absolute value by

$$\sum_{m=0}^{n} \frac{2(n-m+1)(2n+1)!}{m!(2n-m+2)!} = \frac{(2n+1)!}{n!(n+1)!}. \quad \text{(A5)}$$

It follows that in region I,

$$|R_n^{(1)}(z)| \leqslant \frac{\Gamma(n+\frac{3}{2})|z|}{2\Gamma(\frac{1}{2})\Gamma(n+2)} \left|\frac{4z}{(1+z)^2}\right|^{n+1},$$

which vanishes in the limit $n \to \infty$ by virtue of (A4). The argument goes through in similar fashion for $R_n^{(2)}$ in region II. In this region, $|z| \geqslant 1$, and the sum in (A3) is again bounded in absolute value by (A5). So in region II,

$$|R_n^{(2)}(z)| \leqslant \frac{\Gamma(n+\frac{3}{2})}{2\Gamma(\frac{1}{2})\Gamma(n+2)|z|} \left|\frac{4z}{(1+z)^2}\right|^{n+1},$$

which vanishes in the limit $n \to \infty$ by virtue of (A4).

We conclude that with the replacements $\mathrm{Re}A_l(s) \to A_R(s)$ and $\mathrm{Im}A_l(s) \to A_I(s)$ the expansion (7) converges for complex values of $s$ such that $z$ [Eq. (A1)] lies in region I and the expansion (8) holds for values of $z$ in region II.

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# Asymptotic Behavior of the $n$-Point Function and Some Applications*

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A generalization of the Bjorken limit (for the two-point function) to the three-point and four-point functions is given. Some general features of the asymptotic behavior of the $n$-point function are also discussed. These results show that in calculating the various Ward identities for the $n$-point function all currents are "asymptotically conserved." We derive generalized Weinberg sum rules for the three-point functions (these results can be generalized to the $n$-point functions). We show that the $K_L^0$-$K_S^0$ mass difference (in the universal Fermi theory) is quadratically divergent. Making a saturation assumption, we calculate the coefficient of the quadratic divergency and we get a weak-interaction cutoff $\Lambda=4$ BeV, suggesting that weak interactions are strongly nonlocal. By means of a simple power-counting argument, we find that the $n$th order probably behaves like $n!G(G\Lambda^2)^{n-1}$, and assuming that this is some kind of asymptotic expansion, we find that the series begins to blow up for $n \sim 10^4$. The arguments for this do not constitute a proof. We then study the radiative corrections to the decays $\pi \to e\nu$ and $\pi \to \mu\nu$, which involve a three-point function. We find that these decays cannot be discussed within the framework of current algebra. Finally we show that a somewhat generalized version of the Tamm-Dancoff approximation can be justified if we use our results for the $n$-point function.

## 1. INTRODUCTION

SOME time ago Bjorken proposed[1] a method for calculating the (virtual) asymptotic behavior of the two-point function. This method has been very useful in estimating the radiative corrections to $\beta$ decay[1,2] (coming from high virtual masses) as well as the electromagnetic mass differences.[1,3] In this paper we shall generalize Bjorken's expansion to the three-point function as well as the four-point function; it is possible to obtain general results for the $n$-point function also. Such a generalization is required in order to discuss several interesting physical problems, e.g., the $K_L^0$-$K_S^0$ mass difference (in the current-current interaction). The main results of this paper are the following:

In Sec. 2 we generalize the Bjorken expansion to the three-point and the four-point functions. We also give a method for calculating the $n$-point function.

In Sec. 3 we show that the results obtained in the previous section can be used to prove the following theorem: Assuming the ordinary current algebra, all currents are "asymptotically conserved" in the sense that in calculating Ward identities for the $n$-point function

$$\int\cdots\int d^4x_1\cdots d^4x_n e^{iq_1x_1+\cdots+iq_nx_n} \times\langle A|T(j_{\mu_1}^{\alpha_1}(x_1)\cdots j_{\mu_n}^{\alpha_n}(x_n))|B\rangle, \quad (1.1)$$

it is correct to assume that in time-ordered products

$$\partial^{\mu_m} j_{\mu_m}^{\alpha_m}(x_m)=0 \quad (1.2)$$

for all $\alpha$'s in so far as we are only interested in the leading terms of the $n$-point function. This theorem is evidently of practical importance since it shows that asymptotically the Ward identities allow us to express the $n$-point function entirely in terms of the $(n-1)$-

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[1] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[2] E. S. Abers, R. E. Norton, and D. A. Dicus, Phys. Rev. Letters **18**, 676 (1967); E. S. Abers, D. A. Dicus, R. E. Norton, and H. R. Quinn, Phys. Rev. **167**, 1461 (1968).

[3] M. B. Halpern and G. Segrè, Phys. Rev. Letters **19**, 611 (1967); **19**, 1000 (1967); G. C. Wick and B. Zumino, Phys. Letters **25B**, 479 (1967).

point functions. One can also obtain a somewhat related result which is a generalization of Weinberg's first sum rule[4] (which has been shown to follow from the Bjorken limit for the two-point function).[5] If the states $|A\rangle$ and $|B\rangle$ in (1.1) are both equal to the vacuum state, our result means that asymptotically we have $SU(3)\otimes SU(3)$ symmetry. The idea of asymptotic symmetry was introduced in connection with Weinberg sum rules by Das, Mathur, and Okubo.[6] Intuitively one would expect this to happen in the asymptotic limit because the masses are not expected to play any role in the limit; however, against this argument one can say that our theorems apply to the asymptotic limit $q_{m\mu}\to\infty$ but infinitely off the mass shell, $|q_m{}^2|\to\infty$, and it is somewhat difficult to apply physical intuition infinitely off the mass shell.

In Sec. 4 we apply the technique developed for the three-point function to calculate the $K_L{}^0$-$K_S{}^0$ mass difference in the universal Fermi theory. We show that this mass difference is quadratically divergent with coefficients of the type

$$\langle K^0|A_{\mu 2}{}^3(0)A^{\mu}{}_2{}^3(0)|\bar{K}^0\rangle. \tag{1.3}$$

[$A_\mu(x)$ is the axial-vector current, and we use the tensor notation.[7]] To find the numerical value of the coefficient of the quadratically divergent term, one therefore has to insert a set of intermediate states in (1.3) and do some saturation with a few states. If one only inserts the lowest states, one obtains

$$M^2(K_L{}^0)-M^2(K_S{}^0) =(2.5G/16\pi^2)(G\Lambda^2)\sin^2\theta\cos^2\theta\, M_K{}^2F_K{}^2, \tag{1.4}$$

which gives a cutoff $\Lambda\cong 3$ BeV. Equation (1.4) has recently been obtained by Marshak *et al.*[8] using the Tamm-Dancoff approximation.[9,10] Previously Ioffe and Shabalin[8] have calculated the $K_L{}^0$-$K_S{}^0$ mass difference in the $W$ theory (which is an easier problem than in the Fermi theory if one uses a Ward identity and "asymptotic conservation of currents"), and the result is essentially the same as (1.4).

In Sec. 5 we use our general results on the $n$-point function to study the weak interactions to $n$th order. It turns out that the leading divergency (in the universal Fermi interaction) is of the order (to all orders in strong interactions but neglecting electromagnetic interactions)

$$G(G\Lambda^2)^{n-1}, \tag{1.5}$$

which shows that the behavior of the weak interactions depends on two "coupling constants" $G\approx 10^{-5}/M_n{}^2$ and $G\Lambda^2\approx 10^{-4}$ [from Eq. (1.4)]. From the point of view of principles, this reflects the well-known nonrenormalizable character of the weak interactions. From a more practical point of view, Eq. (1.5) strongly suggests that the perturbation series is an asymptotic expansion (valid only with $G=0$), and since $G\Lambda^2$ is very small from the $K_L{}^0$-$K_S{}^0$ mass difference we can use this series for an evaluation of the weak amplitudes up to a certain order where the series starts blowing up. By simply counting the number of terms contributing to the leading term (1.5), one finds that the series should not begin to blow up before $n=10\,000$. The results in this section are certainly not proved, but should be considered as a conjecture.

In Sec. 6 we discuss radiative corrections to $\pi\to\mu\nu$ and $\pi\to e\nu$ decays, which involve the three-point function. It has been shown by Das and Mathur[11] that the radiative corrections to the branching ratio for these decays are finite. It is not possible to arrive at a definite conclusion concerning the possible infinities in the decays. However, making an approximation of keeping only the lowest intermediate states in a particular term which cannot otherwise be calculated, one finds that in none of the models for the equal-time commutators (including the model of Johnson *et al.* and Cabibbo *et al.*[12]) proposed so far are these radiative corrections finite. However, because of the approximation involved this is certainly not a well-established conclusion.

In Sec. 7 we point out that asymptotically approximations of the type

$$\begin{aligned}\langle A|T(j_{\mu_1}(x)j_{\mu_2}(x)j_{\mu_3}(y)j_{\mu_4}(y))|B\rangle &\approx\langle A|T(j_{\mu_1}(x)j_{\mu_3}(y))T(j_{\mu_2}(x)j_{\mu_4}(y))|B\rangle\\ &+\langle A|T(j_{\mu_1}(x)j_{\mu_4}(y))T(j_{\mu_3}(y)j_{\mu_2}(x))|B\rangle\end{aligned} \tag{1.6}$$

are good approximations if one saturates with a few intermediate states. The approximation (1.6) is a generalized Tamm-Dancoff approximation[9,10] used recently.[8,13]

## 2. ASYMPTOTIC BEHAVIOR OF $n$-POINT FUNCTION

In this section we shall discuss the asymptotic behavior of the $n$-point function, which we define as (we

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[5] H. T. Nieh, Phys. Rev. 163, 1769 (1967).
[6] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters 18, 761 (1967).
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[8] R. N. Mohapatra, J. S. Rao, and R. E. Marshak, Phys. Rev. Letters 20, 1081 (1968); B. L. Ioffe, in *Proceedings of the 1967 International Conference on Particles and Fields* (Interscience Publishers, Inc., New York, 1967), p. 447 and references therein; see also B. L. Ioffe and E. P. Shabalin, Yadern. Fiz. 6, 828 (1967) [English transl.: Soviet J. Nucl. Phys. 6, 603 (1968)].
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[11] T. Das and V. S. Mathur, Rochester report, 1967 (unpublished).
[12] K. Johnson, F. E. Low, and H. Suura, Phys. Rev. Letters 18, 1224 (1967); N. Cabibbo, L. Maiani, and G. Preparata, Phys. Letters 25B, 31 (1967); 25B, 132 (1967).
[13] S. N. Biswas and J. Smith, Phys. Rev. Letters 19, 727 (1967).

