Quantum-Mechanical Description of Two Coupled Harmonic Oscillators

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Systems of interacting harmonic oscillators have recently received considerable attention as models for describing a variety of physical problems. We have investigated the validity of the rotating-wave approximation which constitutes the traditional approach to the solution of the dynamical problem by comparing it with the exact solution. A numerical comparison has been made and the limits of validity of the rotating-wave approximation has been established in terms of the strength of oscillator interaction.

In particular, the time development of dynamical operators and certain transition probabilities have been compared. In the region where the rotating-wave approximation is valid, the time evolution of the quasiprobability distribution $P(\alpha, t)$ of one oscillator is given for several initial conditions. A counting scheme similar to the argument given by Feynman for the driven harmonic oscillator, is proposed for the interpretation of the time-dependent transition amplitudes between number states.

I. INTRODUCTION

In a series of recent publications considerable attention has been directed toward systems of coupled harmonic oscillators[1-13]. The Hamiltonian describing N-harmonic oscillators with bilinear interactions is given by[13]

$$H=\hbar\sum_{i,j}\omega_{ij}a_i^{\dagger}a_j+\tfrac{1}{2}\kappa_{ij}a_i^{\dagger}a_j^{\dagger}+\tfrac{1}{2}\kappa_{ij}^{*}a_ia_j\ , \quad (1.1)$$

where a_i is the annihilation operator of the ith oscillator and the κ_{ij} and ω_{ij} are coupling constants. The information sought about such systems is a knowledge of the time development of the dynamical operators along with certain transition probabilities. In general, the algebraic difficulties associated with such determinations are extreme. When seeking transition probabilities however, a very useful and simplifying approximation can be made by neglecting terms of the form $a_i^{\dagger}a_j^{\dagger}$ and a_ia_j in the Hamiltonian (1.1). The argument customarily given justifying this approximation is the following:[8] looking at the free dynamical behavior of the terms a_ia_j and $a_i^{\dagger}a_j^{\dagger}$, we have

$$\begin{aligned} a_ia_j &\propto \exp[-i(\omega_i+\omega_j)t]\ , \\ a_i^{\dagger}a_j^{\dagger} &\propto \exp[i(\omega_i+\omega_j)t]\ , \end{aligned} \quad (1.2)$$

while the operators of the form $a_ia_j^{\dagger}$ develop like

$$a_ia_j^{\dagger}\propto\exp[-i(\omega_i-\omega_j)t]\ , \quad (1.3)$$

where ω_i and ω_j are the natural frequencies of the free oscillators.

The argument is that for sufficiently small coupling constants κ_{ij} transitions should occur over times long compared with the rapid oscillations of the terms in (1.2). Therefore the terms (1.2) go through many oscillations during the transition period causing their average value to be small by comparison with the slowly varying resonant terms of Eq. (1.3). While the argument is a correct one, one of our purposes here is to establish quantitatively the error incurred in the time development of operators and transition probabilities by this so-called resonant approximation (sometimes called the rotating-wave approximation).

In particular we study the behavior of a system of two coupled harmonic oscillators with identical free frequencies. While the techniques used can certainly handle the case of different frequencies, the algebra involved is somewhat simplified by the selection of identical frequencies. The Hamiltonian is taken to be the one for which the coupling is of the position-position type,

$$H=\hbar\omega(a^{\dagger}a+\tfrac{1}{2})+\hbar\omega(b^{\dagger}b+\tfrac{1}{2})+\hbar\kappa(a^{\dagger}+a)(b^{\dagger}+b)\ , \quad (1.4)$$

where a and b are the annihilation operators for oscillators A and B, respectively, and κ is a constant describing the strength of the coupling. We use the Hamiltonian (1.4) to make an exact determination of the time development of the dynamical operators (Sec. II and III) and transition probabilities (Sec. IV). Corresponding results in the resonant approximation are obtained when the Hamiltonian is taken to be

$$H=\hbar\omega(a^{\dagger}a+\tfrac{1}{2})+\hbar\omega(b^{\dagger}b+\tfrac{1}{2})+\hbar\kappa(a^{\dagger}b+ab^{\dagger})\ . \quad (1.5)$$

By making a numerical comparison of the time development of the position and momentum operators for the approximate and exact Hamiltonians, we establish that there is almost exact agreement for values of the dimensionless coupling constant $\lambda=2\kappa/\omega$ less than 0.1. For a comparison of transition probabilities, we consider the transition probability

$$P\begin{pmatrix} l'm' \\ l\ m \end{pmatrix}$$

that if oscillators A and B are in the respective number states $|l\rangle$ and $|m\rangle$ at time zero that they will be found in the number states $|l'\rangle$ and $|m'\rangle$ at a later time t. In particular we construct a generating function for the transition amplitude

$$A\begin{pmatrix} l'm' \\ l\,m \end{pmatrix}, \text{ where } P\begin{pmatrix} l'm' \\ l\,m \end{pmatrix} = \left| A\begin{pmatrix} l'm' \\ l\,m \end{pmatrix} \right|^2$$

Several numerical comparisons are made for a number of different coupling constants λ as high as 0.9. From the form of the generating function we deduce that there will be excellent agreement for coupling constants which satisfy the condition $N\lambda^2 \ll 1$, where N is the largest of the four occupation numbers l, m, l', m'.

At this point we concentrate on the resonant approximation and discuss in detail the time evolution of a system governed by the Hamiltonian (1.5). Because of the simplified algebra associated with this approximation, we are able to give explicitly the transition amplitude

$$A\begin{pmatrix} l'm' \\ l\,m \end{pmatrix}.$$

The form of this result is such that it suggests a probabilistic counting procedure similar to that given by Feynman for the quantum oscillator driven by a classical force.[14] Following Mollow and Glauber[11,12] we define in Sec. V a reduced density operator for one of the oscillators and determine its time development for a variety of initial states of the system.

II. DYNAMICAL EVOLUTION OF THE COUPLED SYSTEM

The total Hamiltonian of the two coupled harmonic oscillators is assumed to be

$$H = \tfrac{1}{2}(p_a^2 + \omega^2 x_a^2) + \tfrac{1}{2}(p_b^2 + \omega^2 x_b^2) + 2\kappa\omega x_a x_b \ , \tag{2.1}$$

where we consider the natural frequencies of the oscillators to be the same and the masses to be equal to one. By performing the standard transformation

$$a = (\omega x_a + i p_a)/(2\hbar\omega)^{1/2} \quad b = (\omega x_b + i p_b)/(2\hbar\omega)^{1/2}, \tag{2.2}$$

where a and b are the annihilation operators for the uncoupled oscillators, we find that Eq. (2.1) takes the form

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) + \hbar\omega(b^\dagger b + \tfrac{1}{2}) + \hbar\kappa(a^\dagger + a)(b^\dagger + b). \tag{2.3}$$

The Heisenberg equations of motion for $a(t)$ and $b(t)$ can be usefully expressed in matrix notation in the following way.

$$\frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = -i\begin{pmatrix} \omega & \kappa \\ \kappa & \omega \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} - i\begin{pmatrix} 0 & \kappa \\ \kappa & 0 \end{pmatrix}\begin{pmatrix} a^\dagger(t) \\ b^\dagger(t) \end{pmatrix}. \tag{2.4}$$

In the Appendix we snow the details of the calculations leading to the solution of Eq. (2.4). The result is

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = M\{\alpha\Lambda(t)\alpha - \beta\Lambda^*(t)\beta\}M\begin{pmatrix} a \\ b \end{pmatrix} + M\{\alpha\Lambda(t)\beta - \beta\Lambda^*(t)\alpha\}M\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} \ , \tag{2.5}$$

where we have indicated the operators at time $t=0$ by a and b. The matrices M, α, β, and $\Lambda(t)$ are given by

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Lambda(t) = \begin{pmatrix} \exp(-i\omega_x t) & 0 \\ 0 & \exp(-i\omega_y t) \end{pmatrix},$$

$$\alpha = \begin{pmatrix} \alpha_x & 0 \\ 0 & \alpha_y \end{pmatrix} \equiv \begin{pmatrix} \dfrac{\omega_x + \omega}{2\sqrt{\omega_x\omega}} & 0 \\ 0 & \dfrac{\omega_y + \omega}{2\sqrt{\omega_y\omega}} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_x & 0 \\ 0 & \beta_y \end{pmatrix} \equiv \begin{pmatrix} \dfrac{\omega_x - \omega}{2\sqrt{\omega_x\omega}} & 0 \\ 0 & \dfrac{\omega_y - \omega}{2\sqrt{\omega_y\omega}} \end{pmatrix} \ , \tag{2.6}$$

where the normal mode frequencies ω_x and ω_y are

$$\omega_x = (\omega^2 + 2\kappa\omega)^{1/2} \equiv \omega(1+\lambda)^{1/2}, \quad \omega_y = (\omega^2 - 2\kappa\omega)^{1/2} \equiv \omega(1-\lambda)^{1/2}, \quad \lambda = 2\kappa/\omega$$

Since we are considering a pure oscillatory behavior it is clear from Eq. (2.7) that the parameter λ is

bounded between zero and one. We observe that $M^{-1}=M$ and that

$$\alpha^2-\beta^2=I \quad , \tag{2.8}$$

where I is the identity matrix.

