ignore them. The leading terms in these two satisfy conspiracy among themselves, paralleling the well-known case of axial-vector exchange, which gives

$$f_{00,1/2-1/2}{}^{t}=\frac{(t+m_V{}^2-\mu^2)(t-4M^2)^{1/2}}{4\sqrt{t}m_VM(t-m_A{}^2)},$$

$$\bar{f}_{00,1/2\;1/2}{}^{t}=-\frac{\{[t-(m_V-\mu)^2][t-(m_V+\mu)^2]\}^{1/2}}{(2\sqrt{t})m_Vm_a{}^2}.$$

For the $M=1$ case, we need only cite the behavior of the nucleon Born-term graph in pion photoproduction $\bar{N}\gamma_5(\gamma\cdot k)(\gamma\cdot\epsilon)N$. This gives (for a photon of mass $m_V$)

$$\bar{f}_{00,++}=-pm_V(\cos\theta_t)/M\,,$$
$$\bar{f}_{10,++}-\bar{f}_{-10,++}=\sqrt{2}pk_0/M\,,$$
$$\bar{f}_{10,++}+\bar{f}_{-10,++}=-\sqrt{2}Eq/M\,,$$
$$\bar{f}_{00,+-}=0\,,$$
$$\bar{f}_{10,+-}-\bar{f}_{-10,+-}=\sqrt{2}q\,,$$
$$\bar{f}_{10,+-}+\bar{f}_{-10,+-}=0\,,$$

which show in a very simple way the singularities of the $M=1$ conspiracy discussed in Sec. V, even though the graph does not represent a $t$-channel exchange.

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# Currents as Coordinates in Nonrelativistic Quantum Mechanics*

W. J. PARDEE, L. SCHLESSINGER, AND JON WRIGHT
*Department of Physics, University of Illinois, Urbana, Illinois*

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The formulation of nonrelativistic quantum mechanics using currents and densities as coordinates is investigated. A general solution for a single-particle theory is presented, and several many-body problems are discussed.

## I. INTRODUCTION

RECENTLY, there has been considerable interest in the description of strong interactions in terms of currents. Dashen and Sharp[1] showed that nonrelativistic quantum mechanics could be described by using currents and densities as coordinates rather than the more familiar canonical coordinates, but they left unanswered the question of solvability of such a theory. In this paper we treat systems of $Q$ identical bosons in their formalism, and show how to obtain all of the information that the usual formulation gives. For single particles interacting with a fixed potential or potential scattering of two particles, the new formalism turns out to require the solution of the Schrödinger equation. For more particles it is not clear what the form of the solution is except in some simple solvable examples.

It seems natural to formulate many-body problems in terms of currents and densities, and perhaps this approach might lead to better or different approximation schemes. Although our original motivation for solving problems this way was to learn how to work with descriptions of systems in terms of currents and densities, we also have shown that the formulation is a feasible approach to nonrelativistic problems. There is no pretense made of mathematical rigor, and since we are in fact working with functional integrals—a relatively unexplored area of mathematics—we may occasionally adopt questionable mathematical procedures.

We start with a review of the work of Dashen and Sharp[1,2] and refer the reader to their paper for further details. Section III is devoted to a discussion of a single particle interacting with a fixed potential, and there it is shown how to find solutions to the usual problems. In Sec. IV we treat noninteracting systems and show how to find the many-boson correlation functions. We also discuss the differences between wave functionals that give identical results for a single-particle theory but different results for a many-particle theory. The last section discusses two problems of interacting bosons—coupled harmonic oscillators and a one-dimensional system of particles interacting through $\delta$-function potentials. The form of the exact solutions to these problems suggests approximation schemes for other kinds of interactions, but it sheds no real light on how to obtain solutions to more complicated systems.

## II. REVIEW

In this section we review for completeness the formulation of nonrelativistic quantum mechanics in terms of currents and charge densities as given by Dashen and

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[1] R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968).

[2] D. H. Sharp, Phys. Rev. **165**, 1867 (1968).

[3] We always assume that integration by parts is permissible and that the boundary terms vanish.

Sharp.[1] We consider a system of $Q$ spinless identical bosons interacting pairwise through local central potentials, $V(|\mathbf{x}_i-\mathbf{x}_j|)$, and through a fixed external local central potential, $U(|\mathbf{x}|)$. The usual formulation is written in terms of second-quantized field operators satisfying the canonical commutation rules,

$$[\psi^\dagger(\mathbf{x}),\psi^\dagger(\mathbf{y})]=0,\quad [\psi(\mathbf{x}),\psi(\mathbf{y})]=0,\quad [\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})]=\delta(\mathbf{x}-\mathbf{y}).$$

The density and current operators are given by

$$\rho(\mathbf{x})=\psi^\dagger(\mathbf{x})\psi(\mathbf{x})$$

and

$$\mathbf{J}(\mathbf{x})=(1/2i)[\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x})-\nabla\psi^\dagger(\mathbf{x})\psi(\mathbf{x})].$$

