Phenomenological Analysis of Weak-Radiative Decay of the K^+ Meson and CP-Noninvariant Effects

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A phenomenological analysis of the pion and the photon energy spectra and the photon-pion angular correlation spectrum in the weak-radiative decay, $K^+\to\pi^++\pi^0+\gamma$, is performed by parametrizing the direct electric ($E1$) and magnetic ($M1$) dipole transition amplitudes. A comparison of the calculated π^+ energy spectrum with the experimental spectrum (together with the qualitative information available from the theoretical investigations of the direct amplitudes) makes it possible to put limits on the direct amplitudes. We also make an estimate of the maximum asymmetry in the weak-radiative decays of K^+ and K^- mesons that could be present because of possible CP noninvariance in these decays. This estimate is based on a model in which CP noninvariance arises from a relative phase (other than $\pi\pi$ scattering phase shifts) between the inner bremsstrahlung and the $E1$ amplitudes. The maximum observable asymmetry in the K^+ and K^- radiative decay rates would be about 25%.

I. INTRODUCTION

There have been some speculations on the possibility of observing relatively large CP-noninvariant effects in the weak-radiative decays of K mesons, particularly of charged K mesons.[1-6] As is well known, these weak-radiative decays can proceed through two distinct mechanisms[7]: inner bremsstrahlung and direct emission. The inner-bremsstrahlung amplitude for the decay $K^+(p)\to\pi^+(q)+\pi^0(q')+\gamma(k)$ is usually written as

$$e\left(\frac{\epsilon\cdot q}{k\cdot q}-\frac{\epsilon\cdot p}{k\cdot p}\right)e^{i\delta_2}A_2 \quad \text{(parity violating)}. \quad (1)$$

A_2 is the amplitude for the nonradiative decay $K^+\to\pi^++\pi^0$. The direct-emission electric and magnetic dipole transition amplitudes,[8] denoted by $E1$ and $M1$, respectively, are

$$E1:\quad e\tilde{a}e^{i\delta_1}q_\mu q_\nu' F_{\mu\nu}, \qquad \text{(parity violating)} \quad (2)$$

$$M1:\quad ei\tilde{b}e^{i\delta_1}\epsilon_{\mu\nu\alpha\beta}q_\mu q_\nu' F_{\alpha\beta}, \quad \text{(parity conserving)} \quad (3)$$

where $F_{\mu\nu}=\epsilon_\mu k_\nu-k_\mu\epsilon_\nu$ and ϵ_μ is the photon polarization four-vector. δ_2 is the s-wave, $T=2$, $\pi\pi$ scattering phase shift at two-pion energy equal to the mass of the K meson. δ_1 is the p-wave, $T=1$, $\pi\pi$ scattering phase shift at two-pion energy squared equal to $(q+q')^2$. A_2, $\tilde{a}$, and $\tilde{b}$ are, in general, complex. The $E1$ amplitude will have contribution from two type of processes: the radiative rescattering process, $K^+\to\pi^++\pi^0\to(\pi^++\pi^0+\gamma;\,E1)$, which involves only $|\Delta\mathbf{T}|=\frac{3}{2}$ weak Hamiltonian $H_w^{(3/2)}$, and the direct process[9] $K^+\to(\pi^++\pi^0+\gamma;\,E1)$, which involves both $|\Delta\mathbf{T}|=\frac{1}{2}$ weak Hamiltonian $H_w^{(1/2)}$ and $|\Delta\mathbf{T}|=\frac{3}{2}$ weak Hamiltonian $H_w^{(3/2)}$. Let us write $\tilde{a}=a'(\frac{3}{2})+a(\frac{3}{2})+a(\frac{1}{2})$, where $a'(\frac{3}{2})$ is the contribution of the radiative-scattering process, $a(\frac{3}{2})$ and $a(\frac{1}{2})$ are the contributions to $\tilde{a}$ coming from the direct process through $H_w^{(3/2)}$ and $H_w^{(1/2)}$, respectively.[10] From the study of nonleptonic decays of K mesons and hyperons, we know that the contribution of $H_w^{(3/2)}$ to these decays is very much suppressed (by a factor of 20–30) compared to that of $H_w^{(1/2)}$. It is therefore likely that the contribution $a(\frac{3}{2})$ is also suppressed compared to $a(\frac{1}{2})$. The radiative rescattering contribution $a'(\frac{3}{2})$ depends, in addition to $H_w^{(3/2)}$, on the s-wave, $T=2$, $\pi\pi$ scattering phase shift δ_2, and may be further suppressed relative to $a(\frac{3}{2})$ if δ_2 is not large. Thus, it seems reasonable to expect that, among the three contributions to $\tilde{a}$, the direct contribution $a(\frac{1}{2})$ will be the dominant one. Similarly the $M1$ amplitude will have a contribution from the radiative-rescattering process, $K^+\to3\pi\to(\pi^++\pi^0+\gamma;\,M1)$, in addition to the contribution of the direct process $K^+\to(\pi^++\pi^0+\gamma;\,M1)$. Since in this case the radiative-rescattering process can proceed through $H_w^{(1/2)}$, its contribution may not be negligible, but it is difficult to make a reliable estimate. The correction to the amplitude for $K^+\to\pi^++\pi^0$, denoted by A_2, could arise due to the process $K^+\to(\pi^++\pi^0+\gamma;\,E1)\to(\pi^++\pi^0)$. As we shall see, the amplitude $E1$ is small; the correction to A_2 will therefore be small

The implications of CPT invariance, which relate K^+ and K^- radiative-decay amplitudes, are in general complicated because of the radiative-rescattering pro-

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[2] T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. **16**, 471 (1966).
[3] D. Cline, Phys. Rev. Letters **16**, 367 (1966).
[4] G. Costa and P. K. Kabir, Phys. Rev. Letters **18**, 429 (1967); **18**, 526E (1967).
[5] N. Christ, Phys. Rev. **159**, 1292 (1967).
[6] S. Barshay, Phys. Rev. Letters **18**, 515 (1967).
[7] J. Good, Phys. Rev. **113**, 352 (1959).
[8] The general expressions for the electric and magnetic transition amplitudes will have $\tilde{a}$ and $\tilde{b}$ as functions of the invariants $k\cdot q$ and $k\cdot q'$. The constant terms in the expansion of $\tilde{a}$ and $\tilde{b}$ with respect to k, will correspond to the coefficient of the dipole amplitudes.

