about 0.012 inch.

[32]The angular deflection is equal to $\tan^{-1}[\mu(\partial B/\partial x)l/2E]$, where l is the linear dimension over which the magnetic field gradient $\partial B/\partial x$ exists, and where E is the energy of the particle. For a particle of energy 0.5°K and a magnetic moment of $2\mu_B$, a magnetic field gradient of 8 kG/cm acting over a region 1 cm in length will produce an angular deflection of about 45°.

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[48]This result may be derived from electromagnetic theory using the analogy between magnetic energy and hydrodynamic energy. In our problem, the analogous case is the magnetic energy in a system of two circular loops of radius R carrying equal currents in opposite directions and separated by a distance $2d$. Equation (A2) may be written down immediately from W. R. Smythe, Static and Dynamic Electricity (McGraw-Hill Book Co., Inc., New York, 1939); Eqs. 8.08 (1), 8.10 (2), 8.03 (6), and 8.06 (1).

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Transport Theory in Strong Magnetic Fields

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A unified treatment of transport phenomena in crossed electric and magnetic fields is presented. This theory is limited to strong magnetic fields, $\omega_c\bar{\tau} \gg 1$, where ω_c and $1/\bar{\tau}$ are the cyclotron and mean collision frequencies, respectively. It is not, however, limited to linear response in the electric field nor is a relaxation time approximation introduced. The semiclassical and quantum theories are developed along parallel lines.

An irreversible transport equation is derived for the asymptotic state of the system, $\omega_c\tau \to \infty$, and it is shown that the asymptotic electron distribution is independent of the absolute values of the coupling constants to the scattering system, but dependent on the form of the scattering interaction. A perturbation theory in $1/\omega_c\bar{\tau}$ is performed and a generalized orbit-jump formula for the dissipative current is derived. Explicit expressions are derived for the ohmic case and are applied as an example to polar optical-phonon scattering.

INTRODUCTION

In this paper we present a theory of transport phenomena in crossed electric and magnetic fields. This work is restricted to strong magnetic fields and is essentially a perturbation theory in $1/\omega_c\bar{\tau}$, where ω_c is the cyclotron frequency and $1/\bar{\tau}$ is an appropriate average collision frequency. Our theory is motivated by the observation that the asymptotic state of the system for $\omega_c\bar{\tau} \to \infty$ is one in which the conduction electrons acquire a drift velocity $\vec{V}_d = \vec{E} \times \vec{B}/B^2$ normal to the electric field, and consequently the average power input to the system vanishes. This is true both classically and quantum mechanically as long as the particle orbits in $\vec{K}$ space are closed, which is the only case we treat in this paper. The simplicity and generality of this result suggests that the strong magnetic field problem is best approached by carrying out a perturbation theory in $1/\omega_c\bar{\tau}$ about

this asymptotic state of the system.

We shall treat both the classical and quantum theories in this spirit[1] and shall extend the usual treatments in two directions. The present work is firstly not limited to ohmic conductivity (linear response in the electric field), nor does it assume the existence of a relaxation time[2] to describe the effect of collisions.

The usual semiclassical theories[3] are based on solving the Boltzmann equation, which even in the ohmic case is only possible when a relaxation time approximation is employed. The nonlinear problem[4] is of considerably greater difficulty, since even if a relaxation time approximation is introduced in order to describe the relaxation of the anisotropic part of the electron distribution, the energy relaxation cannot be treated in this simple fashion. The usual[5] quantum theories of transport, on the other hand, are formulated without recourse to the relaxation time approximation, but deal mostly with the linear response problem.

In this paper, we present a unified treatment of both the classical and quantum theories in strong magnetic fields, which embraces both the linear and nonlinear problems without recourse to the relaxation time approximation. In Sec. 1 we establish the semiclassical kinetic equation describing the asymptotic state of the system and carry out the lowest-order perturbation theory in order to determine the dissipative current. This yields an expression for the current which is strikingly similar to the quantum orbit-jump formula first derived by Titeica.[6] In Sec. 2 we discuss the application of these results to the transport coefficients measured in the Hall geometry, where the electric field is not known *a priori*. Explicit expressions for the ohmic transport coefficients are derived in Sec. 3 and are applied to the case of polar optical-mode scattering in Sec. 4. Section 5 deals with the general quantum theory, and the linear results are displayed in Sec. 6.

I. THE ASYMPTOTIC STATE

We seek the steady-state distribution function f for a gas of noninteracting electrons in the presence of crossed electric and magnetic fields, denoted by $\vec{E}$ and $\vec{B}$, respectively. The relevant Boltzmann equation is

$$e(\vec{E}+\vec{V}\times\vec{B})\cdot\nabla_{\vec{p}}f(\vec{p})=\hat{C}f=\int d\vec{p}'\,[f(\vec{p}')T_{\vec{p}'\vec{p}}-f(\vec{p})T_{\vec{p}\vec{p}'}]\,. \tag{1}$$

Here $\vec{p}$ represents the electron crystal momentum, $\hat{C}$ is the collision operator characterized by transition rates $T_{\vec{p}\vec{p}'}$ between states $\vec{p}$ and $\vec{p}'$, and $\vec{V}=\nabla_{\vec{p}}\epsilon(\vec{p})$, where ϵ is the electron energy. In seeking the asymptotic state of the system for $\omega_c\tau\to\infty$, one is tempted to neglect the collision term at the outset and to solve the resulting steady state Liouville equation:

$$e(\vec{E}+\vec{V}\times\vec{B})\cdot\nabla_{\vec{p}}f=0\,. \tag{2}$$

One is then faced with a rather fundamental difficulty in that the resulting equation is completely reversible and admits an infinity of physically acceptable solutions. We see, for example, that $p_z=\vec{p}\cdot\vec{B}/B$ and $\epsilon-\vec{p}\cdot\vec{V}_d$ are constants of motion, with $\vec{V}_d=\vec{E}\times\vec{B}/B^2$:

$$dp_z/dt=(d\vec{p}/dt)\cdot\vec{B}/B=e(\vec{E}+\vec{V}\times\vec{B})\cdot\vec{B}/B=0,\quad (d/dt)(\epsilon-\vec{p}\cdot\vec{V}_d)=e(\vec{E}\cdot\vec{V}-e\vec{V}\times\vec{B}\cdot\vec{V}_d)=0\,, \tag{3}$$

and therefore any function $g(p_z,\epsilon-\vec{p}\cdot\vec{V}_d)$ solves Eq. (2).

It is in fact the collision term which is responsible for the irreversible behavior of the system and which is therefore responsible for the existence of unique steady state solutions of the transport equation. We must therefore approach the $\omega_c\tau\to\infty$ limit in a gentler manner, so as not to discard the irreversible nature of the system.