It can be shown that the solution $\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ of the Heisenberg equations satisfies the Bose commutation relations at any time t by performing the following unitary transformation.

$$\begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix} = M \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} . \tag{2.9}$$

The primed operators must satisfy the same commutation relations as the unprimed ones, since they are related by a unitary transformation. Multiplying (2.5) by M and making use of the definition (2.9), we have

$$\begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix} = \{\alpha\Lambda(t)\alpha-\beta\Lambda^*(t)\beta\}\begin{pmatrix} a' \\ b' \end{pmatrix} + \{\alpha\Lambda(t)\beta-\beta\Lambda^*(t)\alpha\}\begin{pmatrix} a'^{\dagger} \\ b'^{\dagger} \end{pmatrix} . \tag{2.10}$$

The matrices given above are all diagonal and as a result, the matrix products are trivial. Performing the operations in (2.10) and using the property (2.8), the commutation relations for $a'(t), b'(t)$ and therefore $a(t)$, $b(t)$ are easily verified.

III. DYNAMICAL EVOLUTION IN THE RESONANT APPROXIMATION

Let us assume now that the parameter $\lambda=2\kappa/\omega$ is small enough as to justify the approximation

$$\omega_x \simeq \omega+\kappa, \quad \omega_y \simeq \omega-\kappa \quad . \tag{3.1}$$

Within this approximation the matrix α becomes the identity matrix and the matrix β becomes the null matrix [in both cases the correction is $O(\lambda)$]. The solution (2.5) now takes the form

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = M\Lambda(t)M \begin{pmatrix} a \\ b \end{pmatrix} . \tag{3.2}$$

As a result of this approximation we have the desirable fact that the annihilation and creation operators are no longer coupled. The question now is; what Hamiltonian has Eq. (3.2) and its Hermitian conjugate as solutions of the Heisenberg equations of motion? It is easy to show that the following Hamiltonian is the required one

$$H=\hbar\omega(a^{\dagger}a+\tfrac{1}{2})+\hbar\omega(b^{\dagger}b+\tfrac{1}{2})+\hbar\kappa(a^{\dagger}b+b^{\dagger}a) \quad . \tag{3.3}$$

If we consider the unitary transformation

$$\begin{pmatrix} A \\ B \end{pmatrix} = M\begin{pmatrix} a \\ b \end{pmatrix} \quad , \quad \begin{pmatrix} A^{\dagger} \\ B^{\dagger} \end{pmatrix} = M\begin{pmatrix} a^{\dagger} \\ b^{\dagger} \end{pmatrix} , \tag{3.4}$$

the Hamiltonian (3.3) describes the two decoupled normal modes of oscillation

$$H=\hbar(\omega+\kappa)A^{\dagger}A+\hbar(\omega-\kappa)B^{\dagger}B \quad , \tag{3.5}$$

where the zero-point energy has been neglected for simplicity. The normal-mode operators $A(t)$ and $B(t)$ satisfy the following equation

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \Lambda(t)\begin{pmatrix} A \\ B \end{pmatrix} , \tag{3.6}$$

where A and B are the initial operators. The application of the inverse transformation M^{-1} leads to the required solution.

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = M\Lambda(t)M \begin{pmatrix} a \\ b \end{pmatrix} . \tag{3.7}$$

We point out that the solution (3.7) which we have obtained from the exact Eq. (2.5) by assuming $2\kappa/\omega \ll 1$, is also the exact solution of the Hamiltonian (3.3). Thus (3.7) is the solution to the Heisenberg equations of motion in the resonant approximation. It is worth noting that our assumption $\lambda=2\kappa/\omega \ll 1$ is not the same as a perturbation expansion in powers of the coupling constant κ, since κ is neglected only in comparison to ω but all powers of $\kappa t=\frac{1}{2}\lambda\omega t$ are retained.

In order to make a direct comparison between the systems described by the exact and approximate Hamiltonians we have performed a numerical investigation of the time evolution of the expectation values of the position operator $x(t)$ for different values of λ.

A. Exact Hamiltonian

By making use of the fact that

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \left(\frac{\omega}{2\hbar}\right)^{1/2} \begin{pmatrix} x_a(t) \\ x_b(t) \end{pmatrix} + i\left(\frac{1}{2\hbar\omega}\right)^{1/2} \begin{pmatrix} p_a(t) \\ p_b(t) \end{pmatrix} \tag{3.8}$$

and

$$\begin{pmatrix} a^\dagger(t) \\ b^\dagger(t) \end{pmatrix} = \left(\frac{\omega}{2\hbar}\right)^{1/2} \begin{pmatrix} x_a(t) \\ x_b(t) \end{pmatrix} - i\left(\frac{1}{2\hbar\omega}\right)^{1/2} \begin{pmatrix} p_a(t) \\ p_b(t) \end{pmatrix} \tag{3.9}$$

it is easily shown that the position operators (and their expection values)for the two oscillators in the exact case are related to the initial positions and momenta by the following relation

$$\begin{pmatrix} x_a(t) \\ x_b(t) \end{pmatrix} = \begin{pmatrix} \cos\omega_a t \cos\omega_b t - \sin\omega_a t \sin\omega_b t & \\ -\sin\omega_a t \sin\omega_b t & \cos\omega_a t \cos\omega_b t \end{pmatrix} \begin{pmatrix} x_a \\ x_b \end{pmatrix} + \frac{1}{\omega_x} \begin{pmatrix} \sin\omega_a t \cos\omega_b t & \cos\omega_a t \sin\omega_b t \\ \cos\omega_a t \sin\omega_b t & \sin\omega_a t \cos\omega_b t \end{pmatrix} \begin{pmatrix} p_a \\ p_b \end{pmatrix}$$

$$+ \frac{1}{2} \frac{\omega_x - \omega_y}{\omega_x \omega_y} \sin\omega_y t \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} p_a \\ p_b \end{pmatrix}. \tag{3.10}$$

For simplicity we have used the symbols

$$\omega_a = \tfrac{1}{2}(\omega_x + \omega_y), \quad \omega_b = \tfrac{1}{2}(\omega_x - \omega_y) \ . \tag{3.11}$$

B. Resonant Approximation

In the approximate case we use the same technique to show that the position operators $x_a(t)$ and $x_b(t)$ (and their expectation values) are related to the initial positions and momenta by the equation

$$\begin{pmatrix} x_a(t) \\ x_b(t) \end{pmatrix} = \begin{pmatrix} \cos\omega_a t \cos\omega_b t - \sin\omega_a t \sin\omega_b t & \\ -\sin\omega_a t \sin\omega_b t & \cos\omega_a t \cos\omega_b t \end{pmatrix} \begin{pmatrix} x_a \\ x_b \end{pmatrix} + \frac{1}{\omega} \begin{pmatrix} \sin\omega_a t \cos\omega_b t & \cos\omega_a t \sin\omega_b t \\ \cos\omega_a t \sin\omega_b t & \sin\omega_a t \cos\omega_b t \end{pmatrix} \begin{pmatrix} p_a \\ p_b \end{pmatrix}$$

Again ω_a and ω_b are, respectively, one-half of the sum and difference of the normal-mode frequencies, i.e.,

$$\omega_a = \tfrac{1}{2}(\omega_x + \omega_y) = \omega, \quad \omega_b = \tfrac{1}{2}(\omega_x - \omega_y) = \kappa \, . \tag{3.13}$$

In Fig. 1 a comparison is given between the exact solution $x_a(t)$ given by Eq. (3.10) and the approximate one given by Eq. (3.12). We have assumed the particularly simple initial condition

$$\langle p_a \rangle = \langle p_b \rangle = \langle x_b \rangle = 0.$$

These initial conditions eliminate the last term in Eq. (3.10) which has an amplitude of order λ. It is interesting to observe that even for relatively high values of $\lambda = 2\kappa/\omega$, e.g., $\lambda = 0.1$ the difference between the exact and approximate solution does not exceed a few parts per thousand. We have also investigated the strong coupling situation ($\lambda = 0.9$). An appreciable error is very evident in this case. The two solutions become rapidly dephased from each other and cross the horizontal axis at different points.