[9] The direct process includes the contribution of all the intermediate states except those which are included in the radiative rescattering process. Since parity is conserved in strong and electromagnetic interactions, a 3π intermediate state will not contribute to the $E1$ amplitude, and a 2π intermediate state will not contribute to the $M1$ amplitude.
[10] In the limit of time-reversal invariance, $a'(\frac{3}{2})$ would be imaginary and $a(\frac{1}{2})+a(\frac{3}{2})$ real.

cess.[11] We have seen above that the radiative-rescattering contributions to A_2 and $\tilde{a}$ are expected to be small. We shall therefore neglect these contributions as an approximation. With this simplification, CPT invariance states that the amplitude for K^- decay can be obtained from that of K^+ decay by replacing A_2 and $\tilde{a}$ with their complex conjugates. To obtain the $M1$ amplitude for K^- we are to replace $(i\tilde{b})$ with $(i\tilde{b})^*+\Delta b$, where Δb depends on the contribution of the radiative-rescattering process, $K^+\rightarrow 3\pi\rightarrow(\pi^++\pi^0+\gamma;M1)$. For the sake of simplicity we shall ignore Δb also. CP invariance requires that the decay rates, energy, and angular spectra (summed over photon polarization states) should be the same for K^+ and K^- decays. In the context of the approximation we have made, this can be so only if A_2 and $\tilde{a}$ are relatively real. In case A_2 and $\tilde{a}$ are not relatively real, asymmetries in the rates and spectra (summed over photon polarization states) of K^+ and K^- decays will be present. We shall confine our attention to the case in which photon polarization is not observed. In such an experimental situation the amplitude $M1$ will not contribute to the CP-noninvariant effects.[12] The existence of such effects will then depend on (neglecting radiative-rescattering contributions) (i) the presence of an $E1$ amplitude, (ii) $|\delta_1-\delta_2|$ being different from zero and, (iii) the relative phase between A_2 and $\tilde{a}$, denoted by ϕ, being different from zero or π.[13] Furthermore, the magnitude of CP-noninvariant effects in the rates, photon and pion energy, and angular correlation spectra will depend on the contribution of a term due to the interference between the $E1$ and the inner bremsstrahlung amplitudes.

On the experimental side, no data are available on radiative K^- decay. The available data on the general shape of the π^+ energy spectrum in $K^+\rightarrow\pi^++\pi^0+\gamma$ decay is compatible with the inner-bremsstrahlung amplitude, Eq. (1). The experimental value of the branching ratio[2,3]

$$R_{\text{expt}}=\Gamma(K^+\rightarrow\pi^++\pi^0+\gamma)/\Gamma(\text{total})$$
$$=(2.2\pm0.7)\times10^{-4} \quad (4)$$

for π^+ kinetic energy between 55 and 80 MeV is also not inconsistent, within the large experimental errors, with the theoretical estimate[14]

$$R_{\text{br}}=1.36\times10^{-4} \quad (5)$$

for the same π^+ energy interval, obtained from the inner-bremsstrahlung amplitude alone.

Several attempts[15,16] have been made to calculate the contributions of the direct-emission terms. An optimistic estimate of $E1$ contribution based on a loop model,[16] assuming positive interference with the inner-bremsstrahlung amplitude, gives R_{E1} much less than 10^{-4}. Rather elaborate calculations of the $M1$ amplitude based on a boson-pole model[15,16] again give a small contribution to the branching ratio. A recent estimate[17] of the $M1$ contribution based on current-algebra techniques (in the soft-pion limits) gives $R_{M1}\sim0.3\times10^{-5}$. These estimates, however, are subject to considerable uncertainty due to many not-so-well-known parameters like $K\pi$ scattering amplitude, and various coupling constants like $g_{K^*K\gamma}$, $g_{\rho\pi\gamma}$, etc.

Phenomenological analysis of the asymmetry, due to CP-noninvariant effects of the type discussed above, in the charged-pion energy spectra in K^+ and K^- radiative decays, has already been made in Refs. 4 and 5. Our aim in this paper is to study the problem in greater detail by using the available experimental data on K^+ radiative decay, and to make an estimate of the relative contributions of the direct electric ($E1$) and magnetic ($M1$) dipole transition amplitudes and the inner-bremsstrahlung amplitudes by studying the various distributions of the decay products in K^+ radiative decay. With this motivation, we have made (assuming CP invariance) a phenomenological analysis of the pion and the photon energy spectra and the photon-pion angular correlation spectrum under various restrictive conditions on the direct amplitudes, consistent with the experimental value of the branching ratio, Eq. (4). A comparison of the calculated and the experimental π^+ energy spectra, together with the qualitative information available from the theoretical investigations of the direct amplitudes mentioned above, enables us to put limits on the direct amplitudes. Experimental study of the photon energy spectrum and the photon-pion angular correlation spectrum is suggested to confirm our estimate of the direct amplitudes. In the same manner, we reinvestigate the problem by introducing a maximal CP violation (i.e., $\phi=\frac{1}{2}\pi$) and calculate the magnitudes of the direct amplitudes which are consistent with the experimental data on K^+ decay.[18] Using the values of the direct amplitudes thus obtained, we estimate the maximum asymmetry that could be present in the K^+ and K^- radiative decays. A possibility of the existence of CP-violation effects of an alternative type[12] is briefly

[11] For a detailed discussion on the implications of CPT, CP, and T invariance see Ref. 5.

[12] In case radiative-rescattering contribution to the $M1$ amplitude is not small and there does exist a large CP-violating phase between the radiative-rescattering amplitude and the direct amplitude, then $|M1|^2$ for K^+ and K^- will be different. A possibility of this type for $E1$ amplitude has been considered in Ref. 6.

[13] Our arguments on CP-noninvariant effects are essentially those of Refs. 2, 4, and 5.