In order to accomplish this we use the path variable transformation[7] of the Boltzmann equation. The collision-free trajectories are calculated by solving the semiclassical equation:

$$d\vec{p}/dt=e(\vec{E}+\vec{V}\times\vec{B});\quad \vec{V}=\nabla_{\vec{p}}\epsilon\,, \tag{4}$$

and are introduced into the Boltzmann equation in the manner described in Ref. 7. The resulting equation

$$f(\vec{p})=\int_0^\infty ds\int d\vec{p}'f(\vec{p}')T_{\vec{p}'\vec{p}(s)}\exp[-\int_0^s dy/\tau(y)],\quad 1/\tau(s)\equiv\int d\vec{p}'\,T_{\vec{p}(s)\vec{p}'} \tag{5}$$

is identical in form to Eq. (10) of Ref. 7, the only difference being that the collision-free trajectories are now determined by Eq. (4). The physical interpretation is again the same as before; $T_{\vec{p}'\vec{p}(s)}ds$ is the probability of a particle scattering from state $\vec{p}'$ to $\vec{p}(s)$ in a time interval ds near s, where $\vec{p}(s)$ is the momentum of a particle which would drift into the state $\vec{p}$ under the influence of the applied fields if it suffered no collisions in the subsequent s seconds. This latter survival probability is represented by the exponential factor, while $f(\vec{p}')$ takes account of the occupancy probability of the initial state $\vec{p}'$. The function $f(\vec{p})$ is then constructed by summing over all initial states $\vec{p}'$ and over all points of entry along the collision-free trajectory. We emphasize here that Eq. (5) is an exact transformation of the Boltzmann Eq. (1).

Since the orbits are periodic in $\vec{p}$ space we have $\vec{p}(s)=\vec{p}(s+T)$ where $T=2\pi/\omega_c$ is the cyclotron period.

Introducing this periodicity explicitly in Eq. (5), we obtain

$$f(\vec{p}) = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T'} ds \int d\vec{p}' f(\vec{p}') T_{\vec{p}'\vec{p}(s)} \exp[-\int_0^s dy/\tau(y)]$$

$$= \sum_{n=0}^{\infty} \int_0^T ds \int d\vec{p}' f(\vec{p}') T_{\vec{p}'\vec{p}(s)} \exp[-\int_0^s dy/\tau(y)] \exp[-n\int_0^T dy/\tau(y)]$$

$$= \int_0^T ds \int d\vec{p}' f(\vec{p}') T_{\vec{p}'\vec{p}(s)} \exp[-\int_0^s dy/\tau(y)]\{1-\exp[-\int_0^T dy/\tau(y)]\}^{-1} . \quad (6)$$

At this stage, Eq. (6) is still exact. We now introduce the approximation $1/\omega_c\bar{\tau} \approx T/\bar{\tau} \ll 1$. The integrals appearing in the exponentials are therefore small, and expanding the exponentials we obtain

$$f(\vec{p}) = \int_0^T ds \int d\vec{p}' f(\vec{p}') T_{\vec{p}'\vec{p}(s)} / \int_0^T ds/\tau(s) , \quad (7a)$$

$$f(\vec{p}) \int d\vec{p}' \int_0^T ds\, T_{\vec{p}(s)\vec{p}'} = \int d\vec{p}' f(\vec{p}') \int_0^T ds\, T_{\vec{p}'\vec{p}(s)} . \quad (7b)$$

Equation (7) represents the asymptotic limit $\omega_c\bar{\tau} \to \infty$ of the Boltzmann equation and is one of our essential results. It is very much like the thermal equilibrium equation $\hat{C}f = 0$, except that the transition rates are replaced by an average over one period of the collision-free motion, and are consequently field dependent. The asymptotic distribution function f_{adf}, determined by Eq. (7) is obviously independent of any absolute coupling constants to the scattering system, since Eq. (7) is homogeneous in the transition rates $T_{\vec{p}\vec{p}'}$, although it does depend on the form of the interaction. The independence of f_{adf} of the absolute coupling constants is similar to the weak-coupling theory of thermal equilibrium, where the thermal equilibrium distribution exhibits this same property. This is a point worth emphasizing. Although f_{adf} is independent of the absolute coupling constants, it is not simply related to the thermal equilibrium distribution and depends on the detailed nature of the scattering system. This is discussed further in the Appendix.

We now verify that f_{adf} determined by Eq. (7) also satisfies the Liouville Eq. (2). This follows directly from the following property:

$$e(\vec{E} + \vec{V}\times\vec{B}) \cdot \nabla_{\vec{p}} \int_0^T g(\vec{p}(t))dt = \int_0^T \frac{dg(\vec{p}(t))}{dt} dt = g(\vec{p}(T)) - g(\vec{p}(0)) = 0 . \quad (8)$$

Equation (7) defines f_{adf} as the ratio of two trajectory averages, each of which solves Eq. (2) in the manner shown in Eq. (8). Since the Liouville equation is of first order, it follows that the ratio of the two trajectory averages, and consequently f_{adf}, is a solution of Eq. (2). We use this property to recast Eq. (7) in a slightly different form. Since f_{adf} solves Eq. (2), it is constant along the trajectory $f(\vec{p}) = f(\vec{p}(s))$ and therefore Eq. (7) may be written

$$\int_0^T ds[\hat{C}f]_{\vec{p}(s)} = 0 \quad \text{with } f(\vec{p}) = f(\vec{p}(s)) . \quad (9)$$

The results are readily generalized to take account of Fermi statistics, in which case the appropriate collision operator becomes

$$\hat{C}f = \int d\vec{p}' \{f(\vec{p}')[1-f(\vec{p})]T_{\vec{p}'\vec{p}} - f(\vec{p})[1-f(\vec{p}')]T_{\vec{p}\vec{p}'}\} . \quad (10)$$

Application of the same path variable transformation as above then yields in place of Eq. (7),

$$f(\vec{p}) \int d\vec{p}'[1-f(\vec{p}')] \int_0^T ds\, T_{\vec{p}(s)\vec{p}'} = \int d\vec{p}' f(\vec{p}') \int_0^T ds\, T_{\vec{p}'\vec{p}(s)}[1-f(\vec{p}(s))] . \quad (11)$$

All the properties of the f_{adf} discussed above remain unchanged, the only new feature being the presence of the exclusion factors $[1-f(\vec{p}')]$ multiplying the transition rates $T_{\vec{p}\vec{p}'}$.

In order to calculate the current density $\vec{j}$ in the asymptotic state, we make use of the fact that f_{adf} solves Eq. (2). We take $\vec{B}$ in the z direction and $\vec{E}$ in the x direction and multiply Eq. (2) by p_α, $\alpha = x, y, z$. Integrating over $\vec{p}$ then determines the current density. This is in fact simply the moment equation describing the conservation of momentum. Consider, for example, p_y:

$$e\int d\vec{p}\, p_y \left((E + V_y B)\frac{\partial f}{\partial p_x} - V_x B \frac{\partial f}{\partial p_y}\right) = 0 = -eB \int d\vec{p} f \left(\frac{\partial}{\partial p_x}(p_y V_y) - \frac{\partial}{\partial p_y}(p_y V_x)\right)$$

$$= eB \int d\vec{p}\, V_x f = Bj_x \equiv Bne\langle V_x\rangle , \quad (12)$$

where we have integrated by parts and have assumed that f vanishes sufficiently rapidly as $p \to \infty$. The normalization used throughout is $\int d\vec{p} f = n$, the electron density. A similar procedure yields $\langle V_y\rangle = -E/B$

$\equiv V_d$. Nothing further can be said about $\langle V_z \rangle$, since the p_z moment equation vanishes because p_z is a constant of motion. We take $\langle V_z \rangle = 0$ for convenience if it is not so by symmetry.

