tially, all atoms are in the ground level and no radiation is present. It may then be easily shown that $F_1^R(t)$ remains zero for all time. This is expected, because in this model there is no way to get atoms out of the ground state and to create photons. Therefore nothing happens.

[11]In both Case I and Case II, it can be noted that $4ac \ll b^2$, thereby allowing the expansion $(b^2-4ac)^{1/2} \simeq |b|(1-2ac/b^2)$.

[12]J. Premanand and D. L. Falkoff, Phys. Rev. 163, 178 (1967).

[13]The factor 1.65 has arbitrary origin. It is the value taken by the Boltzmann factor, $(g_l/g_u)\exp(h\nu_{ul}/\theta)$ for $\theta = 2h\nu_{ul}$ in a hypothetical nondegenerate two-level system ($g_u = g_l = 1$). It is perhaps best to view 1.65 as nothing more than an arbitrary factor, inasmuch as the notion of a temperature is as often an encumbrance as an aid when the system is not at or near equilibrium.

[14]Recall that $f_1^R(\nu, t)$ is defined in the continuum. See Eq. (11) *et seq*.

[15]Cases B were included primarily because they represent a set of initial conditions that are more likely to exist, physically, than the conceptually useful Cases A (Cases A correspond to the relaxation of an initial distribution of atoms with an extreme measure of population inversion). Cases B may be viewed to correspond, at least crudely, to a system of atoms (initially distributed in the ratio of 1.65) that is instantaneously irradiated by an arbitrary light source at $t=0$.

[16]It may finally be noted that, for any of the examples in Table I and/or Figs. 1–6, the asymptotic radiation density can be used, after the fact, to define a Planck temperature that also turns out to satisfy Boltzmann's distribution law for the corresponding asymptotic atomic level populations. This calculated asymptotic temperature is generally found to be different from any of the initial temperatures (the initial temperatures being determined from the initial atomic level populations and the initial radiation densities). At any rate, temperature is simply a defined quantity here and is in no way essential to the basic formulation and solution of our kinetic equations. However, if one were to persist with a wish to keep track of time-dependent temperatures for the various components of a given system, it would still be necessary to first solve for the distribution functions, *per se*. Then, with this knowledge and a set of appropriate (and reasonably well-agreed-upon) definitions, the corresponding evolution of temperatures could be computed. It has, of course, been demonstrated in this section that such an evolution is very much dependent upon system parameters such as linewidths, statistical weights, and particle and radiation densities. With the availability of large, high-speed computers, the approach that we are presenting is feasible for much more complicated physical systems as well.

[17]T. Suyehiro, Lawrence Radiation Laboratory, Livermore, California, C.I.C. Report No. D2.3-001, 1965 (unpublished).

[18]The results of the above cases, in which higher-order terms were neglected (thereby reducing to a set of three simultaneous differential equations), have also been verified by this numerical method.

[19]This point may also have been noticed by Premanand and Falkoff.[12] For they noted that the fluctuations may be non-negligible, and that the higher-order terms should, in general, be included in conventional detailed-balance equations for systems not at equilibrium. They did not, however, indicate that the mean values (singlet densities) *must*, in general, change noticeably by the inclusion of these additional terms (although it is suspect). Indeed, the present work explicitly shows that, not only asymptotically but at all times, the singlet densities of photons and atoms are essentially unaffected by the inclusion of, or the neglect of, the higher-order terms in the simple system models under present consideration.

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Coupling of Density and Spin Fluctuations to Quasiparticles in a Fermi Liquid*

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The general interaction between quasiparticles and particle-hole excitations in a neutral Fermi system at $T=0$ is investigated. Density and spin-density fluctuations and collective modes are treated on an equal basis. Use is made of Ward identities to relate the vertex function Γ to the quasiparticle self-energy Σ, and logarithmic corrections to the self-energy are obtained. The coefficients of the logarithmic terms are calculated analytically in terms of known Landau parameters.

I. INTRODUCTION

The microscopic theory of a normal Fermi liquid,[1-4] formulated in terms of Green's functions, rests on certain assumptions about the regularity of the mass operator, Σ (single-particle self-energy). Specifically, it is assumed that the real part of the mass operator, $\mathrm{Re}\Sigma \equiv M$, can be expanded in a power series near the Fermi surface, i.e., when $|\vec{q}| = q \to k_F$, $\epsilon \to 0$:

$$M(\vec{q}, \epsilon) = M(k_F, 0) + \epsilon \frac{\partial M(k_F, \epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} + (q - k_F)\frac{\partial M(\vec{q}, 0)}{\partial q}\bigg|_{q=k_F} + \cdots , \qquad (I.1)$$

for an isotropic system. From this, one finds that the quasiparticle energy is proportional to $(q - k_F)$, which leads to a specific heat C_V, linear in temperature T as $T \to 0$.[5] One would expect C_V to attain this asymptotic behavior at temperatures which are an order of magnitude below the effective degeneracy temperature, about 1°K in He^3.

However experiments on liquid He^3, especially those of the Illinois group,[6] indicate that $C_V T^{-1}$ still increases noticeably as T decreases in the interval from 100 m°K to a few m°K.[7]

Several years ago Anderson[8] suggested that the mass operator may behave in an unexpectedly singular way near the Fermi surface, because of the exchange of collective excitations between quasiparticles. Balian and Fredkin[9] tried to estimate the coupling of zero sound to quasiparticles and found a term in the mass operator proportional to $\epsilon \ln \epsilon$, which in turn leads to a specific head $C_V \propto T \ln |B/T|$. Subsequently, Engelsberg and Platzman[10] pointed out that the result of Balian and Fredkin requires a "piezoelectric" form of quasiparticle–zero-sound coupling which leads to inconsistencies. They suggested that the coupling should be of the "deformation-potential" form, which in analogy to the electron-phonon coupling in metals[11] results in a correction to Eq. (I.1) proportional to $\epsilon^3 \ln \epsilon$ and a $T^3 \ln T$ correction[12] to the linear specific heat. However, the observed temperature variation of C_V in liquid He^3 cannot be explained on this basis. The coefficient of this logarithmic term has the wrong sign, leading to a specific heat $C_V T^{-1}$ which decreases rather than increases with decreasing temperature.

The interaction between quasiparticles and virtual zero-sound quanta is not the only source of a $\epsilon^3 \ln \epsilon$ term in M. A second-order calculation of M shows that terms of this form are present [compare also Ref. 20]. In a Fermi liquid the long-wavelength particle-hole spectrum consists of coherent collective modes and incoherent density and spin density fluctuations. The entire spectrum is contained in the poles and branch cuts of the vertex function Γ, when considered as a function of the energy (ω) and momentum ($\vec{k}$) transfer in a scattering process involving two quasiparticles. In the limit $\omega \to 0$, $\vec{k} \to 0$, Γ depends on ω and $|\vec{k}| = k$ only as ω/k. It is precisely this nonanalytic behavior in k which leads to logarithmic terms in the mass operator.

Recently Berk and Schrieffer[13] and Doniach and Engelsberg[14] pointed out that liquid He^3 may be regarded as a nearly ferromagnetic Fermi system. The low-temperature spin susceptibility of He^3, particularly near the melting pressure, is greatly enhanced over that of a noninteracting system.[5] Consequently, liquid He^3 may be expected to exhibit large, incoherent spin density fluctuations. The virtual interaction between quasiparticles and these "paramagnons" leads to logarithmic corrections in C_V with the correct sign and order of magnitude.[14] These calculations hint at a possible explanation of the anomalous specific heat of liquid He^3.

In this paper we investigate in general the interaction between quasiparticles and particle-hole excitations in a neutral Fermi system at zero temperature. The "paramagnons" and density fluctuations (including zero sound) are treated on an equal basis. As previously mentioned, the dependence of the particle-hole spectrum on ω/k, in the long-wavelength limit, leads to logarithmic terms in the mass operator. Apart from this, we also find that short-wavelength particle-hole excitations ($k \simeq 2k_F$) yield additional logarithmic terms of the same order.

The calculation of the coefficients of all the logarithmic terms is carried out in the framework of the microscopic Landau theory.[1] The coefficients are expressed in terms of Landau parameters. In this analysis we shall use Ward identities to relate the vertex function Γ to the mass operator Σ and the crossing symmetries of Γ which follow from the Pauli principle.

A brief preliminary report of this work has appeared earlier.[15]

II. FORMAL PRELIMINARIES

The Green's functions are defined as ground-state expectation values of time-ordered products of fermion creation and annihilation operators ψ and $\psi^\dagger$. Specifically, the one- and two-particle Green's functions have the form

$$G_{\alpha\beta}(x_1, x_2) = -i\langle | T\psi_\alpha(x_1)\psi_\beta^\dagger(x_2)|\rangle = \int \frac{d^4Q}{(2\pi)^4} G_{\alpha\beta}(Q) e^{iQ(x_1 - x_2)} , \quad G_{\alpha\kappa\beta\lambda}(x_1; x_1'; x_2, x_2')$$

$$= (-i)^2 \langle | T\psi_\alpha(x_1)\psi_\kappa(x_1') \; \psi_\lambda^\dagger(x_2')\psi_\beta^\dagger(x_2)|\rangle \quad (II.1)$$

$$= \int \frac{d^4K}{(2\pi)^4} \int \frac{d^4Q_1}{(2\pi)^4} \int \frac{d^4Q_2}{(2\pi)^4} G_{\alpha\kappa\beta\lambda}(K; Q_1, Q_2) \times e^{i[K(x_1 - x_2') + Q_1(x_1 - x_1') + Q_2(x_1' - x_2')]} , \quad (II.2)$$

where $|\rangle$ is the ground-state vector, T the Wick time-ordering operator, $x = (\vec{r}, t)$, $K = (\vec{k}, \omega)$, $Q_i = (\vec{q}_i, \epsilon_i)$, $Kx = \vec{k} \cdot \vec{r} - \omega t$, and the Greek subscripts label spins. We shall also use unit vectors $\vec{e}$ and $\vec{n}_i$ to write $\vec{k} = \vec{e}k$, $\vec{q}_i = \vec{n}_i q$.

The two-body force between particles is assumed to be spin-independent. Then, in the absence of an external magnetic field, the spin dependence of the Green's functions simplifies considerably

$$G_{\alpha\beta} = G\delta_{\alpha\beta}$$
$$G_{\alpha\kappa\beta\lambda} = \tfrac{1}{2}G^{(1)}\delta_{\alpha\beta}\delta_{\kappa\lambda} + \tfrac{1}{2}G^{(2)}\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\kappa\lambda} . \qquad (II.3)$$

The product of the Pauli matrices in (II.3) can also be written in the form $\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\kappa\lambda} = 2\delta_{\alpha\lambda}\delta_{\beta\kappa} - \delta_{\alpha\beta}\delta_{\kappa\lambda}$.

