be approximately the same as that given by Eq. (12). Also the introduction of the usual subtraction constant to correct the threshold behavior is not physically acceptable because the subtraction constant may contribute a force different from that of Eq. (12). If we were to treat the formally subtracted dispersion relation with corrected threshold behavior, the dynamics would be meaningful only were we to include a short-range force corresponding to $V_{l,3}{}^{I}(s)$, because the contributions from the subtraction constant may be smaller than those coming from the dynamical short-range force $V_{l,3}{}^{I}(s)$. On the other hand, if the Regge parameters (19c) turn out to be the correct ones, it would be necessary to introduce either a ρ' Regge pole with very large reduced residue function near $s=0$, or unknown parameters such as CDD poles. In conclusion, with the present phenomenological determination of the ρ and ρ' Regge poles, we can at least say that there is some possibility that the Frye-Warnock conjecture is actually satisfied and that we can eliminate those unknown parameters used to satisfy the threshold behavior.

ACKNOWLEDGMENT

The author is grateful to Dr. P. Fishbone for his help in improving the manuscript.

PHYSICAL REVIEW VOLUME 175, NUMBER 5 25 NOVEMBER 1968

Crossing-Symmetric Rising Regge Trajectories*

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(Received 22 July 1968)

Corrections due to the exchange of the resonances lying on the leading crossed-channel Regge trajectory are calculated for a linearly rising Regge trajectory in a single-channel, single-trajectory model. The corrections are small, and the equations force no restriction on the slope or intercept of the trajectory. The integral equations for the Regge parameters are derived, and detailed numerical results for the ρ trajectory are given. A method for determining the slope of the trajectory is proposed.

I. INTRODUCTION

FOR several years there has been increasing interest in applying dispersion relations for the Regge parameters to bootstrap calculations as an alternative to the more usual approximations based on the N/D method. We wish to report here some new developments in this general direction.[1]

The basic approach consists of deriving approximate expressions for the imaginary parts of the Regge parameters from unitarity and from a "potential," and inserting these into the dispersion relations. This leads to integral equations for the trajectory which are rather complicated, but which can be solved by computers. The method, at various levels of sophistication, has been extensively tested in potential theory, and is capable of yielding trajectories which are in quite good agreement with the exact ones.[2] The extension to field theory and to bootstrap calculations is, however, considerably more difficult.

Basically, there are four differences between potential theory and a full-relativistic bootstrap model which cause problems. These are (i) the difficulty of constructing a credible field-theoretic "potential," which can be used in the same way as a potential in the Schrödinger equation; (ii) the fact that trajectories apparently rise—perhaps linearly—in the real world, while they approach negative integers in potential theory; (iii) the fact that more trajectories are likely to be numerically important in calculating a field-theoretic amplitude than in potential theory; and (iv) the perennial problem of many channels and multiparticle intermediate states in the relativistic case, which is not present in potential theory.

It is to the solution of the first of these difficulties that we primarily address ourselves in this paper. First, let us elaborate a bit on the other three.

The mechanics of incorporating rising trajectories into the general framework of dispersion relations for the Regge parameters has been understood by Mandelstam and by Epstein and Kaus.[1] However, their prescription, which includes using a twice-subtracted dispersion relation for the Regge trajectory, introduces two subtraction constants, and hence two new param-

* Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract Nos. AT(11-1)-68 and AT-(11-1)-34 for the San Francisco Operations Office, U. S. Atomic Energy Commission.

[1] S. C. Frautschi, P. Kaus, and F. Zachariasen, Phys. Rev. **133**, B1607 (1964); S. Mandelstam, *ibid.* **166**, 1539 (1968); G. Epstein and P. Kaus, *ibid.* **166**, 1633 (1968).

[2] H. Cheng and D. Sharp, Phys. Rev. **132**, 1854 (1963); D. Hankins, P. Kaus, and C. J. Pearson, *ibid.* **137**, B1034 (1965); W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, *ibid.* **140**, B1595 (1965); **141**, 1513 (1966); J. Blue, Ph.D. thesis, California Institute of Technology, 1966 (unpublished).

eters, into the system of equations. (These parameters are essentially the slope and intercept of the trajectory.) Furthermore, they find (and this is true with our improvements as well) that most of the physically interesting consequences of the calculations are controlled by these two parameters, and only fine details in the behavior of the Regge parameters come from the rest of the mathematics. It is, therefore of crucial importance to a successful bootstrap calculation to determine the subtraction constants. Mandelstam[1] attempts to do this by using the "finite-energy sum rules."[3] However, the values of the parameters obtained in this way depend rather sensitively on both the cutoff in the sum rules and on the value of the momentum transfer at which the sum rule is evaluated.[4] This approach, therefore, amounts to little else than replacing the two unknown parameters with two other also unknown parameters.

It seems likely to us (and this will be explored in more detail in Sec. IV) that the values of the subtraction constants have their origin in the last two difficulties on our list—namely, lower trajectories and more channels—so that it is appropriate to turn next to a discussion of these.