We should now like to carry out our perturbation theory in $1/\omega_c\bar{\tau}$. Only the lowest-order terms will be considered here, although the perturbation procedure is readily extended to higher order. The perturbation theory is most readily performed by setting $f = f_{\text{adf}} + f_1$ and inserting this in Eq. (1)

$$e[\vec{E} + \vec{V} \times \vec{B}] \cdot \nabla_{\vec{p}} f_1 = \hat{C}(f_{\text{adf}} + f_1) \approx \hat{C} f_{\text{adf}} \tag{13}$$

since $f_1 \sim O(1/\omega_c\bar{\tau})$. We note that the condition (7) is necessary for the solubility of Eq. (13). This solubility condition and the relation of our work to the linear theory of Lifschitz, Azbel, and Koganov[8] are discussed in the Appendix.

We are particularly interested in calculating the dissipative current (in the electric field direction), which is zero in the asymptotic state. This is readily accomplished by the same type of moment procedure used in Eq. (12). More specifically, the p_y moment of Eq. (13) yields

$$j_x = \int d\vec{p}(p_y/B)\hat{C} f_{\text{adf}} = \int\int d\vec{p}d\vec{p}'(p_y/B)[f(\vec{p}')T_{\vec{p}'\vec{p}} - f(\vec{p})T_{\vec{p}\vec{p}'}], \tag{14}$$

where we drop the adf subscript. Interchanging $\vec{p}$ and $\vec{p}'$ in Eq. (14), adding and dividing by 2, we obtain

$$j_x = e\int\int d\vec{p}d\vec{p}'[(p_y - p_y')/2eB][f(\vec{p}')T_{\vec{p}'\vec{p}} - f(\vec{p})T_{\vec{p}\vec{p}'}] \text{ (classical statistics).} \tag{15a}$$

Here again Eqs. (14) and (15) remain valid for Fermi statistics with the $T_{pp'}$ multiplied by the exclusion factor $1 - f(\vec{p}')$,

$$j_x = e\int\int d\vec{p}d\vec{p}'[(p_y - p_y')/2eB]\{f(\vec{p}')T_{\vec{p}'\vec{p}}[1 - f(\vec{p})] - f(\vec{p})T_{\vec{p}\vec{p}'}[1 - f(\vec{p}')]\} \text{ (Fermi statistics).} \tag{15b}$$

2. THE HALL GEOMETRY

We now discuss some general features of the asymptotic state and their relation to the transport measurements in the Hall geometry. We should first like to point out that f_{adf} depends on the electric and magnetic fields only through V_d. The kinetic Eq. (4) for the collision-free trajectories is rewritten in the following form:

$$d\vec{p}/dt = e\vec{V} \times \vec{B};\quad \vec{V} = \nabla_{\vec{p}}\bar{\epsilon},\quad \bar{\epsilon} = \epsilon - \vec{p} \cdot \vec{V}_d;\quad dp_z/dt = d\bar{\epsilon}/dt = 0\,. \tag{16}$$

The collision-free trajectories are then characterized by p_z = constant and $\bar{\epsilon}$ = constant, the only electric field dependence entering through V_d. Since we consider only the case of periodic orbits in $\vec{p}$ space, the trajectory averages eliminate any field dependence of f_{adf} other than $V_d \equiv -E/B$.

It is important to note, however, that $\vec{E}$, the *total* electric field, is not an independent variable in the Hall geometry (Fig. 1). Here the direction of the current j_l is imposed and a transverse Hall field E_t arises in order that the current j_t be zero. The Hall field E_t is not known *a priori* and must be determined self-consistently, such that $j_t = 0$.

$$j_t = j_y E_l/E + j_x E_t/E = 0,\quad E_t/E_l = -j_y/j_x\,. \tag{17}$$

Inserting $j_y = -neE/B$, and Eq. (14) for j_x we obtain

$$E_t/E_l = (nE/B)/\int d\vec{p}(p_y/eB)\hat{C} f_{\text{adf}},\quad 1 = (E/E_t)E_l/\int d\vec{p}(p_y/ne)\hat{C} f_{\text{adf}} = (E/E_t)E_l/\alpha(E/B)\,, \tag{18}$$

where the last form emphasizes the fact that the integral appearing in this equation is some function of E/B, which we denote by α. Now since $E_t/E_l \approx \omega_c\bar{\tau}, E \cong E_t$ to order $1/\omega_c^2\tau^2$, and Eq. (18) becomes

$$1 = E_l/\alpha(E_t/B). \tag{19}$$

Thus E_t/B is independent of B for given E_l, and therefore f_{adf} is independent of B. The current is, in the same approximation, $j_l = neE_t/B$, which is again independent of B. Therefore in the Hall geometry all the semiclassical transport coefficients saturate (are B independent) for large magnetic fields, just as they do in the linear theory. The Hall coefficient is given by $R = E_t/j_l B = 1/ne$, since $j_l = neE_t/B$. An explicit expression for the conductivity or Hall mobility cannot be given in the general non-ohmic case, since this requires solving Eq. (19) for the Hall field. It is, however, possible to derive such expressions for the linear case, which we present in Sec. 3.

We cite, as an example, results obtained for the case of electrons interacting with acoustic phonons. The asymptotic distribution function is readily obtained in the diffusion approximation[9,10]:

$$f_{\text{adf}} = e^{-\epsilon/kT^*}[1 + (p_y V_d/kT)],\quad T^* = T[1 + \tfrac{1}{3}(V_d/c)^2]\,, \tag{20}$$

where T is the lattice temperature, and c is the velocity of sound. Here we see the independence of f_{adf} on the coupling constants and the field dependence entering only through V_d. For the experimental geometry of Fig. 1, the Hall field is determined from Eq. (19) to be

$$E_t/B = c\{[1 + \tfrac{4}{3}(9\pi/32)^2(uE_l/c)^2]^{1/2} - 1\}^{1/2}(\tfrac{3}{2})^{1/2}\,, \tag{21}$$

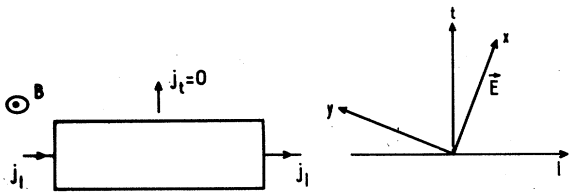

FIG. 1. Hall geometry. The current flow is in the l direction, while the total electric field $\vec{E}$ is in the x direction and the magnetic field is in the z direction.

where u is the ohmic mobility for acoustic phonon scattering. The reappearance of the coupling constants, through u, is clearly seen to be due to the fact that the total field is u-dependent in the Hall geometry.