[14] Other authors have always overestimated this number by using a higher branching ratio for the $K^+\rightarrow\pi^++\pi^0$ mode. The value we have used for that ratio is 21%.

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[16] S. V. Pepper and Y. Ueda, Nuovo Cimento **33**, 1614 (1964); **34**, 1842E (1964).

[17] C. Itzykson, M. Jacob, and G. Mahoux, Nuovo Cimento Suppl. **5**, 978 (1967).

[18] Our approach differs from that of Ref. 5 in that we commit ourselves neither to only one sign (positive) of the interference between the $E1$ and the inner-bremsstrahlung amplitudes nor to only one fixed ratio of the two direct amplitudes. Further, we make use of the available experimental data on K^+ decay to fix the magnitude of the direct amplitudes.

discussed, and a way to check such effects experimentally by comparing the angular distributions, particularly in the backward direction, in the K^+ and K^- decays is suggested.

In Sec. II, we give details of the analysis performed and present the various spectra obtained. In Sec. III, we discuss the effect of the $E1$ amplitude on these spectra. In Sec. IV, we make an estimate of the possible CP-noninvariant effects in the radiative decays of K^+ and K^- mesons. The phase-space integrals for the pion and the photon energy spectra and the photon-pion angular correlation spectrum are defined and tabulated in the Appendix.

II. PHENOMENOLOGICAL ANALYSIS

Since our aim in this section is to estimate the magnitude of the $E1$ and $M1$ amplitudes in $K^+ \to \pi^+ + \pi^0 + \gamma$ decay we shall assume CP invariance in this decay process and therefore use $\phi = 0$. However, we do take account of a reasonable value for the $\pi\pi$ phase shifts, $\delta = \delta_1 - \delta_2 \sim 10^0$.[19] Inclusion of a small CP-violating phase, of the order of δ, in the analysis does not produce any appreciable change in the results. It is known[20] that there are no off mass-shell corrections to the inner bremsstrahlung amplitude, Eq. (1), to zeroth order in k. Any corrections of order k and higher, if present, will also be neglected. With these simplifications and after a slight rewriting of the $E1$ and $M1$ amplitudes we have the expression for the decay amplitude[21]

$$\mathfrak{M} = e\left[\left(\frac{\epsilon \cdot q}{q \cdot k} - \frac{\epsilon \cdot p}{p \cdot k}\right) A_2 + \frac{a|A_2|}{\mu^4} e^{i\delta} q_\mu q_\nu' F_{\mu\nu} + \frac{ib|A_2|}{\mu^4} e^{i\delta} \epsilon_{\mu\nu\alpha\beta} q_\mu q_\nu' F_{\alpha\beta}\right] e^{i\delta_2}. \quad (6)$$

p, q, q', and k are, respectively, the four-momenta of K^+, π^+, π^0, and γ. A_2 is now the on-mass-shell amplitude for the nonradiative decay $K^+ \to \pi^+ + \pi^0$. μ is the mass of the charged pion. In the expression for $E1$ and $M1$ amplitudes we have introduced a factor $1/\mu^4$ to make the parameters a and b dimensionless. The factor $|A_2|$ is introduced for reasons of symmetry between all the three terms. The parity of the $M1$ amplitude is opposite to that of the $E1$ and inner-bremsstrahlung amplitudes. Its interference with these terms will therefore drop out on performing the sum over the photon polarization states. The interference between the $E1$ and inner-bremsstrahlung amplitudes will survive. Performing the sum over the photon polarization states, we have

$$\sum_{\text{pol}} |\mathfrak{M}|^2 = |\bar{\mathfrak{M}}|^2 = 4\pi\alpha |A_2|^2 [F_{\text{br}} + 2a(\cos\delta) F_{\text{int}} + (a^2 + 4b^2) F_e], \quad (7)$$

where

$$F_{\text{br}} = \frac{2(p\cdot q)}{(p\cdot k)(q\cdot k)} - \frac{\mu^2}{(q\cdot k)^2} - \frac{m^2}{(p\cdot k)^2}, \quad (8a)$$

$$F_{\text{int}} = \frac{1}{\mu^4 (p\cdot k)(q\cdot k)} [2(p\cdot k)(q\cdot k)(p\cdot q) - \mu^2 (p\cdot k)^2 - m^2 (q\cdot k)^2], \quad (8b)$$

$$F_e = F_m = \frac{1}{\mu^8} [2(p\cdot k)(q\cdot k)(p\cdot q) - \mu^2 (p\cdot k)^2 - m^2 (q\cdot k)^2]. \quad (8c)$$

α is the fine-structure constant ($\alpha = 1/137$) and m is the mass of K^+ meson. The metric is defined by the scalar product $a \cdot b = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$. Performing the phase-space integrations with proper normalization factors (see Appendix), we finally have the expression for the decay rate,

$$\Gamma(X_1, X_2) = \frac{\alpha |A_2|^2}{\mu} [I_{\text{br}}(X_1, X_2) + 2a(\cos\delta) I_{\text{int}}(X_1, X_2) + (a^2 + 4b^2) I_e(X_1, X_2)]. \quad (9)$$

X_1 and X_2 are, respectively, the lower and upper limits of the appropriate kinematic variable (π^+ kinetic energy, photon energy, or angle between photon and π^+), between which we are interested in calculating the decay rate. I_{br}, I_{int}, and I_e are the *dimensionless* integrals of the inner-bremsstrahlung, interference, and $E1$ ($= M1$) terms, respectively. These integrals are defined and tabulated in the Appendix. To include CP-noninvariant effects in the analysis, one has to replace $a \cos\delta$, the coefficient of I_{int}, with $a \cos(\delta \pm \phi)$ in Eq. (9), for $K^\pm$ decay.

Now to perform the phenomenological analysis, we use the experimental information on the branching ratio, Eq. (4). This one number depends on two parameters a and b which we are to determine. Of the two parameters we choose a as an independent parameter and relate b to a in the following four ways.

(i) $b = 0$: This corresponds to the amplitude $M1$ being absent.