The mass operator $\Sigma(Q)$ is defined by Dyson's equation

$$G^{-1}(Q)=\epsilon-(q^2/2m)+\mu-\Sigma(Q), \qquad (\text{II}.4)$$

where μ is the exact chemical potential for $T=0$, and m the bare fermion mass. $\Sigma(Q)$ has a spectral representation of the form

$$\Sigma(\vec{q},\epsilon)=\Sigma^{\text{HF}}(\vec{q})+\int_0^{\infty}\frac{d\epsilon'}{2\pi}\frac{\gamma(\vec{q},\epsilon')}{\epsilon-\epsilon'+i\eta}+\int_{-\infty}^{0}\frac{d\epsilon'}{2\pi}\frac{\gamma(\vec{q},\epsilon')}{\epsilon-\epsilon'-i\eta}, \qquad (\text{II}.5)$$

where $\eta>0$ and $\Sigma^{\text{HF}}(\vec{q})$ is the frequency-independent Hartree-Fock self-energy.

The vertex function Γ is obtained from the correlated part of the two-particle Green's function;

$$G(1,1';2,2')=G(1,2)G(1',2')-G(1,2')G(1',2)+iG(1,\bar{3})G(1'\bar{3}')\Gamma(\bar{3},\bar{3}';\bar{4},\bar{4}')G(\bar{4},2)G(\bar{4}',2'). \qquad (\text{II}.6)$$

The "variables" 1, 2, ... are abbreviations for $1\equiv(\vec{r}_1,t_1,\alpha_1)$ etc. Integration and spin summation over repeated barred variables is implied in (II.6). The Fourier transform $\Gamma_{\alpha\kappa\beta\lambda}(K;Q_1,Q_2)$ of the vertex function has a form analogous to (II.2) (see Fig. 1), and $\Gamma_{\alpha\kappa\beta\lambda}$ can be decomposed as in (II.3) into spin-symmetric and spin-antisymmetric parts $\Gamma^{(1)}$ and $\Gamma^{(2)}$, respectively. When Q_1 and Q_2 are on the Fermi surface $[Q_i=(\vec{n}_i k_F,\epsilon_i=0)]$, the vertex functions are denoted by $\Gamma^{(j)}(K;\vec{n}_1,\vec{n}_2)$, $j=1,2$.

III. PROPERTIES OF THE MASS OPERATOR: QUASIPARTICLES

We recall some well established properties of $\Sigma(Q)$ which are valid in normal Fermi systems at zero temperature.

The Real Part of $\Sigma(Q)$

The Hugenholtz-Van Hove theorem[16] relates the chemical potential μ, the mean binding energy per particle of an isotropic, homogeneous fermion

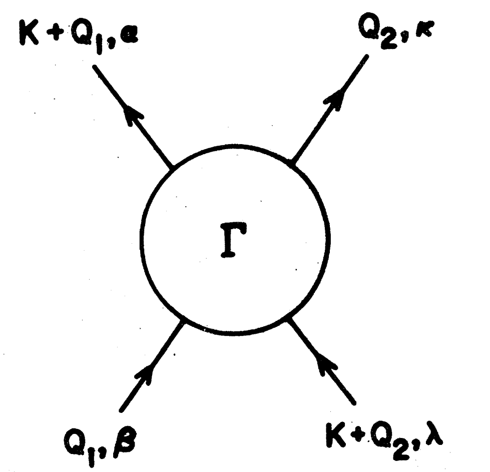

FIG. 1. Momentum and spin variables for the vertex function $\Gamma_{\alpha\kappa\beta\lambda}(K;Q_1,Q_2)$.

system at $T=0$, to $M(Q)$ evaluated on the Fermi surface:[17]

$$\mu=(k_F{}^2/2m)-M(k_F,0), \qquad (\text{III}.1)$$

with $k_F{}^3=3\pi^2\rho$, $\rho=N/\Omega$ the particle density. The nontrivial point to notice is that (III.1) holds for the ideal-gas value of k_F.

The energy dispersion law for $E_{\vec{q}}$ of the single-particle excitations is obtained from

$$E_{\vec{q}}-(q^2/2m)+\mu-M(\vec{q},E_{\vec{q}})=0. \qquad (\text{III}.2)$$

Solution of (III.2) requires knowledge of the analytic properties of M. Assuming that $M(Q)$ can be expanded in a Taylor series near the Fermi surface [see (I.1)] we find from (III.2)

$$E_q=(k_F/m^*)(q-k_F)+O((q-k_F)^2). \qquad (\text{III}.3)$$

The effective mass m^* is defined in terms of derivatives of M evaluated on the Fermi surface

$$a_F{}^{-1}\frac{k_F}{m^*}=\frac{k_F}{m}+\left(\frac{\partial M(\vec{q},0)}{\partial q}\right)_{k_F=q}, \qquad a_F{}^{-1}=1-\left(\frac{\partial M(\vec{k}_F,\epsilon)}{\partial\epsilon}\right)_{\epsilon=0}. \qquad (\text{III}.4)$$

Migdal[18] has shown that the "renormalization constant" a_F is equal to the discontinuity at k_F of the momentum distribution function $n_{\vec{k}}$ of the interacting particles, hence

$$0\le a_F\le 1.$$

A basic hypothesis of Fermi liquid theory is that $a_F>0$.

The Imaginary Part of $\Sigma(Q)$

The imaginary part of $\Sigma(Q)$, which is related to the spectral function $\gamma(Q)$ of (II.5) by

$$\text{Im}\Sigma(\vec{q},\epsilon)=-(\epsilon/2|\epsilon|)\gamma(\vec{q},\epsilon), \qquad (\text{III}.5)$$

is a measure of the inverse lifetime of the single-particle excitations of momentum $\vec{q}$ and energy ϵ. Luttinger[19] has shown that for $\epsilon\ll\mu$

$$\gamma(\vec{q},\epsilon)=\gamma_q\epsilon^2+O(\epsilon^3), \qquad (\text{III}.6)$$

where γ_q is a smooth function of q at least for $q\simeq k_F$. Equation (III.6), valid to all orders of perturbation theory, is essentially a result of phase space restrictions on the decay modes of single-particle excitations, deriving from the Pauli principle.

The denominator of the Green's function (II.4) may now be expanded around $\epsilon=E_q$ with the result

$$G(\vec{q},\epsilon)=a_F/(\epsilon-e_q+i\eta_q)+G^{\text{inc}}(\vec{q},\epsilon), \qquad (\text{III}.7)$$

where $e_q=v_F(q-k_F)$, $v_F=k_F/m^*$. The infinitesimal η_q has the sign of $q-k_F$. The spectral density, $-(\epsilon/|\epsilon|)\text{Im}G^{\text{inc}}(\vec{q},\epsilon)$, of the incoherent part of G vanishes at the Fermi surface, whereas the spectral density of the first term on the right-hand side of (III.7) (the quasiparticle contribution) is $2\pi a_F\delta(\epsilon-e_q)$. Equation (III.7) is valid for the low-lying excitations (near the Fermi surface) where the quasiparticle lifetime increases as $(q-k_F)^{-2}$ according to (III.6). Verifications of Landau's theory of a normal Fermi liquid[2-4] use (III.7) as a starting point.

IV. PROPERTIES OF THE VERTEX FUNCTION: PARTICLE-HOLE EXCITATIONS

The two-particle Green's function, defined in (II.2), has the following rather obvious symmetry property,

$$G(1,1';2,2') = -G(1'1;2,2') = -G(1,1';2',2) \qquad \text{(IV.1)}$$

which is a consequence of the Pauli principle. From its definition (II.6), we see that this symmetry is shared by the vertex function. In terms of momentum variables, the symmetry relation becomes

$$\Gamma^{(j)}(K;Q_1,Q_2) = -\sum_{j'=1}^{2} C_{jj'}\Gamma^{(j')}(Q_1-Q_2;Q_2+K,Q_2) \qquad \text{(IV.2a)}$$

$$= \Gamma^{(j)}(-K;Q_2+K,Q_1-K). \qquad \text{(IV.2b)}$$

where $C_{jj'} = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}_{jj'}$.

The relation (IV.2a) is of some importance in what follows. We discuss its implications in some detail by looking at Feynman diagrams for Γ.

The diagrams of Fig. 2 are representative of the four mutually exclusive classes into which the set of all diagrams of Γ may be divided. A given diagram of Γ may be split into two parts by cutting a pair of lines (i) with four-momentum transfer K [Fig. 2(a)]. (ii) with four-momentum transfer Q_1-Q_2 [Fig. 2(b)]. (iii) with total four-momentum $K+Q_1+Q_2$ [Fig. 2(c)], or finally, (iv) it may not be possible to split the diagram into two parts by cutting just two lines [Fig. 2(d)]. The complete set of diagrams for Γ may be generated by starting with any one of the above four classes of diagrams. If we denote by ${}^{(1)}\Gamma$ all diagrams which do not belong to the class (i), i.e., which do not have a "K cut", we obtain the Bethe-Salpeter equation for Γ by iteration:

$$\Gamma^{(j)}(K;Q_1,Q_2) = {}^{(1)}\Gamma^{(j)}(K;Q_1,Q_2) - i\int\frac{d^4Q_3}{(2\pi)^4} \times {}^{(1)}\Gamma^{(j)}(K;Q_1,Q_3)R(K;Q_3)\Gamma^{(j)}(K;Q_3,Q_2), \qquad \text{(IV.3)}$$

where $R(K,Q_3) = G(K+Q_3)G(Q_3)$. Notice that (IV.3) holds separately for the spin-symmetric ($j=1$) and spin-antisymmetric ($j=2$) part of Γ. The integral equation (IV.3) is depicted schematically in the top line of Fig. 3.

Clearly we may also start with a kernel ${}^{(2)}\Gamma$ which contains no diagrams of class (ii) (no "Q_1-Q_2 cuts") and generate an equation for Γ with intermediate lines $R(Q_1-Q_2,Q_3)$. See the lower line of Fig. 3. The equations for these two-particle hole channels are completely equivalent. In the first channel the momentum transfer K is a parameter; in the second channel the parameter is Q_1-Q_2. The kernels ${}^{(1)}\Gamma$ and ${}^{(2)}\Gamma$ are related through

$${}^{(1)}\Gamma^{(j)}(K;Q_1,Q_2) = -\sum_{j'=1}^{2} C_{jj'}\,{}^{(2)}\Gamma^{(j)}(Q_1-Q_2;Q_2+K,Q_2). \qquad \text{(IV.4)}$$

Thus we see that the crossing symmetry (IV.2a) is a statement of the equivalence of the K and Q_1-Q_2 dependence of Γ.