3. THE LINEAR THEORY

In this section we present the linear version of our theory, which is most readily derived from Eq. (9). We seek a function near the thermal equilibrium distribution $f_0(\epsilon)$, which is constant along the collision-free trajectories. $f_0(\epsilon-\vec{p}\cdot\vec{V}_d)$ satisfies this condition since $\epsilon-\vec{p}\cdot\vec{V}_d$ is a constant of motion, Eq. (3). We therefore take as a trial function

$$f=f_0(\epsilon-\vec{p}\cdot\vec{V}_d)\approx f_0(\epsilon)-\vec{p}\cdot\vec{V}_d\,\partial f_0/\partial\epsilon\,. \tag{22}$$

Equation (9) then becomes

$$\int_0^T ds\,[\hat{C}(f_0-\vec{p}\cdot\vec{V}_d\,\partial f_0/\partial\epsilon)]_{\vec{p}_0(s)}=0\,, \tag{23}$$

where $\vec{p}_0(s)$ indicates that the bracketed quantity is to be evaluated along the trajectory in the *absence* of the electric field, since the former is already linear in E.

Our distribution (22) is thus a valid linear solution under the conditions in which Eq. (23) is satisfied. It is immediately obvious that this is indeed the case for any isotropic system, since in this case $\hat{C}(f_0-\vec{p}\cdot\vec{V}_d\,\partial f_0/\partial\epsilon)=p_y h(\epsilon)$, where $h(\epsilon)$ is some function of energy. This simply means that the collisions do not change the direction of the anisotropic distribution, and follows from the fact that the Legendre polynomials are eigenfunctions of the collision operator in this case. Since $h(\epsilon)$ is constant along the trajectories with $E=0$, we are left with the trajectory average of $p_y(s)$ which vanishes for closed orbits and $E=0$. Equation (23) is also valid for the case where the scattering is anisotropic but only mixes simple anisotropies in the plane normal to $\vec{B}$, i.e.,

$$\hat{C}(f_0-\vec{p}\cdot\vec{V}_d\partial f_0/\partial\epsilon)=p_y h(\epsilon)+p_x g(\epsilon)\,. \tag{24}$$

The additional term in p_x again satisfies Eq. (23) by the same arguments as above. This condition is valid for a large class of scattering systems and in no way implies the existence of a relaxation time, as can be seen, for example, in the case of polar optical-phonon scattering. Here the strong dependence of the scattering cross section on the scattering angle precludes the applicability of the relaxation time approximation, although the scattering is isotropic (dependent only on the scattering angle) and Eq. (23) is satisfied. Polar optical-phonon scattering is considered in detail in Sec. 4.

Inserting Eq. (22) into Eq. (15), we obtain

$$j_x=(e^2E/2kT)\int\int d\vec{p}'d\vec{p}[(p_y-p_y')/eB]^2 f_0(\vec{p}')T_{\vec{p}'\vec{p}}\quad\text{(classical statistics)}, \tag{25a}$$

where detailed balancing has been used $f_0(\vec{p})T_{\vec{p}\vec{p}'}=f_0(\vec{p}')T_{\vec{p}'\vec{p}}$. This result is strikingly similar to the quantum orbit-jump formula which we derive in Eq. (55).

When Fermi statistics are applicable, Eq. (15B) becomes

$$j_x=(e^2E/2kT)\int\int d\vec{p}'d\vec{p}[(p_y-p_y')/eB]^2 f_0(\vec{p}')T_{\vec{p}'\vec{p}}[1-f_0(\vec{p})]\quad\text{(Fermi statistics)}, \tag{25b}$$

where here again detailed balancing for the Fermi-Dirac function has been used; $f_0(\vec{p})T_{\vec{p}\vec{p}'}[1-f_0(\vec{p}')]=f_0(\vec{p}')T_{\vec{p}'\vec{p}}[1-f_0(\vec{p})]$.

Although we have established a fairly wide applicability for Eq. (22), we would like to suggest that our resulting current formula, Eq. (25), has an even wider range of validity. Let us write the linear solution of Eq. (9) as

$$f=f_0-\vec{p}\cdot\vec{V}_d\,\partial f_0/\partial\epsilon+Eg(\vec{p})\,.$$

Since the last term is linear in E we must require that it be constant along the trajectory in the absence of E. We therefore have $g=g(\epsilon)$, where we ignore any possible p_z dependence of g. The contribution of this term to the current is proportional to $\int d\vec{p}\,p_y\hat{C}g(\epsilon)$. We note that if $\hat{C}g(\epsilon)$ is some function of energy this integral is zero, since all the orbits are closed and Eq. (25) remains valid. In order to obtain a nonzero result for this integral, we need a scattering system which tends to create anisotropic distribution out of isotropic ones. This implies that transient currents could arise from initially noncurrent-carrying, nonequilibrium states, even in the absence of applied fields. This seems somewhat unlikely and therefore

suggests that Eq. (25) has an even wider range of validity than we have established here.

For the sake of completeness we display the saturation value of the Hall mobility u_H, which by Eqs. (17) and (25) is given by

$$u_H \equiv R\sigma = Rj_l/E_l = E_t/BE_l = ekT\left[\int\int d\vec{p}d\vec{p}'\,\tfrac{1}{2}(p_y - p_y')^2 n^{-1} f_0(\vec{p}')T_{\vec{p}'\vec{p}}\right]^{-1}. \tag{27}$$

For Fermi statistics, the $T_{\vec{p}'\vec{p}}$ appearing in Eq. (27) is to be multiplied by $[1 - f_0(\vec{p})]$.

4. POLAR OPTICAL-MODE SCATTERING

In this section we apply our linear results to the case of polar optical-phonon scattering. Since a relaxation time does not exist in this case, previous solutions of the Boltzmann equation have been numerical ones,[11] usually based on variational methods.

We treat the case of a standard band $\epsilon = p^2/2m$ and first consider classical statistics. The transition rates are given by[12]

$$T_{\vec{p}\vec{p}'} = \frac{2\pi}{\hbar}\frac{2\pi\hbar^4 eE_0/mV}{|\vec{p}'-\vec{p}|^2}\left[(1+N_0)\delta(\epsilon'-\epsilon+\hbar\omega)+N_0\delta(\epsilon'-\epsilon-\hbar\omega)\right], \quad N_0 = (e^{\hbar\omega/kT}-1)^{-1} \tag{28}$$

where $\hbar\omega$ is the constant optical-phonon energy, V is the sample volume, and E_0 is related to the coupling constants.[12] The evaluation of $\hat{C}(\vec{p}\cdot\vec{V}_d\,\partial f_0/\partial\epsilon)$ is straightforward and we obtain

$$\hat{C}\left(\vec{p}\cdot\vec{V}_d\frac{\partial f_0}{\partial\epsilon}\right) = \frac{eE_0N_0\vec{p}\cdot\vec{V}_d f_0}{kT(2m\epsilon)^{1/2}}\left\{\left(\frac{\epsilon+h\omega}{\epsilon}\right)^{1/2} + e^{\delta}\left(\frac{\epsilon-h\omega}{\epsilon}\right)^{1/2} + \frac{h\omega}{\epsilon}\left[e^{\delta}\sinh^{-1}\left(\frac{\epsilon}{\hbar\omega}-1\right)^{1/2} - \sinh^{-1}\left(\frac{\epsilon}{\hbar\omega}\right)^{1/2}\right]\right\}, \tag{29}$$

where $\delta = \hbar\omega/kT$ and $[(\epsilon/h\omega)-1]^{1/2}$ is taken to be zero for $\epsilon/h\omega < 1$.