We have taken the mass of the bosons to be unity. It is easily verified that the commutators of these operators are given by

$$[\rho(\mathbf{x}),\rho(\mathbf{y})]=0,$$

$$[\rho(\mathbf{x}),J_k(\mathbf{y})]=-i\frac{\partial}{\partial x^k}[\delta(\mathbf{x}-\mathbf{y})\rho(\mathbf{x})],\tag{2.1}$$

and

$$[J_k(\mathbf{x}),J_m(\mathbf{y})]=-i\frac{\partial}{\partial x^m}[\delta(\mathbf{x}-\mathbf{y})J_k(\mathbf{x})]+i\frac{\partial}{\partial y^k}[\delta(\mathbf{x}-\mathbf{y})J_m(\mathbf{y})].$$

The total momentum operator is given by

$$\mathbf{P}=\int \mathbf{J}(\mathbf{x})d^3x\tag{2.2}$$

and the Hamiltonian by

$$H=H_0+V+U,\tag{2.3}$$

where

$$H_0=\int d^3x\,H_0(\mathbf{x})=\tfrac{1}{8}\sum_{k=1}^{3}\int d^3x[\nabla_k\rho(\mathbf{x})-2iJ_k(\mathbf{x})]\times\frac{1}{\rho(\mathbf{x})}[\nabla_k\rho(\mathbf{x})+2iJ_k(\mathbf{x})],\tag{2.4}$$

$$V=\tfrac{1}{2}\int \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})V(|\mathbf{x}-\mathbf{y}|)\psi(\mathbf{x})\psi(\mathbf{y})d^3xd^3y$$

$$=\tfrac{1}{2}\int \rho(\mathbf{x})\rho(\mathbf{y})V(|\mathbf{x}-\mathbf{y}|)d^3xd^3y+\tfrac{1}{2}\int\rho(\mathbf{y})V(0)d^3y,$$

and

$$U=\int d^3x\,\rho(\mathbf{x})U(|\mathbf{x}|).\tag{2.5}$$

The term $\frac{1}{2}\int\rho(\mathbf{y})V(0)d^3y$ is a constant (possibly infinite) and may be neglected in any calculation. The total number of particles is a constant and is given by

$$Q=\int d^3x\,\rho(\mathbf{x}).\tag{2.6}$$

Following Sharp,[2] one now introduces a functional representation of the algebra given by Eqs. (2.1)–(2.3). An abstract state in the Hilbert space is represented by its components along a basis formed by a complete set of eigenvectors of a commuting set of current operators. Because the eigenvectors of the operator $\hat{\rho}(\mathbf{x})$ form such a set, we consider states labeled by the eigenvalues of $\hat{\rho}(\mathbf{x})$,

$$\hat{\rho}(\mathbf{x})|\rho\rangle=\rho(\mathbf{x})|\rho\rangle.$$

The set of components of $|\Psi\rangle$ along such a basis is then a wave functional

$$\Psi(\rho)=\langle\rho|\Psi\rangle.$$

In this basis, the action of $\hat{\rho}(\mathbf{x})$ on $|\Psi\rangle$ is just multiplication of $\Psi(\rho)$ by the eigenvalue of $\rho(\mathbf{x})$ and the action of $J_k(\mathbf{x})$ on these states may be represented by the following functional derivative:

$$J_k(\mathbf{x})\to -i\rho(\mathbf{x})\frac{\partial}{\partial x^k}\frac{\delta}{\delta\rho(\mathbf{x})}.\tag{2.7}$$

The energy spectrum of the system is then determined by

$$H\Psi(\rho)=E\Psi(\rho).$$

The scalar product in such a functional representation is denoted by

$$\langle\Psi|\Phi\rangle=\int\Psi^\dagger(\rho)\Phi(\rho)D(\rho),$$

where the integral is over all functions $\rho$ such that $\rho(\mathbf{x})>0$ and $\int\rho(\mathbf{x})d^3x=Q$, the number of particles. $D(\rho)$ denotes a measure on the space of all functions satisfying the constraints. We know of no way to construct the measure $D(\rho)$ for a general functional integral. For quantum mechanics the measure must be chosen so that the expectation value of an observable is real. The form of the solutions to these functional equations suggests that the correct measure is built into the wave functional. Although we have not been able to do any of the functional integrals prescribed by the theory, we are able to find the expectation values of operators by making use of the reality condition on observables.

## III. SINGLE-PARTICLE QUANTUM MECHANICS

In this section we show how to recover the solutions to the ordinary Schrödinger equation for a particle interacting with a fixed potential. We use only solutions corresponding to $\int\rho(\mathbf{x})d^3x=1$, which represents a single-particle theory. The potential is given by

$$U=\int U(|\mathbf{x}|)\rho(\mathbf{x})d^3x.$$

As a first example, consider a free particle at rest which satisfies

$$H_0\Psi_0=0. \tag{3.1}$$

The free Hamiltonian is positive definite, and can be written in the form

$$H_0=\tfrac{1}{2}\sum_{k=1}^{3}\int d^3x\, A_k^\dagger(\mathbf{x})\frac{1}{\rho(\mathbf{x})}A_k(\mathbf{x}), \tag{3.2}$$

where

$$A_k(\mathbf{x})=\tfrac{1}{2}[\nabla_k\rho(\mathbf{x})+2iJ_k(\mathbf{x})]$$

and

$$A_k^\dagger(\mathbf{x})=\tfrac{1}{2}[\nabla_k\rho(\mathbf{x})-2iJ_k(\mathbf{x})]. \tag{3.3}$$