(ii) $b = 0.5a$: This corresponds to the contributions of the $E1$ and $M1$ amplitudes to the decay rate being equal in the absence of inner bremsstrahlung.

(iii) $b = a$: This corresponds to the equality of the matrix elements of the parity-conserving lowest-order weak-electromagnetic effective Hamiltonian $H_{\text{WE}}^{\text{p.c.}}$ and the parity-violating Hamiltonian $H_{\text{WE}}^{\text{p.v.}}$.

(iv) $b = 1.5a$: An enhancement of the $M1$ amplitude over the $E1$ amplitude.

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[19] See footnote 6 in Ref. 3: For experimental information on δ_2 see, E. Malamud and P. E. Schlein, Phys. Rev. Letters **19**, 1056 (1967).

[20] J. Pestieau, Phys. Rev. **160**, 1555 (1967).

[21] See Refs. 4, 5, and 7. Our parameters a and $2b$ are related to the parameters E and M in Ref. 5, by a factor $(\mu/m)^4 |A_0/A_2| \simeq 0.12$; i.e., $a = 0.12E$ and $2b = 0.12M$.

TABLE I. Positive and negative values of the parameter a for various values of R and the ratio b/a. $\delta=10°$ and $\phi=0$. a and b are, respectively, the coefficients of the $E1$ and $M1$ amplitudes defined in Eq. (6).

10^4R \ $b/a=$	0.0	0.5	1.0	1.5
1.6	0.086	0.080	0.068	0.063
	−0.963	−0.518	−0.243	−0.188
1.9	0.179	0.158	0.126	0.113
	−1.056	−0.597	−0.301	−0.238
2.2	0.259	0.225	0.170	0.152
	−1.137	−0.661	−0.346	0.277
2.5	0.332	0.279	0.209	0.185
	−1.209	−0.718	−0.384	−0.310
2.8	0.398	0.330	0.242	0.214
	−1.275	−0.768	−0.418	−0.340

The interference between the $E1$ and inner-bremsstrahlung amplitudes could be constructive or destructive. Constructive interference corresponds to positive a and destructive interference to negative a. For each of the above four cases, and $\delta=10^0$, we determine two values of the parameter a (positive and negative) which reproduce the required value of the branching ratio

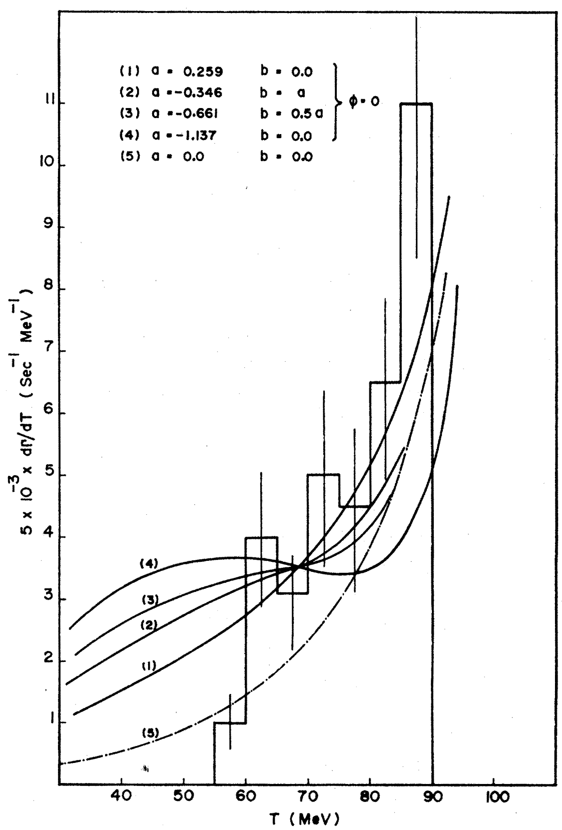

FIG. 1

FIG. 1. Calculated curves for π^+ energy spectrum in the weak-radiative decay, $K^+\rightarrow\pi^++\pi^0+\gamma$, for various values of the parameters a and b (see text). T is the π^+ kinetic energy in the rest system of K^+ meson. The experimental histogram of Ref. 3 is suitably scaled to reproduce the mean value of the branching ratio $R=$ $=2.2\times10^{-4}$.

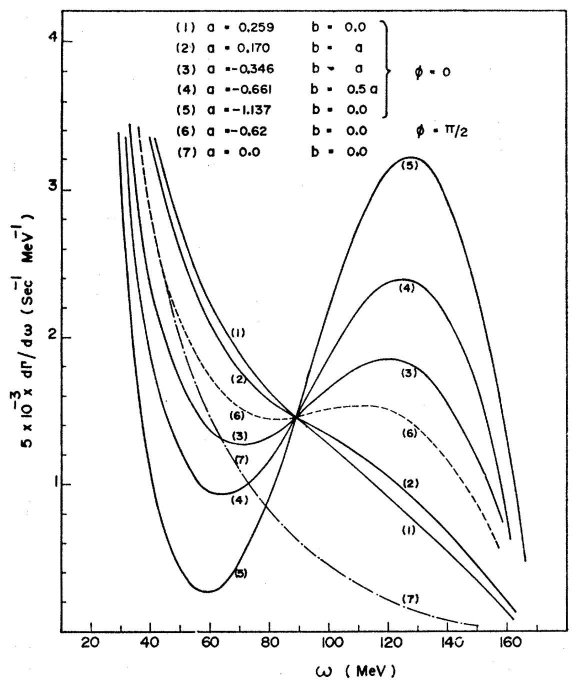

FIG. 2

FIG. 2. Calculated curves for the photon energy spectrum for various values of the parameters a and b (see text).

$\Gamma(K^+\rightarrow\pi^++\pi^0+\gamma)/\Gamma(\text{total})$ for π^+ kinetic energy T between 55 and 80 MeV, which we denote by R. The values of a obtained for five values of R, which are within the experimental range, Eq. (4), are summarized in Table I. The values of the parameter a are not so sensitive to small variations of δ (or the use of small CP-violating phase ϕ, of the order of δ). For example, the values of a for $\delta=0$ or 20° differ from those for $\delta=10°$ by at most 3%.