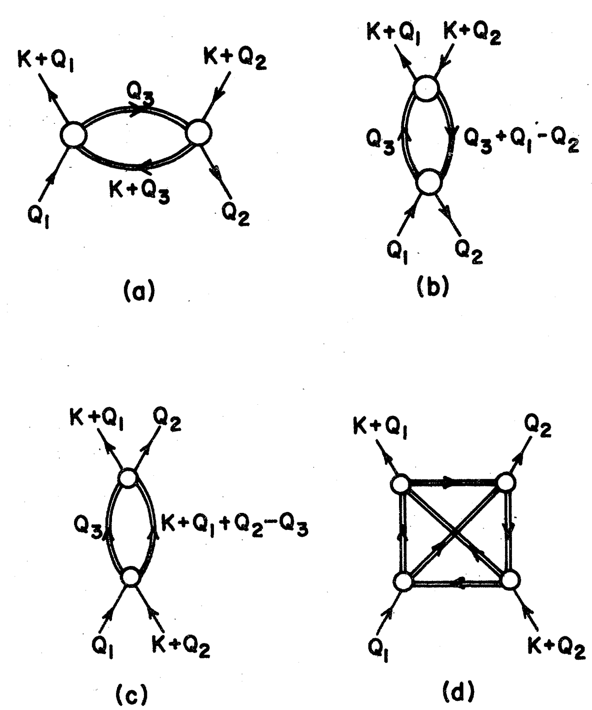

FIG. 2. The four classes of diagrams for the vertex function. ⇒ represents the single-particle Green's function. ○ represents arbitrary diagrams contributing to Γ; Q_3 is an integration variable.

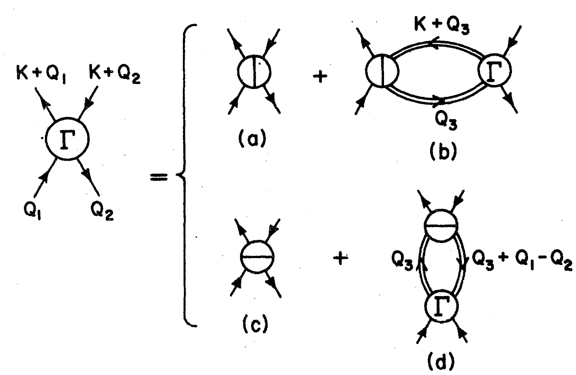

FIG. 3. Bethe-Salpeter equations for the first, "K cut," and second, "Q_1-Q_2 cut," particle-hole channels. The diagrams (a) and (c) represent the irreducible kernels ${}^{(1)}\Gamma$ and ${}^{(2)}\Gamma$, respectively. The diagram (d) is contained in (a) and diagram (b) is contained in (c).

The spectrum of particle-hole excitations is obtained from the singularities of $\Gamma^{(1)}$ (density fluctuations) and $\Gamma^{(2)}$ (spin density fluctuations) in the variable K, or equivalently from the singularities in $Q_1 - Q_2$. If $\Gamma^{(j)}(K; Q_1, Q_2)$ are calculated from (IV.3) the singularities in $Q_1 - Q_2$ are contained in ${}^{(1)}\Gamma^{(j)}(K; Q_1, Q_2)$ exclusively.

Of course the particle-particle channel [class (iii) above] can also be used to generate an equation for Γ with the total four-momentum $K + Q_1 + Q_2$ as the parameter. This equation is transformed into itself under the variable transformation $K \to Q_1 - Q_2$, $Q_1 \to Q_2 + K$, $Q_2 \to Q_2$, corresponding to (IV.2a), since this leaves $K + Q_1 + Q_2$ invariant.

As a rule, the Bethe-Salpeter equation (IV.3) cannot be solved. Neither the "irreducible" kernels ${}^{(1)}\Gamma^{(j)}$ nor the particle-hole propagator $R(K, Q)$ are known in general. However, if we are concerned only with the long-wavelength, low-frequency behavior, it is only necessary to know $\Gamma^{(j)}(K; Q_1, Q_2)$ in the limit $K \to 0$ (i.e., $k \ll k_F$, $\omega \ll \mu$). Since ${}^{(1)}\Gamma^{(j)}$ is, by construction, a smooth function of K, we can set $K = 0$ in ${}^{(1)}\Gamma^{(j)}$ in this limit. The K dependence of the full vertex function is then governed by the behavior of $R(K, Q)$ for $K \to 0$. Using (III.7) for the single-particle Green's function, we obtain an expression for $R(K, Q)$ for $K \to 0$ which depends on the ratio ω/k^3:

$$R(K, Q) = 2\pi i a_F^2 \frac{\omega\delta(e_q)\delta(\epsilon)}{\omega - V_{\vec{q}} \cdot \vec{k} + i\alpha} + R^k(Q), \qquad \text{(IV.5)}$$

where $\operatorname{sgn}\alpha = \operatorname{sgn}(\vec{q} \cdot \vec{k})$. $R^k(Q)$ is the "k limit" of $R(K, Q)$:

$$\lim_{k \to 0} \lim_{\omega \to 0} R(K, Q) = R^k(Q). \qquad \text{(IV.6a)}$$

The "ω limit" is, at zero temperature, identical to $R(0, Q)$.[21]

$$\lim_{\omega \to 0} \lim_{k \to 0} R(K, Q) = R^\omega(Q) \equiv R(0, Q). \qquad \text{(IV.6b)}$$

$\Delta(Q)$, the difference between $R^\omega(Q)$ and $R^k(Q)$, is seen from (IV.5) to be

$$\Delta(Q) = R^\omega(Q) - R^k(Q) = 2\pi i a_F^2 \delta(\epsilon)\delta(e_q), \qquad \text{(IV.7)}$$

and is nonzero only at the Fermi surface. For $K \to 0$, using (IV.5), the Bethe-Salpeter equation (IV.3) can be written in renormalized form as first shown by Landau.[1]

$$T^{(j)}(K; Q_1, Q_2) = T^{k(j)}(Q_1, Q_2) + \int \frac{d\vec{n}_3}{4\pi} \times T^{k(j)}(Q_1, \vec{n}_3) \frac{s}{s - \vec{e} \cdot \vec{n}_3 + i\alpha} T^{(j)}(K; \vec{n}_3, Q_2) \qquad \text{(IV.8)}$$

with $s = \omega/v_F k$. The integral in (IV.8) is over the Fermi surface. $T^{(j)}$ is the renormalized vertex function

$$T^{(j)}(K; Q_1, Q_2) = Z_F \Gamma^{(j)}(K; Q_1, Q_2),$$
$$Z_F = m^* k_F a_F^2 / 2\pi^2, \qquad \text{(IV.9)}$$

and $T^{k(j)}(Q_1, \vec{n}_3)$ is the k limit of $T(K; Q_1, \vec{n}_3)$ for $Q_3 = (0, \vec{n}_3 k_F)$. In the following we will only need $T^{(j)}$ for $Q_i = (0, \vec{n}_i k_F)$, where $i = 1, 2$.

$T^{k(j)}(\vec{n}_1, \vec{n}_2)$, the forward scattering amplitude for quasiparticles on the Fermi surface, can be expanded in a Legendre series

$$T^{k(j)}(\vec{n}_1, \vec{n}_2) = \sum_{l=0}^{\infty} A_l^{(j)} P_l(\vec{n}_1 \cdot \vec{n}_2). \qquad \text{(IV.10a)}$$

Similarly

$$T^{\omega(j)}(\vec{n}_1, \vec{n}_2) = \sum_{l=0}^{\infty} F_l^{(j)} P_l(\vec{n}_1 \cdot \vec{n}_2). \qquad \text{(IV.10b)}$$

Taking the ω limit ($s \to \infty$) of (IV.8) we find

$$T^{\omega(j)}(Q_1, Q_2) = T^{k(j)}(Q_1, Q_2) + \int \frac{d\vec{n}_3}{4\pi} T^{k(j)}(Q_1, \vec{n}_3) T^{\omega(j)}(\vec{n}_3, Q_2) \qquad \text{(IV.11)}$$

which, using (IV.10), leads immediately to a relation between $F_l^{(j)}$ and $A_l^{(j)}$:

$$A_l^{(j)} = \frac{F_l^{(j)}}{1 + F_l^{(j)}/(2l+1)} \qquad \text{(IV.12)}$$

The crossing symmetry relation (IV.2a) can be used to obtain the sum rule

$$\sum_{l=0}^{\infty} [A_l^{(1)} + A_l^{(2)}] = 0, \qquad \text{(IV.13)}$$

which simply states that the forward scattering amplitude for two quasiparticles of equal spin is zero. Since there are some subtleties involved in obtaining (IV.13) from (IV.2a), we discuss the derivation in Appendix A.

The $F_l^{(j)}$ are the usual real, dimensionless Landau parameters.[22] Three of them are directly connected with observable quantities:

$$\text{Effective mass:} \quad \frac{C_V}{C_V^{(0)}} = \frac{m^*}{m} = 1 + \tfrac{1}{3} F_1^{(1)},$$

$$\text{First sound velocity:} \quad c_1^2 = \frac{k_F^2}{3mm^*}(1 + F_0^{(1)}), \qquad \text{(IV.14)}$$

$$\text{Spin susceptibility:} \quad \frac{\chi}{\chi^{(0)}} = \frac{m^*/m}{1 + F_0^{(2)}},$$

where $C_V^{(0)}$ and $\chi^{(0)}$ are the specific heat and spin susceptibility of the noninteracting system. At present, in liquid He^3 there is no information available on the values of the remaining parameters.

The Landau equation (IV.8) can in principle be solved for any number of parameters. But, for, comparison with experiment, the usual procedure is to assume $F_l^{(j)} = 0$ for $l \geq 2$.[23] In Sec. VIII we solve for $T^{(j)}(K; \vec{n}_1, \vec{n}_2)$ in this model and determine $F_1^{(2)}$ from (IV.13).

V. WARD IDENTITIES: COUPLING OF PARTICLE-HOLE EXCITATIONS WITH QUASIPARTICLES

The equations of motion for the single-particle Green's function relate the mass operator to the two-particle Green's function and thus to the vertex function. For the purpose of calculating the coupling between particle-hole excitations and quasiparticles this relationship is not very useful because (i) it involves the bare potential and we want to express the coupling entirely in terms of Landau parameters, i.e., in terms of the effective interaction between quasiparticles and (ii) the integrals include regions of four-momentum space in which the vertex function is not known.