leave out one integration since this is what one usually encounters in practice)

$$T_{\mu_1\cdots\mu_n}{}^{\alpha_1\cdots\alpha_n}(q_1\cdots q_{n-1})$$
$$=(-i)^{n-1}\int\cdots\int d^4x_1\cdots d^4x_{n-1}e^{iq_1x_1+\cdots+iq_{n-1}x_{n-1}}$$
$$\times\langle A|T(j_{\mu_1}{}^{\alpha_1}(x_1)\cdots j_{\mu_{n-1}}{}^{\alpha_{n-1}}(x_{n-1})j_{\mu_n}{}^{\alpha_n}(0))|B\rangle. \tag{2.1}$$

The case of the (virtual) asymptotic behavior of the two-point function is the well-known Bjorken limit.[1] In order to illustrate our technique, we shall briefly indicate a simple method to obtain this limit. Consider

$$T_{\mu_1\mu_2}{}^{\alpha_1\alpha_2}(q)=-i\int d^4x\, e^{iqx}$$
$$\times\langle A|T(j_{\mu_1}{}^{\alpha_1}(x_1)j_{\mu_2}{}^{\alpha_2}(0))|B\rangle. \tag{2.2}$$

By considering the time integration and doing a partial integration we obtain for one of the two terms in Eq. (2.2)

$$\int dx_0\theta(x_0)e^{iq_0x_0}\langle A|j_{\mu_1}{}^{\alpha_1}(x)j_{\mu_2}{}^{\alpha_2}(0)|B\rangle$$
$$=(1/iq_0)[\langle A|j_{\mu_1}{}^{\alpha_1}(x)j_{\mu_2}{}^{\alpha_2}(0)|B\rangle e^{iq_0x_0}e^{-\epsilon x_0}]_{x_0=0}{}^{x_0=\infty}$$
$$-\frac{1}{iq_0}\int_0^\infty dx_0 e^{iq_0x_0}\langle A|\frac{\partial}{\partial x_0}j_{\mu_1}{}^{\alpha_1}(x)j_{\mu_2}{}^{\alpha_2}(0)|B\rangle$$
$$=(i/q_0)\langle A|j_{\mu_1}{}^{\alpha_1}(0,\mathbf{x})j_{\mu_2}{}^{\alpha_2}(0)|B\rangle$$
$$-\frac{1}{q_0^2}\langle A|\frac{\partial}{\partial x_0}j_{\mu_1}{}^{\alpha_1}(x)j_{\mu_2}{}^{\alpha_2}(0)|B\rangle_{x_0=0}$$
$$+O(1/q_0^3). \tag{2.3}$$

Treating the second term in Eq. (2.2) in the same way we obtain in the limit $q_0\to\infty$, $\mathbf{q}$ finite,

$$T_{\mu_1\mu_2}{}^{\alpha_1\alpha_2}(q)$$
$$=\frac{1}{q_0}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle A|[j_{\mu_1}{}^{\alpha_1}(0,\mathbf{x}),j_{\mu_2}{}^{\alpha_2}(0)]|B\rangle$$
$$+\frac{i}{q_0^2}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle A|\left[\frac{\partial}{\partial x_0}j_{\mu_1}{}^{\alpha_1}(x),j_{\mu_2}{}^{\alpha_2}(0)\right]_{x_0=0}|B\rangle$$
$$+O(1/q_0^3). \tag{2.4}$$

This technique (which was first introduced in this connection by Domokos and Karplus[14]) evidently requires the commutator of the two currents to be smooth on the light-cone. It can be shown that if the commutator is not more singular than $\delta(x^2)$ and derivatives of $\delta(x^2)$, Eq. (2.4) is a consequence of local field theory.[15] Instead of using the technique in Eq. (2.3), one can also use the Low equation satisfied by the two-point function.[1] If the spectral functions in the Low equation goes to zero sufficiently fast (faster than $1/q_0$) as $q_0\to\infty$, Eq. (2.4) follows. In the local field theory discussed elsewhere,[15] this is the case if there are no contributions from Schwinger terms (e.g., because $|A\rangle\neq|B\rangle$ and the Schwinger terms are $c$ numbers). In the following we shall therefore assume that Schwinger terms give no contributions. This will be the case in the applications in Secs. 4 and 5 if the Schwinger terms are $c$ numbers. In principle it is possible to take these terms into account,[15] but in practice the resulting formalism is very complicated. With no further excuse we leave out the Schwinger terms, and we also assume that the light-cone singularities are sufficiently smooth for the operations in Eq. (2.3) to be allowed.

It is then immediately obvious that the technique exhibited in Eq. (2.3) can be applied to the $n$-point function (2.1). All we have to do is to apply partial integration consecutively to the $n-1$ integrations in Eq. (2.1). Since we are ignoring Schwinger terms we put

$$\mathbf{q}_1=\mathbf{q}_2=\cdots=\mathbf{q}_{n-1}=0, \tag{2.5}$$

and let all the energies go to infinity. Let us consider the three-point function

$$T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)=-\int\int d^4xd^4y\, e^{iqx+ipy}$$
$$\times\langle A|T(j_{\mu_1}{}^{\alpha_1}(x)j_{\mu_2}{}^{\alpha_2}(y)j_{\mu_3}{}^{\alpha_3}(0))|B\rangle. \tag{2.6}$$

Doing the $x_0$ integration [as in Eq. (2.3)] we get

$$T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)\underset{q_0\to\infty}{\longrightarrow}-\frac{i}{q_0}\int d^4y\int d^3x$$
$$\times\langle A|T([j_{\mu_1}{}^{\alpha_1}(x),j_{\mu_2}{}^{\alpha_2}(y)]_{x_0=y_0}$$
$$\times j_{\mu_3}{}^{\alpha_3}(0))|B\rangle e^{i(q+p)y}-\frac{i}{q_0}\int d^4y\int d^3x\, e^{ipy}$$
$$\times\langle A|T([j_{\mu_1}{}^{\alpha_1}(x),j_{\mu_3}{}^{\alpha_3}(0)]_{x_0=0}j_{\mu_2}{}^{\alpha_2}(y))|B\rangle$$
$$+O(1/q_0^2). \tag{2.7}$$

Now we see that we can perform the $y_0$ integration in the same way if $p_0\neq-q_0$. Assuming this to be the case, we get by use of the ordinary Bjorken expansion (2.4)

$$T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)\to\frac{1}{q_0(q_0+p_0)}\int\int d^3xd^3y\,\langle A|[[j_{\mu_1}{}^{\alpha_1}(x),j_{\mu_2}{}^{\alpha_2}(y)],j_{\mu_3}{}^{\alpha_3}(0)]|B\rangle_{x_0=y_0=0}$$
$$+\frac{1}{p_0q_0}\int\int d^3xd^3y\langle A|[j_{\mu_2}{}^{\alpha_2}(y),[j_{\mu_1}{}^{\alpha_1}(x),j_{\mu_3}{}^{\alpha_3}(0)]]|B\rangle_{x_0=y_0=0}$$
$$+O(1/q_0^2p_0)+O(1/p_0^2q_0),\quad q_0\to\infty,\quad p_0\to\infty,\quad q_0\nrightarrow -p_0. \tag{2.8}$$

[14] G. Domokos and R. Karplus, Phys. Rev. **153**, 1492 (1967).
[15] P. Olesen, Phys. Rev. **172**, 1461 (1968).

We emphasize that this result is correct only if $q_0 \neq -p_0$ asymptotically.[16] In other words, $p$ and $q$ have to be independent variables.

Above we have first performed the $x_0$ integration, then the $y_0$ integration. In order for the method to make sense it is evidently necessary that the order of integration be immaterial (which means that it is immaterial in which order we obtain the limits $p_0 \to \infty$, $q_0 \to \infty$). Since this point is perhaps not quite trivial let us do the $y_0$ integration first,

$$T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p) \xrightarrow[p_0\to\infty]{} -\frac{i}{p_0}\int d^4x\int d^3y\langle A|T([j_{\mu_2}{}^{\alpha_2}(y),j_{\mu_1}{}^{\alpha_1}(x)]_{x_0=y_0}\, j_{\mu_3}{}^{\alpha_3}(0))|B\rangle e^{i(q+p)x} - \frac{i}{p_0}\int d^4x\int d^3y\, e^{iqx}$$

$$\times\langle A|T([j_{\mu_2}{}^{\alpha_2}(y),j_{\mu_3}{}^{\alpha_3}(0)]_{y_0=0}\, j_{\mu_1}{}^{\alpha_1}(x))|B\rangle \xrightarrow[q_0\to\infty,\,p_0\to\infty]{} \frac{1}{p_0(q_0+p_0)}\int\int d^3x\, d^3y$$

$$\times\langle A|[[j_{\mu_2}{}^{\alpha_2}(y),j_{\mu_1}{}^{\alpha_1}(x)],j_{\mu_3}{}^{\alpha_3}(0)]|B\rangle_{x_0=y_0=0} + \frac{1}{p_0q_0}\int\int d^3x\, d^3y\langle A|[j_{\mu_1}{}^{\alpha_1}(x),[j_{\mu_2}{}^{\alpha_2}(y),j_{\mu_3}{}^{\alpha_3}(0)]]|B\rangle_{x_0=y_0=0}. \quad (2.9)$$

Using

$$\frac{1}{p_0(p_0+q_0)} = \frac{-1}{q_0(p_0+q_0)} + \frac{1}{p_0q_0}, \quad (2.10)$$

as well as the Jacobi identity

$$[j_{\mu_1}{}^{\alpha_1}(x),[j_{\mu_2}{}^{\alpha_2}(y),j_{\mu_3}{}^{\alpha_3}(0)]] + [[j_{\mu_2}{}^{\alpha_2}(y),j_{\mu_1}{}^{\alpha_1}(x)],j_{\mu_3}{}^{\alpha_3}(0)] = [j_{\mu_2}{}^{\alpha_2}(y),[j_{\mu_1}{}^{\alpha_1}(x),j_{\mu_3}{}^{\alpha_3}(0)]], \quad (2.11)$$

it is seen that Eq. (2.9) is identical to Eq. (2.8). This can also be shown for the $O(1/q^2p)$ and $O(1/p^2q)$ terms. Hence

$$\lim_{p_0\to\infty}\lim_{q_0\to\infty} T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p) = \lim_{q_0\to\infty}\lim_{p_0\to\infty} T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p), \quad (2.12)$$

provided that $q_0+p_0 \to \infty$. The resulting asymptotic expression for the tensor $T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)$ is, of course, not covariant since the limit is not covariant. Because we have assumed that Schwinger terms are absent, the special values (2.5) of the vectors $\mathbf{q}_k$ can all be replaced by finite values of $\mathbf{q}_k$ without altering the asymptotic values (2.12). In this way we can define a Lorentz frame where all time components are infinite and the space components are finite. The question is then whether it is possible to generalize the tensor (2.12) in a covariant way. In the case of the two-point function such a generalization is known to exist at least if the light-cone commutator is "sufficiently" smooth.[15] Similar arguments can be constructed for the three-point function, but because of the very complicated nature of these arguments we shall only give a recipe for the covariant generalization of the limit (2.12). To see how to proceed, let us return to the first Eq. (2.9) where only the limit $p_0 \to \infty$ has been performed, whereas $q$ is arbitrary. In the time-ordered products the equal-time commutators can be evaluated from current algebra (the quark model or the gauge-field model) and the limit

$$\lim_{p_0\to\infty} T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p) \quad (2.13)$$

contains terms of the type

$$-\frac{i}{p_0}\delta_{\mu_2 0} f^{\alpha_2\alpha_1\alpha_4}\int d^4x\, e^{i(q+p)x}\langle A|T(j_{\mu_1}{}^{\alpha_4}(x)j_{\mu_3}{}^{\alpha_3}(0))|B\rangle,$$

which can be made covariant in only one way, namely, by writing

$$\frac{-i}{p^2}p_{\mu_2}f^{\alpha_2\alpha_1\alpha_4}\int d^4x\, e^{i(q+p)x}\langle A|T(j_{\mu_1}{}^{\alpha_4}(x)j_{\mu_3}{}^{\alpha_3}(0))|B\rangle.$$

Proceeding in this manner with the remaining $q_0+p_0 \to \infty$ limit, the covariant generalization of (2.12) can be obtained. In many practical applications this somewhat cumbersome method is not necessary. For example, in order to estimate divergence or convergence of various quantities, the special limit (2.12) is sufficient (since the order of magnitude is unchanged).