The expectation value of the Hamiltonian in a state $\Psi$ can be written in a form that explicitly exhibits the positive definite character:

$$\langle\Psi|H_0|\Psi\rangle=\tfrac{1}{2}\sum_{k=1}^{3}\int d^3x\left\langle\Psi\left|A_k^\dagger(\mathbf{x})\frac{1}{\rho(\mathbf{x})}A_k(\mathbf{x})\right|\Psi\right\rangle.$$

Clearly the zero-energy eigenstates will be solutions of

$$A_k(\mathbf{x})|\Psi_0\rangle=0, \tag{3.4}$$

or, using (2.7) for $J_k(\mathbf{x})$,

$$\left(\nabla\rho(\mathbf{x})+2\rho(\mathbf{x})\nabla\frac{\delta}{\delta\rho(\mathbf{x})}\right)\Psi_0(\rho)=0. \tag{3.5}$$

Relying on the similarity to an ordinary differential equation, we find the solution

$$\Psi_0(\rho)=\exp\left(-\tfrac{1}{2}\int\rho(\mathbf{x})\ln\rho(\mathbf{x})d^3x\right). \tag{3.6}$$

$\Psi_0(\rho)$ appears in every wave functional and perhaps represents a measure since we shall see that all wave functionals can be written as

$$\Psi(\rho)=\Psi_0(\rho)\Phi(\rho). \tag{3.7}$$

The wave functional for a free particle with momentum $p$ is easily found by looking for solutions of

$$\int\mathbf{J}(\mathbf{x})d^3x\,\Psi(\rho)=-i\int\rho(\mathbf{x})\nabla\frac{\delta}{\delta\rho(\mathbf{x})}\Psi(\rho)d^3x$$
$$=\mathbf{p}\Psi(\rho). \tag{3.8}$$

Since $\int\rho(\mathbf{x})d^3x=1$, we can insert it on the right-hand side and in that form it is easy to guess a solution

$$\Phi(\rho)\sim\exp\left(i\mathbf{p}\cdot\int\mathbf{x}\rho(\mathbf{x})d^3x\right).$$

This $\Phi$ does not satisfy $H\Phi=\frac{1}{2}p^2\Phi$, but multiplying it by $\Psi_0$ will correct this. Multiplication by $\Psi_0$ does not change the property $\mathbf{J}\Psi=\mathbf{p}\Psi$ since[3]

$$\int\mathbf{J}(\mathbf{x})d^3x\,\Psi_0=\tfrac{1}{2}i\int\nabla\rho(\mathbf{x})d^3x\,\Psi_0=0.$$

Thus a free particle of momentum $\mathbf{p}$ is described by the wave functional

$$\Psi(\rho)=\exp\left(i\mathbf{p}\cdot\int\mathbf{x}\rho(\mathbf{x})d^3x\right)\Psi_0(\rho). \tag{3.9}$$

There is another state vector which also satisfies the above conditions:

$$\Psi(\rho)=\int e^{i\mathbf{p}\cdot\mathbf{x}}\rho(\mathbf{x})d^3x\,\Psi_0(\rho). \tag{3.10}$$

For a single particle these solutions seem to be equivalent, but for many particles they are different. If we let $\int\rho(\mathbf{x})d^3x=Q$, then the first solution describes $Q$ particles all with momentum $\mathbf{p}$ while the second solution describes one particle with momentum $\mathbf{p}$ and the other $Q-1$ particles at rest.

It is instructive to solve the problem of the free particle in a one-dimensional box as it illustrates both how probability densities can be found and how boundary conditions can be applied. We start out by guessing the form of the solution by analogy with the free particle. The solution will be taken to be of the form

$$\Psi(\rho)=\Psi_0(\rho)\exp\left(\tfrac{1}{2}\int\ln[\sin^2(kx)]\rho(x)dx\right). \tag{3.11}$$

It is easily verified that $H\Psi=\frac{1}{2}k^2\Psi$, and the problem is to find the allowed values of $k$. The idea is to impose the condition that the expectation value of $\rho$ must vanish at the boundaries of the box, where

$$\langle\rho(x)\rangle\equiv\langle\Psi|\rho(x)|\Psi\rangle/\langle\Psi|\Psi\rangle. \tag{3.12}$$

One way of computing (3.12) would be to do a functional integration. An alternative is to use the fact that $J(x)$ is Hermitian and therefore its expectation values must be real;

$$\langle\Psi|J(x)|\Psi\rangle=\text{real function},$$

$$J(x)\Psi(\rho)=-i\rho(x)\nabla_x\frac{\delta}{\delta\rho(x)}\left[\exp\left(-\tfrac{1}{2}\int\rho(x)\ln\rho(x)dx\right)\right.$$
$$\left.\times\exp\left(\tfrac{1}{2}\int\rho(x)\ln[\sin^2(kx)]dx\right)\right]$$
$$=-i\rho(x)\nabla_x\{-\tfrac{1}{2}\ln\rho(x)-\tfrac{1}{2}$$
$$+\tfrac{1}{2}\ln[\sin^2(kx)]\}\Psi(\rho)$$
$$=i[\tfrac{1}{2}\nabla_x\rho(x)-k\rho(x)\cot(kx)]\Psi(\rho).$$

The expectation value of $J$ is therefore pure imaginary and must vanish, giving the differential equation

$$\left(\tfrac{1}{2}\frac{d}{dx}-k\cot(kx)\right)\langle\rho(x)\rangle=0. \qquad (3.13)$$

The solution is $\langle\rho(x)\rangle\sim\sin^2 kx$ with the proportionality constant determined by $\int\rho(x)dx=1$. The requirement that $\langle\rho(x)\rangle=0$ at $x=0$ and $x=L$ gives the usual allowed values of $k$.

