Analyticity and Broken $O(4)$ Symmetry*

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We investigate the extent to which broken $O(4)$ results can be obtained by requiring that unequal-mass scattering amplitudes be analytic at zero energy. All $O(4)$ constraints upon the trajectory functions can be obtained in this way.

In this paper we study the relation between the analyticity and group-theory approaches to daughter Regge trajectories. For this purpose, we consider the scattering of spinless particles in the s channel, which for convenience we assume to be an annihilation channel with pairwise equal masses in the initial and final states. We study Regge poles in the u channel which contribute to the high-s behavior. If the scattering particles in the u channel have equal masses, then the amplitude in the u channel has $O(4)$ symmetry at $Q_u=0$, where Q_u is the total four-momentum in the u channel.[1] The additional symmetry at this point strongly suggests that poles should be classified according to irreducible representations of $O(4)$; this means that for each parent Regge pole at α ($u=0$) there must be a sequence of daughters with a spacing of two units at $\alpha(0)-2K$ ($K=1, 2, \cdots$) with definite coupling relations among them. In the unequal-mass case, $O(4)$ symmetry does not apply to the scattering amplitude, but Domokos and Domokos and Suranyi have shown that the symmetry does apply to the bound-state Bethe-Salpeter equation, which does not depend upon the external masses at $Q_u=0$.[2] This again suggests that poles should be classified according to $O(4)$. However, now the Regge daughters have a spacing of one unit occurring at $\alpha(0)-K$ ($K=1, 2, \cdots$). There also follow definite rules as to how the symmetry is broken for $u\neq 0$.

The presence of daughter trajectories in the unequal-mass case can also be deduced by analyticity arguments. A single Regge-pole contribution in the u channel is not analytic at $u=0$, in violation of the Mandelstam representation. The analyticity can be restored only by assuming the presence of a daughter sequence of trajectories at $\alpha(0)-K$, ($K=1, 2, \cdots$).[3]

The group-theoretic and analyticity approaches thus lead to the same sequence at $u=0$, but they have different starting points and content. In the present paper we are interested in the analyticity approach, and the extent to which it can recover symmetry results. We cannot expect to recover the symmetry itself, since the unequal-mass amplitude does not possess the symmetry. What we can do is see how many group-theoretic results we can obtain. The principal results from $O(4)$ are the following:

(1) The existence of the daughter sequence and the integer spacing at $u=0$. This has already been obtained by others from analyticity.[4]

(2) For the trajectory functions $\alpha_K(u)$, all the results for $O(4)$ symmetry are broken by $u\neq 0$ (mass formulas). These we can obtain in unaltered form from analyticity because the properties of the trajectories do not depend upon the masses of the particles to which they couple. This contrasts with the derivation given in Domokos and Suranyi[2] where the Bethe-Salpeter equation and bound-state perturbation theory are employed.

(3) For the daughter residues, the fact that when the parent passes through a non-negative integer, all the daughters for $\alpha(0)-K<0$ are absent.[5] This we can verify for the most singular part of the daughter residues in the unequal-mass case.[6]

(4) We might hope to show that daughters must be present in equal-mass scattering. The equal-mass amplitude is analytic at $u=0$, but the (finite) residues of the daughters in this case might be inferred by considering equal-mass scattering as the limit of unequal-mass scattering. This hope does not materialize for reasons we discuss later.[7]

We incorporate known results in studying analyticity. These are that to obtain analyticity at $u=0$, at least one daughter must be present at $\alpha(0)-K$ when $u=0$, with a residue behaving like u^{-K}. One daughter is all that is required, and to use more would be to insert spurious, accidently degenerate trajectories. The con-

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[1] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

[2] G. Domokos, Phys. Rev. **159**, 1387 (1967); G. Domokos and P. Suranyi, Central Research Institute for Physics, Budapest Report, 1968 (unpublished).

[3] We assume that the background integral can be pushed far enough to the left so that it cannot provide the terms which restore analyticity.

[4] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).

[5] In the equal-mass case this follows from the fact that the $O(4)$ quantum number $M=0$ for spinless particles. See also, L. Jones and H. Shepard [Phys. Rev., this issue **175**, 2117 (1968)] who also discuss this question from the point of view of analyticity.