IV. TRANSITION PROBABILITIES

Another method of examining the validity of the resonant approximation is to compare transition probabilities. That is, probabilities for exchange of quanta between the two oscillators, for the exact and approximate cases. The amplitude for a transition from an initial state at $t = 0$ with l quanta in oscillator A and m in oscillator B to a final state at time t with l' quanta in A and m' quanta in B is given by

$$A\begin{pmatrix} l'm' \\ l\ m \end{pmatrix} = \langle l'm' | \, U(t) \, | lm \rangle \, , \tag{4.1}$$

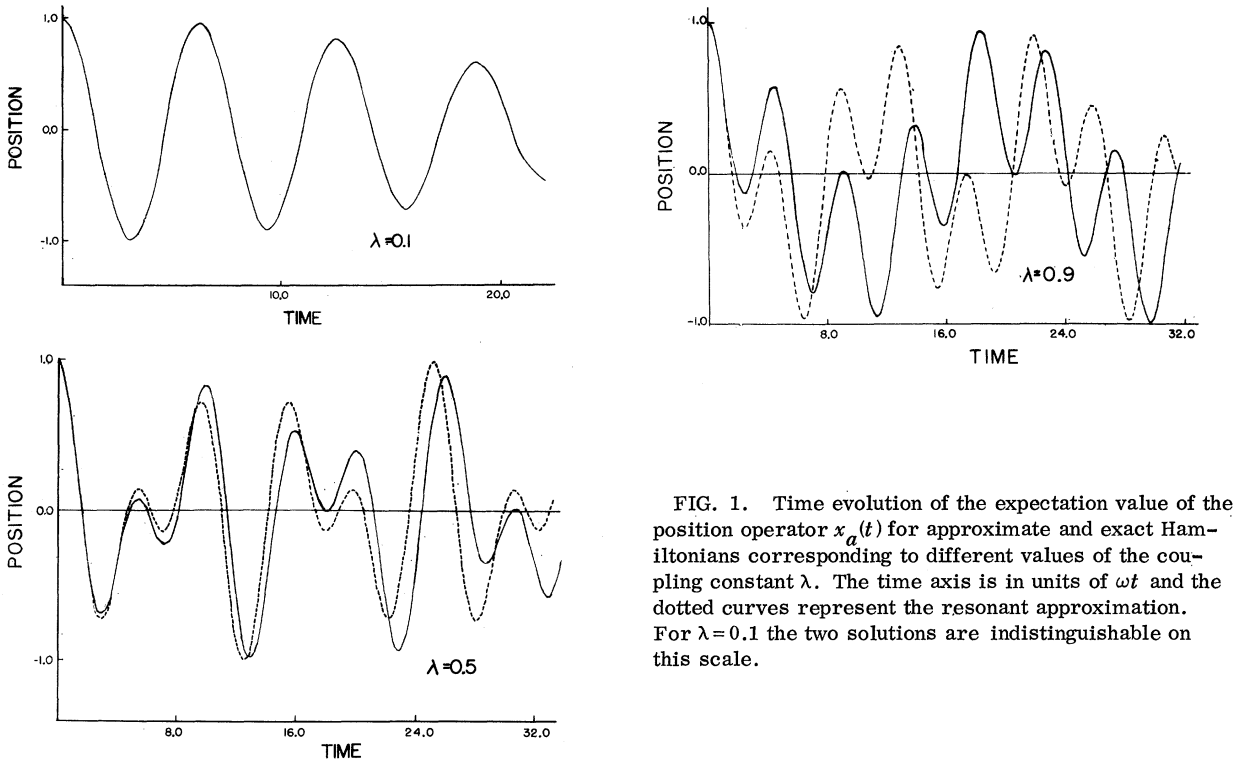

FIG. 1. Time evolution of the expectation value of the position operator $x_a(t)$ for approximate and exact Hamiltonians corresponding to different values of the coupling constant λ. The time axis is in units of ωt and the dotted curves represent the resonant approximation. For $\lambda = 0.1$ the two solutions are indistinguishable on this scale.

where $U(t)$ is the time evolution operator

$$U(t) = \exp[-(i/\hbar)Ht] \tag{4.2}$$

and H is the total Hamiltonian, Eq. (2.3) for the exact case or Eq. (3.3) for the approximate case.

The matrix element can be evaluated by writing Eq. (4.1) in the coordinate representation and evaluating the resulting integrals with the aid of the generating functions for Hermite polynomials. The transition amplitude for the exact case is given by

$$A\begin{pmatrix} l'm' \\ l\ m \end{pmatrix} = A\begin{pmatrix} 0\,0 \\ 0\,0 \end{pmatrix}\left(\frac{l'!m'!l!m!}{2^{l'+m'+l+m}}\right)^{1/2}\left[\frac{1}{l'!m'!l!m!}\left(\frac{\partial}{\partial t_1}\right)^{l'}\left(\frac{\partial}{\partial t_2}\right)^{m'}\left(\frac{\partial}{\partial t_3}\right)^{l}\left(\frac{\partial}{\partial t_4}\right)^{m}\right.$$
$$\left.\times\, G(t_1, t_2, t_3, t_4)\right] t_1 = t_2 = t_3 = t_4 = 0 \tag{4.3}$$

where

$$G(t_1, t_2, t_3, t_4) = \exp[-(t_1^2 + t_2^2 + t_3^2 + t_4^2)]$$
$$\times \exp\left((t_1 + t_2 + t_3 + t_4)^2 \frac{\omega}{2} \frac{1 - e^{i\omega_x t}}{\omega + \omega_x - (\omega - \omega_x)e^{i\omega_x t}}\right) \exp\left((t_1 + t_2 - t_3 - t_4)^2 \frac{\omega}{2} \frac{1 + e^{i\omega_x t}}{\omega + \omega_x + (\omega - \omega_x)e^{i\omega_x t}}\right)$$
$$\times \exp\left((-t_1 + t_2 - t_3 + t_4)^2 \frac{\omega}{2} \frac{1 - e^{i\omega_y t}}{\omega + \omega_y - (\omega - \omega_y)e^{i\omega_y t}}\right) \exp\left((-t_1 + t_2 + t_3 - t_4)^2 \frac{\omega}{2} \frac{1 + e^{i\omega_y t}}{\omega + \omega_y - (\omega - \omega_y)e^{i\omega_y t}}\right) \tag{4.4}$$

and $A\left(\begin{smallmatrix} 0\,0 \\ 0\,0 \end{smallmatrix}\right)$ is the probability amplitude that if the system starts in the state with no quanta in A and B, it remains in the state at time t,

$$A\begin{pmatrix} 0\,0 \\ 0\,0 \end{pmatrix} = \frac{4\omega(\omega_x\omega_y)^{1/2}e^{-i\omega t}}{[(\omega_x + \omega)^2 - (\omega_x - \omega)^2 e^{2i\omega_x t}]^{1/2}[(\omega_y + \omega)^2 - (\omega_y - \omega)^2 e^{2i\omega_y t}]^{1/2}}; \tag{4.5}$$

ω_x and ω_y are defined in Eq. (2.7). The four fold differentiation in Eq. (4.3) may be performed explicitly but the result is too algebraically complicated to be worth writing down.