We next consider the harmonic oscillator potential

$$U=\tfrac{1}{2}\omega^2\int x^2\rho(\mathbf{x})d^3x. \qquad (3.14)$$

For this potential the Hamiltonian density is

$$H=\int d^3x\,H(\mathbf{x})=\sum_{k=1}^{3}\int d^3x\Big(\tfrac{1}{8}[\nabla_k\rho(\mathbf{x})-2iJ_k(\mathbf{x})]\frac{1}{\rho(\mathbf{x})}$$
$$\times[\nabla_k\rho(\mathbf{x})+2iJ_k(\mathbf{x})]+\tfrac{1}{2}\omega^2x_k\rho(\mathbf{x})\frac{1}{\rho(\mathbf{x})}x_k\rho(\mathbf{x})\Big). \qquad (3.15)$$

The eigenstates of this Hamiltonian may be found using raising and lowering operators similar to those used in the ordinary formulations. Defining

$$A_k(\mathbf{x})=\tfrac{1}{2}[\nabla_k\rho(\mathbf{x})+2iJ_k(\mathbf{x})+2\omega x_k\rho(\mathbf{x})]$$

and (3.16)

$$A_k^{\dagger}(\mathbf{x})=\tfrac{1}{2}[\nabla_k\rho(\mathbf{x})-2iJ_k(\mathbf{x})+2\omega x_k\rho(\mathbf{x})],$$

the Hamiltonian (3.15) may be written as

$$H=\int d^3x\,H(\mathbf{x})=\int d^3x$$
$$\times\left\{\tfrac{1}{2}\sum_{k=1}^{3}A_k^{\dagger}(\mathbf{x})\frac{1}{\rho(\mathbf{x})}A_k(\mathbf{x})-\tfrac{1}{2}\omega\mathbf{x}\cdot\nabla\rho(\mathbf{x})\right\}; \qquad (3.17)$$

integrating the last term by parts,[3] the Hamiltonian density becomes

$$H(\mathbf{x})=\tfrac{1}{2}\sum_{k=1}^{3}A_k^{\dagger}(\mathbf{x})\frac{1}{\rho(\mathbf{x})}A_k(\mathbf{x})+\tfrac{3}{2}\omega\rho(\mathbf{x}). \qquad (3.18)$$

Because the first term in Eq. (3.18) is positive, the ground-state wave functional, $\Psi_g(\rho)$, for the harmonic oscillator must satisfy

$$\mathbf{A}(\mathbf{x})\Psi_g(\rho)=0. \qquad (3.19)$$

And clearly

$$H\Psi_g(\rho)=\tfrac{3}{2}\omega\int\rho(\mathbf{x})d^3x\,\Psi_g(\rho)=\tfrac{3}{2}\omega\Psi_g(\rho).$$

Assuming $\Psi_g(\rho)=\Psi_0\Phi_g(\rho)$, we have

$$\nabla\left(\frac{\delta}{\delta\rho(\mathbf{x})}+\tfrac{1}{2}\omega x^2\right)\Phi_g(\rho)=0. \qquad (3.20)$$

Equation (3.20) has the solution

$$\Phi_g(\rho)=\exp\left(-\tfrac{1}{2}\omega\int\rho(\mathbf{x})x^2d^3x\right),$$

therefore

$$\Psi_g(\rho)=\Psi_0(\rho)\exp\left(-\tfrac{1}{2}\omega\int\rho(\mathbf{x})x^2d^3x\right). \qquad (3.21)$$

The commutation relations of the operators $A_k$ and $A_k^{\dagger}$, where $A_k=\int d^3x\,A_k(\mathbf{x})$, with the Hamiltonian, are easily found to be

$$[A_k,H]=\omega A_k,$$
$$[A_k^{\dagger},H]=-\omega A_k^{\dagger}. \qquad (3.22)$$

Thus the excited states of the harmonic oscillator may be obtained by well-known methods. The $n$th excited state of the oscillator obtained in this way is

$$\Psi_n(\rho)=H_n\left[\int\left(\frac{\omega}{Q}\right)^{1/2}x_k\rho(\mathbf{x})d^3x\right]\Psi_g(\rho), \qquad (3.23)$$

where $H_n$ is the $n$th Hermite polynomial. For the single-particle case there are many other solutions, two of which are

$$\Psi_n(\rho)=\int\rho(\mathbf{x})H_n[\omega^{1/2}x_k]d^3x\,\Psi_g(\rho),$$

and

$$\Psi_n(\rho)=\exp\left(\tfrac{1}{2}\int\ln[H_n^2(\omega^{1/2}x_k)\rho(\mathbf{x})d^3x]\right)\Psi_g(\rho).$$

These are equivalent for a single particle, but differ for the many-particle system. This point will be discussed later.