[6] For unequal-mass scattering, the Kth daughter has a u^{-k} singularity at $u=0$. This is what restores analyticity to the sum of Regge amplitudes.

[7] For a later development on this point see J. B. Bronzan and C. E. Jones, Phys. Rev. Letters **21**, 564 (1968), and also Ref. 5.

tribution of the Kth daughter is

$$T_K(s,u)=[\gamma_K(u)/u^K]s^{\alpha_K(u)} \times F(-\alpha_K(u),-\alpha_K(u);-2\alpha_K(u);-4q_u^2/s),$$
$$q_u^2=[u-(m+\mu)^2][u-(m-\mu)^2]/4u, \quad (1)$$
$$\alpha_K(0)=\alpha_0(0)-K,$$

where we have combined various factors, including signature into the reduced residue, and represented $Q_{-\alpha_K(u)-1}(-Z_u=-1-s/4q_u^2)$ as a hypergeometric function. The u-channel center-of-mass momentum is q_u, m and μ are masses, and the singularity u^{-k} has been removed from the residue. We sum over daughters, and write the complete amplitude in the form

$$T(s,u)=\sum_{K=0}^{\infty} T_K(s,u)=\sum_{K=0}^{\infty} s^{\alpha_K(u)}\sum_{n=0}^{\infty} C(K,n,u) \times s^{-n}u^{-n-K},$$
$$C(K,n,u)=\gamma_K(u)\{[u-(m+\mu)^2][u-(m-\mu)^2]\}^n \times\left[\frac{\Gamma(\alpha_K(u)+1)}{\Gamma(\alpha_K(u)-n+1)}\right]^2\frac{\Gamma(2\alpha_K(u)-n+1)}{\Gamma(2\alpha_K(u)+1)n!}. \quad (2)$$

Our program is to examine Eq. (2) for large s and demand that the coefficient of each term (like $s^{\alpha(0)-r}\ln s$, $s^{\alpha(0)-r}$, etc.) be analytic at $u=0$. Since the sum over n in Eq. (2) converges only when $|4q_u^2/s|<1$, and q_u^2 diverges like u^{-1} at $u=0$, one might question whether these constraints are correct. However, one can proceed in a manifestly correct, but laborious way to get the same results. Namely, like Goldberger and Jones,[8] one can use the Mandelstam representation to deduce the corrections which must be added to the parent to correct its analyticity at $u=0$. The correction terms are cancelled by the first daughter, which in turn requires new correction terms to correct its analyticity at $u=0$, etc. This procedure generates constraints equivalent to those we find more easily.

We need the expansions[9]

$$C(K,n,u)=\sum_{i=0}^{\infty} C_i(K,n)u^i,$$
$$s^{\alpha_K(u)}=s^{\alpha_0-K}\{1+(\ln s)[u\alpha_K'+\tfrac{1}{2}u^2\alpha_K''+\cdots] +(\ln s)^2[\tfrac{1}{2}u^2(\alpha_K')^2+\tfrac{1}{2}u^3\alpha_K'\alpha_K''+\cdots] +(\ln s)^3[\tfrac{1}{6}u^3(\alpha_K')^3+\cdots]+\cdots\}. \quad (3)$$

Thus,

$$T(s,u)=\sum_{r=0}^{\infty} s^{\alpha_0-r}\sum_{K=0}^{r}\sum_{i=0}^{\infty} C_i(K,r-K)u^{i-r} \times\{1+(\ln s)[u\alpha_K'+\tfrac{1}{2}u^2\alpha_K''+\cdots]+\cdots\}. \quad (4)$$

We now select for study five subsets of the infinite set of relations implied by Eq. (4). In parentheses we indicate the term whose coefficient must vanish in order for Eq. (4) to be analytic at $u=0$.