Using the values of a and b thus obtained, we have studied the π^+ and the photon energy spectra and the photon- π^+ angular correlation spectrum. The π^+ energy spectrum and the photon energy spectrum for a few selected values of a (and the corresponding values of the ratio b/a) which reproduced the experimental mean value of R are given in Figs. 1 and 2, respectively. The maximum value of the π^+ kinetic energy T is roughly 108 MeV. Near this value of T, the inner bremsstrahlung dominates everything else. At low values of T the contributions of the inner-bremsstrahlung and the interference terms (the latter being essential for distinguishing $E1$ from $M1$) become very small compared to those of the $|E1|^2$ and $|M1|^2$ terms. For these reasons we need to study the photon-π^+ angular correlation in a π^+ energy range where all the terms have comparable contributions. The experimental branching ratio is quoted for T between 55 and 80 MeV. However, we feel that it would be better for an experimental study to have a larger range of T so as to include sufficiently many events for better statistics. The photon-π^+ angular

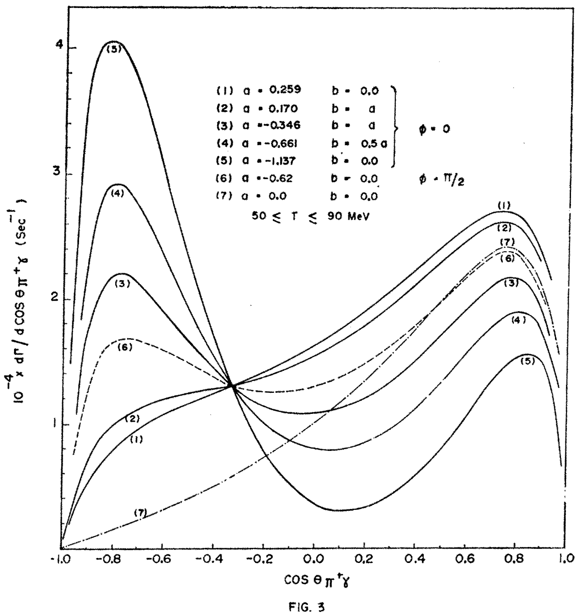

FIG. 3. Calculated curves for the photon-π^+ angular correlation spectrum for various values of the parameters a and b (see text), for π^+ kinetic energy T between 50 and 90 MeV.

correlation spectrum curves for T between 50 and 90 MeV and for five values of a and the corresponding values of the ratio b/a are given in Fig. 3. All the curves in these figures refer to $R=2.2\times10^{-4}$. The curves for other values of R can be easily calculated from the tables of integrals given in the Appendix.

It is interesting to note that the spectrum curves for various values of the ratio b/a (in all three spectra Figs. 1–3) interesct at one point. This is due to the condition[22] we had imposed on the parameters to reproduce a fixed value of the branching ratio R.

III. EFFECT OF THE $E1$ AMPLITUDE ON THE SPECTRA

With the present knowledge of the radiative K^+ decay, we can put only broad limits on the value of the parameter a [see Table I; a and b are, respectively, the coefficients of the $E1$ and $M1$ amplitudes defined in Eq. (6)]. For positive interference, the parameter a (consistent with the present experimental value of the branching ratio) could be anything between 0.1 and 0.4 depending on the ratio b/a. For negative interference it could be as large as -1.2. Such a large value of a would not be consistent with the theoretical estimate of the $E1$ amplitude mentioned in the introduction. A comparison of the calculated π^+ energy spectrum with the experimental spectrum (see Fig. 1) also indicates that such large values of a (due to negative interference) could be ruled out. The positive values of a given in Table I and a small negative value of the order of -0.4 would be consistent with the present experimental data. For positive interference the π^+ energy spectrum curves for different values of the ratio b/a are hardly distinguishable from the curve (1) for $b=0$ given in Fig. 1; this is the reason for not showing them separately in Fig. 1.

The photon energy and the photon-π^+ angular correlation spectra for large negative values of a have a shape much different from those due to inner bremsstrahlung alone. The photon energy spectrum (see Fig. 2) for negative values of a has a dip around 60–80 MeV and a peak around 120–140 MeV, whereas for positive values of a the photon spectrum has a smooth character similar to that of inner bremsstrahlung spectrum. A study of the photon spectrum will, therefore, provide a much better test of the sign of the $E1$ amplitude and of its magnitude if it is negative. The photon-π^+ angular correlation spectrum shown in Fig. 3 for π^+ kinetic energy in the experimentally feasible range is highly peaked in the backward direction (negative values of $\cos\theta_{\pi^+\gamma}$) for negative values of a. For positive values of a the peak in the backward directions is appreciably lower and broader, but still distinguishable from the inner bremsstrahlung spectrum. As in the π^+ energy spectrum, the angular correlation spectrum for positive values of a does not depend much on the ratio b/a [see curves (1) and (2) in Fig. 3]. This is due to the fact that the interference term which distinguishes the $E1$ and $M1$ amplitudes is itself small. This difficulty can be resolved by studying the polarization of the emitted photons. An asymmetry in the left- and right-circularly polarized photons would provide a measure of the $M1$ amplitude.

So far, we have discussed the spectra obtained by assuming that the CP-violating phase ϕ is zero. As we have noted in Sec. II, the values of the parameter a for small values of ϕ (of the order of δ) are not much different from those for $\phi=0$. Therefore, the main features of the spectra discussed above are also true for small values of ϕ. However, as we shall see in the next section, even the largest possible value of ϕ produces only a small asymmetry in the K^+ and K^- decay rates; it would therefore be of interest of know the effect of large values of ϕ on the spectra in the K^+ decay. For $\phi=\frac{1}{2}\pi$ (and $b/a=0$) the largest negative value of a is substantially smaller than its corresponding value for $\phi=0$ [see Eq. (10) below]. As a result, the backward peak in the angular distribution, and the large dip and the peak in the photon spectrum are considerably reduced [see curve (6) in Figs. 2 and 3], and the spectra

