)$ and $v_2(x)$, respectively, are widely separated so that in the region where v_1 is significant v_2 vanishes, and conversely. Hence we are led to approximate $\mathcal{S}(x)$ by

$$\mathcal{S}(x) = [1+s^2 v_1(x)]^{-1} + [1+s^2 v_2(x)]^{-1} - 1. \quad \text{(A8)}$$

Now the evaluation of K is quite simple:

$$K(s^2, \Delta, \delta_1) = \int_{-\infty}^{\infty} \frac{dx}{1+(x-\delta_1)^2} \mathcal{S}(x). \quad \text{(A9)}$$

This leads directly to the result (40).

$\Delta = 0$

In this case, the saturation factor $\mathcal{S}(x)$ becomes (without approximation)

$$\mathcal{S}(x) = [1 + 2s^2 v_1(x)]^{-1} \quad \text{(A10)}$$

and the integral (A9) is again very simple; the result is given in Eq. (42). That the result (A4) for arbitrary Δ reduces to (42) when $\Delta = 0$ can be shown readily. To show that (A4) reduces to (40) when $\Delta \gg w$ is not trivial, but it can be verified. However, when $\Delta \gg w$ by far the simplest approach is to use the approximation (A8).

APPENDIX B: GAIN FUNCTION $\chi''(\omega_s)$

The integral for $\chi''(\omega_s)$ was established in Eqs. (47)–(50) as

$$\chi''(\omega_s) = \frac{\chi_0''}{\pi} (4\pi\alpha)^{1/2} \int_{-\infty}^{\infty} du \exp(-4\alpha u^2) \times (\zeta + u^2)/[(\zeta+u^2)^2 - u^2\Delta^2 + 2s^2(\zeta+u^2)], \quad \text{(B1)}$$

where, as before,

$$\zeta = 1 + \tfrac{1}{4}\Delta^2. \quad \text{(B2)}$$

In the limit of a strongly inhomogeneous line $(\alpha \ll 1)$

we may factor the slowly varying exponential[26] as $\exp(-\alpha\Delta^2)$ [since the response function (45) is resonant at $x=0$, i.e., $u=-\frac{1}{2}\Delta$; the form (B1) obtains only after making the transformation (48) and a few obvious algebraic simplifications]. Then we can write

$$\chi''(\omega_s)=\chi_0''(4\pi\alpha)^{1/2}\exp(-\alpha\Delta^2)\mathcal{g}(s^2,\Delta^2)\qquad(\alpha\ll 1),\tag{B3}$$

where

$$\mathcal{g}(s^2,\Delta^2)=\int_{-\infty}^{\infty}\frac{du}{\pi}\frac{\zeta+u^2}{(\zeta+u^2)^2-u^2\Delta^2+2s^2(\zeta+u^2)},\tag{B4}$$

which can be evaluated directly to give

$$\mathcal{g}(s^2,\Delta^2)=(1+\zeta/r)/[2(\lambda+r)]^{1/2}.\tag{B5}$$

[The parameters λ and r were defined above, Eq. (A5).] This is the result shown in Eqs. (51) and (52).

We can evaluate the integral (B1) for arbitrary values of α by factoring the denominator (there are poles at $u^2=-h_\pm^2$) and expanding the result into two partial fractions. The basic integral which must be evaluated is

$$J(\alpha,c)=2\int_0^\infty du\,\frac{\exp(-4\alpha u^2)}{u^2+c}.\tag{B6}$$

The change of variable $u^2=x$ brings this into the standard form for the Stieltjes transform of $x^{-1/2}\exp(-4\alpha x)$ which is given in the Bateman Tables.[27] We then have

$$J(\alpha,c)=\pi c^{-1/2}\exp(4\alpha c)\,\mathrm{Erfc}[(4\alpha c)^{1/2}],\tag{B7}$$

provided that $|\arg c|<\pi$, where

$$\mathrm{Erfc}(z)=2\pi^{-1/2}\int_z^\infty dt\exp(-t^2)$$
$$=1-\mathrm{Erf}(z).\tag{B8}$$

This allows us to evaluate $\chi''(\omega_s)$ for arbitrary α according to

$$\chi''(\omega_s)=\frac{\chi_0''(4\pi\alpha)^{1/2}}{h_+^2-h_-^2}\left(\frac{\zeta-h_-^2}{h_-}\exp(4\alpha h_-^2)\,\mathrm{Erfc}[(4\alpha)^{1/2}h_-]-\frac{\zeta-h_+^2}{h_+}\exp(4\alpha h_+^2)\,\mathrm{Erfc}[(4\alpha)^{1/2}h_+]\right).\tag{B9}$$

[See Eq. (A5) for definitions of $h_\pm$.]