The introduction of additional trajectories into the problem poses no difficulties of principle. However, practical calculations, even in potential theory and with the simplest approximation to the unitarity relation, become prohibitively complicated with more than two trajectories.[5] It is possible that families of parallel trajectories can be handled, and perhaps this is worthy of further study, but it is unclear whether such a picture of the trajectories has much to do with nature. Essentially the same dismal picture applies to the introduction of large numbers of additional channels. Numbers of two-body channels can be handled in principle, but practice is another matter, and contributions from multiparticle states cannot even be written down as yet. There is the possibility of including a phenomenological description of other channels through an inelasticity factor,[6] but this introduces unknown parameters and defeats any attempt at a complete bootstrap calculation. It seems clear that the most promising way to proceed is to find approximations in which other features, such as analyticity, crossing, and unitarity, are handled crudely enough that the resulting simplicity of the equations permits a many-channel, many-trajectory calculation. If this can be done, then it may be possible to calculate the two subtraction constants for a given trajectory, and insert those values into the kind of approximation method that we are describing here, in order to calculate (hopefully with some reliability) the detailed behavior of the trajectory.

Let us now leave the depressing subject of what we cannot handle and return to a discussion of difficulty (i): how to introduce an analog of the potential in the Schrödinger equation. It is evident that this is to be done through the use of crossing symmetry, which we use to calculate the partial-wave amplitude in the direct (s) channel, as explained in Sec. II. The partial-wave amplitude is written by the Froissart-Gribov formula as a product of the imaginary part of the amplitude and a Q function, integrated from $z=z_0$ to $z=\infty$, where z is the cosine of the s-channel scattering angle. We then make the assumption that the amplitude can be described, for z less than some value N, in terms of narrow-width resonances in the crossed channel, and, for z greater than N, in terms of the s-channel Regge poles. This assumption is in the same spirit as the finite-energy sum rules, but because of the presence of the Q function in the integrand, our "sum rule" is much more convergent than the finite-energy sum rules themselves.[7] The resulting representation of the partial-wave amplitude, together with the fact that the partial-wave amplitude is known through unitarity at $l=\alpha^*$, where α is some Regge pole, allows us to express the imaginary part of the trajectory and its residue in terms of the resonances in the crossed channel. Finally, if we assume that there is only one crossed-channel trajectory and that the crossed-channel resonances all lie on it, then we obtain a closed system of equations.

Our explicit calculation and results will be for the ρ trajectory, and we keep the $\pi\pi$ channel only. Thus we assume that the $\pi\pi$ amplitude is controlled asymptotically by the ρ trajectory, that all low-energy resonances in the $\pi\pi$ system are due to states lying on the ρ trajectory, and that the amount of any of these states leaking out into other channels is small.

These are evidently drastic assumptions. We therefore prefer to look on our calculation more as a test of how important improvements in difficulty (i) are as compared to improvements in difficulties (ii)–(iv), rather than as a reliable calculation of the properties of the ρ trajectory.

Our results show that the ρ trajectory produced by our integral equations is still almost a straight line, and therefore still largely controlled by the two subtraction constants which we must specify as input parameters. Solutions do in fact exist for all values of these parameters, so they are not restricted in any way, and therefore are not predicted by the theory. Agreement with experimentally known properties of the ρ trajectory is good, but this is because the experimental trajectory is also basically straight, and agreement is achieved only by adjusting the two input parameters.

[3] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters **19**, 402 (1967).
[4] B. R. Desai, Y. Shan, and P. Kaus (to be published).
[5] C. J. Pearson, Ph.D. thesis, University of California at Riverside, 1965 (unpublished).
[6] See Epstein and Kaus (Ref. 1).
[7] F. Arbab and R. C. Slansky, Phys. Rev. **171**, 1785 (1968).

The primary conclusion is, therefore, that our more reliable way of putting in crossing—that is, of constructing a "potential"—is not a particularly important improvement, and any of the cruder and simpler methods, such as the "universal-trajectory approximation,"[1] is just as good. This, incidentally, is in complete contrast to the situation in potential theory, where improvements to the universal approximation are crucial to obtaining good agreement with the exact trajectories.[2] As a consequence, we believe that the most important direction for future work is to attempt to simplify treatments of analyticity, unitarity, and (perhaps) crossing, to the point where many channels and many trajectories can be successfully handled in practice.

II. EQUATIONS FOR TRAJECTORY FUNCTIONS

The integral equations for the Regge parameters are derived using the analyticity properties derived from potential theory[1] for $\alpha(s)$ and the residue $\beta(s)$, and unitarity, which relates $\alpha(s)$ and $\beta(s)$. The assumptions and equations for the single rising trajectory are as follows:

(a) The trajectory function $\alpha(s)$ is real analytic with only a right-hand cut, and does not rise faster than linearly in s. It satisfies the dispersion relation

$$\alpha(s)=A+B(s-s_0)+\frac{s-s_0}{\pi}\int_{s_t}^{\infty}\frac{ds'}{(s'-s)(s'-s_0)}\,\mathrm{Im}\alpha(s'). \quad (1)$$

Here A and B are the (arbitrary) subtraction parameters, s_0 is the subtraction point, and s_t is the s-channel threshold.

