# Threshold Behavior of Partial-Wave Dispersion Relation and Short-Range Force*

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In a paper on the existence of a ghost-free solution of the unsubtracted partial-wave dispersion relation, Frye and Warnock conjectured that the positive-definite value at threshold of the unitarity integral on the physical region must be canceled by the short-range force (i.e., multiparticle-exchange contribution) arising from the first and second double spectral functions in order to guarantee the required $p$ wave and higher threshold behavior. We examined this conjecture explicitly in an unsubtracted dispersion relation for the $I=l=1$ pion-pion scattering amplitude. With certain choices of the $\rho$ Regge parameters, we show in our approximation scheme that this conjecture is nearly satisfied, and we can conclude that the so-called subtraction constant or threshold factor which is usually arbitrarily introduced to correct the threshold behavior could be expressed in terms of known physical quantities. However, with some other choice of $\rho$ Regge parameters, our numerical result shows that this conjecture is not satisfied by our one-$\rho$-Regge-pole approximation and indicates a need for additional Regge poles or Castillejo-Dalitz-Dyson poles.

## I. INTRODUCTION

In the unsubtracted $l$th partial-wave dispersion relation, each potential coming from single-particle exchange in the crossed channel generally has the required threshold behavior, but the term involving the unitarity integral over the physical region has a positive-definite value at threshold. Therefore, if we construct the dynamics with only the above two terms, we cannot obtain a ghost-free solution (except for the $s$ wave), because this partial-wave amplitude does not behave as $\sim(q_s^2)^l$ as $s$ approaches the threshold, where $q_s^2$ is the three-momentum squared in the c.m. system.

Of course, any model which violates the threshold behavior can be corrected by introducing a finite number of subtractions, by adding a function with a finite number of poles, or by adding the special type of Castillejo-Dalitz-Dyson (CDD) poles at threshold in the $N/D$ equations.[1]

But several questions remain unsolved at present. For example, what is the physical meaning and origin of these subtraction constants or pole parameters? Or should these parameters be explained in terms of the other known physical quantities at all? Usually we speculate that the subtraction constant might represent some effect of the inner region of the strong interaction, and if we decide that the threshold behavior can be corrected by adding the special type of CDD poles at threshold, we can also speculate that this threshold correction might represent the effect of an elementary-particle pole. This pole may be a bound state of, say, a nucleon-antinucleon or quark-antiquark pair whose channel is switched off from the relevant dynamics at some stage. From these considerations, we see that even in the subtracted dispersion relation with correct threshold behavior, we cannot definitely conclude that the output of some dynamics is the consequence of input alone or that it is the consequence of input plus some unknown effects represented by the subtraction constant.

In fact, Simmons[2] showed that in the $I=J=\frac{3}{2}$ $\pi$-$N$ scattering amplitude the choices of different forms of threshold correction factors very sensitively affect the resulting phase shift and resonance-pole position, and he argued that any dynamical calculation cannot be accepted as physically meaningful unless the origin or meaning of these threshold corrections is given uniquely.

Frye and Warnock[1] conjectured that in the Cini-Fubini approximation of the Mandelstam representation the short-range force coming from the first and second double-spectral function guarantees the required threshold behavior by cancelling the positive unitarity integral at threshold.

Since the term of the unitarity integral is always positive at threshold, the suggested short-range force must therefore always be negative definite at threshold (except for the $s$-wave amplitude).

Although Frye and Warnock argued that this short-range force might not be zero at threshold in the pion-nucleon scattering amplitude, it seems that there has been no clear example which demonstrates explicitly this suggested mechanism.

As an attempt to determine the origin of the threshold factor and its energy dependence, we show in this paper that in $p$-wave pion-pion scattering, with appropriate choices of the $\rho$ Regge parameters, such a negative short-range force not only exists but approximately cancels the positive value of the unitarity integral at threshold, and that this short-range force can be consistently determined by the dynamics. However, it will also be shown that with another possible choice of the $\rho$ Regge-pole parameters this conjecture is not satisfied by our one-$\rho$-Regge-pole approximation. In

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[1] G. Frye and R. L. Warnock, Phys. Rev. 130, 478 (1963).