The density can be found in the same way that we found the density for a free particle in a box. The requirement that expectation values of $J(x)$ be real leads to the differential equation (for the ground state)

$$(\tfrac{1}{2}\nabla+\omega\mathbf{x})\langle\rho(\mathbf{x})\rangle=0, \qquad (3.24)$$

with the solution

$$\langle\rho(\mathbf{x})\rangle=(\omega/\pi)^{3/2}e^{-\omega x^2}. \qquad (3.25)$$

Having built up some necessary techniques, we consider the general problem of a single particle in a potential, and look for solutions of the form

$$\Psi(\rho)=\Psi_0(\rho)\exp\left(-\int f(\mathbf{x})\rho(\mathbf{x})d^3x\right). \qquad (3.26)$$

The equation

$$H\Psi(\rho)=E\int\rho(\mathbf{x})d^3x\,\Psi(\rho)$$

leads to

$$\int d^3x[\tfrac{1}{2}\nabla^2f(\mathbf{x})+\tfrac{1}{2}\nabla f(\mathbf{x})\cdot\nabla f(\mathbf{x})$$
$$+U(\mathbf{x})-E]\rho(\mathbf{x})=0. \qquad (3.27)$$

Since $\rho(\mathbf{x})$ is essentially arbitrary, the integrand must be 0, i.e.,

$$[\tfrac{1}{2}\nabla^2 f(\mathbf{x})+\tfrac{1}{2}\nabla f(\mathbf{x})\cdot\nabla f(\mathbf{x})+U(\mathbf{x})-E]=0. \tag{3.28}$$

The substitution $f(\mathbf{x})=-\ln g(\mathbf{x})$ in (3.28) gives the Schrödinger equation

$$[-\tfrac{1}{2}\nabla^2+U(\mathbf{x})]g(\mathbf{x})=Eg(\mathbf{x}). \tag{3.29}$$

Thus the general solution for an arbitrary potential $U(\mathbf{x})$ is

$$\Psi(\rho)=\exp\left[-\tfrac{1}{2}\int\rho(\mathbf{x})\ln\left(\frac{\rho(\mathbf{x})}{g^2(\mathbf{x})}\right)d^3x\right], \tag{3.30}$$

where $g(\mathbf{x})$ is the solution of the Schrödinger equation. Taking $g(\mathbf{x})=g_R(\mathbf{x})+ig_I(\mathbf{x})$, the expectation value of $\rho(x)$ and $\mathbf{J}(x)$ in such a state is

$$\langle\rho(\mathbf{x})\rangle=ce^{-2f_R(\mathbf{x})}=c[g_R^2(\mathbf{x})+g_I^2(\mathbf{x})],$$
$$\langle\mathbf{J}(\mathbf{x})\rangle=-c\nabla f_I(\mathbf{x})e^{-2f_R(\mathbf{x})}=c\,\mathrm{Im}g^*(\mathbf{x})\nabla g(\mathbf{x}),$$

where $c$ is determined by the condition $\int\rho(\mathbf{x})d^3x=1$. With this solution one may determine boundary conditions, and define the $S$ matrix, etc., in the usual way.

## IV. MANY-PARTICLE NONINTERACTING SYSTEMS

In this section we briefly extend the results of the previous section to include many particles interacting with a central potential but not with each other. The physically measurable quantities in this case are the densities and correlation functions, and they are immediately obtainable from the wave functionals. We shall also see the differences between the wave functionals which gave identical results for one particle.

Consider first the problem of describing $Q$ free particles of various momenta. The state with $Q$ particles all having momentum $\mathbf{k}$ is just the same as in the single-particle case. One can verify, by operating with the Hamiltonian and the total momentum operator, that the state

$$\Psi(\rho)=\exp\left(i\mathbf{k}\cdot\int\mathbf{x}\rho(\mathbf{x})d^3x\right)\Psi_0 \tag{4.1}$$

has energy $E=\frac{1}{2}Qk^2$ and momentum $\mathbf{P}=Q\mathbf{k}$.

It is somewhat harder to construct the states with momenta $\mathbf{k}_1\cdots\mathbf{k}_n$ and energy $\frac{1}{2}(k_1^2+k_2^2+\cdots+k_r^2)$, and we shall simply give the prescription without the proof. To simplify the algebra it is convenient to define creation and destruction operators that do not act on the $\Psi_0$ part of the wave functional. The creation operator subject to this restriction is then

$$A(\mathbf{k})=\int e^{i\mathbf{k}\cdot\mathbf{x}}\rho(\mathbf{x})\exp\left(-\frac{\delta}{\delta\rho(\mathbf{x})}\right)d^3x, \tag{4.2}$$

and the destruction operator is

$$B(\mathbf{k})=\int e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\exp\left(\frac{\delta}{\delta\rho(\mathbf{x})}\right)-1\right]d^3x. \tag{4.3}$$

