Theory of Pion-Pion Scattering*

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(Received 5 December 1968; revised manuscript received 31 May 1968)

Strong-interaction dynamics is formulated on two main assumptions. The first is similar to the one used in the Regge-pole–resonance interference model, and the other is that the potentials arising from the exchange of a few dominant particles and resonances in crossed channels can be accepted physically in the low-energy regions and that the convergent high-energy behavior of the potentials should be guaranteed by the Regge-pole theory. A long-range potential which satisfies both the low-energy and the high-energy boundary conditions stated above is introduced. The dynamical model with only Regge-pole exchange potentials (as in the new form of the strip approximation) does not produce low-energy resonance behavior as a dynamical output. We show that a potential which does produce the main low-energy resonance behavior as output can be constructed by an appropriate superposition of single-particle exchange potentials and Regge-pole exchange potentials according to our second assumption. Also, two other potentials arising from the large-t parts of the Mandelstam double spectral functions are defined under the first assumption.

I. INTRODUCTION

In this paper we present new approximation methods to the dynamical theory of strong interactions based on the following two main assumptions. The first is similar to that used in the Regge-pole–resonance interference model,[1] in which Regge poles and resonances are assumed to overlap in a broad intermediate-energy region. In contrast to the strict strip approximation,[2,3] this first approximation enables us to calculate directly the contributions to the scattering amplitude which arise from the large-t parts of the Mandelstam double spectral functions, not only the usual small-t parts but their corner regions as well. Two new potentials corresponding to the above large-t parts are defined. The large-t parts have usually been neglected without any quantitative justification,[4,5] or they have been simply treated as unknown parameters to be determined.[6] The second main assumption is based on the observations that a partial-wave long-range potential which is given approximately by the sum of exchanges of a few dominant particles and resonances in crossed channels has some physically acceptable meaning and reality, at least at and near elastic threshold[7]; that on the other hand, the partial-wave potential must satisfy the convergent high-energy behavior which is derived from the Regge-pole-dominance hypothesis at high energies[8] and that in a broad intermediate-energy region single-particle exchange potentials and Regge-pole exchange potentials overlap.

We apply these two basic assumptions to the construction of the potential in the dynamical equation in Sec. II. In Sec. III we give several justifications of our theory from both an experimental and a theoretical point of view. We shall also qualitatively discuss the comparison of the present theory with other theories.

II. FORMULATION OF DYNAMICS

We will treat pion-pion scattering as an example. The lth partial-wave amplitude for isotopic spin I is given by

$$A_l^I(s)=\frac{1+(-1)^{I+l}}{2\pi q_s^2}\int_{t_0}^{\infty}dt'A_t^I(s,t')Q_l\left(1+\frac{t'}{2q_s^2}\right),\quad(1)$$

where $A_t^I(s,t)$ is the t absorptive part, and we use the usual notation for the Mandelstam variables with $q_s^2=\frac{1}{4}s-\mu_\pi^2$.[9] In deriving Eq. (1), we used the symmetry property

$$A_u^I(s,t)=(-1)^I A_t^I(s,t).\quad(2)$$

From Eq. (1), we define a function

$$\bar{A}_l^I(s)\equiv\frac{1}{\pi(q_s^2)^{l+1}}\int_{t_0}^{\infty}dt'A_t^I(s,t')Q_l\left(1+\frac{t'}{2q_s^2}\right),\quad(3)$$

which can be uniquely continued to the complex l plane and coincides with physical partial-wave amplitudes divided by $(q_s^2)^l$ when $I+l$ is an even integer. $A_t^I(s,t)$ is expressed in terms of the Mandelstam double spectral functions by

$$A_t^I(s,t)=\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{A_{st}^I(s',t)}{s'-s}+\frac{1}{\pi}\int_{u_0}^{\infty}du'\frac{A_{tu}^I(t,u')}{u'-u}.\quad(4)$$

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* Work supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the U.S. Atomic Energy Commission under Contract No. AT(11-1)-881, COO-881-131.