[2] L. M. Simmons, Jr., Phys. Rev. 144, 1157 (1966). Further references on the problems of the threshold behavior of the partial-wave amplitude can be found in this paper.

this case we must add other Regge poles with the same quantum number as the $\rho$ Regge pole, or CDD poles. Although we cannot conclude which of these alternatives is true because of the lack of the experimental results on the $\rho$ Regge parameters, we can at least say that the Frye-Warnock mechanism is a possible one.

In Sec. II we derive the unsubtracted partial-wave dispersion relation with both long-range and short-range potentials for pion-pion systems under assumptions similar to those applied in our previous paper.[3] In Sec. III we show numerically for the $I=l=1$ pion-pion state how the unitarity-integral term and the short-range force nearly cancel to guarantee the threshold behavior. In Sec. IV, we discuss several physical implications of our result. In this paper we treat only the physical partial-wave amplitude and use the same notation as that used in I; the unit of energy is $\mu_\pi{}^2$.

## II. UNSUBTRACTED DISPERSION RELATION

To derive the unsubtracted partial-wave dispersion relation for pion-pion scattering, we apply the same assumptions and procedures as in I.

The $l$th partial-wave amplitude for isotopic spin $I$ in pion-pion scattering is given by

$$A_l{}^I(s)=\frac{1+(-1)^{I+l}}{2\pi q_s{}^2}\int_{t_0}^\infty dt\, A_t{}^I(s,t')Q_l\left(1+\frac{t'}{2q_s{}^2}\right), \quad (1)$$

where $A_t{}^I(s,t)$ is the $t$-channel absorptive part and we use the usual notation $s$, $t$, and $u$ for Mandelstam's variables with $q_s{}^2=\frac{1}{4}s-1$. We also used the following symmetry relation to derive Eq. (1):

$$A_u{}^I(s,t)=(-1)^I A_t{}^I(s,t). \quad (2)$$

Since we consider only the physical partial waves, Eq. (1) can be written as

$$A_l{}^I(s)=\frac{1}{\pi q_s{}^2}\int_{t_0}^\infty dt' A_t{}^I(s,t')Q_l\left(1+\frac{t'}{2q_s{}^2}\right). \quad (1')$$

Using Eq. (1′) and applying the same procedures and assumptions used to derive Eq. (27) in I, we obtain the following unsubtracted partial-wave dispersion relation:

$$A_l{}^I(s)=\frac{1}{\pi}\int_{s_0}^\infty ds'\frac{\rho(s')|A_l{}^I(s')|^2}{s'-s} +V_{l,1}{}^I(s)+V_{l,2}{}^I(s)+V_{l,3}{}^I(s), \quad (3)$$

where

$$\mathrm{Im}A_l{}^{I,\mathrm{el}(s)}(s)=\rho(s)|A_l{}^I(s)|^2 \quad (4)$$

and

$$\rho(s)=[(s-4)/s]^{1/2}. \quad (5)$$

$V_{l,1}{}^I(s)$ has the form of Eq. (12) in I multiplied by $(q_s{}^2)^l$ and represents the contributions coming from the small-$t$ parts of the first and third double spectral functions, while $V_{l,2}{}^I(s)$ has the form of Eq. (13) in I multiplied by $(q_s{}^2)^l$ and comes from the large-$t$ part of the third double spectral function. $V_{l,3}{}^I(s)$ is given by

$$V_{l,3}{}^I(S)=\frac{1}{\pi}\int_{-\infty}^{s_0-t_{\mathrm{in}}}\frac{ds'}{s'-s}\int_{t_{\mathrm{in}}}^{s_0-s'}dt'\frac{(-1)^l}{-2(-q_{s'}{}^2)}\frac{1}{\pi} \times\int_{s_0}^\infty ds''\frac{A_{st}{}^{I,\mathrm{el}(s)}(s'',t')}{s''-s'}P_l\left(-1-\frac{t'}{2q_{s'}{}^2}\right). \quad (6)$$

This is the left-hand cut contribution obtained by separating the $s$-channel elastic part of the first double spectral function into the right-hand cut unitarity integral term and the above left-hand cut term. It is evident that both $V_{l,1}{}^I(s)$ and $V_{l,2}{}^I(s)$ behave like $(q_s{}^2)^l$ when $s$ approaches threshold value. However, the first and last terms in Eq. (3) apparently do not vanish individually at threshold. As is well known, the $l$th partial-wave amplitude should behave like $(q_s{}^2)^l$ near threshold. Therefore if the ghost-free solution for Eq. (3) really exists, the first term and $V_{l,3}{}^I(s)$ must cancel each other (except for the special case of the $s$-wave amplitude).