The transition amplitude in the resonant approximation can be obtained by making the approximations

$$\omega_x \simeq \omega(1+\tfrac{1}{2}\lambda), \quad \omega_y \simeq \omega(1-\tfrac{1}{2}\lambda) \tag{4.6}$$

in Eqs. (4.4) and (4.5) and retaining only lowest-order terms in λ except where λ appears multiplied by t. Equation (4.4) becomes

$$G(t_1,t_2,t_3,t_4) = \exp[-(t_1t_3+t_2t_4)2\cos\tfrac{1}{2}\lambda\omega t\, e^{i\omega t}]\exp[-(t_2t_3+t_1t_4)2\sin\tfrac{1}{2}\lambda\omega t e^{i\omega t}], \tag{4.7}$$

while Eq. (4.5) becomes

$$A\binom{0\,0}{0\,0} = \exp(-i\omega t). \tag{4.8}$$

The differentiation may be explicitly performed and the transition amplitude is

$$A\binom{l'\,m'}{l\;\;m} = A\binom{0\,0}{0\,0}(l'!m'!l!m!)^{1/2}\sum_{d=0}^{\infty}\frac{(-\cos\tfrac{1}{2}\lambda\omega t\, e^{i\omega t})^{2d+l-m'}(-i\sin\tfrac{1}{2}\lambda\omega t\, e^{i\omega t})^{m+m'-2d}}{d!(d+l-m')!(m-d)!(m'-d)!} \tag{4.9}$$

with the subsidiary condition that

$$l+m = l'+m'. \tag{4.10}$$

That is, in the resonant approximation the total number of quanta is conserved, since the coupling term in Eq. (3.3) commutes with the unperturbed Hamiltonian of the two oscillators.

Equation (4.9) can be rewritten in a particularly symmetric and revealing way

$$\begin{aligned} A\binom{l'\,m'}{l\;\;m} = A\binom{0\,0}{0\,0}(l'!m'!l!m!)^{-1/2}\sum_{d=0}^{\infty}&\frac{m'!}{d!(m'-d)!}\,\frac{m!}{d!(m-d)!}(-i\sin\tfrac{1}{2}\lambda\omega t\, e^{i\omega t})^{m-d} \\ &\times(m-d)!(-\cos\tfrac{1}{2}\lambda\omega t\, e^{i\omega t})^{d}d!\;\frac{l'!}{(l'-m+d)!(m-d)!}\,\frac{l!}{(l-m'+d)!(m'-d)!} \\ &\times(-i\sin\tfrac{1}{2}\lambda\omega t\, e^{i\omega t})^{m'-d}(m'-d)!\,(-\cos\tfrac{1}{2}\lambda\omega t\, e^{i\omega t})^{l-m'+d}(l-m'+d)!\,. \end{aligned} \tag{4.11}$$

The total amplitude is constructed out of two amplitudes. First

$$A\binom{0\,1}{1\,0} = A\binom{0\,0}{0\,0}(-i\sin\tfrac{1}{2}\lambda\omega t\, e^{i\omega t}) = A\binom{1\,0}{0\,1}$$

the amplitude that, if the first (second) oscillator starts with one quantum and the second (first) with zero, at a time t later the first (second) oscillator has no quanta and the second (first) has one, i.e., the amplitude for exchange of a single quanta between the oscillators. Second,

$$A\binom{1\,0}{1\,0} = A\binom{0\,0}{0\,0}(-\cos\tfrac{1}{2}\lambda\omega t\, e^{i\omega t}) = A\binom{0\,1}{0\,1}$$

the amplitude that, if one oscillator has one quantum and the other none, at a time t later the situation is unchanged. The total amplitude can be built up out of these two amplitudes by imagining the process as occurring in the following way (see Fig. 2). Oscillator 1 emits $m'-d$ quanta which are absorbed by oscillator 2, and retains $l-m'+d$ quanta. Oscillator 2 emits $m-d$ quanta which are absorbed by oscillator 1, and retains d quanta. The net transfer is $m-m'$, so in the final state oscillator 2 has m' quanta while oscillator 1 has l' (with $l'=l+m-m'$, since quanta are conserved). Note that the value of d is immaterial since it does not affect the final state, so for the total transition amplitude, d must be summed over.

The factors in Eq. (4.11) may now be understood in the following way: $(-ie^{i\omega t}\sin\tfrac{1}{2}\lambda\omega t)^{m-d}$ is the amplitude for emission of $m-d$ quanta from oscillator 2 and absorption by oscillator 1. These $m-d$ can be chosen from the original m in $m!/d!(m-d)!$ ways. The $m-d$ quanta can be rearranged in $(m-d)!$ ways. Oscillator 2 receives from 1 $(m'-d)$ quanta which can be arranged among the final m' in $m'!/d!(m'-d)!$ ways. Oscillator 2 retains d quanta, hence $(-e^{i\omega t}\cos\tfrac{1}{2}\lambda\omega t)^{d}$; these can be chosen in only one way, once the emitted $m-d$ are chosen, but can be rearranged among themselves in $d!$ ways.

Oscillator 1 emits $m'-d$ which are absorbed by 2, giving $(-ie^{i\omega t}\sin\tfrac{1}{2}\lambda\omega t)^{m'-d}$. These can be chosen from the initial l in $l!/(l-m'+d)!(m'-d)!$ ways and rearranged in $(m'-d)!$ ways. Oscillator 2 absorbs $m-d$ from 2; these can be arranged among the final l' in $l'!/(l'-m+d)!(m-d)!$ ways. Oscillator 1 retains $l-m'+d$, hence $(-e^{i\omega t}\cos\tfrac{1}{2}\lambda\omega t)^{l-m'+d}$ which can be rearranged in $(l-m'+d)!$ ways.

The factor $(l!m!l'!m'!)^{-1/2}$ is the correction for Bose statistics. In the transition probability a state with l quanta present should receive a statistical weight of 1 rather than $l!$.

The argument given here is similar to that of Feynman[14] in the problem of the quantum oscillator driven by a classical source. The same construction is possible for the exact transition probability with the addi-

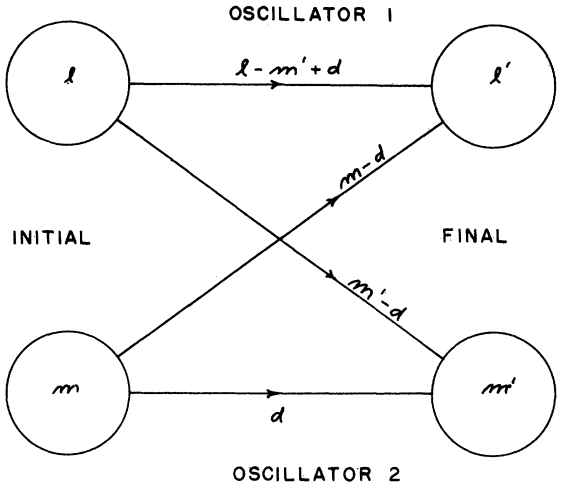

FIG. 2. Schematic representation of the exchange of quanta between the two oscillators.

tional complication that the coupling mechanism may serve as a source or sink of energy.

We have done several numerical examples to examine the validity of the resonant approximation. The simplest transition probability is the probability that, if at $t = 0$ the two oscillators have occupation number zero, at time t they will be in the same state. In the resonant approximation this probability is identically one. For the exact case it is given by

$$P\binom{0\,0}{0\,0} = \left(1+\frac{\lambda^2}{4(1+\lambda)}\sin^2[\omega(1+\lambda)^{1/2}t]\right)^{-1/2}\left(1+\frac{\lambda^2}{4(1-\lambda)}\sin^2[\omega(1+\lambda)^{1/2}t]\right)^{-1/2} \tag{4.12}$$

This differs from one because the $|00\rangle$ state is not an energy eigenstate of the exact Hamiltonian. Figure 3 shows $P\binom{0\,0}{0\,0}$ as a function of time for the exact and approximate cases for several values of the coupling parameter λ. Even for values of λ as high as 0.1, we see that the resonant approximation is excellent.