Σ and Γ are also related through Ward identities which follow from conservation laws.[2-4] We will employ the following two pairs of identities:

$$\frac{\partial\Sigma(Q)}{\partial\epsilon}=i\int\frac{d^4Q_2}{(2\pi)^4}\,\Gamma^{\omega(1)}(Q,Q_2)R^{\omega}(Q_2), \qquad (\text{V.1a})$$

$$\frac{\partial\Sigma(Q)}{\partial\mu}=i\int\frac{d^4Q_2}{(2\pi)^4}\,\Gamma^{k(1)}(Q,Q_2)R^{k}(Q_2), \qquad (\text{V.1b})$$

and

$$\vec{q}\,\frac{\partial\Sigma(Q)}{\partial\epsilon}=i\int\frac{d^4Q_2}{(2\pi)^4}\,\Gamma^{\omega(1)}(Q,Q_2)R^{\omega}(Q_2)\vec{q}_2, \qquad (\text{V.1c})$$

$$m\frac{\partial\Sigma(Q)}{\partial\vec{q}}=-i\int\frac{d^4Q_2}{(2\pi)^4}\,\Gamma^{k(1)}(Q,Q_2)R^{k}(Q_2)\vec{q}_2. \qquad (\text{V.1d})$$

Using (III.4), (V.1a), and (V.1c) we obtain, for Q on the Fermi surface, the renormalization relations

$$1-a_F^{-1}=i\int\frac{d^4Q_2}{(2\pi)^4}\,\Gamma^{\omega(1)}(\vec{n},Q_2)R^{\omega}(Q_2), \qquad (\text{V.2a})$$

$$\vec{n}k_F(1-a_F^{-1})=i\int\frac{d^4Q}{(2\pi)^4}\,\Gamma^{\omega(1)}(\vec{n},Q_2)R^{\omega}(Q_2)\vec{q}_2. \qquad (\text{V.2b})$$

If (V.1b) is subtracted from (V.1a), the right-hand side may be simplified using (IV.7), (IV.11), and (V.2a):

$$i\int\frac{d^4Q_2}{(2\pi)^4}[\Gamma^{\omega(1)}(Q,Q_2)R^{\omega}(Q_2)-\Gamma^{k(1)}(Q,Q_2)R^{k}(Q_2)]$$
$$=i\int\frac{d^4Q_2}{(2\pi)^4}\Big(Z_F\int\frac{d\vec{n}_3}{4\pi}\Gamma^{k(1)}(Q,\vec{n}_3)$$
$$\times\Gamma^{\omega(1)}(\vec{n}_3,Q_2)R^{\omega}(Q_2)+\Gamma^{k(1)}(Q,Q_2)\Delta(Q_2)\Big)$$
$$=-a_F^{-1}\int(d\vec{n}_2/4\pi)T^{k(1)}(Q,\vec{n}_2),$$

where we have used the definition of $T^{(j)}$, (IV.9). Hence,

$$a_F\left\{\frac{\partial\Sigma(Q)}{\partial\epsilon}-\frac{\partial\Sigma(Q)}{\partial\mu}\right\}=-\int\frac{d\vec{n}_2}{4\pi}T^{k(1)}(Q,\vec{n}_2). \qquad (\text{V.3})$$

A formally identical procedure, used after adding (V.1c) and (V.1d), leads, with (V.2b), to

$$a_F\left(\frac{q}{k_F}\frac{\partial\Sigma(Q)}{\partial\epsilon}+\frac{m}{k_F}\frac{\partial\Sigma(Q)}{\partial q}\right)$$
$$=-\int(d\vec{n}_2/4\pi)T^{k(1)}(Q,\vec{n}_2)\vec{n}\cdot\vec{n}_2. \qquad (\text{V.4})$$

This renormalization procedure is well known in Fermi liquid theory.[3,4] For example Eq. (V.4) with $Q=(\vec{n}k_F,0)$ reduces to the effective mass equation $m^*/m=1+F_1^{(1)}/3$. However, (V.3) and (V.4) are valid for quite general values of Q.

As we discussed in Sec. IV, the spectrum of particle-hole excitations is contained in the Q_1-Q_2 momentum transfer singularities [here $Q_1-Q_2\to(\epsilon,\vec{q}-\vec{n}_2k_F)$] of $\Gamma^{k(1)}(Q,\vec{n}_2)$. In order to exhibit this dependence of $T^{k(1)}$ explicitly, we employ (IV.2a)

$$T^{k(1)}(Q,\vec{n}_2)=\lim_{\vec{k}\to 0}T^{(1)}(\vec{k},Q,\vec{n}_2)$$
$$=-\sum_{j=1}^{2}C_{1j}\lim_{\vec{k}\to 0}T^{(j)}(\epsilon,\vec{q}-\vec{n}_2k_F;\vec{k}+\vec{n}_2k_F,\vec{n}_2)$$
$$=-\sum_{j=1}^{2}C_{1j}\lim_{\vec{n}_1\to\vec{n}_2}T^{(j)}(\epsilon,\vec{q}-\vec{n}_2k_F;\vec{n}_1,\vec{n}_2). \qquad (\text{V.5})$$

As shown in Appendix A, the limit $\vec{n}_1\to\vec{n}_2$ of $T^{(j)}$ in (V.5) is different from the value of $T^{(j)}$ at $\vec{n}_1=\vec{n}_2$. Nevertheless, in the limit $\vec{n}_1\to\vec{n}_2$, $T^{(j)}$ in the third line of (V.5) only depends on the angle between $\vec{q}$ and $\vec{n}_2$. Thus we expand $T^{(j)}$ in Legendre polynomials

$$\lim_{\vec{n}_1\to\vec{n}_2}T^{(j)}(\epsilon,\vec{p};\vec{n}_1,\vec{n}_2)=\sum_{l=0}^{\infty}T_l^{(j)}(\epsilon,\vec{p})P_l\left(\frac{\vec{p}}{p}\cdot\vec{n}_2\right),$$
$$\vec{p}=\vec{q}-\vec{n}_2k_F,\quad|\vec{p}|=p. \qquad (\text{V.6})$$

Inserting (V.6) in (V.5) and using this in (V.3), we find for the real part of the mass operator [with $\mathrm{Re}\,T_l^{(j)}\equiv\tilde{T}_l^{(j)}$]

$$a_F\left(\frac{\partial M(Q)}{\partial\epsilon}-\frac{\partial M(Q)}{\partial\mu}\right)$$
$$=\sum_{j=1}^{2}C_{1j}\int\frac{d\vec{n}_2}{4\pi}\sum_{l=0}^{\infty}\tilde{T}_l^{(j)}(\epsilon,\vec{p})P_l\left(\frac{\vec{p}}{p}\cdot\vec{n}_2\right). \qquad (\text{V.7})$$

Similarly

$$a_F\left(\frac{q}{k_F}\frac{\partial M(Q)}{\partial\epsilon}+\frac{m}{k_F}\frac{\partial M(Q)}{\partial q}\right)$$
$$=\sum_{j=1}^{2}C_{1j}\int\frac{d\vec{n}_2}{4\pi}\sum_{l=0}^{\infty}\tilde{T}_l^{(j)}(\epsilon,\vec{p})P_l\left(\frac{\vec{p}}{p}\cdot\vec{n}_2\right)\vec{n}\cdot\vec{n}_2. \qquad (\text{V.8})$$

(V.7) and (V.8) constitute a system of differential equations for the real part of the mass operator; the spectrum of particle-hole excitations enters through the $\tilde{T}_l^{(j)}$. From the solution of these equations we obtain the coupling between density and spin-density fluctuations and quasiparticles.

In the next section we discuss the form of the solutions of (V.7) and (V.8) which contribute logarithmic terms of the self-energy. In Sec. VII we explicitly evaluate $\tilde{T}_l^{(j)}$ and find the coefficients of these logarithmic terms.

VI. LOGARITHMIC TERMS IN THE MASS OPERATOR

We are interested in the mass operator $M(\vec{q}, \epsilon)$ for small values of $\epsilon(\epsilon \ll \mu)$ and for $|q-k_F| \ll k_F$. In Eqs. (V.7) and (V.8) ϵ is a parameter, whereas the integral over the Fermi surface covers the region

$$|q-k_F| \leq p = |\vec{q}-\vec{n}_2 k_F| \leq q+k_F ,$$

i.e., we need to know $\tilde{T}_l^{(j)}(p, \epsilon)$ for small ϵ and $0 \leq p \leq 2k_F$. Consider Eq. (V.7) first. We introduce a cutoff momentum k_L with $|q-k_F| K_L \ll k_F$, and divide the integration domain into three regions:

(i) "Landau region" (LR): $p < k_L$,
(ii) "Middle region" (MR): $k_L \leq p < 2k_F - k_L$,
(iii) "Far region" (FR): $2k_F - k_L \leq p < k_F + q$.

LR: In this region we insert for $\tilde{T}_l^{(j)}(\epsilon, p)$ the solution of the Landau equation (IV.8), defined in (V.6), which depends on ϵ and p only through $s=\epsilon/v_F p$. The integral on $\vec{n}_2$ in (V.7) may be transformed into an integral on s:

$$\int_{\mathrm{LR}} \frac{d\vec{n}_2}{4\pi} \tilde{T}_l^{(j)}(\epsilon, p) P_l\left(\frac{\vec{p}}{p}\cdot\vec{n}_2\right) \simeq \frac{1}{2}\left(\frac{\epsilon}{v_F k_F}\right)^2 \int_{|\epsilon/\epsilon_L|}^{|\epsilon/e_q|} \frac{ds}{s^3} \tilde{T}_l^{(j)}(s) P_l\left(s\frac{e_q}{\epsilon} - \frac{1}{s}\frac{\epsilon}{2v_F k_F}\right), \tag{VI.1}$$

where $\epsilon_L = v_F k_L$ and $e_q = v_F(q-k_F)$. We show in the next section that $\tilde{T}_l^{(j)}(\epsilon, p) = (-1)^l \tilde{T}_l^{(j)}(-\epsilon, p)$; this is also the parity of the right-hand side of (VI.1). The interesting case is $0 < |\epsilon/\epsilon_L| \ll 1$ and for the moment we put $q=k_F$, i.e., $e_q=0$ in order to simplify the discussion.

Consider the contribution to the right-hand side of (VI.1) from the lower limit of the integral. In this interval, I, given by $0 < |\epsilon/\epsilon_L| < s < s_0 \ll 1$, we expand $\tilde{T}_l^{(j)}(s)$ in powers of s.[25] For even l, the general term in $\tilde{T}_l^{(j)}(s)$ is $\propto s^{2n}$ $(n=0,1,2\cdots)$ and in P_l it is $\propto(\epsilon/s)^{2m}$ $(m=0,1\cdots l/2)$ (since we put $e_q=0$). The integrand, multiplied by the ϵ^2 in front, is $\propto s[2(n-m)-3]\epsilon[2m+2]$. Hence for $n-m \neq 1$ we obtain terms from interval $I \propto \epsilon^{2n}$, $(n=0,1,2\cdots)$. In I, for $n-m=1$ we obtain a term $\propto \epsilon^{2m+2}\ln|\epsilon/\epsilon_L|$ $(m=0,1\cdots)$ from the lower limit. Analogously, for odd l the logarithmic terms are $\propto \epsilon^{2m+3}\ln|\epsilon/\epsilon_L|$ $(m=0,1,2\cdots)$. Therefore the leading logarithmic contribution arises for even l and $m=0$. In the interval $s_0 < s$, ϵ only enters into the polynomials P_l and there are no further logarithmic terms.