Hereafter we denote the real part of the first term of Eq. (3) as

$$\mathrm{P.V.I.}=\frac{P}{\pi}\int_{s_0}^\infty ds'\frac{\rho(s')|A_l{}^I(s')|^2}{s'-s}. \quad (7)$$

To evaluate the $V_{l,3}{}^I(s)$, we approximate the $s$-elastic double spectral function $A_{st}{}^{I,\mathrm{el}(s)}(s,t)$ by the sum of the $t$-channel inelastic double spectral function[3]:

$$A_{st}{}^{I,\mathrm{el}(s)}(s,t)=\sum_{I'=0}^{2}\beta_{st}{}^{II'}B_{st}{}^{I',\mathrm{in}(t)}(t,s), \quad (8)$$

where $\beta_{st}{}^{II'}$ is the isotopic spin crossing matrix between the $s$ and the $t$ channels and $B_{st}(t,s)$ is the $t$-channel double spectral function. Approximation (8) can be derived under assumptions similar to those used in strip approximation.[3] By using Eq. (8), we can rewrite Eq. (6) as follows:

$$V_{l,3}{}^I(s)=\sum_{I'=0}^{2}\beta_{st}{}^{II'}\frac{1}{\pi}\int_{-\infty}^{s_0-t_{\mathrm{in}}}\frac{ds'}{s'-s}\frac{(-1)^l}{-2(-q_s{}^2)} \times\int_{t_{\mathrm{in}}}^{s_0-s'}dt'\frac{1}{\pi}\int_{s_0}^\infty ds''\frac{B_{st}{}^{I',\mathrm{in}(t)}(t',s'')}{s''-s} \times P_l\left(-1-\frac{t'}{2q_{s'}{}^2}\right). \quad (9)$$

As discussed in I, the inelastic part of the double spectral function can be well approximated by the exchanges of the leading Regge poles in the crossed channel. This argument is strongly supported by the phenomenological evidence that Regge-pole-like be-

[3] N. Masuda, preceding paper, Phys. Rev. 175, 2087 (1968), hereafter referred to as I.

havior exists not only in intermediate- and high-energy regions but even in quite low-energy regions. We shall assume that Regge-pole-like behavior begins at the lowest inelastic threshold.[4] If we assume that the inelastic amplitude rapidly becomes of diffractive type, this approximation may be a good one. By this approximation, Eq. (9) can be expressed in terms of the $s$-channel Regge poles as

$$V_{l,3}{}^I(s)=\sum_{I'=0}^{2}\beta_{st}{}^{II'}\beta_{ts}{}^{I'I}\sum_i\frac{1}{\pi}\int_{-\infty}^{s_0-t_{\rm in}}\frac{ds'}{s'-s}\frac{(-1)^l}{-2(-q_{s'}{}^2)}\times\int_{t_{\rm in}}^{s_0-s'}dt'\xi_iR_i{}^I(s',t')P_l\left(-1-\frac{t'}{2q_{s'}{}^2}\right),\quad(10)$$

where $\xi_i$ denotes the signature factor of the $i$th Regge pole. Also,[5]

$$R_i{}^I(s,t)=\tfrac{1}{2}\pi[2\alpha_i{}^I(s)+1]\gamma_i{}^I(s)(-q_s{}^2)^{\alpha_i{}^I(s)}\times P_{\alpha_i{}^I(s)}(-1-t/2q_s{}^2),\quad(11)$$

where $\alpha_i{}^I(s)$ and $\gamma_i{}^I(s)$ are the $i$th Regge trajectory and its reduced residue function, respectively. Inserting Eq. (11) into Eq. (10) and integrating in $t$, we obtain

$$V_{l,3}{}^I(s)=\sum_i\frac{1}{\pi}\int_{-\infty}^{s_0-t_{\rm in}}ds'\frac{1}{s'-s}\frac{(-1)^l}{-2(-q_{s'}{}^2)}\times\tfrac{1}{2}\pi[2\alpha_i{}^I(s')+1]\gamma_i{}^I(s')\times(-q_{s'}{}^2)^{\alpha_i{}^I(s')}F(\alpha_i{}^I(s'),Z_1(s')),\quad(12)$$