We should note that if the exact result (B9) is taken in the limit of a strongly inhomogeneous line ($\alpha\ll 1$) we do not recover the earlier result given by Eqs. (B3) and (B5), which were derived on the basis of a cruder but nevertheless useful approximation.

The homogeneous limit of large α can be recovered immediately from the integral (B1) by recognizing that

$$\lim_{\alpha\to\infty}(4\alpha/\pi)^{1/2}\exp(-4\alpha u^2)=\delta(u),\tag{B10}$$

so that the u integration is trivial and yields

$$\lim_{\alpha\to\infty}\chi''(\omega_s)=\chi_0''/(\zeta+2s^2)\tag{B11}$$

or

$$\lim_{\alpha\to\infty}\chi''(\omega_s)=\frac{\chi_0''}{1+2s^2+T_2^2(\omega_s-\omega_0)^2}.$$

As expected, in the homogeneous limit the response is Lorentzian. The associated power output is parabolic and of the form $2s^2=\chi_0''Q_c-1-\frac{1}{4}\Delta^2$. The result (B11) is also obtainable from (29) if the homogeneous saturation factor $\mathcal{S}$ is evaluated for atoms of zero velocity.

Finally, we mention the limiting form of $\chi''(\omega_s)$ for $s^2\gg 1$. We have, for $s^2\gg 1$,

$$\lim h_+^2=2s^2,$$
$$\lim h_-^2=\zeta,$$
$$\lim \mathrm{Erfc}[(4\alpha)^{1/2}h_+]=[\exp(-4\alpha h_+^2)/(4\pi\alpha)^{1/2}h_+],\tag{B12}$$

so that from (B9) we find

$$\lim_{s^2\gg 1}\chi''(\omega_s)=\chi_0''/2s^2.\tag{B13}$$

In this limit, the power output [which comes from Eq. (5)] becomes

$$s^2(\Delta)=\tfrac{1}{2}(\chi_0''Q_c)\qquad(s^2\gg 1).\tag{B14}$$

Hence for sufficiently high intensity the power-tuning characteristic is flat, and, as anticipated[13] on the basis of the power broadening of the holes in the gain profile, there is no Lamb notch in this limit.

APPENDIX C: PHASE FUNCTION $\chi'(\omega_s)$

The integral for $\chi'(\omega_s)$ was given in Eq. (71). If we make the transformation (48) and use the form (49) for $g(u)$ we can express $\chi'(\omega_s)$ as

$$\chi'(\omega_s)=\tfrac{1}{2}\Delta(\chi_0''/\pi)(4\pi\alpha)^{1/2}\times\int_{-\infty}^{\infty}du\exp(-4\alpha u^2)\,\frac{u^2-\zeta}{(u^2+\zeta)^2-u^2\Delta^2+2s^2(u^2+\zeta)}.\tag{C1}$$

In obtaining this form, terms of the numerator which are odd in u were dropped since they will not contribute to the integral. Notice that removal of the exponential outside the integral will not destroy the convergence properties of (C1), although from the unsymmetrized form (71) it would appear that the resultant integral would diverge. Despite the favorable convergence

[26] This factored form should not be considered as the leading term in a power-series expansion of the exponential about $u=-\frac{1}{2}\Delta$ since the higher-order integrals would all diverge.

[27] *Tables of Integral Transforms*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1954), Vol. II. See especially p. 217, item (15).

properties of the symmetrized integral (C1) it is still not valid to approximate $\chi'(\omega_s)$, in the limit of a strongly inhomogeneous response ($\alpha \ll 1$), by factoring the exponential outside the integral as was done for $\chi''(\omega_s)$ in Appendix B. In the latter case, the dominant factor in the response is $(1+x^2)^{-1}$ [see Eq. (45)], so that for a slowly varying exponential we could say that this factor behaves something like a δ function, and in this sense we are justified in removing the exponential. However, in the integral (71) the dominant factor is $x(1+x^2)^{-1}$; the behavior of this function is significantly different, and therefore a more careful treatment of the integral is required.