Now we drop the simplifying assumption $e_q=0$. Then e_q appears in (VI.1) in two places: in the upper limit and in the argument of P_l. As before, the logarithmic terms arise from the term $\propto s^2$ in the expansion of the product $\tilde{T}_l^{(j)}(s) P_l(s e_q/\epsilon)$.[26]

The coefficient of s^2 now contains, apart from terms independent of e_q, terms $\propto (e_q/\epsilon)^2$.

We summarize our discussion by writing for the coefficient of s^2

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[\sum_{j=1}^{2} C_{1j}\sum_{l=0}^{\infty} \tilde{T}_l^{(j)}(s) P_l\left(s\frac{e_q}{\epsilon} - \frac{1}{s}\frac{\epsilon}{2v_F k_F}\right)\right]_{s=0} = -2\left[\phi_0 + \phi_1\frac{e_q}{\epsilon} + \phi_2\left(\frac{e_q}{\epsilon}\right)^2\right]. \tag{VI.2}$$

The integral on s in the interval I in (VI.1) yields $\ln|\epsilon/\epsilon_L|$ multiplied by (VI.2) if the upper limit $|\epsilon/e_q|$ is larger than s_0. If, on the other hand, $|\epsilon/e_q| < s_0 \ll 1$, the argument of the logarithm is changed into $|e_q/\epsilon_L|$.

Since we are ultimately interested in the behavior of $M(\vec{q}, \epsilon)$ on the "energy shell," $\epsilon = e_q$, in the following we consider only the case $|\epsilon/e_q| \simeq 1$.

Altogether we obtain for the right-hand side of Eq. (V.7) from LR

$$\sum_{j=1}^{2} C_{1j}\sum_{l=0}^{\infty}\int_{\mathrm{LR}} \frac{d\vec{n}_2}{4\pi}\tilde{T}_l^{(j)}(\epsilon, p) P_l\left(\frac{\vec{p}}{p}\cdot\vec{n}_2\right) = \text{regular terms in } (\epsilon, e_q)$$
$$+ (1/v_F k_F)^2\{\phi_0\epsilon^2 + \phi_1\epsilon e_q + \phi_2 e_q^2\}\ln|\epsilon/\epsilon_L| + O(\epsilon^3). \tag{VI.3}$$

The only difference between the right-hand side of (V.7) and (V.8) is the factor $\vec{n}\cdot\vec{n}_2$ in the latter. In the LR, this factor may be replaced by 1 to obtain the leading logarithmic term. Consequently the right-hand sides of (V.7) and (V.8) are identical and equal to (VI.3) in the LR. We still have to discuss the middle and far regions.

MR: Here we assume that $\tilde{T}_l^{(j)}(\epsilon, p)$ is expandable in powers of ϵ and p. We can check this by studying typical low-order diagrams or infinite sums of selected classes of diagrams (bubble sums, ladder sums). In these approximations the radius of convergence for the power series in ϵ is roughly $v_F k_L$. (In the LR it is $v_F p$ which shrinks as zero as $q \to k_F$). Hence there are no log-terms coming from this region.

FR: The vertex function for small energy ϵ and momentum transfer $p \simeq 2k_F$ corresponds to the scattering amplitude for two quasiparticles on opposite points of the Fermi surface (small total energy and momentum). This is the dangerous region if Cooper pairs are formed. It is beyond the scope of the Landau theory to provide information about the vertex function in this energy-momentum region.[27]

Let us assume for the moment that the logarithmic terms in M are entirely due to the LR, i.e., to long-wavelength particle-hole excitations.

Examination of (VI.3) leads us to the following ansatz for M^{28}:

$$a_F[M(Q) - M_{\rm reg}(Q)] \equiv a_F \delta M(Q) = \left(\frac{1}{v_F k_F}\right)^2 \{\Phi_0 \epsilon^3 + \Phi_1 \epsilon^2 e_q + \Phi_2 \epsilon e_q^2\} \ln\left|\frac{\epsilon}{\epsilon_L}\right| + O(\epsilon^3). \qquad \text{(VI.4)}$$

This ansatz (VI.4) is now inserted into (V.7) and (V.8) and the coefficients of ϵ^2, ϵe_q, and e_q^2 are compared with the right-hand side of (VI.3). The differentiation with respect to the chemical potential need only be applied to e_q in the order in which we are working; $\partial e_q/\partial\mu$ can be replaced by $-v_F \partial k/\partial\mu$.

Now we encounter an inconsistency. The Φ_i obtained from (V.7) are different from those obtained from (V.8). They only agree if either $\phi_1 = \phi_2 = 0$ or $\mu = k_F^2/2m$. Explicit calculation in Sec. VII shows that $\phi_1 \neq 0$. Further, μ is equal to $k_F^2/2m$ only in an ideal Fermi gas. Thus we conclude that additional logarithmic contributions must exist which enter into (V.7) and (V.8) with different coefficients.

If one looks at second-order diagrams of Γ, one finds that our assumption about the FR is not tenable. There are in fact logarithmic terms coming from the FR in every order. To include these we add a term

$$(1/v_F k_F)^2(\lambda_0 \epsilon^2 + \lambda_1 \epsilon e_q + \lambda_2 e_q^2) \ln|\epsilon/\epsilon_L|$$

to the right-hand side of Eq. (VI.3), where the intregal on $\vec{n}_2$ is now understood to run over the LR and FR. Therefore we write

$$\sum_{l=0}^{\infty}\sum_{j=1}^{2} C_{1j} \int \frac{d\vec{n}_2}{4\pi} \tilde{T}_l^{(j)}(\epsilon, p) P_l\left(\frac{\vec{p}\cdot\vec{n}_2}{p}\right)\left(\frac{1}{\vec{n}\cdot\vec{n}_2}\right) = \text{regular terms}$$

$$+ (1/v_F k_F)^2 \left[\begin{pmatrix}\phi_0 + \lambda_0\\ \phi_0 - \lambda_0\end{pmatrix}\epsilon^2 + \begin{pmatrix}\phi_1 + \lambda_1\\ \phi_1 - \lambda_1\end{pmatrix}\epsilon e_q + \begin{pmatrix}\phi_2 + \lambda_2\\ \phi_2 - \lambda_2\end{pmatrix} e_q^2\right] + \ln|\epsilon/\epsilon_L| + O(\epsilon^3). \qquad \text{(VI. 5)}$$

The lower line is obtained from (V.8). Since $\vec{n}\cdot\vec{n}_2 \simeq -1$ in FR, the λ terms enter with a relative minus sign. From (V.7), (VI.4), and the first line of (VI.5) we find

$$\phi_0 + \lambda_0 = 3\Phi_0 + v_F \frac{\partial k_F}{\partial\mu}\Phi_1, \quad \phi_1 + \lambda_1 = 2\Phi_1 + 2v_F \frac{\partial k_F}{\partial\mu}\Phi_2, \quad \phi_2 + \lambda_2 = \Phi_2, \qquad \text{(VI.6)}$$

and from (V.8), (VI.4) and the second line of (VI.5)

$$\phi_0 - \lambda_0 = 3\Phi_0 + \frac{m}{m^*}\Phi_1, \quad \phi_1 - \lambda_1 = 2\Phi_1 + 2\frac{m}{m^*}\Phi_2, \quad \phi_2 - \lambda_2 = \Phi_2. \qquad \text{(VI.7)}$$

The coefficients Φ_i and λ_i are determined unambiguously by the ϕ_i. With the help of the Landau relations (IV.14) for m^*/m and $v_F \partial k_F/\partial\mu = k_F^2/3mm^* c_1^2$ we find

$$\Phi_0 = \tfrac{1}{3}\phi_0 - \tfrac{1}{12}(A_0^{(1)}/F_0^{(1)} + A_1^{(1)}/F_1^{(1)})\phi_1 + \tfrac{1}{12}(A_0^{(1)}/F_0^{(1)} + A_1^{(1)}/F_1^{(1)})^2\phi_2,$$

$$\Phi_1 = \tfrac{1}{2}\phi_1 - \tfrac{1}{2}(A_0^{(1)}/F_0^{(1)} + A_1^{(1)}/F_1^{(1)})\phi_2, \quad \Phi_2 = \phi_2,$$

$$\lambda_0 = \tfrac{1}{2}(A_1^{(1)}/F_1^{(1)} - A_0^{(1)}/F_0^{(1)})\Phi_1, \quad \lambda_1 = (A_1^{(1)}/F_1^{(1)} - A_0^{(1)}/F_0^{(1)})\Phi_2, \quad \lambda_2 = 0. \qquad \text{(VI.8)}$$

VII. EVALUATION OF THE COEFFICIENTS ϕ_i

In order to obtain the ϕ_i's we have to solve the Landau equation (IV.8) for $T^{(j)}(K; \vec{n}_1, \vec{n}_2)$ [Compare Eqs. (V.6) and (VI.2)]. It is convenient to replace the real energy transfer ω by the complex frequency ζ: $T^{(j)}(\omega, \vec{k}; \vec{n}_1, \vec{n}_2) \to \tau^{(j)}(\zeta, \vec{k}, \vec{n}_1, \vec{n}_2)$. The real part of the vertex function is then obtained from

$$\tilde{T}^{(j)}(\omega, \vec{k}; \vec{n}_1, \vec{n}_2) = \lim_{\eta\to 0} \tfrac{1}{2}[\tau^{(j)}(\omega - i\eta, \vec{k}; \vec{n}_1, \vec{n}_2) + \tau^{(j)}(\omega + i\eta, \vec{k}; \vec{n}_1, \vec{n}_2)]. \qquad \text{(VII.1)}$$

The integral equation for τ is (we omit the index j for simplicity)

$$\tau(z, \vec{k}; \vec{n}_1, \vec{n}_2) = A(\vec{n}_1\cdot\vec{n}_2) + \int \frac{d\vec{n}_3}{4\pi} A(\vec{n}_1\cdot\vec{n}_3)\frac{z}{z - \vec{n}_3\cdot\vec{e}}\,\tau(z, \vec{k}; \vec{n}_3, \vec{n}_2) \qquad \text{(VII.2)}$$

with $z = \zeta/v_F k$ and $A(\vec{n}_1\cdot\vec{n}_2) = T^k(\vec{n}_1\cdot\vec{n}_2)$.

We now introduce polar coordinates, with $\vec{e} = \vec{k}/k$ the polar axis, and use the rotational symmetry to write

$$\tau(z, \vec{k}; \vec{n}_1, \vec{n}_2) = \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{[m]} 4\pi\tau_{l_1 l_2}^{m}(z) Y_{l_1 m}^*(\vec{n}_1) Y_{l_2 m}(\vec{n}_2), \qquad \text{(VII.3)}$$

$$A(\vec{n}_1\cdot\vec{n}_2) = \sum_{l=0}^{\infty} A_l P_l(\vec{n}_1\cdot\vec{n}_2) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1} A_l Y_{lm}^*(\vec{n}_1) Y_{lm}(\vec{n}_2), \qquad \text{(VII.4)}$$

$$\frac{z}{z - \vec{n}_3\cdot\vec{e}} = \sum_{l=0}^{\infty}(2l+1) P_l(\vec{n}_3\cdot\vec{e}) Q_l(z). \qquad \text{(VII.5)}$$

The Y_{lm} are the spherical harmonics; P_l and Q_l are Legendre polynomials of the first and second kind, respectively.[29] The symbol $[m]$ implies that the sum on m for fixed l_1, l_2 is restricted to $|m| \leq \min(l_1, l_2)$. In (VII.4) we have used the addition theorem for the Y_{lm}.