(b) The residue $\beta(s)$ is a real analytic function when multiplied by the threshold factor $(s-s_t)^{-\alpha(s)}$. We define a function $b(s)$, which has no zeros and goes asymptotically to a constant, so that we can write a dispersion relation for $\ln b(s)$,

$$b(s)\equiv G(s)\beta(s)(s-s_t)^{-\alpha(s)}=g^2\exp\left(\frac{(s-s_0)}{\pi}\int_{s_t}^{\infty}\frac{ds'}{(s'-s)(s'-s_0)}\times\mathrm{Im}[\ln b(s')]\right). \quad (2)$$

Models for $G(s)$ will locate the zeros of $\beta(s)$ at the poles of $G(s)$ and determine the asymptotic properties of $\beta(s)$.

(c) The residue function $\beta(s)$ is closely linked to $\mathrm{Im}\alpha(s)$ by unitarity. We define a function $F(s)$ by the equations

$$\beta(s)=\frac{\mathrm{Im}\alpha(s)}{\rho(s)}F(s), \quad (3)$$

and

$$\rho(s)=[(s-s_t)/s]^{1/2},$$

where $F(s)$ is calculated from unitarity. Elastic unitarity for the partial-wave amplitude is

$$a(l,s)-a^*(l^*,s)=2i\rho(s)a(l,s)a^*(l^*,s). \quad (4)$$

Taking the residue of the pole in $a^*(l^*,s)$ at $l=\alpha^*$, we obtain

$$1=-2i\rho(s)a(\alpha^*(s),s). \quad (5)$$

For trajectories with small $\mathrm{Im}\alpha(s)$, $\alpha^*(s)$ is expected to be closer to $\alpha(s)$ than to any other singularity in the partial-wave amplitude. The leading term in $a(l,s)$ near $l=\alpha^*$ is

$$a(l,s)=\beta(s)/[l-\alpha(s)], \quad (6)$$

which, when substituted into Eq. (5), gives

$$\beta(s)=[\mathrm{Im}\alpha(s)]/\rho(s). \quad (7)$$

This is the so-called "universal" approximation and corresponds to $F(s)=1$. More generally, this suggests Eq. (3) as a useful form for $\beta(s)$. Better forms for $a(l,s)$ than Eq. (6) lead to functions $F(s)$ different from unity.

In potential theory, where trajectories do not rise but retreat to the left-hand l plane, $\mathrm{Im}\alpha(s)$ does not remain small and Eq. (7) is not a good approximation. Better representations for $a(l,s)$ than Eq. (6), such as the Khuri, modified Khuri, Cheng, and modified Cheng[1,2] representations for $a(l,s)$ provide different functions $F(s)$, which lead progressively to more accurate relationships between the residue $\beta(s)$ and the trajectory $\alpha(s)$. In particular, forcing the correct asymptotic behavior for $a(l,s)$ by smoothly subtracting out the partial-wave Born term proved to be the most important correction. It would seem reasonable after the potential-theory experience to expect that the most important contribution to $F(s)$ would come from the exchange of trajectories.

We now calculate these corrections to $F(s)$ in a model for the ρ trajectory that includes only the $\pi\pi$ channel. The odd-signature partial-wave amplitude is given by

$$a^-(l,s)=\frac{1}{\pi}\int_{z_0}^{\infty}dz_s Q_l(z_s)[A(s,z_s)-A(s,-z_s)]_{z_s}, \quad (8)$$

where $[A(s,z_s)]_{z_s}$ stands for the discontinuity in z_s of the amplitude $A(s,z_s)$, and where $z_s=[1+2t/(s-4m_\pi^2)]$, $z_0=[1+2t_0/(s-4m_\pi^2)]$, and $t_0=4m_\pi^2$.

Our model for $A(s,t)$ is the following: For $t>s$, $A(s,t)$ is given by the leading Regge pole in the s channel; For $t<s$, $A(s,t)$ is given by the leading Regge pole in the t channel, i.e., the resonances lying on the ρ trajectory. This model for $A(s,t)$ is completely determined by the ρ

Regge parameters,

$$A(s,t)=\pi[2\alpha(s)+1]\beta(s)\frac{P_{\alpha(s)}(z_s)-P_{\alpha(s)}(-z_s)}{2\sin\pi\alpha(s)},\quad t>s$$
$$=\pi[2\alpha(t)+1]\beta(t)\frac{P_{\alpha(t)}(z_t)-P_{\alpha(t)}(-z_t)}{2\sin\pi\alpha(t)},\quad t<s. \quad (9)$$

The integral in Eq. (8) now splits into two parts: one from z_0 to N ($t<s$), and the other from N to ∞ ($t>s$), where $N(s)=1+2s/(s-4m_\pi^2)\approx 3$. It should not matter much whether N is chosen at exactly this value or not, since it is just the value of z_s where we switch from one approximation, appropriate for $t\ll s$, to another, appropriate for $t\gg s$. Nevertheless, for reasons of symmetry between the s and t channels, we have generally preferred the choice $N\approx 3$.