The integral (C1) is very similar to the integral (B1) that we had for $\chi''(\omega_s)$ and may be evaluated for arbitrary values of α by the same techniques discussed in Appendix B. We find that

$$\chi'(\omega_s) = -\tfrac{1}{2}\Delta[\chi_0''(4\pi\alpha)^{1/2}/(h_+^2-h_-^2)] \times\left(\frac{\zeta+h_-^2}{h_-}\exp(4\alpha h_-^2)\,\mathrm{Erfc}[(4\alpha)^{1/2}h_-] - \frac{\zeta+h_+^2}{h_+}\exp(4\alpha h_+^2)\,\mathrm{Erfc}[(4\alpha)^{1/2}h_+]\right). \quad \text{(C2)}$$

[See Eq. (A5) for definitions of $h_\pm$.] Although $\chi'(\omega_s)$ is always real, the expression (C2) involves complex quantities $h_\pm$ whenever $r>\lambda$. We can express $\chi'(\omega_s)$ entirely in terms of real quantities[28] by considering separately the cases of $r<\lambda$ and $r>\lambda$. For convenience in applying the result to Eq. (78) we write $\chi'(\omega_s)$ in the form

$$\chi'(\omega_s) = \chi_0''(\Delta/\sigma)\exp[-(\Delta/\sigma)^2]\mathcal{I}(s^2,\Delta^2)M(s^2,\Delta^2), \quad \text{(C3)}$$

where [$\mathcal{I}(s^2,\Delta^2)$ is given in Eq. (52)]

$$M(s^2,\Delta^2) = \tfrac{1}{2}\pi^{1/2}\exp[4\alpha(1+s^2)] \times\left[\left(\frac{\lambda+r}{\lambda-r}\right)^{1/2}H_- + \frac{r-\zeta}{r+\zeta}H_+\right], \qquad r<\lambda \quad \text{(C4)}$$

$$M(s^2,\Delta^2) = \pi^{1/2}\exp[4\alpha(1+s^2)] \times\left\{\mathrm{Erfc}\left[\frac{2x}{\sigma}\right]\left(\frac{r-\zeta}{r+\zeta}\cos 8\alpha xy + \frac{x}{y}\sin 8\alpha xy\right) - 2\pi^{-1/2}\exp[-4x^2/\sigma^2]\int_0^{2y/\sigma} d\eta\,\exp(\eta^2) \times\left[\frac{r-\zeta}{r+\zeta}\sin\left(\frac{4\eta x}{\sigma}-8\alpha xy\right) + \frac{x}{y}\cos\left(\frac{4\eta x}{\sigma}-8\alpha xy\right)\right]\right\}, \qquad r>\lambda \quad \text{(C5)}$$

with

$$\sigma^2 \equiv 1/\alpha,\quad x^2 \equiv \tfrac{1}{2}(r+\lambda),\quad y^2 \equiv \tfrac{1}{2}(r-\lambda), \quad \text{(C6)}$$

and

$$H_\pm \equiv \exp[4\alpha(\lambda^2-r^2)^{1/2}]\,\mathrm{Erfc}(2h_+/\sigma) \pm \exp[-4\alpha(\lambda^2-r^2)^{1/2}]\,\mathrm{Erfc}(2h_-/\sigma). \quad \text{(C7)}$$

To obtain the inhomogeneous limit ($\alpha \ll 1$) we expand these results to first order in $1/\sigma$. As an additional simplification we assume that the intensity s^2 is not extraordinarily large, so that $\alpha s^2 \ll 1$ also.[29] In taking the limit of small α, care must be taken not to treat $\alpha\zeta$, for example, as a small quantity since over the full Doppler width ζ can assume values as large as $1/\alpha$. When Eqs. (C4) and (C5) are taken to first order in $1/\sigma$, these expressions reduce to those given in Eqs. (73)–(75). The condition that $r \lessgtr \lambda$ is equivalent to $\Delta^2 \lessgtr \Delta_0^2$, where

$$\Delta_0^2 = s^4/(1+s^2). \quad \text{(C8)}$$

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[28] The function $\mathrm{Erfc}(x+jy)$ separates into real and imaginary parts according to

$$\mathrm{Erfc}(x+jy) = \mathrm{Erfc}(x) - (2\pi^{-1/2})e^{-x^2}\int_0^y d\eta\, e^{\eta^2}\sin 2x\eta - j(2\pi^{-1/2})e^{-x^2}\int_0^y d\eta\, e^{\eta^2}\cos 2x\eta.$$

[29] For an inhomogeneous line the hole width may typically be $\frac{1}{10}$ of the Doppler width, in which case α is of order 10^{-3}; see Eq. (50).