Next we compare the probabilities that, if at $t = 0$ Oscillator 1 has one quantum and Oscillator 2 has zero, at time t Oscillator 2 has one quantum and Oscillator 1 has none. In the resonant approximation this is given by

$$P\binom{1\,0}{0\,1} = \sin^2(\tfrac{1}{2}\lambda\omega t) \tag{4.13}$$

while in the exact case

$$P\binom{1\,0}{0\,1} = \tfrac{1}{4}P^3\binom{0\,0}{0\,0}\Big[\frac{\lambda^2}{4(1+\lambda)}\sin^2[\omega(1+\lambda)^{1/2}t]+\frac{\lambda^2}{4(1-\lambda)}\sin^2[\omega(1+\lambda)^{1/2}t]$$
$$+2\left(1-\frac{4-\lambda^2}{4(1-\lambda^2)^{1/2}}\right)\sin^2\{\tfrac{1}{2}\omega[(1+\lambda)^{1/2}+(1-\lambda)^{1/2}]t\}+2\left(1+\frac{4-\lambda^2}{4(1-\lambda^2)^{1/2}}\right)\sin^2\{\tfrac{1}{2}\omega[(1+\lambda)^{1/2}-(1-\lambda)^{1/2}]t\}\Big] \tag{4.14}$$

where $P\binom{0\,0}{0\,0}$ is as given in Eq. (4.12). Figure 4 shows these two probabilities for several values of λ. Again the approximation is excellent for $\lambda = 0.1$.

In general it can be seen by retaining the next-higher term in λ in Eq. (4.7) that the error in the resonant approximation transition probabilities

$$P\binom{l'\,m'}{l\;\;m}$$

is given by

$$\epsilon \sim \tfrac{1}{4}N\lambda^2,$$

where N is the largest of the four occupation numbers l, m, l', and m'.

V. EVOLUTION OF ONE OSCILLATOR OF THE COUPLED SYSTEM

The general conclusion of the preceeding sections substantiates the idea that in many problems of interest involving coupled harmonic oscillators, the resonant approximation is an adequate one. In the present section we discuss the resonant approximation in more detail. In what follows we consider the evolution of one oscillator rather than describing the dynamics of the whole coupled system by means of joint distributions. The language used throughout this section will be the coherent-state representation.[15] Extensive use will also be made of several results obtained by Mollow and Glauber in Refs. 11, 12, and 13.

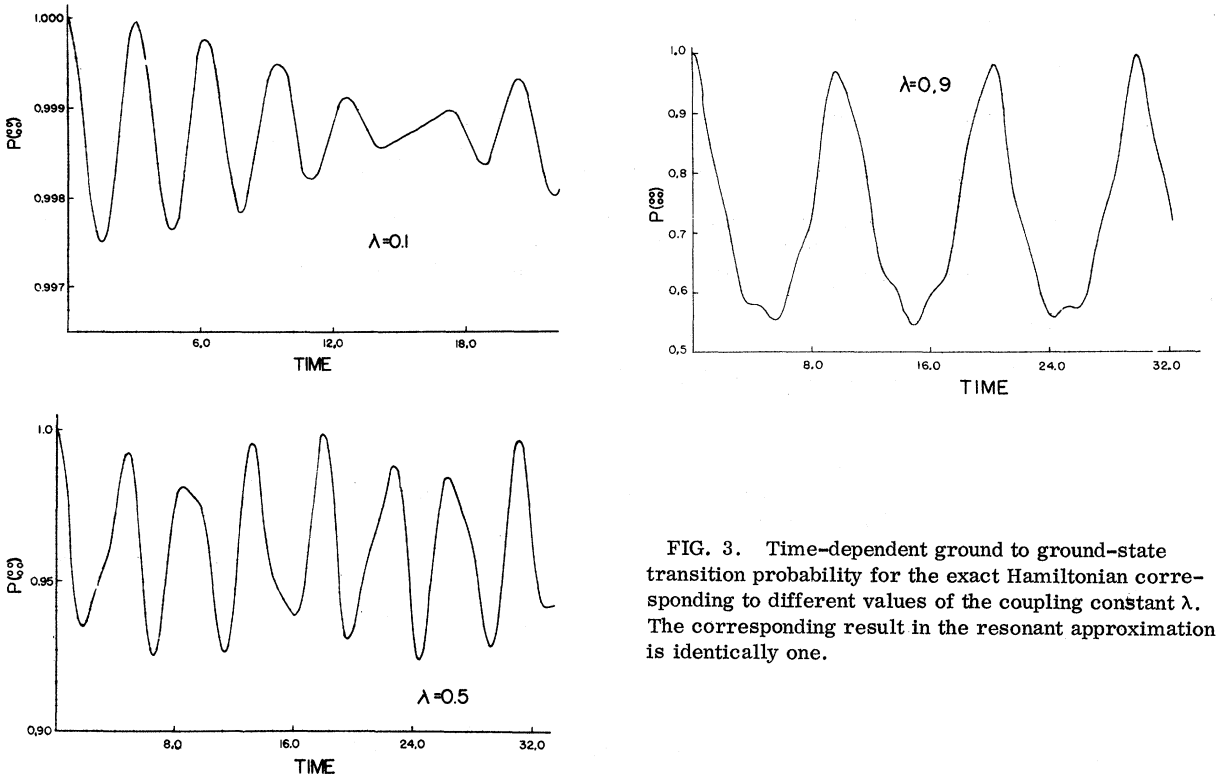

FIG. 3. Time-dependent ground to ground-state transition probability for the exact Hamiltonian corresponding to different values of the coupling constant λ. The corresponding result in the resonant approximation is identically one.

The statistical description of the state of the oscillator can be given in terms of the reduced-density operator $\rho_A(t)$ defined by

$$\rho_A(t) = \mathrm{Tr}_B \rho(t) , \tag{5.1}$$

where the subscript B indicates that the trace is taken only with respect to the variables of the B oscillator and $\rho(t)$ is the total density operator. A very useful representation for the reduced density operator $\rho_A(t)$

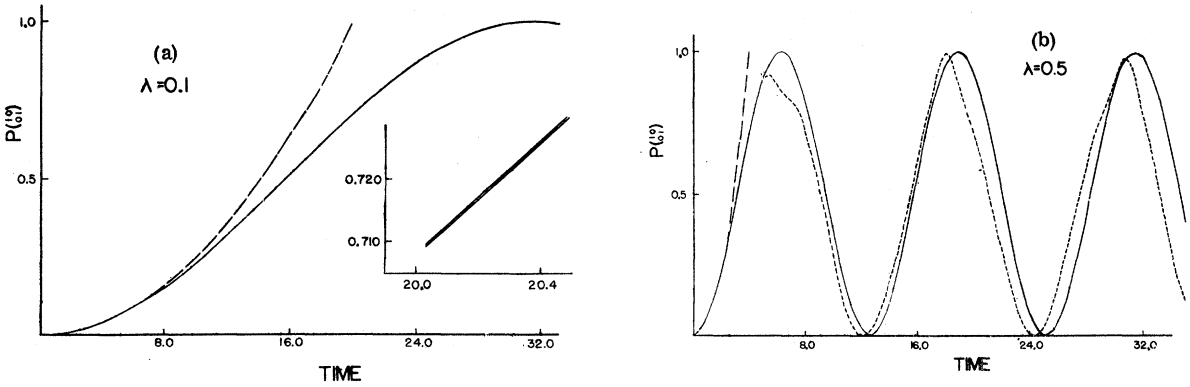

FIG. 4. Time-dependent transition probability between the states $|1,0\rangle$ and $|0,1\rangle$. In Fig. (4a) the exact and approximate solutions are indistinguishable. A small segment of the two curves is shown in the expanded diagram at the right. The dashed curve is the result of first-order perturbation theory. In Fig. (4b) the dotted line represents the exact solution, the dashed curve is the result of first-order perturbation theory.

is the so-called P representation:[16-18]

$$\rho_A(t) = \int d^2\alpha \, |\alpha\rangle \langle \alpha | P(\alpha, t) . \tag{5.2}$$

A question may arise as to whether the expansion (5.2) exists, and in particular if it exists throughout the entire evolution of the system.[12] The P function in some cases may become extremely singular and fail to be interpretable as an ordinary function,[19] or even as a tempered distribution,[20] but this difficulty does not arise for the cases we shall examine here.