The terms neglected in Eq. (2.8) are of the form

$$\frac{i}{q_0^2(q_0+p_0)}\int\int d^3x\, d^3y \times\langle A|\left[\left[\frac{\partial}{\partial x_0}j_{\mu_1}{}^{\alpha_1}(x),j_{\mu_2}{}^{\alpha_2}(y)\right],j_{\mu_3}{}^{\alpha_3}(0)\right]|B\rangle, \quad x_0=y_0=0. \quad (2.14)$$

These terms therefore involve the triple commutator of the three currents with the Hamiltonian.

It is now obvious how to proceed to obtain the asymptotic behavior of the four-point function. One finds

[16] A formula similar to (2.8) has been derived by M. B. Halpern, Phys. Rev. **163**, 1611 (1967), especially Eq. (8). His result is, however, incorrect since an exponential (which is crucial in many applications) has not been taken into account.

$$T_{\mu_1\cdots\mu_4}{}^{\alpha_1\cdots\alpha_4}(q_1q_2q_3)=i\int\int\int d^4x_1d^4x_2d^4x_3\, e^{iq_1x_1+iq_2x_2+iq_3x_3}\langle A\,|\,T(j_{\mu_1}{}^{\alpha_1}(x_1)j_{\mu_2}{}^{\alpha_2}(x_2)j_{\mu_3}{}^{\alpha_3}(x_3)j_{\mu_4}{}^{\alpha_4}(0))\,|\,B\rangle$$

$$\to\frac{1}{(q_1)_0(q_1+q_2)_0(q_1+q_2+q_3)_0}\int\int\int d^3x_1d^3x_2d^3x_3\langle A\,|\,[[[j_{\mu_1}{}^{\alpha_1}(x_1),j_{\mu_2}{}^{\alpha_2}(x_2)],j_{\mu_3}{}^{\alpha_3}(x_3)],j_{\mu_4}{}^{\alpha_4}(0)]\,|\,B\rangle_{(x_1)_0=(x_2)_0=(x_3)_0=0}$$

$$+\frac{1}{(q_1)_0(q_1+q_2)_0(q_3)_0}\int\int\int d^3x_1d^3x_2d^3x_3\langle A\,|\,[j_{\mu_3}{}^{\alpha_3}(x_3),[[j_{\mu_1}{}^{\alpha_1}(x_1),j_{\mu_2}{}^{\alpha_2}(x_2)],j_{\mu_4}{}^{\alpha_4}(0)]]\,|\,B\rangle_{(x_1)_0=(x_2)_0=(x_3)_0=0}$$

$$+\frac{1}{(q_1)_0(q_2)_0(q_1+q_2+q_3)_0}\int\int\int d^3x_1d^3x_2d^3x_3\langle A\,|\,[[j_{\mu_2}{}^{\alpha_2}(x_2),[j_{\mu_1}{}^{\alpha_1}(x_1),j_{\mu_3}{}^{\alpha_3}(x_3)]],j_{\mu_4}{}^{\alpha_4}(0)]\,|\,B\rangle_{(x_1)_0=(x_2)_0=(x_3)_0=0}$$

$$+\frac{1}{(q_1)_0(q_2)_0(q_1+q_3)_0}\int\int\int d^3x_1d^3x_2d^3x_3\langle A\,|\,[[j_{\mu_1}{}^{\alpha_1}(x_1),j_{\mu_3}{}^{\alpha_3}(x_3)],[j_{\mu_2}{}^{\alpha_2}(x_2),j_{\mu_4}{}^{\alpha_4}(0)]]\,|\,B\rangle_{(x_1)_0=(x_2)_0=(x_3)_0=0}$$

$$+\frac{1}{(q_1)_0(q_2)_0(q_3)_0}\int\int\int d^3x_1d^3x_2d^3x_3\langle A\,|\,[j_{\mu_3}{}^{\alpha_3}(x_3),[j_{\mu_2}{}^{\alpha_2}(x_2),[j_{\mu_1}{}^{\alpha_1}(x_1),j_{\mu_4}{}^{\alpha_4}(0)]]]\,|\,B\rangle_{(x_1)_0=(x_2)_0=(x_3)_0}$$

$$+\frac{1}{(q_1)_0(q_2)_0(q_2+q_3)_0}\int\int\int d^3x_1d^3x_2d^3x_3\langle A\,|\,[[j_{\mu_2}{}^{\alpha_2}(x_2),j_{\mu_3}{}^{\alpha_3}(x_3)],[j_{\mu_1}{}^{\alpha_1}(x_1),j_{\mu_4}{}^{\alpha_4}(0)]]\,|\,B\rangle_{(x_1)_0=(x_2)_0=(x_3)_0=0}, \qquad (2.15)$$

where the (covariant) limit is

$$|q_1^2|\to\infty,\quad |q_2^2|\to\infty,\quad |q_3^2|\to\infty,\quad |(q_1+q_2)^2|\to\infty,$$
$$|(q_1+q_3)^2|\to\infty,\quad |(q_2+q_3)^2|\to\infty,\quad |(q_1+q_2+q_3)^2|\to\infty. \qquad (2.16)$$

Again it can be shown [analogous to Eq. (2.12)] that the order in which the limit (2.16) is performed is immaterial.

It should now be obvious in principle how to find the asymptotic behavior of the $n$-point function. In practice it is, however, cumbersome to find the asymptotic behavior of, e.g., the five-point function, although it is a straightforward task to do the four integrations. We shall not write down the result here. Instead we discuss some general features of the $n$-point function which will turn out to be important in Sec. 3.

Considering the $n$-point function (2.1), it is clear that the result of the $(x_1)_0$ partial integration can be written down without much work. Let us consider the time-ordered product of $n$ operators,

$$\langle A\,|\,T(D_1(x_1)D_2(x_2)\cdots D_{n-1}(x_{n-1})D_n(0))\,|\,B\rangle=T_n(x_1\cdots x_{n-1}). \qquad (2.17)$$

In writing out the time-ordered product all permutations of the $D$'s occur multiplied with the relevant $\theta$ functions. Let us, e.g., consider the case where $D_1(x_1)$ is the first operator,

$$\langle A\,|\,D_1(x_1)D_s(x_s)\cdots\,|\,B\rangle, \qquad (2.18)$$

where all possible permutations of $D_2\cdots D_n$ occur. The term (2.18) has to be multiplied by $\theta$ functions, but $(x_1)_0$ only enters in $\theta(x_1-x_s)$. The $(x_1)_0$ integration is then trivial and gives

$$\int_{(x_s)_0}^{\infty} d(x_1)_0e^{i(q_1)_0(x_1)_0}\langle A\,|\,D_1(x_1)D_s(x_s)D_r(x_r)\cdots\,|\,B\rangle\theta(x_s-x_r)\cdots$$
$$=[i/(q_1)_0]\langle A\,|\,D_1(x_1)D_s(x_s)D_r(x_r)\cdots\,|\,B\rangle_{(x_1)_0=(x_s)_0}e^{i(q_1)_0(x_s)_0}\theta(x_s-x_r)\cdots. \qquad (2.19)$$

Now there will always be a term of the type

$$\langle A\,|\,D_s(x_s)D_1(x_1)D_r(x_r)\cdots\,|\,B\rangle \qquad (2.20)$$

also, and this term is multiplied by $\theta(x_s-x_1)\theta(x_1-x_r)$, i.e., $\theta(x_s-x_r)$ with $(x_s)_0>(x_1)_0>(x_r)_0$. Now this gives two terms,

$$\theta(x_s-x_r)\int_{x_r}^{x_s} d(x_1)_0e^{i(q_1)_0(x_1)_0}\langle A\,|\,D_s(x_s)D_1(x_1)D_r(x_r)\cdots\,|\,B\rangle$$
$$=\frac{\theta(x_s-x_r)}{i(q_1)_0}[\langle A\,|\,D_s(x_s)D_1(x_1)D_r(x_r)\cdots\,|\,B\rangle_{(x_1)_0=(x_s)_0}e^{i(q_1)_0(x_s)_0}$$
$$-\langle A\,|\,D_s(x_s)D_1(x_1)D_r(x_r)\cdots\,|\,B\rangle_{(x_1)_0=(x_r)_0}e^{i(q_1)_0(x_r)_0}] \qquad (2.21)$$

and the first term on the right-hand side combines with the term (2.19) to give

$$[i/(q_1)_0]\langle A|[D_1(x_1),D_s(x_s)]_{(x_1)_0=(x_s)_0}D_r(x_r)\cdots|B\rangle\theta(x_s-x_r)\cdots. \tag{2.22}$$

In a similar way, one can see that if $D_1(x_1)$ is not the first operator, the $(x_1)_0$ integration gives rise to an equal-time commutator. Using the fact that the time-ordered product is symmetric in its variables, we find for the $n$-point function (2.1)

$$T_{\mu_1\cdots\mu_n}{}^{\alpha_1\cdots\alpha_n}(q_1\cdots q_{n-1})\xrightarrow[(q_1)_0\to\infty]{}\frac{i(-i)^{n-1}}{(q_1)_0}\int\cdots\int d^4x_2\cdots d^4x_{n-1}\int d^3x_1e^{iq_2x_2+\cdots+iq_{n-1}x_{n-1}}$$
$$\times\sum_{r=2}^{n}\langle A|T([j_{\mu_1}{}^{\alpha_1}(x_1),j_{\mu_r}{}^{\alpha_r}(x_r)]_{(x_1)_0=(x_r)_0}j_{\mu_2}{}^{\alpha_2}(x_2)\cdots[j]_{\mu_r}{}^{\alpha_r}(x_r)\cdots j_{\mu_n}{}^{\alpha_n}(0))|B\rangle e^{iq_1x_r}, \tag{2.23}$$

where $[j]_{\mu_r}{}^{\alpha_r}(x_r)$ means that the current $j_{\mu_r}{}^{\alpha_r}(x_r)$ does not occur in the product.