If we define

$$\tilde{\rho}(\mathbf{k})=\int e^{i\mathbf{k}\cdot\mathbf{x}}\rho(\mathbf{x})d^3x, \tag{4.4}$$

then a one-particle state of momentum $\mathbf{k}$ is given by

$$\Psi(\rho)=\Psi_0(\rho)A(\mathbf{k})=\Psi_0(\rho)\tilde{\rho}(\mathbf{k}), \tag{4.5}$$

where we expand

$$\exp\left(-\frac{\delta}{\delta\rho}\right)=1-\frac{\delta}{\delta\rho}+\cdots,$$

and only the first term contributes, because there are no $\rho$'s to operate on. This corresponds to a state where one particle has momentum $\mathbf{k}$ and the other particles are at rest. Similarly a state with a particle of momentum $\mathbf{p}$ and a second particle of momentum $\mathbf{q}$ is given by

$$\begin{aligned}\Psi(\rho)&=\Psi_0(\rho)A(\mathbf{q})A(\mathbf{p})\\&=\Psi_0(\rho)\{\tilde{\rho}(\mathbf{p})\tilde{\rho}(\mathbf{q})-\tilde{\rho}(\mathbf{p}+\mathbf{q})\}.\end{aligned} \tag{4.6}$$

A state with $n$ particles having momentum $\mathbf{k}_1\cdots\mathbf{k}_n$ is given by

$$\Psi(\rho)=\Psi_0(\rho)A(\mathbf{k}_1)A(\mathbf{k}_2)\cdots A(\mathbf{k}_n). \tag{4.7}$$

One interesting point is the effect of creating more momentum states than there are particles, i.e., if in Eq. (4.7) $n>Q$, what happens? It is easier to see the solution if we place the particles in a box so that the density can be calculated. For example, if we construct a state

$$\begin{aligned}\Psi(\rho)=\Psi_0(\rho)&\int\cos(kx)\rho(x)dx\\&\times\exp\left(\tfrac{1}{2}\int\ln[\sin^2(kx)]\rho(x)dx\right),\end{aligned} \tag{4.8}$$

where $k=\pi/L$, we find a density

$$\begin{aligned}\langle\rho(x)\rangle=[4\sin^2(kx)\cos^2(kx)&\\+(Q-1)\sin^2(kx)]2/L.&\end{aligned} \tag{4.9}$$

This density corresponds to one particle in the first excited state and the other $Q-1$ particles in the ground state. If $Q=1$ then there is only a particle in the first excited state since the coefficient of the second term vanishes. The result is that, although it is not obvious that the wave function takes care of such problems, all expectation values seem to have the appropriate zeroes.

As a further illustration, we consider $Q$ particles interacting with a fixed harmonic-oscillator potential but not with each other. It was shown in the previous section how to calculate expectation values of densities, and we now show how to calculate the correlation functions which are necessary to completely describe the system. Consider $Q$ particles all in the ground state. The wave functional is given by Eq. (3.21) as

$$\Psi(\rho)=\Psi_0(\rho)\exp\left(-\tfrac{1}{2}\omega\int x^2\rho(\mathbf{x})d^3x\right), \qquad (3.21')$$

and the expectation value of the density is

$$\rho_0(\mathbf{x})=\langle\rho(\mathbf{x})\rangle=Q(\omega/\pi)^{3/2}e^{-\omega x^2}. \qquad (3.25')$$

To calculate the two-particle correlation function in the ground state $\langle\rho(\mathbf{x})\rho(\mathbf{y})\rangle_0$, we use the fact that $J_k(y)\rho(x)+\rho(x)J_k(y)$ is Hermitian. This leads to a differential equation

$$-\langle\rho(\mathbf{y})\rangle_0\frac{\partial}{\partial y_k}\delta(\mathbf{y}-\mathbf{x})+\left(\frac{d}{dy_k}+2\omega y_k\right)\langle\rho(\mathbf{y})\rho(\mathbf{x})\rangle_0=0, \qquad (4.10)$$

which has the solution

$$\langle\rho(\mathbf{x})\rho(\mathbf{y})\rangle_0=\rho_0(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})+(1-1/Q)\rho_0(\mathbf{x})\rho_0(\mathbf{y}). \qquad (4.11)$$

The three-particle correlation function is found to be

$$\langle\rho(\mathbf{x})\rho(\mathbf{y})\rho(\mathbf{z})\rangle_0=\rho_0(\mathbf{z})\delta(\mathbf{z}-\mathbf{x})\delta(\mathbf{z}-\mathbf{y})+\left(1-\frac{1}{Q}\right)\{\rho_0(\mathbf{x})\rho_0(\mathbf{y})\delta(\mathbf{z}-\mathbf{x})+\rho_0(\mathbf{x})\rho_0(\mathbf{z})\delta(\mathbf{y}-\mathbf{z})+\rho_0(\mathbf{y})\rho_0(\mathbf{z})\delta(\mathbf{x}-\mathbf{y})\}$$
$$+\left(1-\frac{1}{Q}\right)\left(1-\frac{2}{Q}\right)\rho_0(\mathbf{x})\rho_0(\mathbf{y})\rho_0(\mathbf{z}). \qquad (4.12)$$

To derive the $n$-particle correlation function we use Hermiticity of

$$\mathbf{J}(\mathbf{y})\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\cdots\rho(\mathbf{x}_n)+\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\cdots\rho(\mathbf{x}_n)\mathbf{J}(\mathbf{y}).$$