The operator $\rho_A(t)$ can be found in a natural way by solving the Liouville equation corresponding to the total Hamiltonian and by taking the trace over the unwanted variables (the oscillator B in our case). A considerably simpler procedure is to make use of the characteristic function (CF) of the A oscillator,

$$\chi_A(\eta, t) = \mathrm{Tr}[\rho(t) e^{\eta a^\dagger - \eta^* a}] \tag{5.3}$$

and the normal ordered characteristic function (NOCF)

$$\chi_{NA}(\eta, t) = \mathrm{Tr}[\rho(t) e^{\eta a^\dagger} e^{-\eta^* a}] . \tag{5.4}$$

By applying the Baker-Haussdorf identity to the exponential operator of Eq. (5.3) it is easy to show that the CF is related to the NOCF by the following equation.

$$\chi_A(\eta, t) = e^{-\frac{1}{2}|\eta|^2} \chi_{NA}(\eta, t) . \tag{5.5}$$

If the density operator has a P representation, then the NOCF is the Fourier Transform of the P distribution

$$\chi_{NA}(\eta, t) = \int d^2\alpha \, P(\alpha, t) e^{\eta\alpha^* - \eta^*\alpha} . \tag{5.6}$$

The inverse transform gives the P distribution at any time t once the NOCF is known at all times.

$$P(\alpha, t) = \pi^{-2} \int d^2\eta \, \chi_{NA}(\eta, t) e^{-\eta\alpha^* + \eta^*\alpha} . \tag{5.7}$$

Furthermore the density operator $\rho_A(t)$ possesses the following integral representation[11]

$$\rho_A(t) = \pi^{-1} \int d^2\eta \, \chi_A(\eta, t) e^{\eta^* a - \eta a^\dagger} . \tag{5.8}$$

The complete description of the A oscillator is therefore reduced to the evaluation of the CF at all times. From the definition (5.4) we have,

$$\chi_{NA}(\eta, t) = \mathrm{Tr}[\rho(t) e^{\eta a^\dagger} e^{-\eta^* a}] = \mathrm{Tr}[\rho(0) U^\dagger(t) e^{\eta a^\dagger} e^{-\eta^* a} U(t)] , \tag{5.9}$$

where $U(t)$ is the time-displacement operator corresponding to the Hamiltonian (3.3). Finally by inserting the identity $I = UU^\dagger$ between the exponential operators of Eq. (5.9), we have

$$\chi_{NA}(\eta, t) = \mathrm{Tr}[\rho(0) e^{\eta a^\dagger(t)} e^{-\eta^* a(t)}] . \tag{5.10}$$

The advantage of Eq. (5.10) is that the statistical evolution of the system is described in terms of the initial density operator and the solution of the Heisenberg equations for $a(t)$ and $a\dagger(t)$. As shown in Sec. III the exact solutions of the Heisenberg equations are

$$a(t) = \lambda(t) a + \mu(t) b , \tag{5.11}$$

$$a^\dagger(t) = \lambda^*(t) a^\dagger + \mu^*(t) b^\dagger , \tag{5.12}$$

where the functions of time $\lambda(t)$ and $\mu(t)$ are, respectively, given by

$$\lambda(t) = \exp(-i\omega t) \cos\kappa t , \tag{5.13}$$

$$\mu(t) = -i \exp(-i\omega t) \sin\kappa t . \tag{5.14}$$

Using Eqs. (5.11) and (5.12) the NOCF takes the form

$$\chi_{NA}(t) = \mathrm{Tr}[\rho(0) e^{\eta\lambda^*(t) a^\dagger} e^{-\eta^*\lambda(t) a} e^{\eta\mu^*(t) b^\dagger} e^{-\eta^*\mu(t) b}] . \tag{5.15}$$

Equation (5.15) will be frequently used in the following examples.

A. Coherent Initial State

Let us assume the initial-density operator to be of the form

$$\rho(0) = |\alpha_0\beta_0\rangle\langle\alpha_0\,\beta_0| \ . \tag{5.16}$$

By means of Eq. (5.15) the NOCF is found to be

$$\chi_{NA}(\eta, t) = \exp[\eta\alpha^*(t) - \eta^*\alpha(t)] \ , \tag{5.17}$$

where $\alpha(t)$ is the solution of Hamilton equations corresponding to the classical Hamiltonian (3.3),

$$\alpha(t) = \lambda(t)\alpha_0 + \mu(t)\beta_0 \ . \tag{5.18}$$

As a consequence of Eq. (5.17) the P distribution for the A oscillator is given by

$$P(\alpha, t) = \pi^{-2}\int d^2\eta \ \chi_{NA}(\eta, t)\exp[-\eta\alpha^*(t) + \eta^*\alpha(t)] = \delta^2[\alpha - \alpha(t)] \ . \tag{5.19}$$

The evolution of the P function in the phase space Reα, Imα is easily interpreted. The distribution does not change its shape with time, that is, the A oscillator remains coherent at all times. The center of the distribution however, follows a trajectory whose projections onto the Reα and Imα axis represent the evolution of the position and momentum of the A oscillator.[21]

The probability distribution for the excitation number n is a Poisson distribution with a mean value $|\alpha(t)|^2$

$$P_A(n) = \langle n|\rho_A(t)|n\rangle = e^{-|\alpha(t)|^2}(|\alpha(t)|^{2n}/n!) \ . \tag{5.20}$$

B. Coherent State for A, Chaotic State for B

One oscillator is in contact with a thermal bath and at time $t = 0$ is coupled with the second oscillator which is initially in a coherent state. No energy exchange is allowed for $t > 0$ between the two oscillators and the external surroundings. The initial-density matrix of the system is

$$\rho(0) = |\alpha_0\rangle\langle\alpha_0| \sum_{n_b=0}^{\infty} \frac{1}{1+\bar{n}_b}\left(\frac{\bar{n}_b}{1+\bar{n}_b}\right)^{n_b} |n_b\rangle\langle n_b| \ . \tag{5.21}$$

The NOCF defined by Eq. (5.15) is given by

$$\chi_{NA}(\eta, t) = \exp[\eta\lambda^*(t)\alpha_0{}^* - \eta^*\lambda(t)\alpha_0]\sum_{n_b=0}^{\infty} \frac{\bar{n}_b{}^{n_b}}{(1+\bar{n}_b)^{n_b+1}} L_{n_b}(|\eta|^2|\mu(t)|^2) \ , \tag{5.22}$$

where we have used the fact that

$$\langle n|e^{\beta a^\dagger} e^{-\beta^* a}|n\rangle = L_n(|\beta|^2) \ . \tag{5.23}$$

The series on the rhs of Eq. (5.22) is the generating function for Laguerre polynomials. Therefore,

$$\chi_{NA}(\eta, t) = \exp[-\bar{n}_b|\mu(t)|^2|\eta|^2]\exp[\eta\lambda^*(t)\alpha_0{}^* - \eta^*\lambda(t)\alpha_0] \ . \tag{5.24}$$

The Fourier transform of Eq. (5.24) can be found by using the identity

$$\pi^{-1}\int d^2\gamma \exp[-(|\gamma|^2/\lambda) + \gamma\nu^* + \gamma^*\xi] = \lambda e^{\lambda\nu^*\xi} \quad \mathrm{Re}\lambda > 0 \ . \tag{5.25}$$

The result is

$$P(\alpha, t) = \frac{1}{\pi\bar{n}_b|\mu(t)|^2}\exp\left(-\frac{|\alpha - \lambda(t)\alpha_0|^2}{\bar{n}_b|\mu(t)|^2}\right), \quad |\mu(t)|^2 = \sin^2\kappa t, \ \lambda(t) = e^{-i\omega t}\cos\kappa t \ . \tag{5.26}$$

Let us consider first the initial state of the A oscillator to be the ground state $|\alpha_0 = 0\rangle$. The P-distribution in this case is a Gaussian distribution in the α-phase space whose mean value is zero at all times (Fig. 5). However, the shape of the distribution is a function of time and in fact a periodic one. The variance of the distribution

$$\sigma^2 = \tfrac{1}{2}\bar{n}_b|\mu(t)|^2 = \tfrac{1}{2}\bar{n}_b\sin^2\kappa t \tag{5.27}$$

undergoes a periodic variation whose period is $T = \pi/\kappa$. At time $t = 0$ the P function is a δ function at the origin. As time goes on the state of the A oscillator becomes chaotic. The maximum spread of the dis-