[22] From Eq. (9) one can easily see that that the condition that a given value of the branching ratio R should be reproduced fixes the value of $a^2+4b^2+2a\bar{\rho}\cos\delta=\chi$, where $\bar{\rho}=I_{\text{int}}/I_e$, and $I_i=\int I_i(T)dT$. For example, the choice $R=2.2\times10^{-4}$ for T between 55 and 80 MeV fixes $\chi=0.295$ and $\bar{\rho}=0.445$. The value of $d\Gamma(X)dX$, in general, depends on the parameters a, b, δ, and the ratio $\rho(X)=I_{\text{int}}(X)/I_e(X)$. The variable X stands for any of the three kinematical variables T, $\cos\theta_{\pi^+\gamma}$, and photon energy. It so happens that for one value of X the ratio $\rho(X)$ is equal to $\bar{\rho}$. For that value of X the value of $d\Gamma(X)/dX$ depends only on χ, and therefore is independent of the ratio b/a and δ.

are similar to those for $\phi=0$ and $b/a>1$. The reason for the similarity between the $\phi=0$ and $\phi\neq 0$ spectra is easy to understand. Even though we now have three parameters a, b, and ϕ, the spectra are still controlled by only two effective parameters, namely, $a\cos\bar{\delta}$ and (a^2+4b^2), where $\bar{\delta}=\delta+\phi$. Therefore, the spectra corresponding to a given set of values of a, b and ϕ (ϕ nonzero) can also be reproduced by another set of values of a, b, and $\phi=0$.[23] Consequently, a study of the spectra in K^+ decay *alone* (as was attempted in Ref. 3) is not likely to give information on CP violation unless we have more information about the parameters a, b, and ϕ. One might think that the study of the photon polarization could provide additional information about these parameters. This, however, is not the case. The polarization of the emitted photons, in addition to the above parameters, will depend on another CP-violating phase, namely, the relative phase between the $M1$ and the inner-bremsstrahlung amplitudes. Thus, the only way to get information on CP violation in the radiative decays of the charged K mesons will be to study simultaneously the decay rates and the spectra in the K^+ and K^- decays.

To summarize, we would like to say that the present experimental data on K^+ decay are consistent with small values of the parameters a and b, i.e., both $|a|$ and $|b|\sim 0.2$–0.6. The experimental verification of the absence of the large backward peak in the angular distribution and the large dip and the peak in the photon spectrum (the feature characteristic of large negative a, and b and ϕ both small) would provide a confirmation of the above estimate of the direct amplitudes. Further, the study of the spectra in K^+ decay alone will not give information on CP violation in this decay.

IV. ESTIMATE OF *CP*-NONINVARIANT EFFECTS

As explained in the introduction, the CP-noninvariant effects in the weak-radiative decays of K^+ and K^- mesons could be present if the amplitude $E1$ (characterized by the parameter a) and the inner-bremsstrahlung amplitude are not relatively real, and if the difference in the p-wave $T=1$ and s-wave $T=2$ $\pi\pi$ phase shifts, $\delta=\delta_1-\delta_2$, is nonzero. We have denoted by ϕ the relative phase (other than δ) between the $E1$ and the inner-bremsstrahlung amplitudes. ϕ equal to zero or π corresponds to no CP violation, and ϕ equal to $\frac{1}{2}\pi$ corresponds to maximal CP violation. Since our motive is to make an estimate of the *maximum* CP-noninvariant effects that could occur in weak-radiative decays of charged K mesons, consistent with the experimental data on K^+ radiative decay, we shall choose $\phi=\frac{1}{2}\pi$, i.e., maximum CP violation. The values of the parameter a given in Table I were obtained by assuming $\phi=0$. These values could still be used for small values of ϕ ($\phi\sim\delta$). As our interest is only in the largest value of ϕ, we have reevaluated the parameter a for $\phi=\frac{1}{2}\pi$ and $\delta=10°$.[19] The result is

$$\left.\begin{array}{l} a=0.15,\ -0.31 \text{ for } b=0 \\ \quad 0.08,\ -0.11 \text{ for } b=a \end{array}\right\} \text{ for } R=1.5\times10^{-4}, \tag{10a}$$

$$\left.\begin{array}{l} a=0.47,\ -0.62 \text{ for } b=0 \\ \quad 0.23,\ -0.26 \text{ for } b=a \end{array}\right\} \text{ for } R=2.2\times10^{-4}, \tag{10b}$$

$$\left.\begin{array}{l} a=0.67,\ -0.82 \text{ for } b=0 \\ \quad 0.31,\ -0.34 \text{ for } b=a \end{array}\right\} \text{ for } R=2.9\times10^{-4}. \tag{10c}$$

$R=2.2\times10^{-4}$ is the mean value of the experimental branching ratio for K^+ radiative decay, for π^+ kinetic energy T between 55 and 80 MeV [see Eq. (4)]. The other two values $R=1.5\times10^{-4}$ and 2.9×10^{-4} are respectively the lower and upper limits of the above branching ratio. We define the asymmetry in the K^+ and K^- branching ratios by

$$\eta=|R^+-R^-|/R^+, \tag{11}$$

where R^+ and R^- are, respectively, the branching ratios for K^+ and K^- decays. The estimates of η for some of the above value of a and for T between 50 and 90 MeV are

$$\left.\begin{array}{l} \eta=0.07 \text{ for } a=0.15 \\ \quad 0.14 \text{ for } a=-0.31 \end{array}\right\} \text{ for } b=0 \text{ and } R=1.5\times10^{-4}, \tag{12a}$$

$$\left.\begin{array}{l} \eta=0.15 \text{ for } a=0.47 \\ \quad 0.20 \text{ for } a=-0.62 \end{array}\right\} \text{ for } b=0 \text{ and } R=2.2\times10^{-4}, \tag{12b}$$

$$\left.\begin{array}{l} \eta=0.18 \text{ for } a=0.67 \\ \quad 0.22 \text{ for } a=-0.82 \end{array}\right\} \text{ for } b=0 \text{ and } R=2.9\times10^{-4}. \tag{12c}$$