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[8] If we assume an asymptotic behavior for the amplitude $\sim\beta(t)s^{\alpha(t)}$, with constant $\beta(t)$ and with the linear approximation for a Regge trajectory at $t=0$, $\alpha(t)=\alpha(0)+\alpha'(0)t$, the partial-wave amplitude given by this Regge pole behaves as $\sim1/(s^{1-\alpha(0)}\ln s)$ at high energies.

[9] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

The spectral function $A_{st}^I(s,t)$ is related to the t-channel spectral function $B_{st}^I(t,s)$ by the usual crossing relations,[10]

$$A_{st}^I(s,t)=\sum_{I'=0}^{2}\beta_{st}^{II'}B_{st}^{I'}(t,s)\,, \tag{5}$$

where I' is the total isotopic spin in the t channel and $\beta_{st}^{II'}$ is the isotopic spin crossing matrix between the s and t channels. Similarly, the spectral function $A_{tu}^I(t,u)$ is connected with the t-channel spectral function $B_{tu}^{I'}(u,t)$ and also with the u-channel spectral function $C_{tu}^{I'}(u,t)$ by[10]

$$A_{tu}^I(t,u)=\sum_{I'=0}^{2}\beta_{st}^{II'}B_{tu}^{I'}(u,t)=\sum_{I'=0}^{2}\beta_{su}^{II'}C_{tu}^{I'}(u,t)\,, \tag{6}$$

where

$$\beta_{su}^{II'}=(-1)^{I+I'}\beta_{st}^{II'}\,.$$

$A_{st}^I(s,t)$ can be divided into s-elastic and s-inelastic parts without any ambiguity:

$$A_{st}^I(s,t)=A_{st}^{I,\mathrm{el}(s)}(s,t)+A_{st}^{I,\mathrm{in}(s)}(s,t)\,. \tag{7}$$

Similarly, $A_{tu}^I(t,u)$ can be divided into u- (or t-) elastic and u- (or t-) inelastic parts.

Based on essentially the same concept as the strip approximation,[2,3,10] we now make the following approximation: The elastic part of the spectral function is equal to the inelastic part of the crossed channel spectral function and vice versa. By this approximation we obtain

$$A_{st}^{I,\mathrm{el}(s)}(s,t)=\sum_{I'=0}^{2}\beta_{st}^{II'}B_{st}^{I',\mathrm{in}(t)}(t,s) \tag{8}$$

and

$$A_{st}^{I,\mathrm{in}(s)}(s,t)=\sum_{I'=0}^{2}\beta_{st}^{II'}B_{st}^{I',\mathrm{el}(t)}(t,s)\,. \tag{9}$$

An analogous approximation will be made with respect to $A_{tu}^I(t,u)$. Equations (8) and (9) hold exactly at the two edges of the double spectral functions. With our approximation, we extend these equations to other nonvanishing regions of the double spectral functions. Using Eqs. (8) and (9) we can rewrite Eq. (4) as

$$\begin{aligned}A_t^I(s,t)=&\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{A_{st}^{I,\mathrm{el}(s)}(s',t)}{s'-s}\\&+\sum_{I'=0}^{2}\beta_{st}^{II'}\left(\frac{1}{\pi}\int_{s_{\mathrm{in}}}^{\infty}ds'\frac{B_{st}^{I',\mathrm{el}(t)}(t,s')}{s'-s}\right.\\&\left.+\frac{1}{\pi}\int_{u_{\mathrm{in}}}^{\infty}du'\frac{B_{tu}^{I',\mathrm{el}(t)}(u',t)}{u'-u}\right)\\&+\sum_{I'=0}^{2}\beta_{st}^{II'}\frac{1}{\pi}\int_{u_0}^{\infty}du'\frac{B_{tu}^{I',\mathrm{in}(t)}(u',t)}{u'-u}.\end{aligned} \tag{10}$$