where

$$F(\alpha_i{}^I(s),Z_1(s))=\frac{1}{[l-\alpha_i{}^I(s)][\alpha_i{}^I(s)+l+1]}\{-(l+1)P_{l+1}(Z_1(s))P_{\alpha_i{}^I(s)}(Z_1(s))+[l-\alpha_i{}^I(s)]Z_1(s)P_l(Z_1(s))P_{\alpha_i{}^I(s)}(Z_1(s))+[\alpha_i{}^I(s)+1]P_l(Z_1(s))P_{\alpha_i{}^I(s)+1}(Z_1(s))\}\quad(13)$$

and

$$Z_1(s)=-1-2t_{\rm in}/(s-4).\quad(14)$$

Note also that in Eq. (10) $\sum_{I'=0}{}^2\beta_{st}{}^{I'I'}\beta_{ts}{}^{I'I}\xi_i$ becomes the unit matrix.

In Sec. III we show our numerical results for P.V.I. and $V_{l,3}{}^I(s)$ for the $l=I=1$ pion-pion state at threshold and in other physical regions.

## III. NUMERICAL TEST OF THE FRYE-WARNOCK CONJECTURE

In order to compute Eqs. (7) and (12) numerically, we first determine the various parameters which are to be used in the calculations.

Since in the $p$-wave pion-pion state the $\rho$-meson resonance dominates the amplitude in the low-energy region, we can approximate $A_{l=1}{}^{I=1}(s)$ in Eq. (7) by the Breit-Wigner one-level formula for the $\rho$-meson resonance,

$$A_l{}^I(s)=\frac{(q_s{}^2)^l\Gamma}{s_\rho-s-i(q_s{}^2)^l\rho(s)\Gamma},\quad(15)$$

where $l=I=1$ and $s_\rho$ is the $\rho$-meson mass squared. The reduced width $\Gamma$ is related to the experimental $\rho$-meson decay width $\Gamma_\rho$ by

$$\Gamma=s_\rho{}^{1/2}\Gamma_\rho/(q_{s\rho}{}^2)^l\rho(s_\rho).\quad(16)$$

Experimental data on the resonance energy and decay width of the $\rho$ meson fluctuate between 775 and 780 MeV and between 90 and 150 MeV, respectively.[6] We take tentatively the following two sets of choices:

$$m_\rho=774\text{ MeV},\quad\Gamma_\rho=128\text{ MeV},\quad(17a)$$

and

$$m_\rho=764\text{ MeV},\quad\Gamma_\rho=93\text{ MeV}.\quad(17b)$$

The set (17a) was given by Roos[7] and the set (17b) is the result of the $e^+$-$e^-$ colliding-beam experiment.[6] The convergence of the integral in Eq. (7) can be assured in principle if the elastic total cross section $\sigma^{\rm el}(s)$ decreases at least as $1/\ln s$ at high energies because $\rho(s)|A_l{}^I(s)|^2$ behaves at most as $\sigma^{\rm el}(s)/\ln s$ at high energies. However, if we approximate $A_l{}^I(s)$ in Eq. (7) by the one-level formula (15), this integral diverges logarithmically. We, therefore, cut off the high-energy side of the integral (7) at $s=122$ ($\sqrt{s}=1.5363$ BeV) in our numerical calculation.

As the leading Regge poles contributing to Eq. (12) we may list both the $\rho$ and $\rho'$ Regge poles. Since, however, we have no reliable data for the trajectory and reduced residue function of the $\rho'$ Regge pole and since we can generally say that the contribution of the $\rho'$ Regge pole is smaller than that of the $\rho$ Regge pole, we neglect the $\rho'$ contribution.