Now we can of course carry out the $(x_2)_0$ integration, etc., and it is obvious that we ultimately end up with a (complicated) combination of equal-time commutators [as in the special examples (2.8) and (2.15)]. The author has not succeeded in showing that the order of the limits is immaterial; however, this appears likely since this is so for the three- and four-point functions. The number of terms in the final expression for the $n$-point function can be shown to be

$$(n-1)! \tag{2.24}$$

It is seen that this formula agrees with Eqs. (2.8) and (2.15).

## 3. ASYMPTOTIC CONSERVATION OF CURRENTS AND GENERALIZED WEINBERG SUM RULES

In this section we shall show that in calculating Ward identities one can assume that all currents are conserved as far as the leading asymptotic behavior is concerned.

Again let us consider the two-point function as an illustration,

$$T_{\mu_1\mu_2}{}^{\alpha_1\alpha_2}(q)=-i\int d^4x\,e^{iqx}\langle A|T(j_{\mu_1}{}^{\alpha_1}(x)j_{\mu_2}{}^{\alpha_2}(0))|B\rangle. \tag{3.1}$$

Contracting with respect to $q$, we get

$$q^{\mu_1}T_{\mu_1\mu_2}{}^{\alpha_1\alpha_2}(q)=\int d^4x\,e^{iqx}\delta(x_0)\langle A|[j_0{}^{\alpha_1}(x),j_{\mu_2}{}^{\alpha_2}(0)]|B\rangle -i\int d^4x\,e^{iqx}\langle A|T(\partial^{\mu_1}j_{\mu_1}{}^{\alpha_1}(x)j_{\mu_2}{}^{\alpha_2}(0))|B\rangle. \tag{3.2}$$

Using the theorem[17] that seagull terms [which we have to add to the amplitude (3.1)] cancel Schwinger terms, it follows from the usual current algebra[18] or the algebra of fields[19] that the first term on the right-hand side of Eq. (3.2) is a constant (independent of $q$). Using the Bjorken limit, the second term on the right-hand side of Eq. (3.2) becomes

$$\frac{1}{q_0}\int d^3x\,e^{i\mathbf{q}\cdot\mathbf{x}}\langle A|[\partial^{\mu_1}j_{\mu_1}{}^{\alpha_1}(x),j_{\mu_2}{}^{\alpha_2}(0)]_{x_0=0}|B\rangle. \tag{3.3}$$

Again, since $q^{\mu_1}T_{\mu_1\mu_2}$ is covariant,[17] possible (but unlikely) Schwinger terms in the commutator in Eq. (3.3) combines with seagull terms and drop out. However, it is also rather unlikely that the $O(1/q)$ term contains Schwinger terms. It is then seen that

$$q^{\mu_1}T_{\mu_1\mu_2}{}^{\alpha_1\alpha_2}(q)=\int d^3x\langle A|[j_0{}^{\alpha_1}(x),j_{\mu_2}{}^{\alpha_2}(0)]|B\rangle_{x_0=0}+O(1/q), \tag{3.4}$$

which means that asymptotically the current $j_{\mu_1}{}^{\alpha_1}(x)$ is conserved for all values of $\alpha_1$. It should be emphasized, however, that this is true only for the leading (constant) term. If one is interested also in the $O(1/q)$ term, it is no longer true.

For the $n$-point function we get the Ward identity

$$q_1{}^{\mu_1}T_{\mu_1\cdots\mu_n}{}^{\alpha_1\cdots\alpha_n}=(-i)^{n-2}\int\cdots\int d^4x_1\cdots d^4x_{n-1}e^{iq_2x_2+\cdots+iq_{n-1}x_{n-1}}\sum_{m=2}^{n}\delta(x_{10}-x_{m0})$$
$$\times\langle A|T([j_0{}^{\alpha_1}(x_1),j_{\mu_m}{}^{\alpha_m}(x_m)]j_{\mu_2}{}^{\alpha_2}(x_2)\cdots[j]_{\mu_m}{}^{\alpha_m}(x_m)\cdots j_{\mu_n}{}^{\alpha_n}(0))|B\rangle e^{iq_1x_m}$$
$$+(-i)^{n-1}\int\cdots\int d^4x_1\cdots d^4x_{n-1}e^{iq_1x_1+\cdots+iq_{n-1}x_{n-1}}\langle A|T(\partial^{\mu_1}j_{\mu_1}{}^{\alpha_1}(x_1)j_{\mu_2}{}^{\alpha_2}(x_2)\cdots j_{\mu_n}{}^{\alpha_n}(x_n))|B\rangle. \tag{3.5}$$

[17] L. S. Brown, Phys. Rev. **150**, 1338 (1966); D. G. Boulware and L. S. Brown, *ibid.* **156**, 1724 (1967); R. P. Feynman, in *Proceedings of the 1967 International Conference on Particles and Fields* (Interscience Publishers, Inc., New York, 1967), p. 111.

[18] M. Gell-Mann, Physics **1**, 63 (1964).

[19] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967); T. D. Lee and B. Zumino, Phys. Rev. **163**, 1667 (1967).

Here we assume that the Feynman theorem[17] can be generalized so that seagull terms again cancel Schwinger terms, so that the left-hand side of Eq. (3.5) is covariant. That such a generalization is probably true follows from the fact that this theorem is based on very general arguments.[17] Using Eq. (2.23), the last term on the right-hand side of Eq. (3.5) becomes

$$\frac{i(-i)^{n-1}}{(q_1)_0}\int\cdots\int d^4x_1\cdots d^4x_{n-1}e^{iq_2x_2+\cdots+iq_{n-1}x_{n-1}}$$
$$\times\sum_{m=2}^{n}\delta(x_{10}-x_{m0})\langle A|T([\partial^{\mu_1}j_{\mu_1}{}^{\alpha_1}(x_1),j_{\mu_m}{}^{\alpha_m}(x_m)]$$
$$\times j_{\mu_2}{}^{\alpha_2}(x_2)\cdots[j]_{\mu_m}{}^{\alpha_m}(x_m)\cdots j_{\mu_n}{}^{\alpha_n}(0))|B\rangle$$
$$\times e^{iq_1x_m}. \quad (3.6)$$

Now, by letting all the energies go to infinity the integral (3.6) can be expressed in terms of equal-time commutators, and the integral in (3.6) behaves in the same way as a function of the $q$'s as the first term on the right-hand side of Eq. (3.5) in the same limit (the coefficients of the $q$-dependent terms are of course different). Therefore

$$q_1{}^{\mu_1}T_{\mu_1\cdots\mu_n}{}^{\alpha_1\cdots\alpha_n}(q_1\cdots q_n)$$
$$=q_1{}^{\mu_1}[T_{\mu_1\cdots\mu_n}{}^{\alpha_1\cdots\alpha_n}(q_1\cdots q_n)]$$
$$\text{with } j_{\mu_1}{}^{\alpha_1} \text{ conserved}, \quad (3.7)$$

in the asymptotic limit

$$(q_1)_0\to\infty,\quad (q_2)_0\to\infty,\ \cdots,\ (q_{n-1})_0\to\infty,$$
$$(q_l)_0\not\to-(q_m)_0, \quad (3.8)$$

if all the $q$'s are *independent* (this condition is essential). We can obtain further Ward identities by contracting with respect to other $q$'s. Again the result is

$$q_1{}^{\mu_1}\cdots q_r{}^{\mu_r}T_{\mu_1\cdots\mu_n}{}^{\alpha_1\cdots\alpha_n}(q_1\cdots q_n)$$
$$\to q_1{}^{\mu_1}\cdots q_r{}^{\mu_r}[T_{\mu_1\cdots\mu_n}{}^{\alpha_1\cdots\alpha_n}(q_1\cdots q_n)]$$
$$\text{with } j_{\mu_1}{}^{\alpha_1}\cdots j_{\mu_r}{}^{\alpha_r} \text{ conserved}, \quad (3.9)$$

in the same asymptotic limit (3.8). The right-hand side of Eq. (3.9) gives the leading asymptotic behavior.

Hence, in obtaining all possible Ward identities from the $n$-point function one can assume that all currents are conserved as far as the leading asymptotic behavior is concerned. This result has some implications in connection with a recent work by Schnitzer and Weinberg[20] (using Ward identities for the three-point function), but we shall not discuss this point further in this paper.

It is also possible to obtain results which generalize the first Weinberg sum rule.[4] Let us illustrate this by considering the two-point functions.[5] For definiteness let us consider two states $|A\rangle$ and $|B\rangle$ with the same parity, and define the two-point functions

$$A_{\mu\nu}{}^{\alpha\beta}(q)=-i\int d^4x\, e^{iqx}\langle A|T(A_\mu{}^\alpha(x),A_\nu{}^\beta(0))|B\rangle,$$
$$V_{\mu\nu}{}^{\alpha\beta}(q)=-i\int d^4x\, e^{iqx}\langle A|T(V_\mu{}^\alpha(x),V_\nu{}^\beta(0))|B\rangle, \quad (3.10)$$

where $A_\mu{}^\alpha(x)$ is the axial-vector current and $V_\mu{}^\alpha(x)$ is the vector current. Using the Bjorken limit and the assumption

$$[A_\mu{}^\alpha(x),A_\nu{}^\beta(0)]_{x_0=0}=[V_\mu{}^\alpha(x),V_\nu{}^\beta(0)]_{x_0=0}, \quad (3.11)$$

it follows that

$$\lim_{|q^2|\to\infty}[A_{\mu\nu}{}^{\alpha\beta}(q)-V_{\mu\nu}{}^{\alpha\beta}(q)]=0 \quad [\text{as } O(1/q)]. \quad (3.12)$$

The assumption (3.11) is certainly valid in the current algebra[18] (in the sense of the quark model) and the algebra of fields.[19] In using the Bjorken limit it is not necessary to assume that Schwinger terms are absent. If these terms are present $A_{\mu\nu}{}^{\alpha\beta}$ and $V_{\mu\nu}{}^{\alpha\beta}$ are not covariant, but if Eq. (3.11) is satisfied it still follows from the Bjorken limit that Eq. (3.12) is valid. Alternatively, one can define covariant amplitudes (3.10), and in this case Eq. (3.12) is also valid because the seagull terms [which we have to add to (3.10) in constructing a covariant amplitude] combine with the Schwinger terms in such a way that the asymptotic behavior of the covariant amplitudes is a constant (instead of $1/q$), as has been shown by the author.[15] But if Eq. (3.11) is satisfied, the Schwinger terms are identical for the vector–vector and the axial-vector–axial-vector commutators, and once again we arrive at Eq. (3.12). Hence in all cases Eq. (3.12) is a consequence of Eq. (3.11).