The term in large parentheses in Eq. (10) is equal to the t discontinuity $B_t^{I',\mathrm{el}(t)}(t,s)$ of the elastic part of the t-channel amplitude. Introducing Eq. (10) into Eq. (3), we obtain

$$\begin{aligned}\tilde{A}_l^I(s)=&\frac{1}{\pi(q_s^2)^{l+1}}\int_{t_{\mathrm{in}}}^{\infty}dt'\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{A_{st}^{I,\mathrm{el}(s)}(s',t')}{s'-s}\\&\times Q_l\left(1+\frac{t'}{2q_s^2}\right)+\tilde{V}_{l,1}^I(s)+\tilde{V}_{l,2}^I(s)\,,\end{aligned} \tag{11}$$

where

$$\begin{aligned}\tilde{V}_{l,1}^I(s)=&\sum_{I'=0}^{2}\beta_{st}^{II'}\frac{1}{\pi(q_s^2)^{l+1}}\\&\times\int_{t_0}^{\infty}dt'B_t^{I',\mathrm{el}(t)}(t',s)Q_l\left(1+\frac{t'}{2q_s^2}\right)\end{aligned} \tag{12}$$

and

$$\begin{aligned}\tilde{V}_{l,2}^I(s)=&\sum_{I'=0}^{2}\beta_{st}^{II'}\frac{1}{\pi(q_s^2)^{l+1}}\int_{t_{\mathrm{in}}}^{\infty}dt'\frac{1}{\pi}\int_{u_0}^{\infty}du'\\&\times\frac{B_{tu}^{I',\mathrm{in}(t)}(u',t')}{u'-u}Q_l(1+t'/2q_s^2)\,.\end{aligned} \tag{13}$$

Equation (12) corresponds to the usual long-range potential and Eq. (13) is the contribution coming from the large-t part of the third double spectral function. The first term on the right-hand side of Eq. (11) can be rewritten by applying the Cauchy integral formula in the s plane:

$$\begin{aligned}&\frac{1}{\pi(q_s^2)^{l+1}}\int_{t_{\mathrm{in}}}^{\infty}dt'\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{A_{st}^{I,\mathrm{el}(s)}(s',t')}{s'-s}Q_l\left(1+\frac{t'}{2q_s^2}\right)\\&\qquad=\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{\mathrm{Im}\tilde{A}_l^{I,\mathrm{el}(s)}(s')}{s'-s}+\tilde{V}_{l,3}^I(s)\,,\end{aligned} \tag{14}$$

where

$$\begin{aligned}&\mathrm{Im}\tilde{A}_l^{I,\mathrm{el}(s)}(s)=\tilde{\rho}(s)|\tilde{A}_l^I(s)|^2,\\&\qquad\tilde{\rho}(s)=[(s-4\mu^2)/s]^{1/2}(q_s^2)^l\,,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\tilde{V}_{l,3}^I(s)=&\sum_{I'=0}^{2}\beta_{st}^{II'}\frac{1}{\pi}\int_{-\infty}^{s_0-t_{\mathrm{in}}}\frac{ds'}{s'-s}\int_{t_{\mathrm{in}}}^{s_0-s'}dt'\\&\times\frac{1}{-2(-q_{s'}^2)^{l+1}}\frac{1}{\pi}\int_{s_0}^{\infty}ds''\frac{B_{st}^{I',\mathrm{in}(t)}(t',s'')}{s''-s'}\\&\times P_l\left(-1-\frac{t'}{2q_{s'}^2}\right).\end{aligned} \tag{16}$$

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[10] In this paper, we use the notation A, B, and C for s-, t-, and u-channel scattering amplitudes even though they have well-defined isotopic spins in the respective channels. A similar notation for discontinuity functions and double spectral functions is also applied. See also M. Kretzschmer, Nuovo Cimento **39**, 835 (1965).

To derive Eq. (16), we have also used the relation Eq. (8).

We now make our first assumption: (I) As a function of t for a given value of s, the elastic t-channel discontinuity function $B_t^{\mathrm{el}(t)}(t,s)$ can be approximated by a summation over direct-channel (t) resonances, while the inelastic t-channel discontinuity function $B_t^{\mathrm{in}(t)}(t,s)$ can be well approximated by a sum of crossed-channel (s) Regge poles.