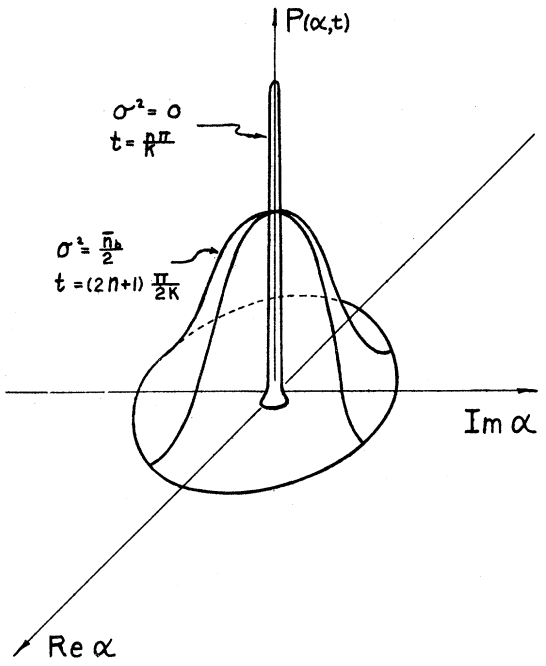

FIG. 5. Time evolution of the P distribution describing the complex amplitude of oscillator A whose initial state is the coherent state $|\alpha=0\rangle$ interacting with an initially chaotic oscillator B.

tribution occurs at time $T=\pi/2\kappa$ when the mean excitation number for the A oscillator is $\bar{n}_b$. At this point A has lost every memory of its initial condition. Then the "chaoticity" flows back to B, and half a period later the initial situation is reproduced. When the coupling constant κ is very small compared with an observation time, i.e., when $\kappa t \ll 1$, the P distribution can be approximated by

$$P(\alpha,t) \cong (\pi \bar{n}_b \kappa^2 t^2)^{-1} \exp\left(-\frac{|a|^2}{n_b \kappa^2 t^2}\right). \tag{5.28}$$

The process described by the P function (5.28) is a diffusion process (in a probabilistic sense) because of the particular choice of the observation time ($t \ll T$). In many cases however, the observation time is naturally subjected to the condition $t \ll T$. Equation (5.28) is therefore an irreversible approximation of the true behavior.

Consider now the general initial condition $\alpha_0 \neq 0$. In this case two different processes take place in the α-phase space. In addition to the periodic variation of the variance $\sigma^2 = \frac{1}{2}\bar{n}_b \sin^2 \kappa t$ the center of the distribution moves according to the law of motion.

$$\bar{\alpha}(t) = \alpha_0 \lambda(t) = \alpha_0 \exp(-i\omega t). \tag{5.29}$$

Such a motion is illustrated by Fig. 6. The rapid rotation with angular velocity ω drives the polar radius $\bar{\alpha}$ around the origin. Meanwhile the slowly varying factor $\cos\kappa t$ modulates its length. At the same time the variance assumes its maximum value $\sigma^2 = \frac{1}{2}\bar{n}_b$ and the oscillator A is completely chaotic with a mean excitation $\bar{n} = \bar{n}_b$. The main difference therefore, between the cases $\alpha_0 = 0$ and $\alpha_0 \neq 0$ is that the state of the oscillator A for $\alpha_0 = 0$ is always chaotic except for $t = n\pi/\kappa$, whereas for $\alpha_0 \neq 0$ the state is a mixture of coherent and chaotic states. The last statement will be clarified if we evalutate the probability distribution for the excitation number n. By definition

$$p_A(n,t) = \langle n|\rho_A(t)|n\rangle = \int d^2\alpha \; P(\alpha,t) e^{-|\alpha|^2} |\alpha|^{2n}/n! \tag{5.30}$$

$$p_A(n,t) = (\pi \bar{n}_b |\mu(t)|^2)^{-1} \int d^2\alpha \exp[-|\alpha - \alpha_0 \lambda(t)|^2/\bar{n}_b |\mu(t)|^2 - |\alpha|^2] |\alpha|^{2n}/n! \tag{5.31}$$

Eq. (5.31) can be easily integrated. The result is

$$p_A(n,t) = \frac{[\bar{n}_b |u(t)|^2]^n}{[1+\bar{n}_b |u(t)|^2]^{n+1}} \exp\left(\frac{|\alpha_0|^2 |\lambda(t)|^2}{1+\bar{n}_b |\mu(t)|^2}\right) L_n\left(-\frac{|\alpha|^2 |\lambda(t)|^2}{n_b |\mu(t)|^2 (1+\bar{n}_b |\mu(t)|^2)}\right) \tag{5.32}$$

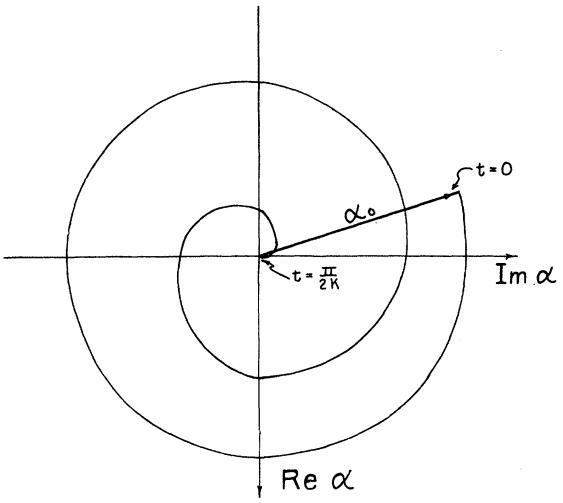

FIG. 6. Time evolution of the center of the P distribution for oscillator A whose initial state is a coherent state $|\alpha_0\rangle$, interacting with an initially chaotic oscillator B.

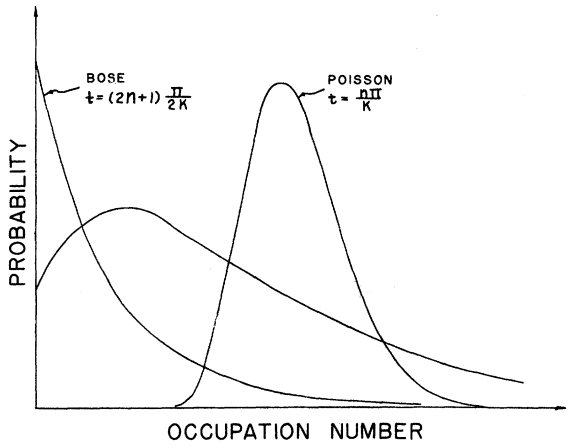

FIG. 7. Time evolution of the probability distribution $p(n,t)$ for the occupation number n of oscillator A whose initial state is the coherent state $|\alpha_0\rangle$ interacting with an initially chaotic oscillator B.

As we mentioned above, for $\alpha_0=0$ the distribution $p_A(n,t)$ is always a single-mode Bose-Einstein distribution with a mean occupation number $\bar{n}_b|\mu|^2$ that for $t=0$ reduces to

$$p_a(n,0)=\delta_{n,0}\ . \tag{5.33}$$

For $\alpha_0\neq 0$ the time evolution of $p_A(n,t)$ is qualitatively described in Fig. 7. It is easy to check by using Eq. (5.32) that at time $t=(\pi/2\kappa)(2n+1)$, $n=0,1,2\ldots$ the distribution is a Bose-Einstein distribution with mean value $\bar{n}_b$ and at time $t=\pi n/\kappa$ the distribution is a Poisson distribution with mean value $|\alpha_0|^2$.