The $n$-particle correlation function is very ugly, but it is useful for computing expectation values in excited states. It is given by

$$\langle\rho(\mathbf{x}_1)\cdots\rho(\mathbf{x}_n)\rangle=\left[\prod_{i=1}^{n-1}\left(1-\frac{i}{Q}\right)\right]\rho_0(\mathbf{x}_1)\cdots\rho_0(\mathbf{x}_n)+\sum_{\alpha=1}^{n-1}\left[\prod_{i=1}^{n-\alpha-1}\left(1-\frac{i}{Q}\right)\right]\sum_{k_1<k_2<k_3\cdots<k_\alpha}$$
$$\times\sum_{\substack{n_1<k_1,\cdots,n_\alpha<k_\alpha\\ n_1\neq k_\alpha,\, i=1,\cdots,\alpha}}\delta(\mathbf{x}_{k_1}-\mathbf{x}_{n_1})\cdots\delta(\mathbf{x}_{k_\alpha}-\mathbf{x}_{n_\alpha})\prod_{m\neq k_1\cdots k_\alpha}\rho_0(\mathbf{x}_m). \qquad (4.13)$$

Although we derived the above result for the harmonic-oscillator ground state, it is true for any system of particles in the same state interacting only with a central potential. Again we point out that, if one asks for a correlation function for more particles than are present in the system, the extra terms are automatically eliminated like $(N-Q)$ in Eq. (4.12).

Having found the correlation functions, we now return to the original problem. In the previous section, we found more than one harmonic-oscillator solution with the same energy and it was not possible to differentiate between them for one particle. With $Q$ particles, however, there is a difference which can be determined by calculating the density of the particles in that state. For example, consider the state

$$\Psi=\int H_n(\omega^{1/2}x_k)\rho(\mathbf{x})d^3x\,\Psi_g,$$

with

$$\Psi_g=\Psi_0\exp\left(-\tfrac{1}{2}\omega\int x^2\rho(\mathbf{x})d^3x\right),$$

and calculate the density using Eq. (4.11) to obtain

$$\langle\rho(\mathbf{x})\rangle=\langle\Psi|\rho(\mathbf{x})|\Psi\rangle$$
$$=\int d^3zd^3y\,H_n(\omega^{1/2}y_k)H_n(\omega^{1/2}z_k)\times\langle\Psi_g|\rho(\mathbf{x})\rho(\mathbf{y})\rho(\mathbf{z})|\Psi_g\rangle$$
$$=\left(1-\frac{1}{Q}\right)\rho_0(\mathbf{x})+\frac{1}{Q}H_n^2(\omega^{1/2}x_k)\rho_0(\mathbf{x}); \qquad (4.14)$$

the energy of the state is

$$E_n=\tfrac{1}{2}Q\omega+n\omega.$$

From this expression it is easily seen that this wave functional corresponds to one particle in the $n$th state and $Q-1$ particles in the ground state.

On the other hand, the density as given by the wave functional of Eq. (3.23) is, for $n=2$,

$$\langle\rho(\mathbf{x})\rangle=\frac{1}{Q^2}[(Q-1)^2+2(Q-1)H_1^2(\omega^{1/2}x_k)+H_2^2(\omega^{1/2}x_k)]\rho_0(\mathbf{x}).$$

This functional corresponds to a mixture of two particles in the first excited state and one particle in the second excited state. The other states corresponding to the same energy can also be constructed and their densities calculated similarly. Of course, the correlation functions are also directly calculable with the use of Eq. (4.12). Because there are obviously many different states with the same energy for the many-particle system, we can now understand why there were several wave functionals describing the same state in the single-particle case discussed in Sec. III. For single particles, all the observables are the same, but for many particles they are now different.

## V. MANY-PARTICLE INTERACTING SYSTEMS

There are very few exactly soluble many-particle problems, and we have not been able to extend the list. It is possible that this new formulation may have some use in approximation schemes; however, we have not attempted to develop any as yet but instead have confined ourselves to finding the exact solutions of two problems—particles interacting through harmonic-oscillator potentials and particles in one dimension interacting through $\delta$-function potentials.

For the harmonic oscillator, the potential is

$$V=\tfrac{1}{4}\omega^2\int\rho(x)\rho(y)(x-y)^2dxdy\,, \tag{5.1}$$

where for simplicity we consider only one dimension. To simplify the notation we introduce a new variable,

$$z(x)=Qx-\int y\rho(y)dy\,,$$

and a new frequency,

$$\nu=\omega/\sqrt{Q}\,.$$

The potential becomes

$$V=\tfrac{1}{2}\nu^2\int\rho(x)z^2(x)dx\,, \tag{5.2}$$

and the Hamiltonian may be written

$$H=\tfrac{1}{2}\int dx\,A^\dagger(x)\frac{1}{\rho(x)}A(x)+\tfrac{1}{2}\nu Q(Q-1)\,, \tag{5.3}$$

where

$$A(x)=\tfrac{1}{2}[\nabla\rho(x)+2iJ(x)+2\nu z(x)\rho(x)]\,. \tag{5.4}$$

As before, we look for solutions of

$$A(x)\Psi_g(\rho)=0$$

and find that

$$\Psi_g(\rho)=\Psi_0(\rho)\exp\!\left(-\frac{\nu}{2Q}\int z^2(x)\rho(x)dx\right)$$
$$=\Psi_0(\rho)\exp\!\left(-\frac{\omega}{4\sqrt{Q}}\int dxdy\times\rho(x)\rho(y)(x-y)^2\right). \tag{5.5}$$