A similar approximation will be applied to other discontinuity functions. Although assumption (I) is applied to the discontinuity function, it is quite similar to that of the successful Regge pole-resonance interference model[1] which assumes the overlapping of the direct-channel resonances and crossed-channel Regge poles not only in the intermediate-energy region but even in the low- and high-energy regions. By dividing the total discontinuity function $B_t(t,s)$ into elastic and inelastic parts and identifying the elastic part with direct-channel resonances and the inelastic part with the crossed-channel Regge poles,[11] we have formally avoided the possible double counting of the same contributions. This problem will be further discussed in Sec. III.

Experimentally, one of the most important features of strong interactions (especially in pion-pion scattering) is the fact that the resonance formation and decay dominate in the elastic channel. In addition, the inelastic cross section may be regarded as having a relatively smooth variation considered as a function of energy even in the low-energy region, with, however, small corrections arising from the inelastic resonances. This relatively smooth behavior of the inelastic cross section strongly suggests that the inelastic amplitude can be well approximated only by the crossed-channel Regge poles.

Assumption (I) is quite different from that used in the new form of the strip approximation[2,3] which is based on the assumption that the scattering is completely elastic up to some critical energy and that scattering is completely diffractive beyond that energy.

Although the expansion of $B_t^{I',\mathrm{el}(t)}(t,s)$ in Eq. (12) into t-channel partial waves converges only up to $s=s_{\mathrm{in}}$, we assume its convergence for all s.

We obtain the following expression from Eq. (12):

$$\tilde{V}_{l,1L}^{I}(s)=\sum_{I'=0}^{2}\beta_{st}^{II'}\frac{1}{\pi(q_s^2)^{l+1}}\int_{t_0}^{\infty}dt'\sum_{l'=0}^{\infty}(2l'+1)\times\mathrm{Im}B_{l'}^{I',\mathrm{el}(t)}(t')P_{l'}\left(1+\frac{s}{2q_{t'}^2}\right)Q_l\left(1+\frac{t'}{2q_s^2}\right),\quad(17)$$

where the subscript L means, as we shall discuss later, that Eq. (17) is valid at least in low-energy regions. By the first half of assumption (I), we approximate Eq. (17) by taking only the dominant particles or resonance poles. As is well known, the partial-wave potential given by the exchange of the particles or resonances has the desired threshold behavior, and furthermore only with such contributions can we calculate scattering lengths for some processes.[7] We can therefore say that our approximation to Eq. (12) using only the dominant particles and resonances is valid in a relatively low-energy region. However, a potential given by the exchange of a particle with spin shows divergent behavior at high energy. Thus we cannot regard such a potential as being physically meaningful at high energy. In order to construct a potential which has both the correct low-energy and high-energy behavior, we first introduce the alternative Regge representation of the long-range potential Eq. (12).

By integrating over t the second term of Eq. (10) and using the crossing symmetries, we obtain the following double dispersion relation from the contributions of the small-t parts of the double spectral functions:

$$V_1^I(s,t)=\frac{1}{\pi}\int_{s_{\mathrm{in}}}^{\infty}ds'\frac{1}{s'-s}\frac{1}{\pi}\int_{t_0}^{\infty}dt'\frac{A_{st}^{I,\mathrm{in}(s)}(s',t')}{t'-t}+\frac{1}{\pi}\int_{u_{\mathrm{in}}}^{\infty}du'\frac{1}{u'-u}\frac{1}{\pi}\int_{t_0}^{\infty}dt'\frac{A_{tu}^{I,\mathrm{in}(u)}(u',t')}{t'-t}.\quad(18)$$

From the last half of assumption (I), Eq. (18) can be expressed by the sum of the t-channel Regge poles as follows:

$$V_1^I(s,t)=\sum_{I'=0}^{2}\beta_{st}^{II'}\sum_j\left(\frac{1}{\pi}\int_{s_{\mathrm{in}}}^{\infty}ds'\frac{R_j^{I'}(s',t)}{s'-s}+\frac{1}{\pi}\int_{u_{\mathrm{in}}}^{\infty}du'\frac{\xi_jR_j^{I'}(u',t)}{u'-u}\right),\quad(19)$$

where

$$R_j^I(s,t)=\tfrac{1}{2}\pi[2\alpha_j^I(t)+1]\gamma_j^I(t)\times(-q_t^2)^{\alpha_j^I(t)}P_{\alpha_j^I(t)}(-1-s/2q_t^2),\quad(20)$$