The discontinuity of $A(s,z_s)$ for $t>s$ is

$$[A(s,z_s)-A(s,-z_s)]_{z_s}=\pi[2\alpha(s)+1]\beta(s)P_{\alpha(s)}(z_s), \quad (10)$$

and can readily be integrated with $Q_l(z_s)$.[8] The discontinuity for $t<s$ is much more complicated. In this region we assume that the amplitude can be represented as a sum of resonances lying on the ρ trajectory, and let the widths shrink to zero. This gives, for $t<s$,

$$[A(s,z_s)-A(s,-z_s)]_{z_s}=\sum_{n\,\mathrm{odd}}2\pi[2n+1]\frac{\beta(t_n)}{\alpha'(t_n)}$$
$$\times P_n\left(1+\frac{2s}{t_n-4m_\pi^2}\right)\left(\frac{2}{s-4m_\pi^2}\right)\delta(t-t_n). \quad (11)$$

The sum on n is over the resonances lying on the ρ trajectory. Thus, $n=1, 3, 5, \cdots$, and $\alpha(t_n)=n$. We now substitute Eqs. (10) and (11) into (8) and find that

$$a(l,s)=I_1(l,s)+\frac{\beta(s)}{l-\alpha(s)}I_2(l,s), \quad (12)$$

where

$$I_1(l,s)=\sum_n^{t_n<s}2(2n+1)\frac{\beta(t_n)}{\alpha'(t_n)}P_n\left(1+\frac{2s}{t_n-4m_\pi^2}\right)\left(\frac{2}{s-4m_\pi^2}\right)$$
$$\times Q_l\left(1+\frac{2t_n}{s-4m_\pi^2}\right) \quad (13a)$$

and

$$I_2(l,s)=-(N^2-1)^{1/2}[P_{\alpha(s)}(N)Q_l^1(N)$$
$$-Q_l(N)P_{\alpha(s)}^1(N)]. \quad (13b)$$

Finally, we substitute this model for $a(l,s)$ into the unitarity relation, Eq. (5):

$$\beta(s)=\frac{\mathrm{Im}\alpha(s)}{\rho(s)}F(s)=\frac{\mathrm{Im}\alpha(s)}{\rho(s)}[1+2i\rho(s)I_1(\alpha^*(s),s)]/$$
$$I_2(\alpha^*(s),s). \quad (14)$$

[8] *Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 170. At any l, the integral is to be considered in the continuation from $l>\mathrm{Re}\alpha$.

It is clear that $I_2(\alpha^*(s),s)$ plays the role of better representations here, like the Khuri or Cheng representations,[9] and $I_1(\alpha^*(s),s)$ plays the role of their "modification." [1,2] Actually, in contrast to potential theory, we find that I_1 is numerically too small to play a really significant role in modifying the rising trajectory (see Sec. III).

The universal approximation is recovered by setting $N=1$, since then the whole amplitude is represented by just the s-channel Regge pole. [For $N=1$, $I_1=0$, $I_2=1$, and $F(s)=1$.]

To completely determine the system of equations in terms of the trajectory, we must specify $G(s)$ in Eq. (2). The trajectory which rises linearly for positive s, falls linearly for negative s. Since the Mandelstam symmetry demands that the residue vanish whenever $\alpha(s)$ is a negative half-integer smaller than $-\frac{1}{2}$, we choose

$$G(s)=\Gamma(\alpha(s)+\tfrac{3}{2})e^{cs}. \quad (15)$$

We assume $\mathrm{Im}\alpha(s)$ vanishes asymptotically, and we must decide whether it vanishes like a power of s or exponentially with s as $s\to\infty$. If we require that $b(s)$ in Eq. (2) go to a constant as $s\to\infty$, the constant c in Eq. (15) controls the asymptotic behavior of $\mathrm{Im}\alpha(s)$. From Eq. (1), the real part of $\alpha(s)$ behaves like

$$\alpha(s)\underset{s\to\infty}{\to}\bar{A}+Bs. \quad (16)$$

Substituting Eq. (3) into Eq. (2), we have

$$b(s)\underset{s\to\infty}{\to}\Gamma(\bar{A}+Bs+\tfrac{3}{2})\frac{\mathrm{Im}\alpha(s)}{\rho(s)}$$
$$\times\exp[cs-(\bar{A}+Bs)\ln(s-s_t)]\to\text{const}. \quad (17)$$

For $\mathrm{Im}\alpha(s)$ vanishing no faster than a power, Eq. (17) implies that

$$c=B-B\ln B. \quad (18)$$

The usual scale of energy s_0 in the Regge formula $(s/s_0)^{\alpha(t)}$ is naturally given by the slope of the trajectory: $s_0=1/B$. If $\mathrm{Im}\alpha(s)$ decays exponentially, then c is an arbitrary parameter which is smaller than that given by Eq. (18). We do not investigate exponentially falling $\mathrm{Im}\alpha$, since the corrections from $F(s)$ would be even smaller than those which we calculate using Eq. (18). The power with which $\mathrm{Im}\alpha(s)$ decays is s^{-1}. A higher-power behavior, such as s^0, can be forced, but then we must introduce another pole into $G(s)$. The corresponding zero in $\beta(s)$ is a new "indeterminacy point," and is at a value of s which is arbitrary, as far as we know. We have not investigated this possibility either.

[9] N. N. Khuri, Phys. Rev. **130**, 429 (1963); H. Cheng, *ibid.* **144**, 1237 (1966).