If the states $|A\rangle$ and $|B\rangle$ are identical to the vacuum state, Eq. (3.12) gives the first Weinberg sum rule.[4] The fact that the first Weinberg sum rule is a consequence of Eq. (3.11) is well known.[5] In this case we can state our result by saying[6] that Eq. (3.11) implies asymptotic $SU(3)\otimes SU(3)$. However, we want to emphasize that the consequences of Eq. (3.11) are much more general than the first Weinberg sum rule, since $|A\rangle$ and $|B\rangle$ can be arbitrary states with the same parity.

We now explore the situation for the $n$-point function. Since we found in Sec. 2 that by an appropriate definition of the asymptotic limit it is possible to express the $n$-point function in terms of $n-1$ equal-time commutators in the asymptotic limit, it is obviously sufficient to consider the three-point function. The results can be extended quite trivially to the $n$-point function. Let us this time consider two states $|A\rangle$ and $|B\rangle$ with different parity and define

[20] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).

$$T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)^{AAA}=-\int\int d^4x d^4y\, e^{iqx+ipy}\langle A|T(A_{\mu_1}{}^{\alpha_1}(x)A_{\mu_2}{}^{\alpha_2}(y)A_{\mu_3}{}^{\alpha_3}(0))|B\rangle,$$

$$T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)^{VVA}=-\int\int d^4x d^4y\, e^{iqx+ipy}\langle A|T(V_{\mu_1}{}^{\alpha_1}(x)V_{\mu_2}{}^{\alpha_2}(y)A_{\mu_3}{}^{\alpha_3}(0))|B\rangle,$$

$$T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)^{VAV}=-\int\int d^4x d^4y\, e^{iqx+ipy}\langle A|T(V_{\mu_1}{}^{\alpha_1}(x)A_{\mu_2}{}^{\alpha_2}(y)V_{\mu_3}{}^{\alpha_3}(0))|B\rangle, \tag{3.13}$$

$$T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)^{AVV}=-\int\int d^4x d^4y\, e^{iqx+ipy}\langle A|T(A_{\mu_1}{}^{\alpha_1}(x)V_{\mu_2}{}^{\alpha_2}(y)V_{\mu_3}{}^{\alpha_3}(0))|B\rangle.$$

Using now the asymptotic limit (2.8) for each of the amplitudes (3.13) one can verify that in the algebra of fields,[19] where

$$[j_0{}^\alpha(x),j_\nu{}^\beta(0)]_{x_0=0}=if^{\alpha\beta\gamma}j_\nu{}^\gamma(0)\delta^3(x)+c\delta^{\alpha\beta}(\partial/\partial x_k)\delta^3(x)g_{\nu k}, \tag{3.14}$$
$$[j_k{}^\alpha(x),j_s{}^\beta(0)]_{x_0=0}=0,$$

where $c$ is a (finite) $c$ number, we get

$$T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)^{AAA}\to T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)^{VVA}\to T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)^{VAV}\to T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(q,p)^{AVV}$$

for

$$|q_0|\to\infty,\quad |p_0|\to\infty,\quad |q_0+p_0|\to\infty. \tag{3.15}$$

In the current algebra we cannot make a similar statement since the Schwinger terms are not known to be $c$ numbers. However, if we assume that Schwinger terms are $c$ numbers, it again follows that Eq. (3.15) is valid in the (quark-model) current algebra. If one wants to formulate quite generally what the condition for the validity of (3.15) is, it turns out that one has to require Eq. (3.11) as well as

$$[A_{\mu_1}{}^{\alpha_1}(x),[A_{\mu_2}{}^{\alpha_2}(y),A_{\mu_3}{}^{\alpha_3}(0)]]_{x_0=y_0=0}=[V_{\mu_1}{}^{\alpha_1}(x),[V_{\mu_2}{}^{\alpha_2}(y),A_{\mu_3}{}^{\alpha_3}(0)]]_{x_0=y_0=0}=[A_{\mu_1}{}^{\alpha_1}(x),[V_{\mu_2}{}^{\alpha_2}(y),V_{\mu_3}{}^{\alpha_3}(0)]]_{x_0=y_0=0}. \tag{3.16}$$

This condition is satisfied in the algebra of fields and (with the assumption about $c$-number Schwinger terms) in the current algebra.

Equation (3.15) shows that we have generalized Weinberg sum rules also for the three-point function. If $|A\rangle$ and $|B\rangle$ are vacuum states it also follows that asymptotic $SU(3)\otimes SU(3)$ is valid for the three-point function. All these results can be trivially extended to the $n$-point function.

Finally we mention that any three-point function satisfies an asymptotic condition which is independent of dynamics (it is, in particular, independent of current algebra). Using Eq. (2.8) and the identity

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0,$$

we get

$$\lim_{|q^2|\to\infty,\ |p^2|\to\infty,\ |(q+p)^2|\to\infty}\{T_{\mu_1\mu_2\mu_3}{}^{\alpha_1\alpha_2\alpha_3}(p,q)+T_{\mu_2\mu_3\mu_1}{}^{\alpha_2\alpha_3\alpha_1}(p,q)+T_{\mu_3\mu_1\mu_2}{}^{\alpha_3\alpha_1\alpha_2}(p,q)\}=0. \tag{3.17}$$

This is in general only true in the leading order. Equation (3.17) can also be shown directly from translational invariance and by using the fact that asymptotically the momenta $p$ and $q$ are much larger than the momenta of the states $|A\rangle$ and $|B\rangle$.

In the arguments above we have only made a first-order asymptotic expansion, and the sum rules (3.15) are therefore only valid to the leading order. If one makes further asymptotic expansions, it is necessary in the case of the two-point function that

$$\left[\frac{\partial}{\partial x_0}A_\mu{}^\alpha(x),A_\nu{}^\beta(0)\right]_{x_0=0}=\left[\frac{\partial}{\partial x_0}V_\mu{}^\alpha(x),V_\nu{}^\beta(0)\right]_{x_0=0} \tag{3.18}$$

in order for Eq. (3.12) to be correct. In the algebra of fields, Eq. (3.16) is correct for $\mu=k$, $\nu=s$. However, for $\mu=0$ Eq. (3.18) is not correct, e.g., if $\alpha=1,2,3$, since in this case the left-hand side of Eq. (3.18) does not vanish whereas the right-hand side does vanish. Therefore we do not believe that the results (3.12) and (3.15) are valid to second order in the asymptotic limit. This means, e.g., that we cannot obtain the second Weinberg sum rule[4] by this method (this is only possible if the $\sigma$ terms vanish).

## 4. $K_L{}^0$-$K_S{}^0$ MASS DIFFERENCE

In this section we shall discuss the $K_L{}^0$-$K_S{}^0$ mass difference in order to learn something about the weak-interaction cutoff $\Lambda$. The mass difference is given by[21]

$$M^2(K_L{}^0)-M^2(K_s{}^0)=-(2\pi)^3\,\mathrm{Re}\,i\int d^4x\langle K^0|T(H_w(x)H_w(0))|\bar{K}^0\rangle, \tag{4.1}$$

[21] V. Barger and E. Kazes, Nuovo Cimento 28, 394 (1963).

where the relevant part of the weak Hamiltonian is

$$H_w(x)=(G/2\sqrt{2})\sin\theta\cos\theta[\{V_{\mu 2}{}^1(x),V^{\mu}{}_1{}^3(x)\}+\{A_{\mu 2}{}^1(x),A^{\mu}{}_1{}^3(x)\}]+(G/2\sqrt{2})\sin\theta\cos\theta\times[\{V_{\mu 2}{}^1(x),A^{\mu}{}_1{}^3(x)\}+\{V_{\mu 1}{}^3(x),A^{\mu}{}_2{}^1(x)\}]+\text{H.c.},\quad(4.2)$$

in the usual tensor notation.[7] In the universal Fermi interaction we are thus led to consider integrals of the type

$$I=\tfrac{1}{2}\int d^4x\langle K^0|T(\{j_\mu{}^1(x),j^{\mu 2}(x)\}j_\nu{}^3(0)j^{\nu 4}(0))|\bar{K}^0\rangle,\quad(4.3)$$

where $j_\mu{}^1\cdots j_\mu{}^4$ are four currents distinguished by the indices $1\cdots 4$, and where we have used a symmetrized ($PC$-invariant) Hamiltonian. It turns out that symmetrization of $j_\nu{}^3(0)j^{\nu 4}(0)$ does not have any effect in our method so we leave it out.

The integral (4.3) is extremely singular because the theory of weak interactions is nonrenormalizable. It is well known (see, e.g., Ref. 22) that in nonrenormalizable theories the time-ordered product is not well defined, because in the neighborhood of $x_0=0$ the commutator is extremely singular so that multiplication with $\theta(x_0)$ becomes a rather meaningless operation. In renormalizable theories the singularities are not so strong[22] and the time-ordered product has therefore more meaning in these theories. In the case of the integral (4.3), one can easily see that a straightforward attack on (4.3) leads to ambiguities in the form of contact terms, so the $\theta$-multiplication problem in nonrenormalizable theories is not only a question of principles. In the following we shall present a method which gives the time-ordered product a more well-defined meaning than Eq. (4.3) (in the sense that contact terms do not occur).

Let us consider the quantity

$$T^{12}(x,x')=T(j_\mu{}^1(x)j^{\mu 2}(x')j_\nu{}^3(0)j^{\nu 4}(0))-\theta(x_0')\theta(x_0-x_0')[j_\mu{}^1(x),j^{\mu 2}(x')]j_\nu{}^3(0)j^{\nu 4}(0)-\theta(-x_0')\theta(x_0-x_0')j_\nu{}^3(0)j^{\nu 4}(0)[j_\mu{}^1(x),j^{\mu 2}(x')].\quad(4.4)$$

By writing out the definition of the time-ordered product in terms of $\theta$ functions and adopting the definitions (which define how we are going to treat the $\theta$-multiplication problem)

$$\lim_{\epsilon\to 0+}\theta(\epsilon)=1,\quad\lim_{\epsilon\to 0+}\theta(-\epsilon)=0,\quad(4.5)$$

we obtain

$$\lim_{x_0-x_0'\to+0}T^{12}(x,x')=\theta(x_0)j_\mu{}^1(x)j^{\mu 2}(x')j_\nu{}^3(0)j^{\nu 4}(0)+\theta(-x_0)j_\nu{}^3(0)j^{\nu 4}(0)j_\mu{}^1(x)j^{\mu 2}(x')-\theta(x_0)[j_\mu{}^1(x),j^{\mu 2}(x')]j_\nu{}^3(0)j^{\nu 4}(0)-\theta(-x_0)j_\nu{}^3(0)j^{\nu 4}(0)[j_\mu{}^1(x),j^{\mu 2}(x')]$$
$$=\theta(x_0)j_\mu{}^2(x')j^{\mu 1}(x)j_\nu{}^3(0)j^{\nu 4}(0)+\theta(-x_0)j_\nu{}^3(0)j^{\nu 4}(0)j_\mu{}^2(x')j^{\mu 1}(x)\quad(4.6)$$

and

$$\lim_{x_0-x_0'\to-0}T^{12}(x,x')=\theta(x_0)j_\mu{}^2(x')j^{\mu 1}(x)j_\nu{}^3(0)j^{\nu 4}(0)+\theta(-x_0)j_\nu{}^3(0)j^{\nu 4}(0)j_\mu{}^2(x')j^{\mu 1}(x).\quad(4.7)$$