The values of η for other values of a, which correspond to $b=a$, are smaller by a factor of ~ 2 than those given above for $b=0$. The asymmetry is almost constant over the range of T we have chosen. For example, the maximum value of η for $a=-0.82$ is about 0.25 around $T\sim 55$–60 MeV, which is to be compared with the average value 0.22 quoted above. We emphasize that the maximum value $\eta\sim 0.25$ is obtained by neglecting the $M1$ amplitude altogether, using the maximum value of the CP-violating phase ϕ, and the maximum value of the branching ratio R, and a destructive interference between the $E1$ and the inner-bremsstrahlung amplitudes. The last condition further maximizes the $E1$ amplitude. Any change in these conditions would only

[23] For example, the spectra for $a=-0.62$, $b=0$, $\phi=90°$ [curve (6) in Figs. 2 and 3] is also reproduced by $a=-0.11$, $b=0.23$, and $\phi=0$.

lower this estimate of η. Thus we see that the maximum asymmetry in the branching ratios of the rather rare radiative decays of the charged K-mesons, in the simplified model, will be about 25% for $\delta \simeq 10°$.

So far we have neglected the possible radiative-rescattering contributions to the direct amplitudes. If such contributions are significant, then there exists also a possibility of sizable CP-noninvariant effects in the contributions of the direct amplitudes. A model to estimate CP-noninvariant effects in the $E1$ amplitude is discussed in Ref. 6 (hereafter this paper will be referred to as S.B. and the model as the S.B. model). In the S.B. model the CP-noninvariant effects in the $E1$ amplitude arise due to a coupling between the nonradiative $K^+ \to \pi^+ + \pi^0$, and the radiative $K^+ \to (\pi^+ + \pi^0 + \gamma; E1)$, decay channels, with CP-violating phase in the eigenamplitudes of the coupled system. As mentioned in the introduction, the radiative-rescattering contribution to the $E1$ amplitude is expected to be small. For the $M1$ amplitude, the radiative-rescattering contribution could be significant. A coupling between the nonradiative $K^+ \to 3\pi$ and the radiative $K^+ \to (\pi^+ + \pi^0 + \gamma; M1)$ decay channels could also produce CP-noninvariant effects (i.e., different values of $|M1|^2$ for K^+ and K^-), analogous to those considered in S.B. for the $E1$ amplitude. An estimate of 25% asymmetry in the K^+ and K^- radiative decay rates is obtained by neglecting the $M1$ amplitude. If the $M1$ amplitude (CP-conserving) is taken to be different from zero, our new estimate of the asymmetry will be less than the above value; but, on the other hand, different contributions of $|M1|^2$ to K^+ and K^- decay rates[12] may either increase or further decrease the new estimate of the asymmetry. It is difficult to make a reliable estimate of this effect. A way to check experimentally the existence of CP-noninvariant effects in the direct amplitudes would be to compare the K^+ and K^- spectra in the region dominated by the direct amplitudes. As we have seen in Sec. III, the photon-pion angular correlation spectrum in the region $\cos\theta_{\pi+\gamma} < -0.5$ is quite sensitive to the size of the direct amplitudes. Thus, a comparison of the angular spectra in the K^+ and K^- radiative decays, particularly in the backward direction, will be useful in making a comparative study of the two types of CP-noninvariant effects.

In conclusion, we would like to say that the available experimental data on K^+ radiative decay, according to our analysis, is consistent with the inner-bremsstrahlung amplitude plus a small amount of the $E1$ and $M1$ amplitudes. The maximum observable asymmetry in the weak-radiative decay rates of K^+ and K^- would be about 25%. A comparison of the photon-pion angular correlation spectra in K^+ and K^- decays, particularly in the backward direction, would provide information about the importance of CP-noninvariant effects in the direct amplitudes.

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APPENDIX

The decay rate for the process $K^+(p) \to \pi^+(q) + \pi^0(q') + \gamma(k)$, in the rest frame of K^+ meson, is given by

$$\Gamma = \frac{1}{2m} \int \frac{d^3Q}{(2\pi)^3 2E} \frac{d^3Q'}{(2\pi)^3 2E'} \frac{d^3K}{(2\pi)^3 2\omega} \times (2\pi)^4 \delta^4(p-q-q'-k) |\mathfrak{M}|^2. \quad \text{(A1)}$$

Using Eq. (7) for $|\mathfrak{M}|^2$, we have

$$\Gamma = \frac{\alpha |A_2|^2}{8m(2\pi)^4} \int \frac{d^3Q}{E} \frac{d^3Q'}{E'} \frac{d^3K}{\omega} \delta^4(p-q-q'-k) \times [F_{\text{br}} + 2a(\cos\delta) F_{\text{int}} + (a^2+4b^2) F_e], \quad \text{(A2)}$$

where F_{br}, F_{int}, and F_e are given in Eq. (8). Integration over d^3Q' can easily be performed by using standard covariant techniques. To integrate over d^3Q and d^3K, we choose the rest frame of the decaying particle and fix $\mathbf{Q}$ along the z axis and $\mathbf{K}$ along any (θ,ϕ) direction. Since our integrand has no ϕ dependence, integration over $d\phi$ gives a factor of (2π). Now we can choose $\mathbf{K}$ in the $x-z$ plane, so that the momentum vectors are
$p = (m, 0)$;

$q = (E, \mathbf{Q})$; $\mathbf{Q}$ is along the z axis;

$k = (\omega, \mathbf{K})$; $\mathbf{K}$ is in the $x-z$ plane,

$$\text{and } \mathbf{Q}\cdot\mathbf{K}/|\mathbf{Q}||\mathbf{K}| = \cos\theta \equiv x. \quad \text{(A3)}$$

Using the relation $d^3X = |\mathbf{X}|^2 d|\mathbf{X}| d\Omega_X$, performing the

TABLE II. Values of the π^+ energy spectrum dimensionless integrals, $10^5 I_i(T_1, T_2)$, defined in Eq. (A6).