$\sum_j$ means the sum over all possible Regge poles, and ξ_j is the signature factor of the jth Regge pole. In Eq. (20), we used the Regge form given by Chew and Jones.[3] The partial-wave projection of Eq. (19) is given by using the formula suggested by Wong,[12]

$$\tilde{V}_{l,1H}^{I}(s)=-\frac{1}{\pi(q_s^2)^{l+1}}\int_{\infty-}^{0}dt\,\mathrm{Im}\left[Q_l\left(1+\frac{t}{2q_s^2}\right)\right]\times\left[\sum_{I'=0}^{2}\beta_{st}^{II'}\sum_j\left(\frac{1}{\pi}\int_{s_{\mathrm{in}}}^{\infty}ds'\frac{R_j^{I'}(s',t)}{s'-s}+\frac{1}{\pi}\int_{u_{\mathrm{in}}}^{\infty}du'\frac{\xi_jR_j^{I'}(u',t)}{u'-u}\right)\right],\quad(21)$$

[11] In this context, it should be noted that there is a possibility of constructing the theory by dividing the discontinuity into resonant and diffractive parts, as opposed to the elastic and inelastic parts. Therefore we might say that s_{in}, t_{in}, and u_{in} do not necessarily mean the exact lowest inelastic threshold but the beginning of the diffractive-type scattering in the total scattering amplitude.

[12] D. Y. Wong (private communication to G. F. Chew); see also Ref. 3.

where the subscript H means that Eq. (21) is valid in a relatively high-energy region. The Regge representation (21) apparently gives us the correct convergent high-energy behavior, but it does not satisfy the usual low-energy threshold behavior. Therefore we regard Eq. (21) as physically acceptable in the high-energy region of the long-range potential; on the other hand, Eq. (17) is acceptable in the low-energy region of the long-range potential. Comparing the two alternative expressions (17) and (21) for the long-range potential which are valid in the low- and high-energy regions, respectively, we now make our second main assumption: (II) The high-energy limit of the summation of exchanges of particles and resonances belonging to the same Regge family should lead to the Regge behavior given by the exchange of the relevant Regge pole.[13] From this assumption we can construct a long-range potential which satisfies both low- and high-energy behavior by superposing the two potentials smoothly in the intermediate-energy region.

As was stressed by Van Hove, in the broad intermediate-energy region the single-particle-exchange potential (17) and the Regge-pole-exchange potential (21) overlap, and the appropriate potential becomes dominant only in the low- or high-energy limit.

There are several possibilities for forming a composite potential by the introduction of parameters. In the following paragraph we show one of the most naive ways to do this. If we denote by $\tilde{V}_{l,1Lj}{}^I(s)$ the potential given by exchanges of the first few dominant particles or resonances belonging to the same Regge trajectory and denote by $\tilde{V}_{l,1Hj}{}^I(s)$ the potential given by the exchange of the relevant Regge pole, we can express the desired long-range potential as follows:

$$\tilde{V}_{l,1}{}^I(s)=\sum_j \left[\tilde{V}_{l,1Lj}{}^I(s)f_{1j}(s)+\tilde{V}_{l,1Hj}{}^I(s)f_{2j}(s)\right], \quad (22)$$

where summation over j runs over all different kinds of Regge families, and

$$f_{1j}(s)=\left[1+\exp\left(\frac{s-s_{1j}}{\Delta_{1j}}\right)\right]^{-1}, \quad (23)$$

$$f_{2j}(s)=\left[1+\exp\left(\frac{s_{2j}-s}{\Delta_{2j}}\frac{s^2}{(s-s_0)^2}\right)\right]^{-1}, \quad (24a)$$

and where s_{1j}, $s_{2j}>s_0$, and Δ_{1j} and Δ_{2j} are positive constants. If we also assume the correct threshold behavior for the Regge-pole exchange potential,[14] we may use the following correction factor:

$$f_{2j}(s)=\left(\frac{q_s{}^2}{q_s{}^2+a_{2j}}\right)^l, \quad (24b)$$

where $a_{2j}>0$. The factor $s^2/(s-s_0)^2$ in Eq. (24a) is inserted to avoid a possible threshold divergence of the potential $\tilde{V}_{l,1Hj}{}^I(s)$.