Thus, by the definition (4.4) we have achieved that $T^{12}(x,x')$ has the same value in the limits $x_0-x_0'\to 0$. The addition of the terms containing the retarded propagator in Eq. (4.4) is absolutely necessary in order to achieve this. Defining $T^{21}(x,x')$ as the same as $T^{12}(x,x')$ with $j_\mu{}^1(x)\to j_\mu{}^2(x)$ and $j_\mu{}^2(x')\to j_\mu{}^1(x')$, it is seen that Eq. (4.3) can be rewritten in the form

$$I=\tfrac{1}{2}\int\int d^4xd^4x'\delta^4(x-x')[T^{12}(x,x')+T^{21}(x,x')].\quad(4.8)$$

Introducing Fourier transforms,

$$T^{12}(x,x')=\frac{1}{(2\pi)^8}\int\int d^4qd^4q'e^{iqx+iq'x'}\Delta^{12}(q,q'),$$
$$\Delta^{12}(q,q')=\int\int d^4xd^4x'e^{-iqx-iq'x'}T^{12}(x,x'),\quad(4.9)$$

we see that Eq. (4.8) becomes

$$I=\frac{1}{2(2\pi)^4}\int d^4q[\Delta^{12}(q,-q)+\Delta^{21}(q,-q)].\quad(4.10)$$

Thus, the problem of extracting the most divergent part of the $K_L{}^0$-$K_S{}^0$ mass difference, is equivalent to studying $\Delta(q,-q)$ for large values of $|q^2|$. In view of the definitions (4.4), this problem can be attacked by use of the technique developed in Sec. 2. At the same time we shall show (asymptotically, at least) that Eq. (4.10) is regular and does not give rise to contact terms.

[22] K. Bardakci and B. Schroer, J. Math. Phys. 7, 10 (1966); 7, 16 (1966).

Using Eq. (2.7), we get to order $1/q_0^2$

$$\Delta^{12}(q,-q)\underset{q_0\to\infty}{\longrightarrow}-\frac{i}{q^0}\int d^4x'\int d^3x\langle K^0|T([j_\mu{}^1(x),j^{\mu 2}(x')]_{x_0=x_0'}j_\nu{}^3(0)j^{\nu 4}(0))|\bar{K}^0\rangle$$
$$-\frac{1}{q_0^2}\int d^4x'\int d^3x\langle K^0|T\left(\left[\frac{\partial}{\partial x_0}j_\mu{}^1(x),j^{\mu 2}(x')\right]_{x_0=x_0'}j_\nu{}^3(0)j^{\nu 4}(0)\right)|\bar{K}^0\rangle$$
$$-\frac{i}{q_0}\int d^4x'\int d^3x\, e^{iqx'}\langle K^0|T([j_\mu{}^1(x),j_\nu{}^3(0)j^{\nu 4}(0)]_{x_0=0}j^{\mu 2}(x'))|\bar{K}^0\rangle$$
$$-\int\int d^4xd^4x'e^{-iqx+iqx'}\{\theta(x_0')\theta(x_0-x_0')\langle K^0|[j_\mu{}^1(x),j^{\mu 2}(x')]j_\nu{}^3(0)j^{\nu 4}(0)|\bar{K}^0\rangle$$
$$+\theta(-x_0')\theta(x_0-x_0')\langle K^0|j_\nu{}^3(0)j^{\nu 4}(0)[j_\mu{}^1(x),j^{\mu 2}(x')]|\bar{K}^0\rangle\}. \quad (4.11)$$

In the last term we can perform an ordinary Bjorken expansion by doing the $x_0$ integration (the retarded commutator give the same result as the time-ordered product in the Bjorken limit[1,15]) and this term then gives asymptotically

$$\frac{i}{q_0}\int d^4x'\int d^3x\{\theta(x_0')\langle K^0|[j_\mu{}^1(x),j^{\mu 2}(x')]_{x_0=x_0'}j_\nu{}^3(0)j^{\nu 4}(0)|\bar{K}^0\rangle$$
$$+\theta(-x_0')\langle K^0|j_\nu{}^3(0)j^{\nu 4}(0)[j_\mu{}^1(x),j^{\mu 2}(x')]_{x_0=x_0'}|\bar{K}^0\rangle\}$$
$$+\frac{1}{q_0^2}\int d^4x'\int d^3x\left\{\theta(x_0')\langle K^0|\left[\frac{\partial}{\partial x_0}j_\mu{}^1(x),j^{\mu 2}(x')\right]_{x_0=x_0'}j_\nu{}^3(0)j^{\nu 4}(0)|\bar{K}^0\rangle\right.$$
$$\left.+\theta(-x_0')\langle K^0|j_\nu{}^3(0)j^{\nu 4}(0)\left[\frac{\partial}{\partial x_0}j_\mu{}^1(x),j^{\mu 2}(x')\right]_{x_0=x_0'}|\bar{K}^0\rangle\right\}. \quad (4.12)$$

It is seen that to order $1/q^2$ the terms (4.12) exactly cancel the two first terms in Eq. (4.11). Hence we are left with the third term in Eq. (4.11), which by a further expansion gives

$$\Delta^{12}(q,-q)\to-\frac{1}{q_0^2}\int\int_{x_0=x_0'=0} d^3xd^3x'\langle K^0|[[j_\mu{}^1(x),j_\nu{}^3(0)j^{\nu 4}(0)],j^{\mu 2}(x')]|\bar{K}^0\rangle. \quad (4.13)$$

By expanding the commutator and adding $\Delta^{21}$, we get (we also use locality for equal-time commutators and ignore Schwinger terms)

$$\Delta^{12}(q,-q)+\Delta^{21}(q,-q)\underset{q_0\to\infty}{\longrightarrow}\frac{2}{q_0^2}\int\int d^3xd^3x'\{\langle K^0|[j_\mu{}^1(x),j_\nu{}^3(0)][j^{\mu 2}(x'),j^{\nu 4}(0)]|\bar{K}^0\rangle$$
$$+\langle K^0|[j_\mu{}^2(x),j_\nu{}^3(0)][j^{\mu 1}(x'),j^{\nu 4}(0)]|\bar{K}^0\rangle\}_{x_0=x_0'=0}$$
$$-\frac{1}{q_0^2}\int\int d^3xd^3x'\{\langle K^0|j_\nu{}^3(0)[[j_\mu{}^1(x),j^{\nu 4}(0)],j^{\mu 2}(x')]|\bar{K}^0\rangle+\langle K^0|j_\nu{}^3(0)[[j_\mu{}^2(x),j^{\nu 4}(0)],j^{\mu 1}(x')]|\bar{K}^0\rangle$$
$$+\langle K^0|[[j_\mu{}^1(x),j_\nu{}^3(0)],j^{\mu 2}(x')]j^{\nu 4}(0)|\bar{K}^0\rangle+\langle K^0|[[j_\mu{}^2(x),j_\nu{}^3(0)],j^{\mu 1}(x')]j^{\nu 4}(0)|\bar{K}^0\rangle\}_{x_0'=x_0=0}. \quad (4.14)$$

It is seen that this expression does not contain and contact terms of the form

$$[j_1(x),j_2(x)], \quad (4.15)$$

etc. The reason for this is the following. Let us consider the contribution of the terms containing the retarded propagators (4.4) to the mass difference. Calculating this contribution to the integral

$$\int d^4q\Delta^{12}(q,-q),$$

we find

$$\int d^4x\{\theta(x_0)\langle K^0|[j_\mu{}^1(x),j^{\mu 2}(x)]j_\nu{}^3(0)j^{\nu 4}(0)|\bar{K}^0\rangle$$
$$+\theta(-x_0)\langle K^0|j_\nu{}^3(0)j^{\nu 4}(0)[j_\mu{}^1(x),j^{\mu 2}(x)]|\bar{K}^0\rangle\}, \quad (4.16)$$

which is a contact term. As we saw in Eqs. (4.11) and (4.12), these contact terms are exactly cancelled by contact terms coming from the time-ordered product in the definition (4.4) of $T^{12}(x,x')$. Hence the net result is that the contact terms never contribute. This has of course only been shown to order $1/q^2$.

Inserting the expression (4.14) into Eq. (4.10) we see that the mass difference is quadratically divergent. This result is based on the expansion in Sec. 2 and the assumption that Schwinger terms do not enter.[1,15] No other assumptions have been made. It is very unlikely that Schwinger terms enter. If they are $c$ numbers with the same structure as in the algebra of fields,[19] it can be shown that these terms do not enter.

Using the equal-time commutators in the algebra of fields,[19] it can be seen that Eq. (4.14) reduces to the matrix elements of two currents. Hence, to calculate the numerical coefficient of the quadratic divergency it is necessary to saturate. Saturating with the lowest possible state, namely, the vacuum state, and using the explicit form of the Hamiltonian (4.2), we get

$$M^2(K_L{}^0)-M^2(K_s{}^0)$$
$$=\frac{2.5\sin^2\theta\cos^2\theta}{16\pi^2}G^2\Lambda^2F_K{}^2M_K{}^2, \quad (4.17)$$

which gives $\Lambda\cong 4$ BeV. This is a very nice result in the sense that $\Lambda$ is far below the unitarity limit. In Sec. 5 we shall discuss the physical importance of this.

Recently Mohapatra, Rao, and Marshak[8] have calculated the $K_L{}^0$-$K_S{}^0$ mass difference using a generalized Tamm-Dancoff approximation.[9,10] The result is the same as Eq. (4.17). In Sec. 7 we shall show why the Tamm-Dancoff method gives the same result as the asymptotic expansion if one keeps only the vacuum state.

In general, Eq. (4.14) leads to the expression

$$M^2(K_L{}^0)-M^2(K_S{}^0)=(G/\sqrt{2})^2\sin^2\theta\cos^2\theta$$
$$\times(\Lambda^2/16\pi^2)\langle K^0|\{j_\mu{}^{K^0}(0),j^{\mu K^0}(0)\}|\bar{K}^0\rangle, \quad (4.18)$$

where $j_\mu{}^{K^0}(x)$ is the neutral isotopic partner of the current. We shall not attempt any estimate of contributions from other states than the vacuum state. It is easy to see, however, that the matrix element on the right-hand side of Eq. (4.18) has a behavior which depends very much on the dynamics of the strong interactions. Assuming a dipole form factor for the vertex

$$\langle K^0|j_\mu{}^{K^0}(0)|\pi^0\rangle, \quad (4.19)$$

the matrix element in Eq. (4.18) is convergent, whereas it diverges if one assumes a single pole form factor for the vertex (4.19).