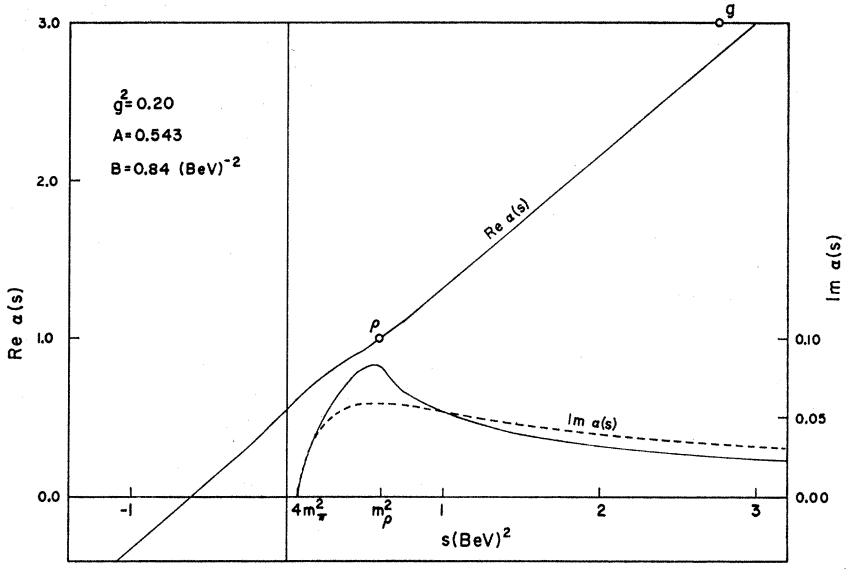

FIG. 1. Reα(s) and Imα(s). Solution of Eqs. (19) and (20) corresponding to the ρ trajectory. The subtraction point is $s_0=0$. The other parameters are $A=0.543$, $B=0.84$ BeV^{-2}, and $g^2=0.20$. The dashed curve is the solution for Imα(s) in the universal approximation, Eq. (7), with the same parameters. Both solutions for Reα(s) are nearly identical. The point g at Reα=3 and $s^{1/2}=1.660$ BeV corresponds to the possible Regge recurrence of the ρ.

Finally, we substitute Eqs. (1) and (3) into Eq. (2) and obtain

$$\mathrm{Re}\alpha(s)=A+(s-s_0)\times\left(B+\frac{1}{\pi}P\int_{s_t}^{\infty}\frac{ds'}{(s'-s)(s'-s_0)}\mathrm{Im}\alpha(s')\right) \quad (19)$$

and

$$\mathrm{Im}\alpha(s)=g^2\frac{\rho(s)(s-s_t)^A}{|G(s)F(s)|}\times\exp\left\{(s-s_0)\left[B\ln(s-s_t)+\frac{1}{\pi}P\int_{s_t}^{\infty}\frac{ds'}{(s'-s)(s'-s_0)}\times\left(\mathrm{Im}\alpha(s')\ln\frac{s-s_t}{s'-s_t}+\phi_F(s')+\phi_G(s')\right)\right]\right\}, \quad (20)$$

where

$$F(s)\equiv|F(s)|e^{i\phi_F(s)}$$

is given by Eq. (14), and

$$G(s)\equiv|G(s)|e^{i\phi_G(s)}$$

is given by Eq. (15).

Summing up, Eqs. (19) and (20) result from assuming that $\alpha(s)$ and $\beta(s)(s-s_t)^{-\alpha(s)}$ are real-analytic functions with only a right-hand cut and that the trajectory rises linearly. We have made the following approximations. In the neighborhood of $l=\alpha$, the partial-wave amplitude $a(l,s)$ is forced to obey elastic unitarity. (No inelastic channels are explicitly included.) The amplitude $A(s,t)$ is approximated by a single Regge trajectory in the s channel for $t>s$ and the same trajectory of resonances in the t channel for $s>t$. The narrow-width-resonances approximation is used for the t-channel trajectory to calculate the discontinuity in z_s. We have also used the option of letting $\mathrm{Im}\alpha(s)\to s^{-1}$ as $s\to\infty$, since all other options imply more arbitrary parameters.

In Sec. III we see that the "potential" coming from exchanging a single trajectory in a single channel is too weak to force or explain the slope of the trajectory.

III. NUMERICAL RESULTS

Equations (19) and (20), together with Eqs. (14) and (15), form a closed set of integral equations, which can be solved by iteration in terms of the three parameters A, B, and g^2. For a given set of A, B, and g^2, we start the iteration by putting $\mathrm{Im}\alpha=0$ and by calculating Reα from Eq. (19). Then we obtain $F(s)$ from Eqs. (13) and (14) and $G(s)$ from Eq. (15). The new Imα can now be calculated from Eq. (20) and the iteration starts over again.