The phenomenological damping functions (23) and (24) were introduced only because there is no unique formula to connect analytically $\tilde{V}_{l,1Lj}{}^I(s)$ and $\tilde{V}_{l,1Hj}{}^I(s)$ at present, although they are, respectively, low- and high-energy limits of the same potential. The parameters in the damping functions should be determined by imposing boundary conditions such as the output resonance position and width, output Regge trajectories, and high-energy behavior of the partial-wave amplitude. When these parameters are determined with the boundary conditions, one can see that $\tilde{V}_{l,1Lj}{}^I(s)$ and $\tilde{V}_{l,1Hj}{}^I(s)$ overlap at intermediate energy and that they approach pure $\tilde{V}_{l,1Lj}{}^I(s)$ and $\tilde{V}_{l,1Hj}{}^I(s)$ behavior at low and high energies, respectively.

We now turn to the new potentials (13) and (16) which arise from the large-t parts of the double spectral functions. By applying the last half of assumption (I), we obtain for $\tilde{V}_{l,2}{}^I(s)$

$$\tilde{V}_{l,2}{}^I(s)=\sum_{I''=0}^{2}\beta_{st}{}^{II''}\sum_{I'=0}^{2}\beta_{tu}{}^{I''I'}\frac{1}{\pi(q_s{}^2)^{l+1}} \times\int_{t_{\mathrm{in}}}^{\infty}dt'\sum_k \xi_k R_k{}^{I'}(t',u)Q_l\left(1+\frac{t'}{2q_s{}^2}\right), \quad (25)$$

and for $\tilde{V}_{l,3}{}^I(s)$

$$\tilde{V}_{l,3}{}^I(s)=\sum_{I'=0}^{2}\beta_{st}{}^{II'}\beta_{ts}{}^{I'I}\int_{-\infty}^{s_0-t_{\mathrm{in}}}ds'\frac{1}{s'-s} \times\int_{t_{\mathrm{in}}}^{s_0-s'}dt'\frac{\sum_i R_i{}^I(t',s')P_l(-1-t'/2q_{s'}{}^2)}{-2(-q_{s'}{}^2)^{l+1}}, \quad (26)$$

where $R_k{}^{I'}(t,u)$ and $R_i{}^I(t,s)$ are given by Eq. (20). The sum in k and i in Eqs. (25) and (26) run over all possible families of the Regge poles in the u and s channels, respectively. Both potentials (25) and (26) have finite values at threshold and converge at high energies.

Combining Eqs. (11), (14), (22), (25), and (26), we obtain our final equation:

$$\tilde{A}_l{}^I(s)=\frac{1}{\pi}\int_{s_0}^{\infty}ds\frac{\tilde{\rho}(s')|\tilde{A}_l{}^I(s')|^2}{s'-s} +\tilde{V}_{l,1}{}^I(s)+\tilde{V}_{l,2}{}^I(s)+\tilde{V}_{l,3}{}^I(s). \quad (27)$$

[13] Although this fact had been generally believed, L. Van Hove devised a simple model which clearly shows the above fact. L. Van Hove, Phys. Letters **24B**, 183 (1967). See also L. Durand, III, Phys. Rev. **161**, 1610 (1967).

[14] In Ref. 3 the same potential was used at the elastic threshold $s=s_0$ without a threshold correction.

In (27), $\tilde{V}_{l,1}{}^{I}(s)$ is the long-range potential, and $\tilde{V}_{l,2}{}^{I}(s)$ and $\tilde{V}_{l,3}{}^{I}(s)$ are short-range potentials. The last two are newly introduced under assumption (I). In Eq. (27), the most important point is the fact that all potentials satisfy the required properties at both low and high energies.