Inserting (VIII.3)–(VIII.5) into (VII.2) and comparing the coefficients of $Y_{lm}{}^*(\vec{n}_1)Y_{l_2m}(n_2)$, we deduce

$$\tau_{l_1l_2}{}^m(z) - \frac{A_{l_1}}{2l_1+1} z \sum_{l_3=0}^{\infty} J_{l_1l_3}{}^m(z)\tau_{l_3l_2}{}^m(z) = \frac{A_{l_1}}{2l_1+1}\delta_{l_1l_2} \tag{VII.6}$$

with $J_{l_1l_2}{}^m(z) = \sum_{l=0}^{\infty} [4\pi(2l+1)]^{1/2} Q_l(z) \int d\vec{n}\, Y_{l_1m}(\vec{n}) Y_{l0}(\vec{n}) Y_{l_2m}{}^*(\vec{n})$

$$= [(2l_1+1)(2l_2+1)]^{1/2} \sum_{l=0}^{\infty} (2l+1)Q_l(z) C^{l_1\ l\ \ l_2}_{m\ 0\ -m}\,. \tag{VII.7}$$

The coefficients $C^{l_1\ l\ l_2}_{m\ 0 - m}$ are related to the Wigner 3-j symbols,[30] $\begin{pmatrix} l_1 & l & l_2 \\ -m & 0 & m\end{pmatrix}$, by

$$C^{l_1\ l\ \ l_2}_{m\ 0-m} = (-1)^m \begin{pmatrix} l_1 & l & l_2 \\ m & 0 & -m\end{pmatrix}\begin{pmatrix} l_1 & l & l_2 \\ 0 & 0 & 0\end{pmatrix}. \tag{VII.8}$$

From the properties of the 3-j symbols we find: (i) $J_{l_1l_2}{}^m(z) = J_{l_2l_1}{}^m(z)$, (ii) the coefficient of Q_l on the right-hand side of (VII. 7) vanishes unless $(-1)^l = (-1)^{l_1+l_2}$. Furthermore

$$Q_l(-z) = (-1)^{l+1}Q_l(z) \tag{VII.9}$$

hence $J_{l_1l_2}{}^m(-z) = (-1)^{l_1+l_2+1} J_{l_1l_2}{}^m(z)$. (VII.10)

Taking the forward limit of τ in (VII.3) [compare (V.6)], we arrive at

$$\lim_{\vec{n}_1\to\vec{n}_2} \tau(z,\vec{k};\vec{n}_1,\vec{n}_2) = \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{[m]} 4\pi\tau_{l_1l_2}{}^m(z) Y_{l_1m}{}^*(\vec{n}_1)Y_{l_2m}(\vec{n}_1) \equiv \sum_{l=0}^{\infty} \tau_l(z)P_l(\vec{n}_1\cdot\vec{e}) \tag{VII.11}$$

from which we obtain $\tau_l(z)$ in the form

$$\tau_l(z) = (2l+1)\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{[m]} [(2l_1+1)(2l_2+1)]^{1/2} C^{l_2\ l\ \ l_1}_{m\ 0-m}\tau_{l_1l_2}{}^m(z). \tag{VII.12}$$

Again, only those $\tau_{l_1l_2}{}^m$ enter in (VII.12) for which $(-1)^{l_1+l_2} = (-1)^l$. From (VII.6) and (VII.10) we have

$$\tau_{l_1l_2}{}^m(-z) = (-1)^{l_1+l_2}\tau_{l_1l_2}{}^m(z) \text{ and } \tau_l(-z) = (-1)^l\tau_l(z). \tag{VII. 13}$$

Before we proceed, let us summarize our method for obtaining the coefficients ϕ_i from the unwieldly formulas above. Given the A_l, we calculate $\tau_{l_1l_2}{}^m(z)$ from (VII.6). Inserting the result into (VII.12) gives the $\tau_l(z)$ from which we obtain $\tilde{T}_l(s)$ through the analog of (VII.1). The $\tilde{T}_l(s)$ is then inserted into (VI.2). The ϕ_i's are obtained by comparing the coefficients of ϵ^2, ϵe_q, and $e_q{}^2$.

This program can be carried out with any number of A_l's. Since only three of the Landau parameters ($A_0^{(1)}$, $A_1^{(1)}$, $A_0^{(2)}$) are known for He^3, we proceed using a model in which

$$A_l^{(j)} = 0 \text{ for } l \geq 2 \tag{VII.14}$$

and where $A_1^{(2)}$ is obtained from Eq. (IV.13). In this model $\tau_{l_1l_2}{}^m = 0$ for $l_1l_2 \geq 2$. This follows from (VII.6) and the symmetry $\tau_{l_1l_2}{}^m = \tau_{l_2l_1}{}^m$. The solutions of Eq. (VII.6) are (we again suppress the index j)

$$\begin{aligned} &\tau_{00}{}^0(z) = A_0[1 - \tfrac{1}{3}A_1zJ_{11}{}^0(z)]D^{-1}(z), \quad \tau_{10}{}^0(z) = \tau_{01}{}^0(z) = \tfrac{1}{3}A_0A_1zJ_{10}{}^0(z)D^{-1}(z),\\ &\tau_{11}{}^0(z) = \tfrac{1}{3}A_1[1 - A_0zJ_{00}{}^0(z)]D^{-1}(z), \quad \tau_{11}{}^1(z) = \tau_{11}{}^{-1}(z) = \tfrac{1}{3}A_1[1 - \tfrac{1}{3}A_1zJ_{11}{}^1(z)]^{-1},\\ &D(z) = [1 - A_0zJ_{00}{}^0(z)][1 - \tfrac{1}{3}A_1zJ_{11}{}^0(z)] - \tfrac{1}{3}A_0A_1z^2[J_{01}{}^0(z)]^2. \end{aligned} \tag{VII.15}$$

The $J_{l_1l_2}{}^m(z)$ are given in Eq. (VII.7). Using the value of the 3-j symbols, we arrive at

$$\begin{aligned} &J_{00}{}^0(z) = Q_0(z) = \tfrac{1}{2}\ln[(z+1)/(z-1)], \quad J_{01}{}^0(z) = \sqrt{3}Q_1(z) = \sqrt{3}[zQ_0(z) - 1],\\ &J_{11}{}^0(z) = 3\ Q_1(z), \quad J_{11}{}^1(z) = \tfrac{3}{2}[Q_0(z) - zQ_1(z)]. \end{aligned} \tag{VII.16}$$

From (VII.12) we get

$$\tau_0=\tau_{00}{}^0+\tau_{11}{}^0+2\tau_{11}{}^1, \quad \tau_1=2\sqrt{3}\tau_{10}{}^0=2\sqrt{3}\tau_{01}{}^0,$$
$$\tau_2=2(\tau_{11}{}^0-\tau_{11}{}^1), \quad \tau_l=0 \text{ for } l\geq 3, \tag{VII.17}$$

from which we obtain $\tilde{T}_l(s)$ according to (VII.1). The $\tau_{l_1 l_2}{}^m(z)$ and $\tau_l(z)$ are analytic functions of z for $\operatorname{Im} z\neq 0$. They have a branch cut along $-1<\operatorname{Re} z<1$ resulting from the logarithm in $J_{l_1 l_2}{}^m(z)$. In addition, isolated poles may exist on the real axis $|\operatorname{Re} z|>1$ at the zeros of the denominators in (VII.15). The appearance of these poles depends on the numerical values of the A_l's. The branch cut corresponds to the continuum of particle-hole excitations; the discrete poles give the spectrum of collective excitations. In any case, whether these poles occur or not, the real functions $\tilde{T}_l(s)$ in (VII.18) are well behaved in the interval $-1<s<1$ and are expandable in powers of s around $s=0$. The formulas (VII.15)–(VII.18) hold for both values of j. If we now insert $\tilde{T}_l(s)$ of this model into (VI.2) we obtain the ϕ_i by straightforward, but somewhat tedious algebra. The result is (with the index j reinserted)[31]

$$\phi_0=-\tfrac{1}{2}\sum_{j=1}^{2} C_{1j}[(A_0{}^{(j)})^2(1+A_1{}^{(j)}-\tfrac{1}{4}\pi^2 A_0{}^{(j)})+(A_1{}^{(j)})^2(1-\tfrac{1}{16}\pi^2 A_1{}^{(j)})],$$
$$\phi_1=\sum_{j=1}^{2} C_{1j}A_0{}^{(j)}A_1{}^{(j)}, \quad \phi_2=0. \tag{VII.18}$$

The vanishing of ϕ_2 is quite general and does not depend on our model assumption (VII.14). From (VII.2) we see that

$$\lim_{\vec{n}_1\to\vec{n}_2} \tau(0,\vec{k};\vec{n}_1,\vec{n}_2)$$

is independent of $\vec{k}\cdot\vec{n}_1$, thus $\tau_l(0)=0$ for $l\neq 0$ and hence $\tilde{T}_l(0)=0$ for $l\neq 0$ which implies $\phi_2=0$.

Equations (VII.18) together with (VI.8) solve our problem of calculating the coefficients of the logarithmic term in the mass operator. The result (VII.18) [but not (VI.8)] depends of course on our model assumption (VII.14). The coefficients ϕ_i in (VII.18) include both the contribution of incoherent density and spin density fluctuations and zero sound. The zero-sound contribution to the Φ_i is calculated explicitly in Appendix B where we also exhibit numerical values of the Φ_i for He^3 at two pressures in Table II.

VIII. CORRECTIONS TO THE ENERGY AND LIFETIME OF A QUASIPARTICLE

The logarithmic corrections (VI.4) to the mass operator lead to a correction $\propto e_q{}^3\ln|e_q/\epsilon_L|$ to the quasiparticle energy (III.3). We now derive the coefficient.