$$0=\sum_{K=0}^{r} C_0(K,r-K), \qquad (r\geq 1) \qquad (s^{\alpha_0-r}/u^r) \quad (5a)$$

$$0=\sum_{K=0}^{r} \alpha_K'C_0(K,r-K), \qquad (r\geq 2) \qquad (s^{\alpha_0-r}\ln s/u^{r-1}) \quad (5b)$$

$$0=\sum_{K=0}^{r} (\alpha_K')^nC_0(K,r-K), \qquad (r\geq n+1) \qquad (s^{\alpha_0-r}(\ln s)^n/u^{r-n}) \quad (5c)$$

$$0=\sum_{K=0}^{r} C_1(K,r-K), \qquad (r\geq 2) \qquad (s^{\alpha_0-r}/u^{r-1}) \quad (5d)$$

$$0=\sum_{K=0}^{r} [\alpha_K'C_1(K,r-K)+\tfrac{1}{2}\alpha_K''C_0(K,r-K)], \qquad (r\geq 3) \qquad (s^{\alpha_0-r}\ln s/u^{r-2}). \quad (5e)$$

We make use of

$$C_0(K,r-K)=\gamma_KB_{K,r}(\alpha_0),$$
$$C_1(K,r-K)=[\gamma_K'-2(r-K)\gamma_K(m^2+\mu^2)/\epsilon]B_{K,r}(\alpha_0) +\alpha_K'\gamma_KB_{K,r}'(\alpha_0), \quad (6)$$

where

$$B_{K,r}(\alpha_0)=\frac{\epsilon^{r-K}}{(r-K)!}\left[\frac{\Gamma(\alpha_0-K+1)}{\Gamma(\alpha_0-r+1)}\right]^2\frac{\Gamma(2\alpha_0-K-r+1)}{\Gamma(2\alpha_0-2K+1)} \quad (7)$$

and

$$\epsilon=(m^2-\mu^2)^2.$$

We observe that Eqs. (5a), which now take the form

$$\sum_{K=0}^{r} \gamma_KB_{K,r}(\alpha_0)=0, \quad r\geq 1$$

determine all the γ_K in terms of γ_0. They are easy to solve recursively because as r increases more and more

[8] M. L. Goldberger and C. E. Jones, Phys. Rev. **150**, 1269 (1966).

[9] Where no argument of a function is written, read $u=0$.

residues enter. We find

$$\gamma_K=\gamma_0\frac{(-\epsilon)^K}{K!}\left[\frac{\Gamma(\alpha_0+1)}{\Gamma(\alpha_0-K+1)}\right]^2\frac{\Gamma(2\alpha_0-2K+2)}{\Gamma(2\alpha_0-K+2)}. \quad (8)$$

We observe that γ_K vanishes for non-negative integer α_0 when $\alpha_0-K<0$, in agreement with the $O(4)$ result (point 3 above). Apparent poles in Eq. (8) for α_0 at half odd integer can be associated with the $(\cos\pi\alpha_0)^{-1}$ which has been lumped into γ_K. Equations (5b) in the form

$$\sum_{K=0}^{r}\alpha_K'\gamma_K B_{K,r}(\alpha_0)=0,\quad r\geq 2 \quad (9)$$

now determine $\alpha_K'\gamma_K$ for all $K\geq 2$ in terms of its values at $K=0$ and $K=1$. Since according to (8) all γ_K are known multiples of γ_0, the latter factors out of (9) entirely, resulting in a mass formula mentioned under point 2 above:

$$\alpha_K'=\alpha_0'+(\alpha_1'-\alpha_0')K(2\alpha_0-K+1)/2\alpha_0. \quad (10)$$

This gives all slopes in terms of those of the parent and first daughter. This formula agrees with Ref. 2, where the two parameters appear as reduced matrix elements after using the $O(4)$ Wigner-Eckhart theorem. Equations (5c) are similar to Eqs. (5b), except that more equations are missing, and higher powers of the slopes appear. They are all satisfied identically. We conjecture that all the constraints imposed by analyticity are contained in the $(\ln s)^0$ and $(\ln s)^1$ terms, with higher powers of $\ln s$ providing no further information.[10]

The solution of Eqs. (5d) proceeds just as above, after $B_{K,r}'(\alpha_0)$ is eliminated by differentiating Eqs. (9) with respect to α_0.