T_1 (MeV)	T_2 (MeV)	$10^5 I_{\text{br}}$	$10^5 I_{\text{int}}$	$10^5 I_e$
10	15	0.267	0.938	3.724
15	20	0.464	1.517	5.882
20	25	0.714	2.158	8.144
25	30	1.021	2.839	10.39
30	35	1.396	3.543	12.53
35	40	1.848	4.253	14.45
40	45	2.395	4.953	16.09
45	50	3.055	5.625	17.36
50	55	3.860	6.252	18.19
55	60	4.850	6.814	18.51
60	65	6.084	7.289	18.28
65	70	7.645	7.651	17.46
70	75	9.675	7.871	16.03
75	80	12.42	7.915	14.03
80	85	16.23	7.735	11.50
85	90	21.89	7.277	8.067
90	95	31.38	6.472	5.573
95	100	49.40	5.212	2.786
100	105	102.0	3.362	0.786

$d\Omega_Q$ integration, and changing the integration variables from $d|\mathbf{Q}|$ and $d|\mathbf{K}|$ to dE and $d\omega$, we finally have

$$\Gamma=\frac{\alpha|A_2|^2}{4m(2\pi)^2}\int dEd\omega dx\, 2|\mathbf{Q}||\mathbf{K}|\delta^0[(p-q-k)^2-\mu_0^2]$$
$$\times[F_{\rm br}+2a(\cos\delta)F_{\rm int}+(a^2+4b^2)F_e]$$
$$\equiv\frac{\alpha|A_2|^2}{\mu}[I_{\rm br}+2a(\cos\delta)I_{\rm int}+(a^2+4b^2)I_e],\quad \text{(A4)}$$

TABLE III. Values of the photon energy spectrum dimensionless integrals, $10^5I_i(\omega_1,\omega_2)$, defined in Eq. (A7).

ω_1 (MeV)	ω_2 (MeV)	$10^5I_{\rm br}$	$10^5I_{\rm int}$	10^5I_e
10	15	46.00	1.501	0.066
15	20	30.99	2.016	0.170
20	25	22.84	2.485	0.345
25	30	17.72	2.909	0.603
30	35	14.21	3.287	0.953
35	40	11.65	3.620	1.399
40	45	9.696	3.907	1.945
45	50	8.168	4.149	2.588
50	55	6.942	4.349	3.324
55	60	5.938	4.505	4.145
60	65	5.100	4.619	5.039
65	70	4.396	4.691	5.993
70	75	3.796	4.723	6.989
75	80	3.280	4.715	8.007
80	85	2.834	4.669	9.025
85	90	2.445	4.584	10.02
90	95	2.105	4.463	10.95
95	100	1.806	4.308	11.81
100	105	1.543	4.119	12.54
105	110	1.310	3.899	13.13
110	115	1.104	3.648	13.54
115	120	0.922	3.371	13.74
120	125	0.760	3.069	13.69
125	130	0.618	2.746	13.36
130	140	0.876	4.453	24.51
140	150	0.495	3.016	19.38
150	160	0.220	1.585	11.86
160	170	0.048	0.404	3.439

where

$$I_i=\frac{\mu}{16m\pi^2}\int dEd\omega dx\, 2|\mathbf{Q}||\mathbf{K}|$$
$$\times\delta^0[(p-q-k)^2-\mu_0^2]F_i.\quad \text{(A5)}$$

The subscript i stands for any of the subscripts br, int, and e. μ_0 is the mass of the π^0 meson. One of the three integrations can be easily performed with the help of the δ function. The integrals for the π^+ energy spectrum

$$I_i(T_1,T_2)=\int_{E(T_1)}^{E(T_2)} dE\left(\frac{\mu}{16m\pi^2}\int d\omega dx\, 2|\mathbf{Q}||\mathbf{K}|\right.$$
$$\left.\times\delta^0[(p-q-k)^2-\mu_0^2]F_i\right)\quad \text{(A6)}$$

are given in Table II. $T=E-\mu$ is the kinetic energy of the π^+ meson. The integrals for photon energy spectrum

$$I_i(\omega_1,\omega_2)=\int_{\omega_1}^{\omega_2} d\omega\left(\frac{\mu}{16m\pi^2}\int dEdx\, 2|\mathbf{Q}||\mathbf{K}|\right.$$
$$\left.\times\delta^0[(p-q-k)^2-\mu_0^2]F_i\right)\quad \text{(A7)}$$

are given in Table III. The integrals for the photon-π^+ angular correlation spectrum

$$I_i(x_1,x_2)=\int_{x_1}^{x_2} dx\left(\frac{\mu}{16m\pi^2}\int dEd\omega\, 2|\mathbf{Q}||\mathbf{K}|\right.$$
$$\left.\times\delta^0[(p-q-k)^2-\mu_0^2]F_i\right)\quad \text{(A8)}$$

for the π^+ kinetic energy between 50 and 90 MeV are given in Table IV.

TABLE IV. Values of the photon-π^+ angular correlation spectrum dimensionless integrals, $10^5I_i(x_1,x_2)$, defined in Eq. (A8), for π^+ kinetic energy between 50 and 90 MeV.

x_1	x_2	$10^5I_{\rm br}$	$10^5I_{\rm int}$	10^5I_e
−1.0	−0.9	0.126	0.967	7.771
−0.9	−0.8	0.390	2.327	14.85
−0.8	−0.7	0.675	3.123	15.83
−0.7	−0.6	0.983	3.588	14.69
−0.6	−0.5	1.317	3.847	12.88
−0.5	−0.4	1.681	3.973	10.99
−0.4	−0.3	2.077	4.009	9.229
−0.3	−0.2	2.511	3.982	7.671
−0.2	−0.1	2.986	3.909	6.325
−0.1	0.0	3.508	3.803	5.178
0.0	0.1	4.083	3.668	4.206
0.1	0.2	4.715	3.509	3.385
0.2	0.3	5.408	3.325	2.693
0.3	0.4	6.162	3.116	2.109
0.4	0.5	6.966	2.877	1.617
0.5	0.6	7.785	2.598	1.202
0.6	0.7	8.524	2.264	0.852
0.7	0.8	8.933	1.848	0.556
0.8	0.9	8.318	1.301	0.306
0.9	1.0	4.458	0.519	0.094
−1.0	1.0	82.0	58.0	122.0