C. Number State for A, Chaotic For B

If we assume the initial state of the A oscillator to be an eigenstate of the unperturbed Hamiltonian and if the initial state of B is a thermal mixture, the density operator is given by

$$\rho(0)=|n_a\rangle\langle n_a|\sum_{n_b=0}^{\infty}\frac{1}{1+\bar{n}_b}\left(\frac{\bar{n}_b}{1+\bar{n}_b}\right)^{n_b}|n_b\rangle\langle n_b|\ . \tag{5.34}$$

By use of Eq. (5.15) and (5.23) we easily find that

$$\chi_{NA}(\eta,t)=L_{n_a}(|\eta|^2|\lambda(t)|^2)\sum_{n_b=0}^{\infty}\frac{\bar{n}_b{}^{n_b}}{(1+\bar{n}_b)^{n_b+1}}L_{n_b}(|\eta|^2|\mu(t)|^2)$$

$$=L_{n_a}(|\eta|^2|\lambda(t)|^2)\exp(-\bar{n}_b|\mu(t)|^2|\eta|^2)\ . \tag{5.35}$$

The P distribution can be found by Fourier transforming Eq. (5.35). By inspection we can easily observe that as long as $|\mu|^2\neq 0$, the Fourier transform of Eq. (5.35) exists and is well defined in the space of square-integrable functions. When $|\mu|=0$ (i.e., for times $t=n\pi/\kappa$) the NOCF becomes a Laguerre polynomial. If we assume $|\mu|\neq 0$ the integration can be done. The result is

$$P(\alpha,t)=\frac{1}{\pi}\frac{\bar{n}_b|\mu(t)|^2-|\lambda(t)|^2}{(\bar{n}_b|\mu(t)|^2)^{n_a+1}}\exp\left(-\frac{|\alpha|^2}{\bar{n}_b|\mu(t)|^2}\right)L_{n_a}\left(\frac{|\lambda(t)|^2|\alpha|^2}{\bar{n}_b|\mu(t)|^2(|\lambda(t)|^2-\bar{n}_b|\mu(t)|^2)}\right)\ . \tag{5.36}$$

We should observe that while this P function is correctly normalized, it is not in general positive definite. For instance, when $|\lambda|^2-\bar{n}_b|\mu|^2>0$, the Laguerre Polynomials take on positive and negative values. Furthermore, even when L_{na} is positive the factor $(\bar{n}_b|\mu|^2-|\lambda|^2)^{n_a}$ can be made negative by choosing an odd

value for the integer n_a. The probability distribution for the occupation number n of the A oscillator can be evaluated by means of the following integral

$$p_A(n,t) = \int d^2\alpha P(\alpha,t) e^{-|\alpha|^2} |\alpha|^{2n}/n! \ . \tag{5.37}$$

The result of a rather involved calculation is

$$p_A(n,t) = \frac{(\bar{n}_b|\mu|^2 - |\lambda|^2)^{n_a}(\bar{n}_b|\mu|^2)^{n-n_a}}{(\bar{n}_b|\mu|^2+1)^{n+1}} \, {}_2F_1\left(-n_a, n+1; 1; \frac{|\lambda(t)|^2}{(\bar{n}_b|\mu|^2+1)(|\lambda|^2 - \bar{n}_b|\mu|^2)}\right), \tag{5.38}$$

where ${}_2F_1$ is the hypergeometric function. The complicated function given by Eq. (5.38) reduces to the familiar Bose-Einstein distribution for times $t = (2n+1)\pi/2\kappa$,

$$p_A(n,t) = (2n+1)\pi/2\kappa) = \bar{n}_b{}^n/(1+\bar{n}_b)^{n+1} \ . \tag{5.39}$$

D. Number State for A and for B

Let the initial-density operator of the uncoupled oscillators be

$$\rho(0) = |n_a, n_b\rangle\langle n_a, n_b| \ . \tag{5.40}$$

This situation is interesting because it represents a particular case of the problem discussed in Sec. III with a different approach. The NOCF is given by

$$\chi_{NA}(\eta,t) = L_{n_a}(|\eta|^2|\lambda|^2) L_{n_b}(|\eta|^2|\mu|^2) \ . \tag{5.41}$$

By means of the NOCF we can calculate the expectation value of normal ordered product of an arbitrary number of factors of $a^\dagger$ and a

$$\langle a^{\dagger m} a^n\rangle = (\partial/\partial\eta)^m(-\partial/\partial\eta^*)^n \chi_{NA}(\eta,t)\big|_{\eta=\eta^*=0} \ . \tag{5.42}$$

The expectation value of the number operator $a^\dagger a$ can be easily found by using Eq. (5.42)

$$\langle a^\dagger a\rangle = (\partial/\partial\eta)(-\partial/\partial\eta^*) L_{n_a}(|\eta|^2|\lambda|^2) L_{n_b}(|\eta|^2|\mu|^2)\big|_{\eta=\eta^*=0} = |\lambda|^2 n_a + |\mu|^2 n_b \ . \tag{5.43}$$

In general, all information concerning the behavior of A oscillator is contained in the distribution function for the excitation number

$$p_A(n,t) = \pi^{-1}\int d^2\eta e^{-|\eta|^2} L_{n_a}(|\eta|^2|\lambda|^2) L_{n_b}(|\eta|^2|\mu|^2) L_n(|\eta|^2) \ . \tag{5.44}$$

The integral (5.44) cannot be evaluated in closed form, except for several special cases. For $n=0$

$$p_A(0,t) = [(n_a+n_b)!/n_a!\,n_b!]\sin^{2n_a}(\kappa t)\cos^{2n_b}(\kappa t) \ . \tag{5.45}$$

The meaning of the factors $\sin^2\kappa t$ and $\cos^2\kappa t$ can be found easily putting $n_a = 1$ and $n_b = 0$. In this case

$$p_A(0,t) = \sin^2(\kappa t) \tag{5.46}$$

and for $n_a = 0$, $n_b = 1$

$$p_A(0,t) = \cos^2(\kappa t) = 1 - \sin^2(\kappa t) \ . \tag{5.47}$$

Equations (5.46) and (5.47) show that $\sin^2\kappa t$ is the probability of emission of one photon by Oscillator A and $\cos^2\kappa t$ is the probability of retention of the photon by oscillator A.

If the A oscillator is initially in its ground state $n_a = 0$ and if B is excited ($n_b \neq 0$), the probability of finding A excited at the n th level is given by

$$p_A(n,t) = [n_b!/(n_b-n)!\,n!]\cos^{2(n_b-n)}(\kappa t)\sin^{2n}(\kappa t), \quad n \leq n_b \ , \tag{5.48}$$

$$p_A(n,t) = 0 , \qquad n > n_b \ . \tag{5.49}$$

We note that the result (5.49) is in complete agreement with the quanta conserving property of the resonant approximation. Making use of our interpretation of (5.46) and (5.47) we can interpret (5.48) in the following way. The probability that A absorbs n quanta is equal to the probability that B emits n quanta ($\sin^{2n}\kappa t$) and retains $n_b - n$ quanta [$\cos^{2(n_b-n)}\kappa t$]. The factor $\binom{n_b}{n}$ is the number of ways the n quanta can be selected from the original n_b.

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APPENDIX

In this Appendix we give the details leading to the solution (2.5). Defining the normal coordinates

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix} = M \begin{pmatrix} x_a \\ x_b \end{pmatrix}; \quad \begin{pmatrix} p_A \\ p_B \end{pmatrix} = M \begin{pmatrix} p_a \\ p_b \end{pmatrix} \tag{A.1}$$

the Hamiltonian (2.1) takes the uncoupled form

$$H = \tfrac{1}{2}(p_A^2 + \omega_x^2 x_A^2) + \tfrac{1}{2}(p_B^2 + \omega_y^2 x_B^2), \tag{A.2}$$

where ω_x and ω_y are the normal-mode frequencies defined in Eq. (2.7). We can now define the normal-mode annihilation operators A and B.

$$\begin{aligned} A &= (\omega_x x_A + i p_A)/(2\hbar\omega_x)^{1/2}. \\ B &= (\omega_y x_B + i p_B)/(2\hbar\omega_y)^{1/2}. \end{aligned} \tag{A.3}$$

In terms of the operators (A.3) the Hamiltonian (A.2) can be written as

$$H = \hbar\omega_x(A^\dagger A + \tfrac{1}{2}) + \hbar\omega_y(B^\dagger B + \tfrac{1}{2}). \tag{A.4}$$

Writing the Heisenberg equations of motion for A and B, we have

$$\frac{d}{dt}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = -i\begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix}. \tag{A.5}$$

This equation is easily integrated and yields the result

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \Lambda(t)\begin{pmatrix} A \\ B \end{pmatrix}, \tag{A.6}$$

where $\Lambda(t)$ is defined in Eq. (2.6). The algebraic operations (A.3) and (A.1) can be inverted to arrive at the original position and momentum coordinates which in turn can be used in the defining Eq. (2.2) to determine the time behavior of the original annihilation operators. The final result of these algebraic manipulations is Eq. (2.5).

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