Inserting this into Eq. (14), we obtain

$$j_x = 4neE_0N_0IE/3B^2(2\pi kT/m)^{1/2}; \quad I = 2\int_0^\infty e^{-x}[x(x+\delta)]^{1/2}dx = \delta e^{\delta/2}K_1(\delta/2) \text{ (classical statistics)}, \tag{30}$$

where K_1 is the first-order modified Bessel function.[13] This result is identical to the classical limit of the Kubo formula, which has been evaluated by Gurevich and Firsov,[14] using Green's-function techniques.

When Fermi statistics are applicable we obtain the same result as Eq. (30) with I replaced by

$$I = (\pi^{1/2}/2)\int_0^\infty dx\{[x(x+\delta)]^{1/2}G_+ - \delta\sinh^{-1}(x/\delta)^{1/2}G_-\}\{\int_0^\infty f_0(x)x^{1/2}dx\}^{-1} \text{ (Fermi statistics)}, \tag{31}$$

where $G_\pm = f_0(x+\delta)e^{\delta}[1-f_0(x)] \pm f_0(x)[1-f_0(x+\delta)]$.

The Hall mobility is given by

$$u_H \equiv R\sigma = Rj_l/E_l = 3(2\pi kt/m)^{1/2}/4E_0N_0I. \tag{32}$$

5. QUANTUM THEORY

The basic ideas presented in our discussion of the classical theory will now be formulated in the quantum mechanical treatment. The linear problem has been studied extensively and has been developed principally in two directions.

The first consists of calculating corrections to the thermal equilibrium state to first order in the driving field, from which the transport coefficients are readily calculated. This method has been largely developed by Kubo[5] who emphasizes the relation between linear response theory and certain correlation functions in the thermal equilibrium state. The second scheme consists of deriving irreversible kinetic equations for the relevant part of the density matrix. This latter approach has been applied to the linear crossed-field problem by Argyres.[15] As we have seen in our classical theory, the asymptotic state of the system when $\omega_c\bar{\tau} \to \infty$ is determined by an irreversible kinetic equation, and therefore it is in this direction that we shall formulate our theory.

The fact that exact eigenstates are available for the crossed-field Hamiltonian[16] has led to some confusion in the literature. Adams and Holstein[16] and Kahn and Frederickse[17] relate the asymptotic state to the thermal equilibrium distribution. In both cases, these authors work in the crossed-field eigenstate representation and argue that the appropriate zero-order (in the coupling constant) state is one in which the density matrix is diagonal and equal to that of the zero-order density matrix in the absence of the electric field. This is equivalent to starting from a thermal equilibrium distribution in the presence of the magnetic field and adiabatically applying the electric field. If collisions are completely neglected the adiabatic theorem assures us that the system evolves into the state proposed by these authors. This completely reversible process fails in general for exactly the same reasons as in the classical case (see Appendix). The zero-order function is not independent of the scattering system; it is only independent of the absolute coupling constants, i.e., the absolute strength of the coupling. The procedure employed by

these authors is however valid for the linear case, which we treat in Sec. 6, and it is to this order that these authors have limited their work. Calecki[18] has unfortunately applied their procedure to the nonlinear case and has obtained erroneous results. In this latter work Calecki does not even obtain the correct classical limits.

Kazarinov and Skobov[9] have presented a nonlinear theory based on solving a kinetic equation for the density matrix. Their theory is based on using the same relation between the dissipative current and the density matrix that holds in the linear theory. This is reminiscent of the diffusion approximation in the classical theory, where it is known to be equivalent to a weak anisotropy approximation. Yamashita[19] has presented a similar theory, where he explicitly expands the diagonal part of the density matrix in the cosine of the angle between the electric field and the momentum vector, and keeps only the two lowest-order Legendre polynomials. In the same spirit he also expands the energy conservation δ functions in the electric field.

In the present work we shall follow the general method discussed in the classical theory. The asymptotic state of the system is determined by the quantum analog of Eq. (7) and a perturbation scheme is used to derive a generalized orbit-jump formula for the dissipative current, in analogy with Eq. (15).

We take, as before, the electric field E in the x direction and the magnetic field B in the z direction, and we work in the Landau gauge $A=[0,Bx,0]$. The one-electron Hamiltonian in the effective mass approximation is given by

$$H_e=[p_x^2+(p_y-eBx)^2+p_z^2]/2m-eEx\,. \tag{33}$$

We consider only the case of spherical energy surfaces and constant mass. The eigenfunctions and eigenvalues of H_e are given by

$$\Psi_{n,k}=\Phi_n(x-\bar{x})\exp[i(k_y y+k_z z)];\quad \bar{x}=\hbar k_y/eB+mE/eB^2\,, \tag{34}$$

$$\epsilon_{n,k}=(n+\tfrac{1}{2})\hbar\omega_c+(\hbar^2k_z^2/2m)-eE\bar{x}+\tfrac{1}{2}m(E/B)^2,\quad \omega_c=|e|B/m\,.$$

The Φ_n are the usual Hermite functions associated with the harmonic oscillator, and are centered at $\bar{x}$. The normalization factors have been dropped in Eq. (34).

We denote by H_S the Hamiltonian of the scattering system (SS) and by V the electron-SS interaction. The set of quantum numbers describing the electronic states will be denoted by lower case Greek letters α, β, etc., which represent the set $\{n,k_x,k_y\}$. The eigenstates of the SS will be denoted by $|S\rangle$, where S is again an appropriate set of quantum numbers. For phonon scattering, the S represent the different phonon states. We also take S to represent symbolically the states of any relevant scattering system, i.e., the distribution of impurities, for example, in the case of impurity scattering. The density matrix for the total electron-SS system is denoted by ρ and satisfies the kinetic equation

$$\partial\rho/\partial t=(i/\hbar)[\rho,H_e+H_S+V]\,. \tag{35}$$

The derivation of the master equation, describing the irreversible evolution of the occupancy probabilities, from Eq. (35) has been studied in great detail. The original treatment of Van Hove[20] is based on certain special properties of the scattering-interaction matrix elements. An alternative derivation of the kinetic equation for electrons interacting with a random distribution of fixed impurities was given by Kohn and Luttinger[21] in connection with the ohmic transport problem. Subsequent work by Zwanzig,[22] Montroll,[23] and Argyres[24] has systematized and clarified this derivation. No new features are introduced by working in the crossed-field representation, Eq. (34), and so we merely state the result

$$\frac{\partial\rho_{\alpha S}}{\partial t}=\sum_{\beta S'}(\rho_{\beta S'}-\rho_{\alpha S})\frac{2\pi}{\hbar}|V_{\alpha S,\beta S'}|^2\delta(E_{\alpha S}-E_{\beta S'}),\quad E_{\alpha S}=\epsilon_\alpha+\epsilon_S\,. \tag{36}$$