We shall not discuss the evaluation of the mass difference in the vector-boson theory. However, we mention that the theorem on "asymptotic conservation" of currents discussed in Sec. 3 cannot be applied in this case since the relevant four-vectors are not independent.[23]

## 5. WEAK-INTERACTION CUTOFF AND HIGHER-ORDER WEAK INTERACTIONS

The main result of Sec. 4, namely, the indication that $\Lambda$ is a "small" quantity (in the sense that it is much smaller than the unitarity limit 300 BeV), is very interesting from a practical point of view. The order of magnitude of the mass difference is then (we disregard the depression coming from $\sin^2\theta$)

$$M^2(K_L{}^0)-M^2(K_S{}^0)\sim G(G\Lambda^2) \quad (5.1)$$

and

$$G\approx 10^{-5}/M_n{}^2, \quad (5.2)$$

$$G'\equiv G\Lambda^2\approx 10^{-4}. \quad (5.3)$$

In this section we would like to give arguments which support (but which do certainly not prove) that as far as the leading (mathematically speaking, divergent) order is concerned one can view the perturbation series as an expansion of the type $G(G')^{n-1}$, involving two "small" coupling constants. Taking this for granted, we can then estimate where this asymptotic series (mathematically valid only for $G=0$) begins to blow up, and we find that this happens for $n\sim 10^4$.

Let us first try to apply simple power-counting arguments to the weak (universal Fermi theory) interaction. We include strong interactions to all orders but completely ignore the electromagnetic interactions. If we include the latter our conclusions are not valid, and one can find definite counter-examples. In this connection we also mention the work by Lee and Yang.[24]

We divide the weak interactions into three classes. The first class consists of the processes involving only leptons as external particles. In this case the power-counting argument has been used by Feinberg and Pais[25] for the $W$ theory. They find that for diagrams containing uncrossed ladders the leading term is of the form

$$G(G\Lambda^2)^{2n-1} \quad\text{or}\quad G(G\Lambda^2)^{2n+1}, \quad (5.4)$$

[23] If one calculates the $K_L{}^0-K_S{}^0$ mass difference in the intermediate $W$-boson theory, a genuine four-point function is involved. We found that not much can be done since the calculation depends on commutators like

$$[(\partial/\partial x_0)J_{\mu 1}{}^3(x),J_{\nu 2}{}^1(y)]_{x_0=y_0},\ [(\partial/\partial x_0)J_{\lambda 1}{}^3(z),(\partial/\partial y_0)J_{\sigma 2}{}^1(y)]_{x_0=y_0},$$

and even in the algebra of fields this leads to matrix elements which we are not able to evaluate. Thus, in the $W$ case we are not able to find support for the calculations in Ref. 8.

[24] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962); T. D. Lee, *ibid.* **128**, 899 (1962).

[25] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963).

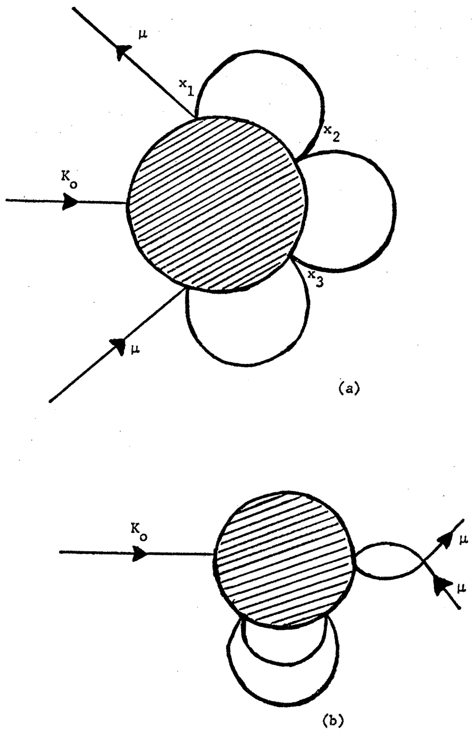


FIG. 1. Typical fourth-order diagrams for a semileptonic process.

the last result being true if the lowest-order process is forbidden. We have translated the results of Feinberg and Pais from the $W$ theory to the universal Fermi interaction, which can be done either by repeating the arguments of Ref. 25 for the latter case or by letting $m_W \to \infty$ in a suitable way. The only complications in this argument come if, e.g., a virtual lepton pair annihilates and then undergoes a semileptonic process, creating, e.g., a neutral pion, which undergoes some complicated process (involving strong interactions) and finally decays to a virtual lepton pair. This type of intermediate process was not considered in Ref. 25. The matrix element for processes of this type involves the vacuum-expectation value of the time-ordered product of a certain number of weak currents. From Sec. 2 we know that this expectation value asymptotically can be expressed in terms of equal-time commutators, and current algebra allows us to reduce these commutators to a sum of various currents. Thus the leading order in an asymptotic expansion vanishes (this is true in the algebra of fields at least, since the space-space commutators vanish), and it then follows that diagrams of this type give rise to less divergent contributions than (5.4).

The second class of processes are the semileptonic processes where at least one of the external particles is not a lepton (we include also $\beta$ decay in this class), but where not all external particles are strongly interaction particles.[26] This class consist of diagrams where the external leptons originate from the strong-intera tion bubble [Fig. 1(a)] as well as diagrams where c e or more external lepton pairs originate from p rely leptonic processes [Fig. 1(b)]. Clearly, throug the Feinberg-Pais argument[25] diagrams of type 1(l can easily be taken care of once we know the behavior of the diagrams of type 1(a). These diagrams are of the type [we consider for definiteness Fig. 1(a)]

$$\int\int\int d^4x_1 d^4x_2 d^4x_3 \times \langle K^0 | T(j_{\mu 1}(x_1) j_{\mu 2}(x_2) j_{\mu 3}(x_3) j_{\mu 4}(0)) | 0 \rangle \times e^{ik_1x_1+ik_2x_2+ik_3x_3}, \quad (5.5)$$

with an asymptotic behavior of the order $1/k^3$. We furthermore have three propagators which behave like $1/k^3$. We have three virtual momentum integrations, so a simple power-counting argument gives the result that the diagram behaves like $G^4\Lambda^6 = G(G\Lambda^2)^3$ in accordance with the conjecture. Figure 1(b) behaves in the same way. The arguments can of course trivially be extended to $n$th order.[27] In the $S$-matrix expansion there is a factor $1/n!$, but it is well known that it is cancelled by the fact that there are $n!$ different distributions of the points $x_1 \cdots x_n$. Hence, according to Eq. (2.24) there are in general $(n-1)!$ terms [some of them can of course vanish, but $(n-1)!$ is an upper limit on the number of terms] in the asymptotic expansion of the Fourier transform of the time-ordered product of $n$ currents. From current algebra each of these terms is of order 1 [at least when compared to $(n-1)!$] and in diagrams of type 1(b) there are $(n-2)!$, $(n-3)!$, etc., terms, so altogether $n$ should be an upper limit on the sum of terms in the $n$th order. Hence, as a rough numerical estimate of the $n$th-order perturbation theory, we have

$$n!G(G\Lambda^2)^{n-1} \sim n^n G(G\Lambda^2)^{n-1}. \quad (5.6)$$

To get a dimensionless quantity we multiply by $M_N^2$, and ask the question: Suppose (5.6) is an asymptotic expansion, when does the perturbation series start to blow up? Taking the values (5.2) and (5.3) and estimating the "blow-up number" from

$$n^n G(G\Lambda^2)^{n-1} \sim \tfrac{1}{10}, \quad (5.7)$$

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[26] The third class consists of only strongly interacting external particles. This class has been discussed by M. B. Halpern, Phys. Rev. **163**, 1611 (1967), and his arguments confirm our hypothesis for the third class also. We shall therefore not discuss this class further.

[27] In estimating the behavior of the $n$th order we have ignored the possibility that Schwinger terms can contribute. If they contribute, the power series would become more divergent (see Refs. 1 and 15), but we could still introduce a "coupling constant" $G\Lambda^2$. The "blow-up number" would, however, be reduced considerably. In some cases Schwinger terms can contribute (even if they are $c$ numbers), and in these cases perturbation theory would not behave so nicely.

we get $n \sim 10^4$! Now this number may of course in some cases be too high,[27] but at least we can take it as an indication that with a cutoff $\Lambda \approx 3$ BeV we do not have to worry about the perturbation series from the point of view of numbers, since the first couple of hundred orders are very small.

From the point of view of principles, the behavior (5.6) is of course not very satisfactory. The nonrenormalizable character of the weak interactions is obvious since we cannot even renormalize the leading term (5.6). In $n$th order one can of course introduce a renormalized coupling constant $G^n\Lambda^{2n-2} \equiv G_{\rm ren}$. But in $(n+1)$th order one has to introduce another renormalized coupling constant, etc., so the number of renormalized constants depends on the order. The most economical way to proceed is to define the two "coupling constants" (5.2) and (5.3) and forget that $\Lambda$ is infinite. We then "renormalize" $\Lambda$ to give the correct $K_L{}^0$-$K_S{}^0$ mass difference. Clearly this is an unsatisfactory theory from the point of view of principles, but because of the lack of a better theory we think that the above estimate of the blow-up number is encouraging from a practical point of view.

If one looks upon the perturbation series as an asymptotic expansion, it follows that the terms of order $G(G\Lambda^2)^{n-1}$ are also the leading terms in each order. This implies that one can calculate the most important contribution in each order in a relatively simple way. If one also wishes to calculate the nonleading terms this is usually an impossible business, since detailed knowledge of the strong interactions is necessary. A simple way to test this view is to predict some other higher-order weak processes with $\Lambda \cong 3$ BeV and compare the result with experiments. At present the experimental situation does not allow us to make such comparisons, and more accurate experiments are therefore badly needed. The program of calculating the weak-interaction cutoff from some process and then using this information for predictions has been discussed by Ioffe in several interesting papers.[8]

Finally we mention that the need for two coupling constants $G$ and $G\Lambda^2$ might be an indication that a more satisfactory theory of weak interactions requires two coupling constants. Such a modification would possibly be nonlocal, since $\Lambda \cong 3$ BeV means a nonlocal theory.[28] Also, let us emphasize once more that the arguments in this section do not intend to prove anything, but should be considered as heuristic.

[28] $\Lambda \cong 3$ BeV indicates a strong nonlocality, since a local theory has $\Lambda = \infty$. This immediately leads to the question: Where does the nonlocality come from? One might think that somehow the strong interactions are able to cut off the weak interactions (in the form of nonlocal "forces"). If this is true, it is reasonable to assume that the mysterious division of the forces (strong, weak, and electromagnetic) among the various particles will find its solution once we really understand the weak interactions. Of course one can produce models of nonlocal theories giving finite weak interactions in terms of one or more "form factors"; however, this does not solve the problem as long as we do not understand the origin of these form factors.