Before presenting the details of the numerical solutions, we discuss how large the deviations from the universal approximation[6] are expected to be. We show that $F(s)$, defined by Eqs. (13) and (14), differs from 1 by a term of order Imα when evaluated using the solutions α and β of the universal approximation. Since Imα is small in the universal approximation, $F(s)$ is of order 1. Thus, unless there is an unexpected instability because of their nonlinearity or complexity, the solutions to Eqs. (19) and (20) should not differ qualitatively from the corresponding old ones.[6] Since for a reasonable range of A, B, and g^2 the old solutions give good phenomenological fits to all the known constraints of the ρ trajectory, we can conclude two things: (a) The solutions to Eqs. (19) and (20) will give a good phenomenological ρ trajectory; and (b) we shall not be able to accomplish a bootstrap calculation since the modification does not impose any restriction on the parameters A, B, and g^2. The reason that $F(s)$ stays close to 1 depends crucially on the fact that Imα is small and $\mathrm{Im}\beta=0$ for the old solutions,[6] which give a ρ meson with an acceptable width.

[10] See Ref. 8, p. 129, Eq. (26), and p. 137, Eq. (44).

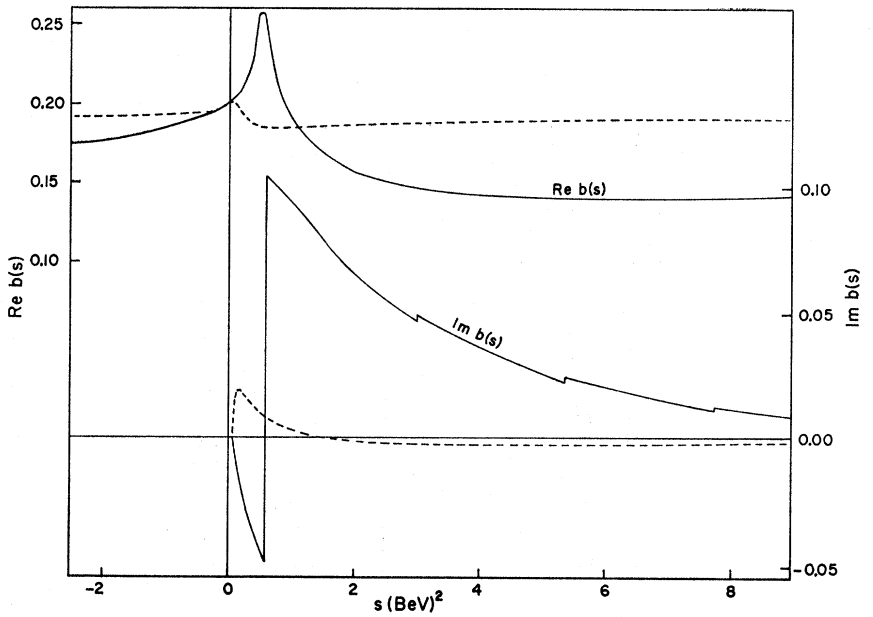

FIG. 2. Re$b(s)$ and Im$b(s)$. The reduced residue $b(s)$ is defined in Eqs. (2) and (14) in terms of the solution of Eqs. (19) and (20). The parameters s_0, A, B, and g^2 are the same as in Fig. 1. The dashed curves are the results of the universal approximation, Eq. (7). The "notches" in Im$b(s)$ result from the notches in $F(s)$. (See Fig. 3.)

Put $l=\alpha^*(s)$ in Eq. (19) for I_1 and I_2. For small Imα, we expand I_1 and I_2 in powers of Imα, using appropriate expansion formulas for $P_l{}^\mu$ and $Q_l{}^\mu$.[10] It is easy to see that ReI_1 is of order Reβ or Imβ Imα (whichever is larger), ImI_1 of order Reβ Imα or Imβ, ReI_2 of order 1 and ImI_2 of order Imα. Now Eq. (13) can be written as a power series of Imα, starting from the universal approximation Re$\beta=$ (Imα)$/\rho$ and Im$\beta=0$. We conclude $|F(s)|$ to be of order 1 and $\phi_F(s)$ of order Imα.

Typical solutions are shown in some detail in Figs. 1–3. In Fig. 1 we plot Reα and Imα as functions of s and compare with the corresponding solution of the universal approximation.[6] We see that there is no significant difference in the two cases. The reduced residue function is shown in Fig. 2 and also compared to the corresponding old solution. In Fig. 3 we plot $|F(s)|$ and $\phi_F(s)$ as functions of s. The function $|F(s)|$ does not deviate very much from unity.

In Fig. 4 some of the main properties of the solutions are represented as functions of the parameters B and g^2. The parameter A is adjusted so that all solutions give a ρ meson with mass $m_\rho=770$ MeV. The figure shows contours of constant width Γ, intercept $\alpha(0)=\alpha(s=0)$ and $\alpha'(0)=d\alpha/ds|_{s=0}$, intercept $\alpha(s)=0$, and $\sqrt{s}$ where $\alpha(s)=3$.