From (III.2) we have, with $e_q{}^{(0)}=q^2/2m-\mu$,

$$E_q=e_q{}^{(0)}+M(\vec{q},E_q)=e_q{}^{(0)}+M_{\text{reg}}(\vec{q},E_q)+\delta M(\vec{q},E_q). \tag{VIII.1}$$

M_{reg} denotes the regular terms in the expansion of M in powers of $E_{\vec{q}}$; δM is given in (VI.4). We put

$$E_q=E_{\text{reg}}+\delta_q E_q, \quad E_{\text{reg}}=e_q+O[(q-k_F)^2]; \tag{VIII.2}$$

then $$M_{\text{reg}}(\vec{q},E_{\text{reg}}+\delta_q E_q)=M_{\text{reg}}(E_{\text{reg}})+\partial M_{\text{reg}}(\vec{q},\epsilon)/\partial\epsilon|_{\epsilon=e_q}\,\delta_q E_q. \tag{VIII.3}$$

Combining (VIII.1)–(VIII.3) and (VI.4) we get for the leading logarithmic term in $\delta_q E_{\vec{q}}$

$$\delta_q E_q=[1-\partial M_{\text{reg}}(\vec{q},\epsilon)/\partial\epsilon|_{\epsilon=e_q}]^{-1}\delta M(\vec{q}E_{\text{reg}}+\delta_q E_q)=a_F\delta M(\vec{q},e_q)+O(e_q{}^3)$$
$$=(1/v_F k_F)^2[\Phi_0+\Phi_1]e_q{}^3\ln|e_q/\epsilon_L|+O(e_q{}^3), \tag{VIII.4}$$

where we used $\Phi_2=0$.

The real and imaginary part of the mass operator Σ are Hilbert transforms of each other as is obvious from the spectral representation (II.5). Especially,

$$M(\vec{q},\epsilon)-M^{\text{HF}}(q)=P\int_{-\infty}^{\infty}(d\epsilon'/2\pi)[\gamma(\vec{q},\epsilon')/(\epsilon-\epsilon')], \tag{VIII.5}$$

where the right-hand side is the principal value integral. The logarithmic corrections (VI.4) to M imply a correction

$$a_F\delta\gamma(\vec{q},\epsilon)=[\pi/(v_F k_F)^2](\Phi_0\epsilon^2+\Phi_1\epsilon e_q)|\epsilon|, \tag{VIII.6}$$

for $\epsilon\to 0$, $q\to k_F$. This may be checked by inserting (VIII.6) into (VIII.5).

The correction (VIII.6) to γ gives a correction $\alpha(\Phi_0+\Phi_1)|e_{\vec{q}}|^3$ to the inverse lifetime of a quasiparticle, the leading term still being given by (III.6).

IX. DISCUSSION

Our principal result is the procedure for obtaining logarithmic corrections to the quasiparticle self-energy, resulting from the exchange of density and spin fluctuations, in terms of Landau parameters. We have calculated the coefficients of these terms explicitly in a model in which $A_l^{(j)}=0,\ l\geq 2$. The approach of a system to a ferromagnetic instability is characterized by $F_0^{(2)}\to -1$ or $A_0^{(2)}\to -\infty$. As the transition is approached, the $(A_0^{(2)})^3$ term in Φ_0, Eq. (VII.18), becomes dominant. This is the "paramagnon" limit.

In liquid He^3 at a pressure of 27 atm $m^*/m\simeq 6$ and the static spin susceptibility is enhanced by a factor of nearly 20 over the value for a noninteracting system. From (IV.14) we then find $A_0^{(2)}\simeq -2.6$. This value is sufficiently large to dominate Φ_0 as may be seen in Table II.

A word of caution is in order concerning the validity of models such as (VII.14) near an instability. There exist a set of inequalities for the parameters $F_l^{(j)}$, which follow from stability requirements on the ground state,[3] namely, $F_l^{(j)}>-(2l+1)$ for all l and j. Suppose that for some $l=l_0$, $F_{l_0}=-(2l_0+1)+x$ with $0<x\ll 1$; then $-A_{l_0}\simeq(2l_0+1)/x\gg 1$. From the sum rule (IV.13), which holds for all $x>0$, we see that the large negative value of A_{l_0} must be compensated by other terms in the sum. But the A_l's are bounded from above by $(2l+1)$ and an increasing number of them will be needed as the system approaches the instability.

In other words, the forward scattering amplitude $T^k(\vec{n}_1,\vec{n}_2)$ becomes strongly angle-dependent. A model in which only a few Legendre coefficients of $T^k(\vec{n}_1,\vec{n}_2)$ are taken into account will not be valid when the system tends toward an instability.

APPENDIX A

We consider the vertex function,

$$\Gamma_{\alpha\kappa,\beta\lambda}(K;Q_1,Q_2)\equiv\Gamma_{\alpha\kappa,\beta\lambda}=\tfrac{1}{2}\Gamma^{(1)}\delta_{\alpha\beta}\delta_{\kappa\lambda}+\tfrac{1}{2}\Gamma^{(2)}\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\kappa\lambda}, \tag{A1}$$

for the case $\alpha=\kappa=\beta=\lambda$

$$\Gamma_{\alpha\alpha,\alpha\alpha}\equiv\Gamma=\tfrac{1}{2}\Gamma^{(1)}+\tfrac{1}{2}\Gamma^{(2)}. \tag{A2}$$

The crossing symmetry (IV.2) implies

$$\Gamma(K;Q_1,Q_2)=-\Gamma(Q_1-Q_2;Q_2+K,Q_2), \tag{A3}$$

and for $K=0,\ Q_1=Q_2$,

$$\Gamma(0;Q_1,Q_2)=0. \tag{A4}$$

Since the "ω limit" of $R(K,Q)=G(K+Q)G(Q)$ is equal to $R(0,Q)$ we infer from the Bethe-Salpeter equation (IV.3) that $\Gamma(0;Q_1,Q_1)=\Gamma^\omega(Q_1,Q_1)$. From (A.4), with Q on the Fermi surface, it follows that that

$$\Gamma^{\omega(1)}(\vec{n},\vec{n})+\Gamma^{\omega(2)}(\vec{n},\vec{n})=0. \tag{A5}$$

Now consider the "ω limit" of (IV.8), with $Q_1=Q_2=(0,\vec{n}k_F)$,

$$T^{\omega(j)}(\vec{n},\vec{n})=T^{k(j)}(\vec{n},\vec{n})+\int(d\vec{n}''/4\pi)T^{k(j)}(\vec{n},\vec{n}'')T^{\omega(j)}(\vec{n}'',\vec{n}). \tag{A6}$$

However $T^{k(j)}(\vec{n},\vec{n})$ may also be written in the form

$$\begin{aligned}T^{k(j)}(\vec{n},\vec{n})&=\lim_{\vec{n}'\to\vec{n}}T^{(j)}(\vec{n}-\vec{n}';\vec{n},\vec{n})=-\lim_{\vec{n}'\to\vec{n}}\sum_{j'=1}^{2}C_{jj'}T^{(j')}(0;\vec{n}',\vec{n})\\&=-\lim_{\vec{n}'\to\vec{n}}\sum_{j'=1}^{2}C_{jj'}T^{\omega(j')}(\vec{n}',\vec{n}).\end{aligned} \tag{A7}$$

Inserting the expansion (IV.10b),

$$T^{\omega(j)}(\vec{n}',\vec{n})\equiv Z_F\Gamma^{\omega(j)}(\vec{n}',\vec{n})=\sum_{l=0}^{\infty}F_l^{(j)}P_l(\vec{n}'\cdot\vec{n}),$$

we obtain

$$T^{k(j)}(\vec{n},\vec{n})=-\sum_{l=0}^{\infty}\sum_{j'=1}^{2}C_{jj'}F_l^{(j')}. \tag{A8}$$

Then, using (A.8) in (A.6) we have

$$T^{\omega(j)}(\vec{n},\vec{n})=\sum_{l=0}^{\infty}\left(\frac{1}{2l+1}F_l^{(j)}A_l^{(j)}-\sum_{j'=1}^{2}C_{jj'}F_l^{(j')}\right). \tag{A.9}$$

From (A.5) the sum on j of (A.9) must vanish and this leads to

$$\sum_{l=0}^{\infty} (A_l^{(1)} + A_l^{(2)}) = 0. \tag{A.10}$$

We emphasize that

$$T^{\omega(j)}(\vec{n}, \vec{n}) \neq \sum_{l=0}^{\infty} F_l^{(j)}.$$

To see this let us assume the contrary. From (A.5) we would obtain

$$\sum_{l=0}^{\infty} [F_l^{(1)} + F_l^{(2)}] = 0. \tag{A.11}$$

Subtracting (A.11) from (A.10) yields

$$\sum_{j=1}^{2} \sum_{l=0}^{\infty} (F_l^{(j)})^2/(2l+1+F_l^{(j)}) = 0. \tag{A.12}$$

Because of the stability criterion $F^{(j)} > -(2l+1)$, every term in this sum is non-negative. (A.12) is only $F_l^{(j)} = 0$ for all l and j. This is only true in the trivial case of noninteracting particles.

Finally we remark that $T^{k(j)}(\vec{n}, \vec{n}')$ is also discontinuous in the forward direction, i.e.,

$$T^{k(j)}(\vec{n}, \vec{n}) \neq \sum_{l=0}^{\infty} A_l^{(j)}.$$

In fact, one finds from (A.8) that

$$\sum_{j=1}^{2} T^{k(j)}(\vec{n}, \vec{n}) = -\sum_{j=1}^{2} \sum_{l=0}^{\infty} F_l^{(j)} \neq 0. \tag{A.13}$$

APPENDIX B

In this section we calculate explicitly the contributions of collective excitations to the coupling constants ϕ_i for the model $A_l^{(j)} = 0$ for $l \geq 2$. The terms involving $\epsilon/2v_F k_F s$ in the argument of P_l in (VI.1) can be ignored, and the sum on l on the right-hand side of (VI.1) becomes

$$\sum_{l=0}^{\infty} \tilde{T}_l(s) P_l\left(s \frac{q^e}{\epsilon}\right) = \tilde{T}_0(s) - \tfrac{1}{2}\tilde{T}_2(s) + \tilde{T}_1(s) s \frac{q^e}{\epsilon} + \tfrac{3}{2}\tilde{T}_2(s) s^2 \frac{q^{e\,2}}{\epsilon^2} \tag{B.1}$$

for each value of j. From (VII.1) and (VII.12) we find

$$\tilde{T}_0(s) - \tfrac{1}{2}\tilde{T}_2(s) = \mathrm{Re}[\tau_{00}{}^0(s) + 3\tau_{11}{}^1(s)], \qquad \tilde{T}_1(s) = 2\sqrt{3}\,\mathrm{Re}\tau_{10}{}^0, \quad \tfrac{3}{2}\tilde{T}_2(s) = 3\,\mathrm{Re}[\tau_{11}{}^0(s) - \tau_{11}{}^1(s)]. \tag{B.2}$$

The analytic properties of $\tau_{l_1 l_2}{}^m(Z)$ are summarized in the spectral representation,

$$\tau_{l_1 l_2}{}^m(z) = \tau_{l_1 l_2}{}^m(\infty) + \int_{-\infty}^{\infty} (ds'/2\pi)[\chi_{l_1 l_2}{}^m(s')/(z - s')], \tag{B.3}$$

with the spectral functions,

$$\chi_{l_1 l_2}{}^m(s) = \lim_{\eta \to 0} \chi_{l_1 l_2}{}^m(s, \eta) = i \lim_{\eta \to 0} [\tau_{l_1 l_2}{}^m(s + i\eta) - \tau_{l_1 l_2}{}^m(s - i\eta)]. \tag{B.4}$$