For the $\rho$ Regge trajectory we use the following form for the negative-$s$ region:

$$\left.\begin{array}{ll}\alpha_\rho(s)=2.42+3/[1-(s/130)], & \text{for }\alpha_\rho(s)\geq-0.99\\ \alpha_\rho(s)=-0.99, & \text{for }\alpha_\rho(s)<-0.99.\end{array}\right\}\quad(18)$$

Our choice of the $\rho$ Regge trajectory is almost equivalent to one determined phenomenologically.[8] Considering the lack of the reliable knowledge of the energy dependence of the $\rho$ Regge reduced residue function, we assume the following three forms:

$$\gamma_\rho(s)=0.063\times100^{\alpha_\rho(0)}\times\frac{(s_0-0.25s)^{\alpha_\rho(s)}}{400^{|\alpha_\rho(s)|}}\frac{1}{(-q_s{}^2)^{\alpha_\rho(s)}},\quad(19a)$$

$$\gamma_\rho(s)=0.063\times100^{\alpha_\rho(0)}\times\frac{(s_0-s)^{\alpha_\rho(s)}}{400^{|\alpha_\rho(s)|}}\frac{1}{1-s/50}\frac{1}{(-q_s{}^2)^{\alpha_\rho(s)}},\quad(19b)$$

[4] See also footnote 11 of Ref. 3.
[5] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).
[6] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).
[7] M. Roos, CERN Reports, 1967 (unpublished).
[8] W. Rarita, R. J. Riddell, Jr., C. B. Chiu, and R. J. N. Phillips, Phys. Rev. **165**, 1615 (1968).

and

$$\gamma_\rho(s)=0.063\times 100^{\alpha_\rho(0)} \times\frac{(s_0-s)^{\alpha_\rho(s)}}{400^{|\alpha_\rho(s)|}}\frac{1}{1-s/10}\frac{1}{(-q_s^2)^{\alpha_\rho(s)}}. \quad (19c)$$

All three reduced residue functions (19) have the numerical coefficient 0.063. This value is obtained by converting the ratio $\gamma_P(0):\gamma_\rho(0)=3.17:4.24$ with trajectory intercept $\alpha_P(0)=1$ and $\alpha_\rho(0)=0.48$ (Barger and Olsson's results[9] with energy scale 1 $\text{GeV}^2$) into our parametrizations (11) with $\alpha_P(0)=1$ and $\gamma_\rho(0)=0.58$ and with energy scale $2\mu_\pi^2$ at $s=0$. The value $\gamma_P(0)=0.01$ roughly corresponds to the pion-pion total cross section 15.7 mb at the high-energy limit. The $s$ dependence of the parametrizations (19) is obtained by converting the results of Rarita *et al.*[8] for $s$ with energy scale 2 $\text{GeV}^2$ to our form (11) under the assumption that the $P$ and $\rho$ Regge reduced residue functions have roughly same $s$ dependence for the negative $s$ region. For the $s$ dependence of the residue function we obtain the factor

$$100^{\alpha_\rho(0)}\times(s_0-s)^{\alpha_\rho(s)}/400$$

from the difference of the energy scales between that of Rarita *et al.* and ours.[10] In Eqs. (19), we have also suppressed the large-$(-s)$ parts of the reduced residue functions. The denominators $(-q_s^2)^{\alpha_\rho(s)}$ in Eqs. (19) are inserted to cancel the same term in Eq. (11).

To test the reliability of the Regge parameters given in Eqs. (18) and (19), we use the unitarity restriction. From the unitarity relation, we obtain[1]

$$\mathrm{Im}A_l^I(s)=\rho(s)\,|A_l^I(s)|^2+\frac{1-\eta^2(s)}{4\rho(s)}, \quad (20)$$

where $\eta(s)$ is the inelasticity factor. By splitting $\mathrm{Im}A_l^I(s)$ into elastic and inelastic parts, we obtain

$$\mathrm{Im}A_l^{I,\mathrm{el}(s)}(s)=\rho(s)\,|A_l^I(s)|^2$$

and

$$\mathrm{Im}A_l^{I,\mathrm{in}(s)}(s)=\frac{1-\eta_l^2(s)}{4\rho(s)}. \quad (21)$$

From Eq. (21), we set the following unitarity restriction for the imaginary part of the inelastic amplitude:

$$0\leq\rho(s)\,\mathrm{Im}A_l^{I,\mathrm{in}(s)}(s)\leq\tfrac{1}{4}. \quad (22)$$