IV. EXTENSIONS, IMPROVEMENTS, AND CONCLUSIONS

Our purpose has been to calculate the corrections to a linearly rising Regge trajectory due to the exchange of the resonances lying on the leading crossed-channel trajectory. These corrections are large in potential

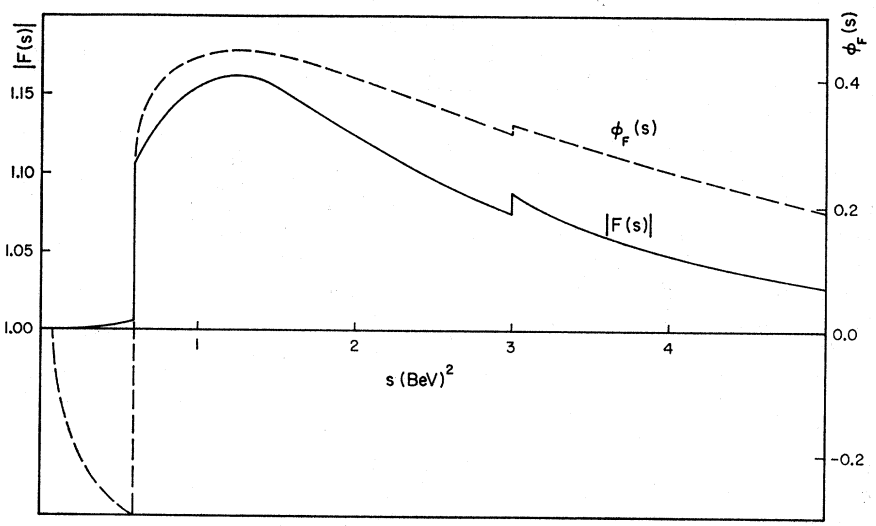

FIG. 3. $|F(s)|$ and $\phi_F(s)$. $F(s)$ is defined by Eqs. (13) and (14). The scale of $\phi_F(s)$ is in radians. Note that $|F(s)|$ does not differ much from 1. The "notches" in $F(s)$ result from the fact that the number of resonances included in I_1 increases with s.

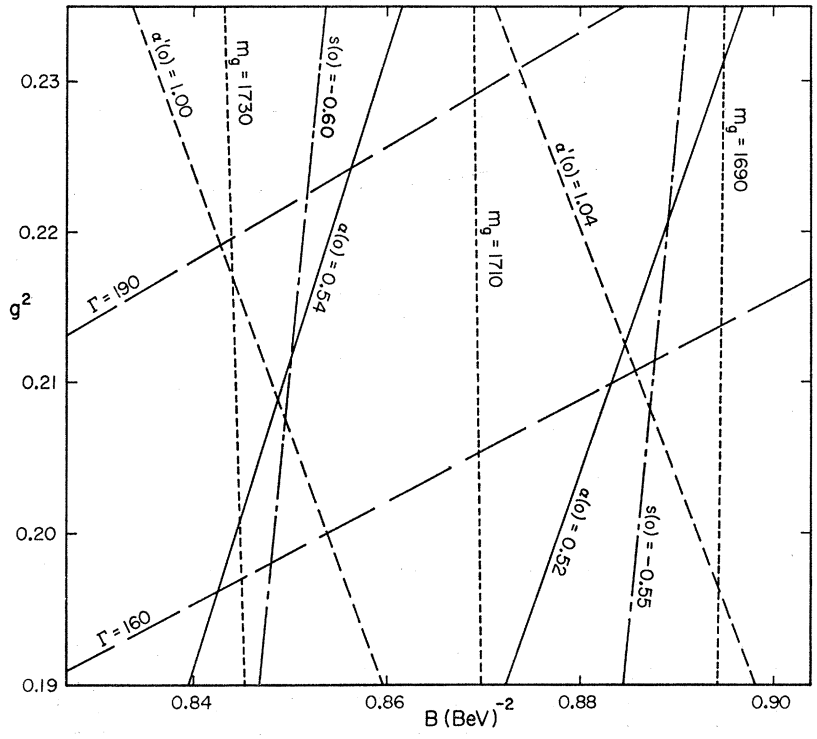

FIG. 4. Properties of solutions corresponding to ρ trajectory as functions of B and g^2. The parameter A, adjusted in each case to give $m_\rho=770$ MeV, is identical to the intercept $\alpha(0)\equiv\alpha(s=0)$ since the subtraction point is $s_0=0$. Also shown are contours of constant ρ width, Γ, in MeV; slope at $s=0$, $\alpha'(0)$, in BeV^{-2}; s for $\alpha=0$, $s(0)$, in BeV^2; and $\sqrt{s}$ for $\alpha=3$, m_g, in MeV. Γ is related to the Regge parameters by

$$\mathrm{Im}\alpha(M_\rho^2)=\alpha'(M_\rho^2)M_\rho\Gamma_\rho.$$

theory where the trajectories asymptotically approach negative integers, but in the rising-trajectory scheme, the trajectory calculated with crossing symmetry, the universal trajectory, or, for that matter, the straight-line trajectory $\alpha(s)=A+Bs$ do not differ much. Moreover, including the exchange forces in the context of a single-channel, single-trajectory model does not restrict the subtraction parameters A and B.

Our results indicate that rising-trajectory calculations which incorporate crossing symmetry even more accurately than we have will still produce trajectories that are very nearly straight lines. Thus the most important problem remaining is how to calculate the subtraction parameters. In this section we describe several approaches to understanding the subtraction parameters and the technical simplifications suggested by our calculations.