## 6. RADIATIVE CORRECTIONS TO $\pi \to e\nu$ AND $\pi \to \mu\nu$

As another application of the analysis in Sec. 2 of the asymptotic behavior of the three-point function we consider the radiative corrections to $\pi \to e\nu$ and $\pi \to \mu\nu$. We do this mainly because this example indicates that the expansion obtained in Sec. 2 cannot be applied to all cases.

Berman[29] observed in 1958 that the radiative corrections to the ratio of the decays $\pi \to e\nu$ and $\pi \to \mu\nu$ is finite in a special phenomenological model. Recently Das and Mathur derived the same result by use of current-algebra methods.[11] The very interesting point in their paper is that this result is true independently of any model for the equal-time commutators. Hence, if one wants to learn what the correct set of commutators is, one has to study the radiative corrections to the individual processes $\pi \to e\nu$ or $\pi \to \mu\nu$.

Das and Mathur[11] have shown that the total matrix element for $\pi^- \to l^-\bar{\nu}_l$ with radiative corrections is given by

$$M^{(l)} = -m_l F_\pi (1+\alpha f_l)\bar{u}_l(p_1)(1+\gamma_5)v(p_2)\,, \quad (6.1)$$

$$f_l = (1/8\pi^3 F_\pi)[A+B_l+C_l]. \quad (6.2)$$

$B_l$ can be calculated from the usual Bjorken limit and $C_l$ is trivial; one then obtains[11,30]

$$B_l = \tfrac{3}{4} i F_\pi (2a+d-11/3)\int \frac{d^4q}{q^4} + \text{convergent part},$$
$$C_l = F_\pi\left[2i\int\frac{d^4q}{q^4} - 11\pi^2\right]. \quad (6.3)$$

The constant $A$, which is independent of the lepton mass, is given by

$$A = \frac{1}{m_\pi{}^2}\int \frac{d^4q}{q^2} k^\sigma T_\mu{}^\mu{}_\sigma(k)\,,$$
$$T_{\mu\nu\sigma}(k,q) = i\int\int d^4x d^4y\, e^{iqx-iky} \quad (6.4)$$
$$\times \langle 0|T(V_\mu{}^{(\rm em)}(x)V_\nu{}^{(\rm em)}(0)A_\sigma{}^{(+)}(y))|\pi-(k)\rangle\,,$$

where $V_\mu{}^{(\rm em)}(x)$ is the electromagnetic current. If we can find the asymptotic behavior of $T_{\mu\nu\sigma}$, we can find whether $f_l$ is finite or not.

Using the methods of Sec. 2, Eq. (6.4) can be written asymptotically as

[29] S. M. Berman, Phys. Rev. Letters 1, 468 (1958).

[30] The constants $a$ and $d$ determine the model for the space-space equal-time commutators. See Ref. 11 for these definitions.

$$T_{\mu\nu\sigma}(k,q)\to\frac{-i}{q_0}\int d^4y\int d^3x\, e^{-iky}\langle 0|T([V_\mu{}^{(\mathrm{em})}(x),V_\nu{}^{(\mathrm{em})}(0)]_{x_0=0}A_\sigma(y))|\pi^-(k)\rangle$$
$$-\frac{i}{q_0{}^2}\int d^4y\int d^3x\, e^{-iky}\langle 0|T\left(\left[\frac{\partial}{\partial x_0}V_\mu{}^{(\mathrm{em})}(x),V_\nu{}^{(\mathrm{em})}(0)\right]_{x_0=0}A_\sigma(y)\right)|\pi^-(k)\rangle$$
$$-\frac{i}{q_0}\int d^4y\int d^3x\, e^{iqy-iky}\langle 0|T([V_\mu{}^{(\mathrm{em})}(x),A_\sigma(y)]_{x_0=y_0}V_\nu{}^{(\mathrm{em})}(0))|\pi^-(k)\rangle. \quad (6.5)$$

The last term is of order $1/q_0{}^2$, as can be seen by further expansion. The first term does not contribute to the constant $A$ for symmetry reasons. The constant $A$ can then be expressed as

$$A=-\frac{i}{m_\pi{}^2}k^\sigma\left\{\int\int\int d^4yd^3x\, e^{-iky}\langle 0|T\left(\left[\frac{\partial}{\partial x_0}V_\mu{}^{(\mathrm{em})}(x),V^{\mu(\mathrm{em})}(0)\right]_{x_0=0}A_\sigma(y)\right)|\pi^-\rangle\right.$$
$$\left.+\int\int_{x_0=y_0=0} d^3xd^3y\langle 0|[[V_\mu{}^{(\mathrm{em})}(x),A_\sigma(y)],V^{\mu(\mathrm{em})}(0)]|\pi^-\rangle\right\}\int d^4q/q^4. \quad (6.6)$$

The interesting question in connection with Eq. (6.6) is the following: In Ref. 12 a model was constructed which gave finite radiative corrections to $\beta$ decay. Does this model also give finite radiative corrections to $\pi\to l\nu$ decay? The point is, as we see from Eq. (6.6), that the problem of $\pi\to l\nu$ does not have much in common with $\beta$ decay, which depends on commutators like

$$[V_\mu{}^{(\mathrm{em})}(x),A_\nu(0)]_{x_0=0} \quad (6.7)$$

but *not* on the commutator

$$\left[\frac{\partial}{\partial x_0}V_\mu{}^{(\mathrm{em})}(x),V_\nu{}^{(\mathrm{em})}(0)\right]_{x_0=0}, \quad (6.8)$$

which enters Eq. (6.6). Hence, without some detailed model of the Hamiltonian, the model in Ref. 12 does not say anything about the finiteness of $\pi\to l\nu$ decay.

In the algebra of fields the commutator (6.8) can be expressed in terms of the product of two currents[19] [only space-space components contribute in (6.6)], but this does not help us since we cannot evaluate the resulting matrix element. One can make an approximation of keeping only the vacuum as intermediate state in the first term of Eq. (6.6), and it is then seen that in all models this term vanishes trivially since $\langle 0|H|0\rangle=0$. One can than calculate the last term in (6.6), and by adding the result to (6.3) one obtains a value for (6.2). It turns out that in none of the models for (6.7) does one get a finite result. Because of the approximation involved this is not a well-established conclusion.

## 7. GENERALIZED TAMM-DANCOFF METHOD

In this section we would like to point out that the formulas for the asymptotic behavior of the $n$-point function developed in Sec. 2 sheds some light on the validity of the generalized Tamm-Dancoff method.[9,10] In its simplest form, this method says that it is a good approximation to write

$$\langle 0|T(j^1(x)j^2(x)j^3(0)j^4(0))|0\rangle$$
$$\approx\langle 0|T(j^1(x)j^3(0))|0\rangle\langle 0|T(j^2(x)j^4(0))|0\rangle$$
$$+\langle 0|T(j^2(x)j^3(0))|0\rangle\langle 0|T(j^1(x)j^4(0))|0\rangle, \quad (7.1)$$

where $j^1,\cdots,j^4$ are four local operators. We shall now show that our formalism in Sec. 2 gives rise to approximations similar to Eq. (7.1) under certain circumstances to be specified later on.

Suppose we want to calculate the quantity

$$I=\int d^4x\langle A|T(\{j^1(x),j^2(x)\}j^3(0)j^4(0))|B\rangle, \quad (7.2)$$

where $|A\rangle$ and $|B\rangle$ are at the moment arbitrary states. The integral $I$ is characteristic for the calculation of second-order mass differences, and the $K_L{}^0$-$K_S{}^0$ mass difference calculation in Sec. 4 is a typical example of Eq. (7.2). In the same way as in Sec. 4 we have [see Eq. (4.10)]

$$I=\frac{1}{2(2\pi)^4}\int d^4q[\Delta^{12}(q,-q)+\Delta^{21}(q,-q)], \quad (7.3)$$

where the $\Delta$'s have the asymptotic behavior [see Eq. (4.14)]

$$\Delta^{12}(q,-q)+\Delta^{21}(q,-q)\xrightarrow[q_0\to\infty]{}\frac{2}{q^2}\int\int d^3x d^3x' \{\langle A|[j^1(x),j^3(0)][j^2(x'),j^4(0)]|B\rangle$$
$$+\langle A|[j^2(x),j^3(0)][j^1(x'),j^4(0)]|B\rangle\}_{x_0=x_0'=0}$$
$$-\frac{1}{q^2}\int\int d^3x d^3x' \{\langle A|j^3(0)[[j^1(x),j^4(0)],j^2(x')]|B\rangle+\langle A|j^3(0)[[j^2(x),j^4(0)],j^1(x')]|B\rangle$$
$$+\langle A|[[j^1(x),j^3(0)],j^2(x')]j^4(0)|B\rangle+\langle A|[[j^2(x),j^3(0)],j^1(x')]j^4(0)|B\rangle\}_{x_0=x_0'=0}. \quad (7.4)$$

Now, let us make the following two approximations: (i) It is a good approximation to keep only the vacuum state in sums over intermediate states in Eq. (7.4). (ii) The matrix elements $\langle A|j^3(0)|0\rangle$ and $\langle 0|j^4(0)|B\rangle$ are either identically zero (e.g., because of quantum numbers) or they are very small.

With these two conditions satisfied, Eq. (7.4) reduces considerably. We can then imagine that the resulting expression is the asymptotic expansion of

$$\Delta^{12}(q,-q)+\Delta^{21}(q,-q)$$
$$\approx 2\int\int d^4x d^4x' \langle A|T(j^1(x)j^3(0))|0\rangle \times\langle 0|T(j^2(x')j^4(0))|B\rangle e^{iqx+i(-q)x'}$$
$$+2\int\int d^4x d^4x' \langle A|T(j^2(x')j^3(0))|0\rangle \times\langle 0|T(j^1(x)j^4(0))|B\rangle e^{iqx-iqx'}, \quad (7.5)$$

which, when inserted into Eq. (7.3), gives

$$I\approx\int d^4x \times\{\langle A|T(j^1(x)j^3(0))|0\rangle\langle 0|T(j^2(x)j^4(0))|B\rangle +\langle A|T(j^2(x)j^3(0))|0\rangle\langle(0|Tj^1(x)j^4(0))|B\rangle\}, \quad (7.6)$$

which is a result very similar to the Tamm-Dancoff type of approximation (7.1).

In the $K_L{}^0$-$K_S{}^0$ mass difference calculation condition (ii) is trivially satisfied because of quantum numbers. The reason why the calculation by Marshak *et al.*[8] gives the same answer as the calculation we performed in Sec. 4 is precisely that, with assumption (i), condition (ii) is trivially satisfied.

The work in this section suggests (but does certainly not prove) that the Tamm-Dancoff approximation is best for small values of $x_0$.

Finally, let us mention that it is easy to extend the considerations in this section to more than four operators. The result again confirms the Tamm-Dancoff type of approximation if (i) is valid and if a generalized version of (ii) is valid.

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