On the other hand, $\mathrm{Im}A_l^{I,\mathrm{in}(s)}(s)$ can be expressed by the $l$th partial-wave projection of the imaginary part of the $t$-channel Regge-pole contributions, i.e., the imaginary part of Eq. (19) in I;

$$\mathrm{Im}A_l^{I,\mathrm{in}(s)}(s)=-\frac{1}{q_s^2}\int_{-\infty}^{0}dt\,\mathrm{Im}\left[Q_l\left(1+\frac{t}{2q_s^2}\right)\right] \times\sum_{I'=0}^{2}\sum_j\beta_{st}^{II'}R_j^{I'}(s,t),\quad \text{for}\quad s\geq s_{\mathrm{in}} \quad (23)$$

where $R_j^{I'}(s,t)$ has the same form as Eq. (11). Using the relation

$$\mathrm{Im}\left[Q_l\left(1+\frac{t}{2q_s^2}\right)\right] =-\tfrac{1}{2}\pi P_l\left(1+\frac{t}{2q_s^2}\right),\quad\text{for}\quad -1<1+\frac{t}{2q_s^2}<1$$
$$=\sin\pi l\;Q_l\left(-1-\frac{t}{2q_s^2}\right),\quad\text{for}\quad 1+\frac{t}{2q_s^2}<-1 \quad (24)$$

we rewrite Eq. (23) as

$$\mathrm{Im}A_l^{I,\mathrm{in}(s)}(s) =\frac{\pi}{2q_s^2}\int_{-s+s_0}^{0}dt\sum_{I'=0}^{2}\sum_j\beta_{st}^{II'}R_j^{I'}(s,t)P_l\left(1+\frac{t}{2q_s^2}\right) -\frac{\sin\pi l}{2q_s^2}\int_{-\infty}^{-s+s_0}dt\sum_{I'=0}^{2}\sum_j\beta_{st}^{II'}R_j^{I'}(s,t) \times Q_l\left(-1-\frac{t}{2q_s^2}\right). \quad (25)$$

The last term of Eq. (25) vanishes for the physical partial waves. Using Eq. (11) and the $\rho$ Regge parameters (18) and (19), we can numerically calculate the finite integral of the first term of Eq. (25). In Fig. 1, the calculated values of $\rho(s)\,\mathrm{Im}A_l^{I,\mathrm{in}(s)}(s)$ are shown as function of $s$ in three sets of the $\rho$ Regge parameters.

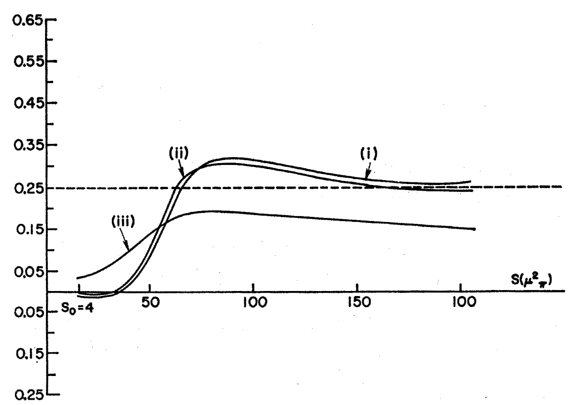


FIG. 1. Unitarity restriction of the Regge parameters. The values $\rho(s)\,\mathrm{Im}A_l^{I,\mathrm{in}(s)}(s)$ calculated with the $\rho$ Regge trajectory, Eq. (18) and the reduced residue functions, Eqs. (19), are shown. Line (i) is the value with the reduced residue function, Eq. (19a). Line (ii) is similarly the value with Eq. (19b). Line (iii) is similarly the value with Eq. (19c).

[9] V. Barger and M. Olsson, Phys. Rev. **146**, 1080 (1966).

[10] For the diffraction width of pion-pion scattering, see T. T. Chou and C. N. Yang, in Proceedings of the Conference on High-Energy Physics and Nuclear Structure, Weizmann Institute, 1967 (unpublished).