Here $\rho_{\alpha S}\equiv\langle\alpha S|\rho|\alpha S\rangle$ is the diagonal element of the density matrix and represents the occupation probability of the state αS, while $\epsilon_\alpha,\epsilon_S$ represent the electron and SS energies, respectively. We take the statistical properties of the SS to be given by the diagonal density matrix P and neglecting correlations between the electrons and SS set $\rho=fP$, where f is the electronic density matrix. Inserting this in Eq. (36) and taking the trace over the SS variables, we obtain

$$\frac{\partial f_\alpha}{\partial t}=\sum_\beta(f_\beta T_{\beta\alpha}-f_\alpha T_{\alpha\beta});\quad f_\alpha\equiv\langle\alpha|f|\alpha\rangle,\quad \text{with } T_{\alpha\beta}=\frac{2\pi}{\hbar}\sum_{S,S'}P(S)|V_{\alpha S,\beta S'}|^2\delta(E_{\alpha S}-E_{\beta S'}), \tag{37}$$

where we have used $\sum_S P(S)=1$. We take $P(S)$ to be given by the thermal equilibrium distribution for the case of phonon scattering, while the trace over S is taken to mean an appropriate ensemble average[21] for a random distribution of fixed impurities. We shall not consider any higher order (in V) kinetic equations or the multiple scattering problem.

The steady-state solution of Eq. (37) determines the asymptotic distribution function f_α, and is the quantum analog of Eq. (7):

$$f_\alpha\textstyle\sum_\beta T_{\alpha\beta}=\sum_\beta f_\beta T_{\beta\alpha}\quad\text{(classical statistics)}. \tag{38a}$$

If we continue to neglect electron-electron interactions but allowing for Fermi statistics, the $T_{\alpha\beta}$ are to be multiplied by $1-f_\beta$:

$$f_\alpha \sum_\beta T_{\alpha\beta}(1-f_\beta) = \sum_\beta f_\beta T_{\beta\alpha}(1-f_\alpha) \quad \text{(Fermi statistics)}. \tag{38b}$$

That the exclusion factors appear in the expected places has been demonstrated explicitly by Argyres.[25] The only difference here is that in the present context the electronic states are given by Eq. (34) rather than by the free-electron states used in Argyres's proof.

We see here again the independence of the asymptotic state on the absolute coupling constants. Furthermore, to this order the density matrix has only diagonal elements f_α, which means that it commutes with the unperturbed Hamiltonian and is therefore a constant of motion in analogy with the same property of the f_{adf}, Eq. (9).

Similarly, the current flow is again identical to the classical case. Since the average velocity corresponding to the eigenstates (34) is $\langle\alpha|V_x|\alpha\rangle = 0$ and $\langle\alpha|V_v|\alpha\rangle = -E/B$ we have

$$j_x = e\,\mathrm{Tr}\rho V_x = e\sum_\alpha f_\alpha \langle\alpha|V_x|\alpha\rangle = 0, \quad j_y = e\,\mathrm{Tr}\rho V_y = e\sum_\alpha f_\alpha \langle\alpha|V_y|\alpha\rangle = -neE/B. \tag{39}$$

It is important to realize that Eq. (37) *also* describes thermal equilibrium in a uniform electric field, and therefore admits thermal-equilibrium-type solutions $f_\alpha \sim \exp(-\epsilon_\alpha/kT)$. This is also true of the classical transport Eq. (1), with $f \sim \exp\{-[\epsilon(p)-eEx]/kT\}$. These solutions are of no interest in the transport problem where we are interested in nearly uniform electron distributions, as is also the case in the linear problem,[16] while the thermal-equilibrium-type solutions correspond to strong concentration gradients. More specifically, we seek k_y-independent solutions f_α, since k_y is the quantum number defining the orbit-center x coordinate.

We must now compute the lowest-order correction to the density matrix in order to calculate the dissipative current, which is zero in the asymptotic state. Since $\langle\alpha|V_x|\alpha\rangle = 0$ for the eigenstates (4), we shall need the off-diagonal elements of the density matrix, which we compute to lowest order in V. Before doing this, however, we now employ the same moment method used in Eq. (12) in order to simplify the calculation of the dissipative current. We multiply Eq. (35) by V_y and take the trace

$$(d/dt)\,\mathrm{Tr}\rho V_y = (d/dt)\langle V_y\rangle = (i/\hbar)\,\mathrm{Tr}\{[\rho, H_e]V_y + [\rho, V]V_y\} \tag{40}$$

since $[V_y, H_S] = 0$. Now

$$(i/\hbar)\,\mathrm{Tr}[\rho, H_e]V_y = (i/\hbar)\,\mathrm{Tr}\rho[H_e, V_y] = -(eB/m)\,\mathrm{Tr}\rho V_x = -(B/m)j_x.$$

The steady-state form of Eq. (40) then becomes

$$j_x = (ie/\hbar\omega_c)\,\mathrm{Tr}\rho[V, V_y] = (i/\hbar B)\,\mathrm{Tr}\rho[V, p_y]. \tag{41}$$

It is readily verified that the commutator algebra involved here is valid, since x is nonsingular in the crossed-field representation.

The most convenient way to obtain the steady-state off-diagonal elements of ρ is by use of the Laplace transform technique of Kohn and Luttinger.[21] Using their definition of the Laplace transform of the density matrix $\bar\rho \equiv s\int_0^\infty e^{-st}\rho(t)dt$, the transformed kinetic Eq. (35) becomes

$$s\bar\rho - \rho_{\text{adm}} = (i/\hbar)[\bar\rho, H_e + H_S] + (i/\hbar)[\bar\rho, V], \tag{42}$$

where $\rho_{\text{adm}} = fP$ is the asymptotic diagonal density matrix, the f_α being determined by Eq. (37) and P being given. To lowest order in V, the off-diagonal elements are determined by

$$s\langle\alpha S|\bar\rho|\beta S'\rangle \equiv s\bar\rho_{\alpha S,\beta S'} = (i/\hbar)(E_{\beta S'} - E_{\alpha S})\bar\rho_{\alpha S,\beta S'} + (i/\hbar)\langle\alpha S|[\rho_{\text{adm}}, V]|\beta S'\rangle. \tag{43}$$

The steady-state off-diagonal elements are given by

$$\begin{aligned}\rho_{\alpha S,\beta S'} &= \lim_{s\to 0^+}(\rho_{\alpha S} - \rho_{\beta S'})V_{\alpha S,\beta S'}/(E_{\alpha S} - E_{\beta S'} - is\hbar) \\ &= (\rho_{\alpha S} - \rho_{\beta S'})V_{\alpha S,\beta S'}[\pi i\delta(E_{\alpha S} - E_{\beta S'}) + \hat P(E_{\alpha S} - E_{\beta S'})^{-1}],\end{aligned} \tag{44}$$

where $\hat P$ denotes the principal part and where we have dropped the adm subscript.