From (VII.15) and (VII.16)

$$\tau_{00}{}^0(z) = F_0\left(1 + \frac{F_0 \omega(z)}{\bar{D}_0(z)}\right), \quad \tau_{10}{}^0(z) = \frac{F_0 A_1}{\sqrt{3}} \frac{z\,\omega(z)}{\bar{D}_0(z)}, \quad \tau_{11}{}^0(z) = \tfrac{1}{3}A_1\left(1 + \frac{A_1 z^2 \omega(z)}{\bar{D}_0(z)}\right), \quad \tau_{11}{}^1(z) = \tfrac{2}{3}\frac{1}{\bar{D}_1(z)}, \tag{B.5}$$

$$\text{where } \bar{D}_0(z) = 1 - (F_0 + A_1 z^2)\omega(z), \quad \bar{D}_1(z) = \frac{2}{A_1} - 1 + (z^2 - 1)\omega(z), \quad \omega(z) = \frac{z}{2}\ln\frac{z+1}{z-1} - 1. \tag{B.6}$$

The spectral functions are obtained using (B.4). We find

$$\chi_{00}{}^0(s) = \pi F_0{}^2 \omega(s)\, \mathrm{sgn}\left(\frac{\partial \bar{D}^0(s)}{\partial s}\right) \delta[\bar{D}_0'(s)], \quad \chi_{10}{}^0(s) = \chi_{01}{}^0(s) = (1/\sqrt{3})(A_1/F_0) s \chi_{00}{}^0(s),$$

$$\chi_{11}{}^0(s) = \tfrac{1}{3}(A_1/F_0)^2 s^2 \chi_{00}{}^0(s), \quad \chi_{11}{}^1(s) = \tfrac{2}{3}\pi\, \mathrm{sgn}\,[\partial \bar{D}_1'(s)/\partial s]\ \delta[\bar{D}_1'(s)], \tag{B.7}$$

$$\text{where } \bar{D}_0'(s) = \mathrm{Re}\bar{D}_0(s + i\eta), \quad \bar{D}_1'(s) = \mathrm{Re}\bar{D}_1(s + i\eta). \tag{B.8}$$

The existence of zeros for the arguments of the δ functions depends on the numerical values of the Landau parameters. In Table I we list the Landau parameters for liquid He^3 at two pressures from Ref.

6. $F_1^{(2)}$ is obtained by using (IV. 12) and (IV. 13).[32]

$j=1$: At both pressures $F_0^{(1)}+A_1^{(1)}s^2$ is positive. A solution of the equation $(F_0^{(1)}+A_1^{(1)}s^2)\omega(s)=1$ exists for $s>1$ if $1-m/m^*=\frac{1}{3}A_1^{(1)}<1$. The dispersion law for this collective mode is $\omega=v_F s_0 k=c_0 k$ corresponding to longitudinal zero sound (azimuthal index $m=0$). There is also a transverse, spin-symmetric mode ($m=1$) with velocity $v_F s_0'$, obtained from the solution of $(s^2-1)\omega(s)=1-2/A_1^{(1)}$ when $A_1^{(1)}>2$ ($F_1^{(1)}>6$), which is present at least at high pressure.

$j=2$: $F_0^{(2)}$ and $A_1^{(2)}$ are negative at both pressures. Consequently there are no collective spin waves in our model and the spectral functions vanish identically for $|s|>1$.

Using $\delta(f(x))=\Sigma_\alpha|f'(x_\alpha)|^{-1}\delta(x-x_\alpha)$, $f(x_\alpha)=0$, $f'(x_\alpha)=\partial f(x)/\partial x|_{x=x_\alpha}$, we have the final form of the spectral functions from (B.7) for $j=1$:

$$\chi_{00}{}^0(s)=Q(s_0)[\delta(s-s_0)-\delta(s+s_0)],\quad \chi_{10}{}^0(s)=\frac{1}{\sqrt{3}}\left(\frac{A_1^{(1)}}{F_0^{(1)}}\right)Q(s_0)[\delta(s-s_0)+\delta(s+s_0)],$$

$$\chi_{11}{}^0(s)=\tfrac{1}{3}\left(\frac{A_1^{(1)}}{F_0^{(1)}}\,s_0\right)^2 Q(s_0)[\delta(s-s_0)-\delta(s+s_0)],\quad \chi_{11}{}^1(s)=Q'(s_0')[\delta(s-s_0)-\delta(s+s_0)],\tag{B.9}$$

where

$$Q(s_0)=\frac{2\pi F_0^{(1)2}s_0(s_0^2-1)}{s_0^4A_1^{(1)}(A_1^{(1)}-3)+s_0^2(3A_1^{(1)}+2A_1^{(1)}F_0^{(1)}-F_0^{(1)})+F_0^{(1)}(F_0^{(1)}+1)},$$

$$Q'(s_0')=\tfrac{2}{3}\pi A_1^{(1)}s_0'(s_0'^2-1)/[s_0'^2(A_1^{(1)}-3)+1].\tag{B.10}$$

Using (B.9) in the spectral representation (B.3), it is a simple matter to obtain $\mathrm{Re}\,\tau_{l_1l_2}{}^m(s)$ by taking the principal value of the integral. We can then obtain $\tilde{T}_l(j)$ and find the coefficients ϕ_i in (VI.3). The results of this procedure are

$$\phi_0{}^{\text{coll}}=\frac{1}{4\pi}\left[\frac{Q(s_0)}{s_0^3}+\frac{Q'(s_0')}{s_0'^3}\right],\quad \phi_1{}^{\text{coll}}=\frac{1}{2\pi}\left(\frac{A_1^{(1)}}{F_0^{(1)}}\right)\frac{Q(s_0)}{s_0},$$

$$\phi_2{}^{\text{coll}}=\frac{1}{4\pi}\left[\left(\frac{A_1^{(1)}}{F_0^{(1)}}\right)^2Q(s_0)s_0-\frac{Q'(s_0')}{s_0'}\right].\tag{B.11}$$

The contribution of the incoherent particle-hole excitations is obtained from $\phi_i-\phi_i{}^{\text{coll}}$.

The values of Φ_i, using the Landau parameters of Ref. 6, are given in Table II. Also shown are the contributions of the longitudinal zero sound, $\Phi_i{}^{zs}$, computed from (B.11) with $Q'(s_0')=0$, the "paramagnon" contribution, $\Phi_0{}^{\text{pm}}$, and the λ_i from the short-wavelength excitations. $\Phi_0{}^{\text{pm}}$ is the value of Φ_0 with only $A_0^{(2)}\neq 0$. Notice that $\Phi_2{}^{zs}\neq 0$, but is always cancelled identically by the contributions of the tranverse modes and particle-hole continuum. Φ_i and λ_i are obtained from Eqs. (VII.18) and (VI.8).

The longitudinal zero-sound velocity from Ref. 6 is $c_0=193.6$ m/sec ($s_0=3.6$) at 0.28 atm., and $c_0=389.5$ m/sec ($s_0=12.2$) at 27 atm. For the transverse collective mode we obtain $s_0'=1.003$ at 0.28 atm and $s_0'=1.202$ at 27 atm.

TABLE I. The Landau parameters at two pressures taken from Ref. 6.

P (atm)	$F_0^{(1)}$	$F_0^{(2)}$	$F_1^{(1)}$	$F_1^{(2)}$
0.28	10.77	−0.67	6.25	−0.72
27	75.63	−0.72	14.35	−0.66

TABLE II. Numerical values of the coefficients Φ_i, λ_i, using the Landau parameters of Ref. 6. Notice that $\Phi_2=\lambda_1=\lambda_2=0$.

P (atm)	Φ_0	$\Phi_0{}^{\text{pm}}$	$\Phi_0{}^{zs}$	Φ_1	$\Phi_1{}^{zs}$	$\Phi_2{}^{zs}$	λ_0
0.28	− 5.67	− 5.16	0.04	1.91	0.22	1.06	−0.228
27.0	−11.40	−10.48	0.03	2.24	0.34	2.95	−0.018

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[22]In Appendix B our notation for the Landau parameters is compared with that of other authors.
[23]The zero-sound velocity depends on all of the $F_l^{(1)}$'s and in a model in which $F_l^{(1)}=0$ for $l\geq 2$ the theoretical value of c_0 (see Appendix B) is in good agreement with experiment. (See Ref. 24.)
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[26]We have dropped terms proportional to ϵ/s in the argument of P_l since these yield higher than quadratic powers of ϵ preceeding the logarithm, and we are confined henceforth to the leading terms as $\epsilon\rightarrow 0$, $q\rightarrow k_f$.
[27]A procedure analogous to the one used for examining the LR Eq. (IV. 8), could be used in the FR. However the solution would require new parameters.
[28]We do not include a term $\propto\Phi_3 e_q{}^3 \ln|\epsilon/\epsilon_L|$. This term requires the presence of a term $\alpha\, e_q{}^2/\epsilon$ on the right-hand side of (VI. 1) which does not exist; $\Phi_3=0$.
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[30]M. Rotenberg, R. Bivius, N. Metropolis, and J. K. Wooten, Jr., The 3-j and 6-j Symbols (The Technology Press, Massachusetts Institute of Technology, Massachusetts, 1959).
[31]It is interesting to note that no matter how many A_l's are taken into account, they enter the Φ_i's at most cubically.
[32]The notation of Ref. 6 is related to ours by $F_l=F_l^{(1)}$, $Z_l=4F_l^{(2)}$. In Ref. 15 we used $F_l^s=F_l^{(1)}$, $F_l^a=F_l^{(2)}$.

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Logarithmic Terms in the Specific Heat of a Normal Fermi Liquid*

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Using the properties of the mass operator and vertex function derived previously, we calculate corrections of the form $T^3\ln T$ to the term in the specific heat C_V that is linear in the temperature. The connection of this calculation with the microscopic theory is pointed out and the limitations are discussed.

I. INTRODUCTION

In a previous paper[1] we studied the corrections to the single-particle mass operator of a normal Fermi liquid, which arise from the interaction of quasiparticles with density and spin density fluctuations at $T=0$. For a quasiparticle with momentum q near the Fermi momentum k_F, the corrections were found to be proportional to $(q-k_F)^3 \times \ln|q-k_F|$. The coefficient of this logarithmic contribution was obtained in a completely renormalized form in terms of Landau parameters.

In this paper, which supplements I, we calculate the change in the low-temperature specific heat resulting from the above corrections to the mass operator. The calculation is performed within the phenonenological Landau theory.[2] The result is a $T^3\ln T$ correction to the leading linear term in the specific heat.