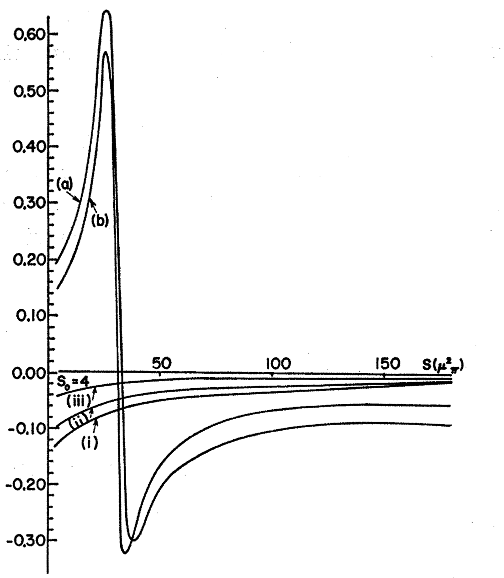


FIG. 2. Energy dependences of P.V.I. and $V_{l,3}{}^{I}(s)$. Line (a) is P.V.I. with $\rho$-resonance parameters, Eq. (17a). Line (b) is P.V.I. with $\rho$-resonance parameters, Eq. (17b). Line (i) is $V_{l,3}{}^{I}(s)$ calculated by the $\rho$ Regge trajectory, Eq. (18) and its reduced residue function, Eq. (19a). Line (ii) is $V_{l,3}{}^{I}(s)$ with Eqs. (18) and (19b). Line (iii) is $V_{l,3}{}^{I}(s)$ with Eqs. (18) and (19c).

From Fig. 1 we can say the combination of trajectory (18) and reduced residue function (19c) completely satisfies the restriction (22). The combinations of trajectory (18) and reduced residue function (19a), or (19b), slightly violate the restriction (22) in some regions, but we regard these latter two choices of Regge parameters as not differing very much.

Even though the integral of the Eq. (12) with the $\rho$ Regge parameters (18) and (19) is highly convergent, our actual numerical calculations were done by taking the lower limit of the integration region at $s=-1694$ in all cases.

In Fig. 2 the computed values of P.V.I. and $V_{l,3}{}^{I}(s)$ are shown as a function of $s$.

Considering the fact that the one-level formula (15) suppresses the low-energy side of the resonance peak and strongly enhances the high-energy side of the peak, it may be said that the threshold value of P.V.I. lies between about 0.1 and 0.2. As for the threshold value of $V_{l,3}{}^{I}(s)$, we cannot derive definite results because the calculated values depend strongly on the choices of the reduced residue functions. It seems, however, that the value of $V_{l,3}{}^{I}(s)$ at threshold probably lies between about $-0.05$ and $-0.13$.

As an immediate consequence of our numerical computations, Fig. 2 thus shows that at least all the threshold value of $V_{l,3}{}^{I}(s)$ have the required negative sign and that the absolute values of both P.V.I. and $V_{l,3}{}^{I}(s)$ at threshold do not differ by very much.

Considering possible further contributions to $V_{l,3}{}^{I}(s)$ from the $\rho'$ Regge pole, we can say that in the choices of (19a) and (19b) of the $\rho$ Regge reduced residue functions the major part of P.V.I. at threshold is canceled by $V_{l,3}{}^{I}(s)$, and that even if there remains some contribution of P.V.I. which is not canceled, this part would be small compared to the canceled one.

The energy dependence of $V_{l,3}{}^{I}(s)$ near threshold in Fig. 2 agrees with the result calculated by Igi and Kawai[11] by assuming the strict strip approximation.[5,12] Igi and Kawai treated the amplitude

$$\tilde{A}_{l}{}^{I}(s)=A_{l}{}^{I}(s)/(q_s{}^2)^l$$

and calculated the contribution coming from the large-$t$ part of the first double spectral function. Their short-range force $\tilde{V}_{l,3}{}^{I}(s)$ can be expressed in terms of $V_{l,3}{}^{I}(s)$ as

$$\tilde{V}_{l,3}{}^{I}(s)=[V_{l,3}{}^{I}(s)-V_{l,3}{}^{I}(s_0)]/(q_s{}^2)^l. \qquad (26)$$

They obtained a positive constant for this quantity. But from Fig. 2, we can say that in our approximation $\tilde{V}_{l,3}{}^{I}(s)$ decreases sharply as $s$ increases, in contradiction to the Igi-Kawai result.[11] Since the strict strip approximation applied in their paper is very doubtful[3] except at threshold, we do not consider the difference serious. They also discussed the effect of this term on the dynamical output. Because of the rather strong energy dependence for $\tilde{V}_{l,3}{}^{I}(s)$ which we find, contrary to their numerical results, reexamination of the effect of the short-range potential seems necessary. Numerical results with the choice (19c) for the $\rho$ Regge reduced residue function (Fig. 2) shows that $V_{l,3}{}^{I}(s)$ using only the $\rho$ Regge pole cannot cancel the value of P.V.I. at threshold. In order to preserve the correct threshold behavior of the partial-wave amplitude, we must in this case introduce an additional $\rho'$ Regge pole with almost the same contribution to $V_{l,3}{}^{I}(s)$ as that of the $\rho$ Regge pole, or we must introduce the CDD pole parameters into the amplitude.