$$\sum_{K=0}^{r}\left[\gamma_K'-2(\gamma-K)\gamma_K(m^2+\mu^2)/\epsilon-\frac{\partial}{\partial\alpha_0}(\alpha_K'\gamma_K)\right]\times B_{K,r}(\alpha_0)=0,\quad r\geq 2.$$

These determine the γ_K' in terms of γ_0', γ_1', and previously encountered parameters:

$$\gamma_K'=-\frac{(K-1)(2\alpha_0-K)}{2\alpha_0}\frac{\gamma_K}{\gamma_0}\gamma_0'+\frac{K(2\alpha_0-K+1)}{2\alpha_0}\frac{\gamma_K}{\gamma_1}\gamma_1' -\frac{K(K-1)(m^2+\mu^2)}{\alpha_0\epsilon}\gamma_K+\frac{\partial}{\partial\alpha_0}(\alpha_K'\gamma_K-\alpha_1'\gamma_1'). \quad (11)$$

We remark that γ_K' vanishes for positive integer α_0 when $\alpha_0-K<0$, but the property does not go through when $\alpha_0=0$ unless we assume, for $\alpha_0=0$, the relation

$$\gamma_1'=-\tfrac{1}{2}\epsilon\alpha_1'\gamma_0. \quad (12)$$

Equations (5e) can now be solved, giving α_K'' in terms of α_0'', α_1'', α_2'', and previously encountered parameters. We omit the details and merely present the following solution:

$$\alpha_K''=\alpha_0''\frac{(K-1)(K-2)(2\alpha_0-K)(2\alpha_0-K-1)}{4\alpha_0(2\alpha_0-1)}-\alpha_1''\frac{K(K-2)(2\alpha_0-K+1)(2\alpha_0-K-1)}{4\alpha_0(\alpha_0-1)} +\alpha_2''\frac{K(K-1)(2\alpha_0-K+1)(2\alpha_0-K)}{4(2\alpha_0-1)(\alpha_0-1)}+\frac{K(K-1)(K-2)(\alpha_1'-\alpha_0')}{2\alpha_0^2(\alpha_0-1)} \times\left[\alpha_0'\frac{4\alpha_0-K-1}{2\alpha_0-1}+(\alpha_1'-\alpha_0')(2\alpha_0-K+1)\right]. \quad (13)$$

This equation constrains the quadratic terms in the mass formula; it does not appear to have been derived previously.

Equations (8), (10), (11), and (13) provide the pattern for an infinite series of other constraint equations involving higher derivatives of α_K and γ_K. These constraint equations become less restrictive the higher the derivatives. If we use a superscript to denote the order of the derivative, the pattern is that the $\alpha_K^{(n)}$ are completely determined by a known function f_n as follows:

$$\alpha_K^{(n)}=f_n(K;\alpha_0^{(0)},\alpha_0^{(1)},\cdots,\alpha_0^{(n)};\alpha_1^{(1)},\alpha_1^{(2)},\cdots,\alpha_1^{(n)};\cdots;\alpha_n^{(n)}). \quad (14)$$

Thus, the slopes $\alpha_K^{(1)}$ are all determined in terms of the first two, as in Eq. (10), whereas the second derivatives $\alpha_K^{(2)}$ are all determined by the first three second derivatives as well as the first two slopes.

An analogous pattern of constraint conditions emerges for residue derivatives $\gamma_K^{(n)}$, except that the trajectory derivatives also enter as determining parameters. The formula corresponding to (14) is

$$\gamma_K^{(n)}=g_n(K;\gamma_0^{(0)},\gamma_0^{(1)},\cdots,\gamma_0^{(n)};\gamma_1^{(1)},\cdots,\gamma_1^{(n)};\cdots;\gamma_n^{(n)};\alpha_0^{(0)},\cdots,\alpha_0^{(n)};\cdots;\alpha_n^{(n)}). \quad (15)$$

The result embodied in Eqs. (14) and (15) enables us to deal with point 4 above; namely, the possibility of deducing the presence of daughters in the equal-mass

[10] For a demonstration of this, see P. K. Kuo and J. F. Walker, Massachusetts Institute of Technology Report, 1968 (unpublished).

scattering by considering it as the limit of unequal-mass scattering.