The energy of the ground state $\Psi_g$ is

$$E_g=\tfrac{1}{2}\nu Q(Q-1)=\frac{\omega}{2\sqrt{Q}}Q(Q-1)\,. \tag{5.6}$$

The excited states are easily found by analogy with the single-particle case to be

$$\Psi_n(\rho)=\int\rho(x)dxH_n\!\left[\left(\frac{\omega}{Q(Q-1)}\right)^{1/2}z(x)\right]\Psi_g(\rho)\,, \tag{5.7}$$

with energies

$$E_n=\frac{\omega}{2\sqrt{Q}}Q(Q-1)+\frac{n\omega}{\sqrt{Q}}Q\,. \tag{5.8}$$

Although the solutions were found by a trick, it is quite easy to guess the form of the solution and then operate on it with the Hamiltonian to find the correct energy.

The next example is very instructive as it illustrates the problem of the additional term in the Hamiltonian, $\int\rho(x)V(0)dx=QV(0)$. The potential is a one-dimensional $\delta$ function,

$$V=-\tfrac{1}{2}\lambda\int\delta(x-y)\rho(x)\rho(y)\,,$$

and of course $V(0)=\delta(0)$ is infinite. The solution is found by guessing and is

$$\Psi(\rho)=\Psi_0(\rho)\exp\!\left(-\tfrac{1}{4}\lambda\int dxdy|x-y|\rho(x)\rho(y)\right). \tag{5.9}$$

Operating on (5.9) with the Hamiltonian gives a term

$$H\Psi=-\tfrac{1}{8}\lambda^2\int\rho(x)\rho(y)\rho(z)\times\epsilon(z-x)\epsilon(z-y)dxdydz\,\Psi\,, \tag{5.10}$$

where

$$\epsilon(x)=\;\;1,\quad x>0$$
$$=-1,\quad x<0\,.$$

If the expression is symmetrized, it is easily seen to be

$$H\Psi=\left(-\frac{1}{24}\lambda^2\int\rho(x)\rho(y)\rho(z)dxdydz\right)\Psi$$
$$=\left(-\frac{Q^3\lambda^2}{24}\right)\Psi\,. \tag{5.11}$$

Therefore, the energy is

$$E=(-\lambda^2/24)Q^3.$$

However, there is still the problem of $V(0)Q$, which means that there is some arbitrariness in terms linear in $Q$. We require that $E=0$ for $Q=0$ or 1 and thus obtain

$$E=(-\lambda^2/24)(Q^3-Q),$$

which is the correct answer.[4] This seems to be an appropriate way to handle the $V(0)$ term. Note that there is no problem like this for the harmonic oscillator since $V(0)=0$.

The form of the solution for these two examples was very simple. Other potentials will have much more complicated solutions which probably cannot be found exactly. However, it may be useful to find approximate solutions using variational techniques or other methods. We have not yet investigated any of the various possibilities.

## VI. CONCLUSIONS

We have shown how a nonrelativistic theory of bosons based on currents may be solved and how to extract the information from the solutions necessary to give a complete description of single-particle quantum mechanics in terms of currents and densities. Still lacking is a firm mathematical understanding of functional techniques, especially functional integration. Nevertheless, the success we have had in these attempts convinces us that the techniques and the formulation are basically sound. Furthermore, we expect the methods described here and the insight gained into the nature of the solutions of the functional equations will prove useful in the approximate solution of many-body problems and in the solution of relativistic theories, where the formulation in terms of currents may have some real advantages.

*Note added in proof.* Since this paper was written, we have learned how to define a measure and scalar product for the Hilbert space. The functional integration must be done on the set of eigenvalues $\rho(x)$ of the physical density operator $\rho(x)$, and, as indicated by Percus in the book "The Many Body Problem," these are just $\delta$ functions

$$\rho(x)=\sum_{j=1}^{Q}\delta(x-a_i),$$

see also a recent unpublished report by David Gross. The functional integration is simply the integration over the $Q$ coordinates $a_i$. Direct calculation verifies that this yields all expectation values and the configuration-space wave functions.

Gross argues that the existence of the operator $\delta/\delta\rho(x)$ forces certain undesirable features on the theory. Note that the dynamical equations employ functional derivatives only in the form $\rho(x)\nabla[\delta/\delta\rho(x)]$, which does not share the properties that Gross ascribes to $\delta/\delta\rho(x)$. In particular, it does not connect states of different $Q$. In fact, the operator $\delta/\delta\rho(x)$ does not exist for the above-mentioned scalar product—its matrix elements are all infinite.

Because we are working formally in a much larger space than Fock space, there may be the possibility of either not obtaining the entire spectrum of $H$ or obtaining too large a spectrum. One obtains the latter if there exists a set of states $\Psi$ such that $(H-E)\psi=0$ and if all of the states in the set have norm zero. We have not been able to construct such states, and suspect that they do not exist. The other possibility arises if there exist states such that $(H-E)\psi=\chi$ where $\chi$ has zero norm, and there is no $\psi$ satisfying formally $(H-E)\psi=0$. We have not investigated this situation. We would like to thank David Gross for making his report available to us. See also Hong-mo Chan and J. E. Valatin, Nuovo Cimento **19**, 118 (1960), who have done some very similar work.

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[4] C. N. Yang, Phys. Rev. **168**, 1920 (1968).