For the conclusion of this section we reexplain our approximation for the double spectral function $B_{st}(t,s)$ in (5) by contrasting it to the new form of the strip approximation.[2,3]

In the new form of the strip approximation,[2,3] the double spectral function $B_{st}(t,s)$ is separated into the following three regions: (i) s small and t large, (ii) s small and t small, and (iii) s large and t small. The contributions of regions (i) and (iii) were approximated by the s- and t-channel Regge poles, respectively, and the contribution of region (ii) was neglected.

In our theory, however, the contributions of region (ii) plus (i) are approximated by the t-channel resonances plus s-channel Regge poles under assumption (I) without the introduction of any division such as strip width.[2,3] Also the contributions of the region (ii) plus (iii) are approximated by the t-channel resonances plus t-channel Regge poles under assumption (II) also without any division in energy region.

It should be emphasized that *the same potential* (17) *coming from the exchanges of the low-energy t-channel particles and resonances satisfies both assumption (I) with s-channel Regge poles and assumption (II) with t-channel Regge poles.* From this fact one can understand that assumptions (I) and (II) are closely related in the construction of the dynamics and that *both assumptions are necessary and sufficient to describe all regions of the double spectral function.*

Note also that if we construct the potential with only the Regge-pole-exchange contributions[2,3] we cannot obtain a physically meaningful result. This problem will be discussed in Sec. III in connection with discussion of the criticism of the Regge-pole resonance-interference model.

III. DISCUSSION

In Sec. II we constructed the dynamical Eq. (27) with three convergent potentials based on two main assumptions. However, there has been some criticism[15,16] of one of our main assumptions: There may be a possibility of counting the same amplitude twice by adding the Regge-pole exchange amplitudes and direct-channel resonance amplitudes in assumption (I). In this section we discuss this problem and related problems from both experimental and theoretical points of view.

Although we have no data on direct pion-pion scattering at present, we may safely guess the general features of pion-pion scattering from other existing experimental data such as π-p, p-p, and so on.

The most significant evidence for the Regge-pole resonance-interference model is π^--p charge-exchange scattering and π-p backward scattering.[1]

The π^--p charge-exchange data have been analyzed with the ρ Regge pole plus direct-channel resonances down to 0.7 GeV/c incident pion energy.[17,18] The fits with this model to the experimental data are extremely good and it seems that this model works at even lower incident pion energies than 0.7 GeV/c.

On the other hand, Barger and Olsson[19] analyzed the total cross-section data of π-p, K-N, N-N, and $\bar{N}$-N scatterings in the intermediate-energy regions (1–6 GeV/c) in terms of a Regge-pole-exchange model whose parameters were determined in the asymptotic-energy regions. They showed that the data can be well reproduced by the Regge-pole exchange model except for some direct-channel resonance contributions.

The above phenomenological analyses clearly show that even in the quite low-energy regions as well as in the intermediate-energy regions, Regge-like behavior really exists along with the resonance amplitudes.

As for the criticism over the possible double counting of the same amplitude in the Regge-pole resonance-interference model, it is well known that we can construct a consistent theory with both direct-channel resonances and their background terms in nonrelativistic theory.[20] Durand[21] has recently discussed this problem and constructed a modified Regge-pole resonance-interference model which avoids the possible double-counting difficulty.

We may say that there is no theoretical inconsistency in this model except for some technical problems[22] such as the energy dependence of the elasticity factor, the effect of a background phase shift on the Breit-Wigner one-level formula, and the energy dependence of the tails of the one-level formula.

We now turn to some quantitative aspects of our theory. The sum of the contributions of P-, P'-, and ρ-Regge-pole exchanges to the Regge-pole exchange potential (21) in pion-pion scattering is always negative for all states because the P' contribution dominates the real part of the potential (21) according to the phenom-

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[17] G. Höhler *et al.*, Phys. Letters **20**, 79 (1966); A. S. Carroll *et al.*, Phys. Rev. Letters **16**, 288 (1966).

[18] V. Barger and M. Olsson, Phys. Rev. **151**, 1123 (1966).

[19] V. Barger and M. Olsson, Phys. Rev. **148**, 1428 (1966).