## IV. DISCUSSION

In Sec. III we showed numerically that the repulsive force $V_{l,3}{}^{I}(s)$ arising from the large-$t$ part of the first double spectral function can cancel the major part of the value of P.V.I. at threshold.

Because of our crude approximation for both P.V.I. and the parameters used in $V_{l,3}{}^{I}(s)$, we cannot say whether the exact threshold behavior of the unsubtracted partial-wave dispersion relation is really guaranteed by dynamical short-range forces represented only by the $s$-channel Regge poles or whether it is necessary to introduce unknown CDD pole-type contribution. But from our numerical results [using (19a) and (19b) for the $\rho$ Regge reduced residue functions], even if it turns out to be necessary to introduce unknown arbitrary parameters, these contributions may be smaller than those coming from the dynamical origin. Therefore, we can say that in these cases the energy dependence of the threshold correction factor should

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[12] G. F. Chew, Phys. Rev. **129**, 2363 (1963).

be approximately the same as that given by Eq. (12). Also the introduction of the usual subtraction constant to correct the threshold behavior is not physically acceptable because the subtraction constant may contribute a force different from that of Eq. (12). If we were to treat the formally subtracted dispersion relation with corrected threshold behavior, the dynamics would be meaningful only were we to include a short-range force corresponding to $V_{l,3}{}^{I}(s)$, because the contributions from the subtraction constant may be smaller than those coming from the dynamical short-range force $V_{l,3}{}^{I}(s)$. On the other hand, if the Regge parameters (19c) turn out to be the correct ones, it would be necessary to introduce either a $\rho'$ Regge pole with very large reduced residue function near $s=0$, or unknown parameters such as CDD poles. In conclusion, with the present phenomenological determination of the $\rho$ and $\rho'$ Regge poles, we can at least say that there is some possibility that the Frye-Warnock conjecture is actually satisfied and that we can eliminate those unknown parameters used to satisfy the threshold behavior.

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# Crossing-Symmetric Rising Regge Trajectories*

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Corrections due to the exchange of the resonances lying on the leading crossed-channel Regge trajectory are calculated for a linearly rising Regge trajectory in a single-channel, single-trajectory model. The corrections are small, and the equations force no restriction on the slope or intercept of the trajectory. The integral equations for the Regge parameters are derived, and detailed numerical results for the $\rho$ trajectory are given. A method for determining the slope of the trajectory is proposed.

## I. INTRODUCTION

FOR several years there has been increasing interest in applying dispersion relations for the Regge parameters to bootstrap calculations as an alternative to the more usual approximations based on the $N/D$ method. We wish to report here some new developments in this general direction.[1]

The basic approach consists of deriving approximate expressions for the imaginary parts of the Regge parameters from unitarity and from a "potential," and inserting these into the dispersion relations. This leads to integral equations for the trajectory which are rather complicated, but which can be solved by computers. The method, at various levels of sophistication, has been extensively tested in potential theory, and is capable of yielding trajectories which are in quite good agreement with the exact ones.[2] The extension to field theory and to bootstrap calculations is, however, considerably more difficult.

Basically, there are four differences between potential theory and a full-relativistic bootstrap model which cause problems. These are (i) the difficulty of constructing a credible field-theoretic "potential," which can be used in the same way as a potential in the Schrödinger equation; (ii) the fact that trajectories apparently rise—perhaps linearly—in the real world, while they approach negative integers in potential theory; (iii) the fact that more trajectories are likely to be numerically important in calculating a field-theoretic amplitude than in potential theory; and (iv) the perennial problem of many channels and multiparticle intermediate states in the relativistic case, which is not present in potential theory.

It is to the solution of the first of these difficulties that we primarily address ourselves in this paper. First, let us elaborate a bit on the other three.

The mechanics of incorporating rising trajectories into the general framework of dispersion relations for the Regge parameters has been understood by Mandelstam and by Epstein and Kaus.[1] However, their prescription, which includes using a twice-subtracted dispersion relation for the Regge trajectory, introduces two subtraction constants, and hence two new param-

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