It has been proposed that the finite-energy sum rules give at least one relation between A and B. However, calculations of the intercept of the ρ trajectory are very dependent on the value of s at which the finite-energy sum rule is written. Although this s dependence can result from a defective model for $\mathrm{Im}A(s,t)$, it still remains to determine the trajectory slope. Within the context of a single-channel model, it appears very difficult to understand both subtraction parameters.

As stated in the Introduction, we conjecture that the physical origin of the subtraction parameters is the existence of many other channels and trajectories. Of course, the assumption that trajectories rise implies the existence of a great number of channels. So another way of stating the conjecture is that at any value of s the existence of the channels with thresholds above s contributes enough to $\alpha(s)$ to keep it rising linearly. Thus the slope of the trajectory is a phenomenological way of incorporating the higher-mass channels. The "force" which keeps the trajectory rising is *not* the exchange of crossed-channel trajectories, but the existence of many direct channels. We call this the "boost-Imα" model, since Imα receives a small boost at each new threshold. The rising trajectory results from the sum total of all these boosts. To see this more clearly, consider the contribution to Re$\alpha(s)$ from the nth channel with threshold at s_n to be

$$\mathrm{Re}\alpha_n(s)=\frac{1}{\pi}\int_{s_n}^{\infty}\frac{\mathrm{Im}\alpha_n(s')}{s'-s}ds'. \tag{21}$$

When s_n is much larger than s, we can consider Im$\alpha_n(s)$ to be concentrated near s_n and proportional to the coupling of the trajectory into that channel. We then obtain a contribution from channels with $s_n\gg s$ to Re$\alpha(s)$ of the form

$$\sum_{s_n\gg s}\frac{g_n^2}{s_n-s}\approx\sum_n\frac{g_n^2}{s_n}+\sum_n\left(\frac{g_n^2}{s_n^2}\right)s. \tag{22}$$

The first sum on the right-hand side is to be absorbed in the first subtraction constant A, and the second can be thought of as being the slope, B. Thus,

$$B=\sum_{s_n\gg s}\frac{g_n^2}{s_n^2}.$$

Whether this is sensible depends on whether or not this sum is approximately independent of s over a moderate energy range. This, in turn, depends on the slopes of the external trajectories and the couplings of the resulting channels into $\alpha(s)$, which are here represented as g_n^2.

If this conjecture is correct, then this model of infinitely many small boosts of $\alpha(s)$ through many channels bears a striking resemblance to considering only one other channel, say, the quark-antiquark channel, with a potential in that channel which gives rising trajectories, such as a harmonic-oscillator potential. In this sense, one may possibly think of quarks as a lumping together of all the higher-mass channels. Channels with thresholds before or in the neighborhood of s, on the other hand, must always be included explicitly.

Whether we think of the many-channel or the quark-antiquark boost, both pictures suggest that the "force" which causes trajectories to rise is not to be looked for in the exchange of the top trajectory.

In principle, this model of rising trajectories may be tested in the framework of the equations of Sec. II. However, in practice, even the two-channel or two-trajectory models are almost prohibitively difficult to solve. The imaginary and real parts of the Regge functions are so interrelated by unitarity that at first sight one seems forced to use the dispersion relations for the trajectory functions in order to learn anything about them at all. The problem, then, is to learn how to disentangle or suppress those aspects of the equations in Sec. II that force on us the chore of solving integral equations. In essence, we wish to approximate the analytic approach of solving dispersion relations by a more algebraic scheme. In the scheme we outline below, there is hope for solving the many–two-body channel problem, and thus some hope for understanding the subtraction parameters.

The change necessary in the equations of Sec. II to avoid the dispersion-relation approach is simply to take N large enough. The order-of-magnitude relations of Sec. III are changed, and we find that terms containing $\mathrm{Im}\beta$ in the unitarity relations can be neglected. The resulting sum rule, which we call the "unitarity" sum rule, is derived and discussed in detail in Ref. 7.

In the resonance approximation, the unitarity sum rule relates the coupling of the s-channel Regge pole to any one of the s channels to all of the crossed-channel resonance parameters, including the couplings to all the crossed channels. The weighting of the channels (phase space) is strongly dependent on the value of s at which new channels appear. Since the external particles also lie on Regge trajectories, the consistency requirements between external towers of channels and trajectories of bound states should lead to restrictions of the subtraction parameters for the trajectories involved. We conjecture that this set of algebraic equations determines the Regge parameters.

The technical simplification suggested by the results of Sec. III is that straight-line trajectories are excellent first approximations. However, there are two difficulties which are common to this type of calculation. (a) If we keep N fixed as the range of s increases into its asymptotic region, then the narrow-width-resonance model is not consistent with Regge behavior.[11] (b) If we learn how to let $N \to \infty$ as $s \to \infty$ so as to circumvent the theorem in Ref. 11, then it appears likely that lower trajectories make major contributions to $\mathrm{Im}A$.[12] Thus it would be necessary to calculate the lower trajectories simultaneously. Whether the intrinsic error caused by variations in N prevents us from calculating the lower trajectories is not yet clear.[13]

[11] J. E. Mandula and R. C. Slansky, Phys. Rev. Letters **20**, 1402 (1968).

[12] C. J. Goebel (private communication).

[13] One of the authors (RCS) would like to thank Dr. J. E. Mandula for a discussion of this point.