[20] See, e.g., L. D. Landau and E. M. Lipshitz, *Quantum Mechanics*, (Pergamon Press, Inc., New York, 1965), 2nd ed.; J. M. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

[21] L. Durand, III, Phys. Rev. **166**, 1680 (1968).

[22] An excellent discussion of the criticisms of the Regge-pole resonance-interference model was given by V. Barger and L. Durand, III, Phys. Letters **26B**, 588 (1968).

enologically determined Regge parameters.[23,24] This can be easily understood from experiment as follows. The ratios of real to imaginary parts of the forward scattering amplitudes are always negative in the Regge-exchange-dominated energy regions and the amplitudes strongly decrease as $-t$ increases. The fits with the Regge-pole model to the above ratios have been well established. Therefore, the partial-wave projections of amplitudes with the above properties also give negative real parts.[25]

Therefore, with only the Reggeized potential (21) (as in the new form of strip approximation[2,3]), we mostly obtain only the repulsive force due to primarily P'-Regge exchange for all states and we cannot produce a resonance as a dynamical output unless we use unrealistic Regge parameters which enhance the ρ-Regge pole contribution and suppress the P'-Regge contributions.[24,26] In this sense we should stress that the new form of strip approximation[3] which proposes to construct the dynamics with only Regge-pole-exchange potentials is not a correct one.

From the above discussion we see that the main force for producing the low-energy resonance phenomena as dynamical output must come from the particle- and resonance-exchange potential (17) which should coexist with the Regge-pole-exchange potential (21).

In fact, we can quite naturally obtain the experimentally established ρ-meson decay width as an output in our formalism by using a potential constructed with the particle- and resonance-exchange potential (17) and the Regge-exchange potential (21).[27] Usually the calculated meson decay width is several times as large as that of the experimental one which is used as input[5] and this situation seems not to change even when we take into account the multichannel effects.

Besides, the additional potentials such as $\tilde{V}_{l,2}{}^I(s)$ and $\tilde{V}_{l,3}{}^I(s)$ have not been seriously considered in the older treatments. Actually we have found that $\tilde{V}_{l,3}{}^I(s)$ has an appreciable contribution.[28]

Note added in proof. G. R. Bart and R. L. Warnock, Bull. Am. Phys. Soc. **13**, 106 (1968), have also discussed boundary conditions similar to those in this paper. They also claimed that with only the Regge-exchange potential one cannot produce the ρ-meson resonance as a dynamical output. The author is indebted to Professor V. Barger for discussions of the present situation of the Regge-pole resonance-interference model.

ACKNOWLEDGMENTS

The author would like to express his sincere gratitude to Professor L. Durand, III, and Dr. A. R. Swift for their reading of the manuscript and very valuable comments. He is also indebted to Dr. P. Fishbane for his help in improving the manuscript.

[23] For example, V. Barger and M. Olsson [Phys. Rev. **146**, 1080 (1966)] obtained the residues at $t=0$ for P-, P'-, and ρ-Regge poles with trajectory intercepts at $t=0$, $\alpha_P(0)=1$, $\alpha_{P'}(0)=0.39$ and $\alpha_\rho(0)=0.48$. The ratio for the above three Regge-pole residues $\gamma_P(0)$: $\gamma_{P'}(0)$: $\gamma_\rho(0)$ is about 1:9:1.3 in their energy scale of 1 GeV^2. If we use the exact form of our Eq. (20) whose energy scale is $2\mu_\pi^2$ at $t=0$, the above ratio becomes about 1:94:10. The dominant P' residue has also been obtained by Rarita *et al.*, Phys. Rev. **165**, 1615 (1968).

[24] P- and P'-Regge-pole exchanges give the negative real parts and ρ-Regge-pole exchange gives the positive real part for the potential (21).

[25] See also Y. Higuchi and S. Machida, Progr. Theoret. Phys. (Kyoto) **36**, 313 (1966).

[26] P. D. B. Collins and V. L. Teplitz, Phys. Rev. **140**, B663 (1965); P. D. B. Collins, *ibid.* **142**, 1163 (1966).

[27] N. Masuda (unpublished).

[28] N. Masuda, following paper, Phys. Rev. **175**, 2093 (1968).