Inserting this in Eq. (41) we obtain

$$\begin{aligned}j_x &= \frac{i}{\hbar B}\sum_{\alpha S,\beta S'}\rho_{\alpha S,\beta S'}[p_y(\alpha) - p_y(\beta)]V_{\beta S',\alpha S} \\ &= \frac{i}{\hbar B}\sum_{\alpha S,\beta S'}[\rho_{\alpha S} - \rho_{\beta S'}][p_y(\alpha) - p_y(\beta)]\,|V_{\alpha S,\beta S'}|^2\pi i\delta(E_{\alpha S} - E_{\beta S'}),\end{aligned} \tag{45}$$

where the principal part vanishes since it is multiplied by a quantity which is invariant under the interchange $\alpha S \leftrightarrow \beta S'$.

Using our previous definition of $T_{\alpha\beta}$, Eq. (45) becomes

$$j_x = e\sum_{\alpha,\beta}\left[\frac{\hbar k_y(\beta) - \hbar k_y(\alpha)}{2eB}\right][f_\alpha T_{\alpha\beta} - f_\beta T_{\beta\alpha}]$$

$$= e\sum_{\alpha,\beta}\left[\frac{\bar{x}(\beta) - \bar{x}(\alpha)}{2}\right][f_\alpha T_{\alpha\beta} - f_\beta T_{\beta\alpha}] \quad \text{(classical statistics)} \tag{46a}$$

which is the general orbit-jump formula for the dissipative current, and is the quantum analog of Eq. (15). Equation (46a) thus provides us with a relation between the dissipative current and the asymptotic occupation probabilities f_α.

Here again we merely state the result for Fermi statistics:

$$j_x = e\sum_{\alpha,\beta}\left[\frac{\bar{x}(\beta) - \bar{x}(\alpha)}{2}\right][f_\alpha T_{\alpha\beta}(1-f_\beta) - f_\beta T_{\beta\alpha}(1-f_\alpha)] \quad \text{(Fermi statistics)}. \tag{46b}$$

6. THE LINEAR THEORY

We now consider the linear version of our theory and proceed as in Sec. 3. We seek an electronic density matrix near thermal equilibrium which is diagonal in the crossed-field representation, Eq. (34), and solves Eq. (38), both these conditions being satisfied to first order in E. A likely candidate is $f = e^{-\beta H_0}$, where H_0 is the electron Hamiltonian, Eq. (33), with $E = 0$. One readily verifies, using the eigenfunctions (34), that this matrix is indeed diagonal to first order in E and its diagonal elements are

$$f_\alpha = \exp(-\gamma\epsilon_\alpha{}^0), \quad \epsilon_\alpha{}^0 = (n+\tfrac{1}{2})\hbar\omega_c + \hbar^2 k_z{}^2/2m, \quad \gamma = 1/kT. \tag{47}$$

Inserting this trial function in Eq. (38a), we obtain

$$0 = \sum_\beta \exp(-\gamma\epsilon_\alpha{}^0)T_{\alpha\beta} - \exp(-\gamma\epsilon_\beta{}^0)T_{\beta\alpha} = \sum_\beta \{\exp[-\gamma eE\bar{x}(\alpha)] - \exp[-\gamma eE\bar{x}(\beta)]\}\exp(-\gamma\epsilon_\alpha)T_{\alpha\beta}, \tag{48}$$

where we have used detailed balance: $\exp(-\gamma\epsilon_\alpha)T_{\alpha\beta} = \exp(-\gamma\epsilon_\beta)T_{\beta\alpha}$. Since the difference of exponentials is already linear in E we have

$$0 = -\gamma eE\sum_\beta [\bar{x}(\alpha) - \bar{x}(\beta)]\exp(-\gamma\epsilon_\alpha{}^0)T_{\alpha\beta}{}^0, \tag{49}$$

where $T_{\alpha\beta}{}^0 = T_{\alpha\beta}(E=0)$. Our trial function (47) is therefore a valid solution under the conditions for which Eq. (49) is satisfied. Let us consider the case of an isotropic scattering system. It is then to be expected that $T_{\alpha\beta}{}^0$ is invariant under permutation of the orbit centers $\bar{x}(\alpha)$, $\bar{x}(\beta)$, i.e., the sites $\bar{x}(\alpha)$ and $\bar{x}(\beta)$ are equivalent in the absence of an electric field. Since, further, the $\epsilon_\alpha{}^0$ are independent of $\bar{x}(\alpha)$, the sum in Eq. (49) vanishes.

This is readily demonstrated explicitly, and we consider, for example, the case of phonon scattering. Writing the interaction Hamiltonian as

$$V = \sum_{\vec{q}} C(q)b_{\vec{q}}\, e^{i\vec{q}\cdot\vec{r}} + C^*(q)b_{\vec{q}}{}^+ e^{-i\vec{q}\cdot\vec{r}} \tag{50}$$

where b^+, b are phonon creation and annihilation operators. We obtain for $T_{\alpha\beta}{}^0$ in the usual way

$$T_{\alpha\beta}{}^0 = \frac{2\pi}{\hbar}|C(q)|^2 |J_{\alpha\beta}(\vec{q})|^2 (1+n_q)\delta(\epsilon_\alpha{}^0 - \epsilon_\beta{}^0 - \hbar\omega_q)n_q\delta(\epsilon_\alpha{}^0 - \epsilon_\beta{}^0 + \hbar\omega_q) \tag{51}$$

where $J_{\alpha\beta}{}^0(\vec{q}) = \int d\vec{r}\ \Psi_\alpha{}^{0*} e^{i\vec{q}\cdot\vec{r}}\Psi_\beta{}^0$, \tag{52}

and the superscript 0 indicates that E is to be taken as equal to zero. We see from Eq. (34a) that $q_{y,z} = k_{y,z}(\alpha) - k_{y,z}(\beta)$ and that the remaining integral in $J_{\alpha\beta}$ is simply

$$J_{\alpha\beta}{}^0 \sim \int dx\, \Phi_{n_\alpha}[x - \hbar k_y(\alpha)/eB]\exp(iq_x x)\Phi_{n_\beta}[x - \hbar k_y(\beta)/eB]$$

$$= \exp[iq_x\hbar k_y(\alpha)/eB]\int dx\, \Phi_{n_\alpha}(x)\exp(iq_x x)\Phi_{n_\beta}(x - \Delta x); \quad \Delta x = \bar{x}(\beta) - \bar{x}(\alpha). \tag{53}$$