Since in the equal-mass limit there are no analyticity problems, the singular parts of residues vanish as $\epsilon \rightarrow 0$. This fact is verified by Eqs. (8) and (10). The coupling of trajectories in the equal-mass case then is determined by the nonsingular parts of the residues, namely, by $\gamma_K^{(K)}$. But according to Eq. (15) these $\gamma_K^{(K)}$ are completely unconstrained. If the poles in the equal-mass limit are classified by irreducible representations of $O(4)$, a definite ratio of $\gamma_K^{(K)}$ to $\gamma_0^{(0)}$ is required. This means that analyticity cannot eliminate the possibility in the equal-mass problem of a string of integer-spaced Lorentz poles at $n_K(0)=\alpha_K$.[7]

One important remark should be made. Equation (13), as well as Eq. (14) which generalizes it, involves only trajectory functions—not external masses or residues—as it should, since Regge poles couple to many channels. By a slight reorganization of the calculation, it is possible to show that m and μ will never appear in these equations. Even so, the independence of Eq. (13) on residues is not evident *a priori*, and requiring such an independence from residues could lead to a constraint involving among other things $\gamma_1^{(1)}$ and $\gamma_0^{(0)}$, as an examination of the steps leading to Eq. (13) shows. Such new constraints might well determine $\gamma_1^{(1)}/\gamma_0^{(0)}$, thereby indicating the necessity of daughter trajectories in equal-mass scattering. However, the dependence on residues turns out to drop out of Eq. (13) automatically.

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Physical Region on the Plane of Two Invariant Momentum Transfers for a Reaction with Three Particles in the Final State

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The boundary of the kinematically allowed region on the plane of two invariant momentum transfers of a reaction with three particles in the final state is derived and the phase-space distribution in this region determined. The boundary may in the most general case consist of branches of a complicated curve, a parabola, and two straight lines. Applications to the analysis of doubly peripheral reaction mechanisms are discussed.

1. INTRODUCTION

In the analysis of reactions of the type $p_1+p_2=p_3+p_4+p_5$ with three particles in the final state (Fig. 1), one commonly uses the Dalitz and Chew-Low plots. The former is a plot in the two invariant energies

$$s_1=(p_3+p_4)^2, \quad s_2=(p_4+p_5)^2, \tag{1}$$

and it is used mainly for the investigation of particle spectra and intrinsic properties of resonances. The Chew-Low plot is a plot in one energy and one invariant momentum transfer, which are not adjacent (in the sense of Fig. 2) to each other, i.e., defining

$$t_1=(p_1-p_3)^2, \quad t_2=(p_2-p_5)^2, \tag{2}$$

a plot in either s_1 and t_2 or s_2 and t_1. It is used mainly for investigating mechanisms of resonance production. Kinematically, the Dalitz and Chew-Low plots have the common feature that they involve, apart from masses, three invariant variables, one of which is the fixed total energy $s=(p_1+p_2)^2$, and which in the sense of Fig. 2 are not all adjacent to each other. The boundaries of all plots involving variables as in Fig. 2(a) are obtained from the same analytic function, which is also the boundary curve of the physical region of a two-body process.

In this paper, we shall investigate plots involving three adjacent [Fig. 2(b)] invariant variables one of which is fixed. The most interesting one of these is the t_1t_2 plot involving the total energy s and the two momentum transfers t_1 and t_2 [Eq. (2)]. It is convenient to apply, when investigating double peripheralism, i.e., the simultaneous dependence of the reaction mechanism on two momentum transfers. The derivation will be carried out for the t_1t_2 plot, but results for any other plot of the same type, e.g., a plot in s, t_1, and s_1 or s, t_2, and s_2, may be obtained by a suitable reordering of the particles.

Section 2 contains a discussion of some properties of the Dalitz and Chew-Low plots necessary for the understanding of the t_1t_2 plot. In Sec. 3, a technique based

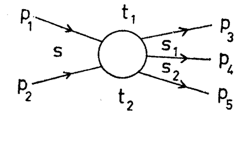

FIG. 1. A reaction with three particles in the final state