If we interchange the orbit centers in Eq. (53), we have

$$J_{\alpha\beta}{}' = \exp[iq_x\hbar k_y(\beta)/eB]\int dx\, \Phi_{n_\alpha}(x)\exp(iq_x x)\Phi_{n_\beta}(x + \Delta x)$$

$$= \exp[iq_x \hbar k_y(\beta)/eB] \int dx\, \Phi_{n_\alpha}(-x) \exp(-iq_x x) \Phi_{n_\beta}(-x + \Delta x)$$

$$= \exp[iq_x \hbar k_y(\beta)/eB] (-1)^{n_\alpha + n_\beta} \int dx\, \Phi_{n_\alpha}(x) \exp(-iq_x x) \Phi_{n_\beta}(x - \Delta x), \tag{54}$$

where in the second line we have changed x to $-x$, and in the third line have used the fact that the parity of Φ_n is n. Since the Φ_n are real it then follows that $|J_{\alpha\beta}|^2 = |J_{\alpha\beta}'|^2$ and therefore $T_{\alpha\beta}{}^0$ is invariant under permutation of the orbit centers. A similar proof applies to the impurity scattering problem. The assumption of an isotropic scattering system can be somewhat relaxed, as discussed in Sec. 3, but we shall not consider this here.

Inserting Eq. (47) into Eq. (46), we obtain for the dissipative current

$$j_x = e \sum_{\alpha,\beta} \left[\frac{\bar{x}(\beta) - \bar{x}(\alpha)}{2}\right] [\exp(-\gamma \epsilon_\alpha{}^0) T_{\alpha\beta} - \exp(-\gamma \epsilon_\beta{}^0) T_{\beta\alpha}]$$

$$= \frac{e^2 E}{kT} \sum_{\alpha,\beta} \frac{[\bar{x}(\beta) - \bar{x}(\alpha)]^2}{2} f_0(\epsilon_\alpha{}^0) T_{\alpha\beta}{}^0 \quad \text{(classical statistics)}, \tag{55a}$$

where we have kept only linear terms in E and have performed the same sort of manipulations as in Eq. (48).

The same sort of procedure applied above works equally well with Fermi statistics and we obtain

$$j_x = \frac{e^2 E}{kT} \sum_{\alpha,\beta} \frac{[\bar{x}(\beta) - \bar{x}(\alpha)]^2}{2} f_0(\epsilon_\alpha{}^0) T_{\alpha\beta}{}^0 [1 - f(\epsilon_\beta{}^0)] \quad \text{(Fermi statistics)}. \tag{55b}$$

SUMMARY

We have presented a unified treatment of transport phenomena in crossed electric and magnetic fields in the limit $\omega_c \bar{\tau} \gg 1$. The asymptotic ($\omega_c \bar{\tau} \to \infty$) state of the system is determined by an irreversible transport equation and is shown to depend on the form of the scattering interaction even though it is independent of the absolute coupling constants. The asymptotic state is, therefore, of zeroth order in the coupling constants, even though it is determined by the detailed nature of the scattering system.

Once the asymptotic state of the system is determined a perturbation theory in $1/\omega_c \bar{\tau}$ allows us to compute the dissipative current in a direct and simple manner. This provides us with a generalized orbit-jump formula for the dissipative current.

These results establish a unified basis for both linear and nonlinear transport phenomena in strong magnetic fields without recourse to a relaxation time approximation. Both the quantum and semiclassical treatments have been established by the same basic perturbation method.

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APPENDIX

We discuss here an alternative method of determining the f_{adf}, which is a generalization of the linear theory of Lifschitz, Azbel, and Kaner.[8] Here we start with the Liouville Eq. (2) and take as a solution $g(\epsilon - \vec{p} \cdot \vec{V}_d)$, where g is to be determined. Now doing perturbation theory we set $f = g + h$ and seek to determine h to lowest order in $1/\omega_c \tau$. The perturbation equation is, as in Eq. (13),

$$e[\vec{E} + \vec{V} \times \vec{B}] \cdot \nabla_p h = \hat{C} g .$$

Integrating this equation over one period of the collision-free trajectory, the left-hand member vanishes by Eq. (8), and we obtain

$$\int_0^T [\hat{C} g]_{\vec{p}(s)} dS = 0, \quad g(\vec{p}) = g(\vec{p}(s)),$$

which is Eq. (9). Thus although any function $g(\epsilon - \vec{p} \cdot \vec{V}_d)$ satisfies the Liouville Eq. (2), the condition Eq. (9) is in fact a necessary condition for the solubility of the Eq. (13) and assures the consistency of the perturbative method.

Lastly, we note that solving the initial value problem is of no assistance in determining the f_{adf}, which is determined by an irreversible transport Eq. (7). If we start, for example, with the thermal equilibrium distribution f_0 and slowly turn on the electric field, the magnetic field always being present, the distribution evolves adiabatically into $f = f_0(\epsilon - \vec{p} \cdot \vec{V}_d(t))$, $\vec{V}_d(t) = \vec{E}(t) \times \vec{B}/B^2$ in the absence of collisions. If we now attempt to do perturbation theory with the collision term, we encounter the sort of problem discussed above, the insolubility of the perturbation equations, or, in general, physically unacceptable solutions. This is the wrong zero-order function. The correct one is determined by Eq. (7) and is not independent of the scattering system, but is on the contrary determined by the detailed nature of the interaction even though there is no dependence on the absolute coupling constants.

It is only in the linear transport problem that this scheme is applicable, since in this limit the system is energetically isolated, in the sense that there is no energy dissipation to first order in E. The linear version of the adiabatically evolving distribution discussed above is then an appropriate zero-order function under the same conditions that Eq. (22) is valid.

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Solution of Boltzmann Equation for Degenerate Fermi Systems*

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A new method of solving the Boltzmann equation for a degenerate Fermi system is described and compared with the approximation of Abrikosov and Khalatnikov and with variational calculations. The first two terms in the expansions of the inverse relaxation times in powers of the absolute temperature are calculated, and the resulting expressions are applied to dilute mixtures of He^3 in liquid He^4 and to nearly ferromagnetic Fermi liquids.

1. INTRODUCTION

It has been pointed out[1,2] that it is necessary to reconsider the derivation of expressions for the transport coefficients of degenerate Fermi systems. At low temperatures, the mean free times may be expanded in the form[1]

$$1/\tau = aT^2 + bT^3, \tag{1.1}$$

where T is the absolute temperature, and a and b are coefficients which depend upon the transport process. Two methods have been used to obtain τ. One is a direct approximate solution of the Boltzmann equation,[1,3] the other a variational calculation,[2,4] and they give different values of a and b. Without further calculation, it is not clear which is more accurate. The discrepancy is particularly important for the thermal conductivity K, and it is large enough to affect the comparison between theory and experiment for dilute mixtures[5,6] of He^3 in liquid He^4, and possibly[1,4,7] for pure He^3.

The purpose of this paper is to describe a more accurate approximation which resolves the difficulty. A brief account of the results has been given already.[8] During the course of this work,