Stress-Tensor–Current Commutators, Electromagnetic and Weak Corrections to Current Commutators, and Sum Rules

R. JACKIW*
CERN, Geneva
and
Jefferson Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 13 June 1968)

Equal-time commutation relations between selected components of the energy-momentum tensor and selected components of a current, arising from internal transformations, are derived in a model-independent fashion. These commutators are then used to establish the following three results: (1) It is shown that current-current commutators do not have the standard form in the presence of electromagnetic and weak interactions. Specifically, it is demonstrated that the space components of an isospin current do not transform as isospin vectors. (2) Weinberg's second sum rule is shown to follow from further assumptions about our commutators, and it is argued that kaon mass corrections must be expected in the $SU(3)\times SU(3)$ generalization of this sum rule. (3) A relation between decay constants in broken $SU(3)\times SU(3)$ is established. It is $F_\kappa\mu_\kappa^2+F_K\mu_K^2=F_\pi\mu_\pi^2$.

1. INTRODUCTION

EQUAL-TIME commutation relations (ETCR) between time components of currents which arise from continuous transformations play a fundamental role in physical theory. Schwinger has emphasized the connection between Lorentz invariance and the ETCR[1]

$$[T^{00}(\mathbf{x},t),T^{00}(\mathbf{y},t)]=-i[T^{0k}(x)+T^{0k}(y)]\times\partial_k\delta(\mathbf{x}-\mathbf{y})\,. \quad (1.1)$$

($T^{\mu\nu}$ is the energy momentum tensor, thus a generalized current.) Gell-Mann[2] has given the now famous current-algebra relations

$$[K_a{}^0(\mathbf{x},t),L_b{}^0(\mathbf{y},t)]=if_{abc}(K\cdot L)_c{}^0(y)\delta(\mathbf{x}-\mathbf{y})\,. \quad (1.2)$$

Here K, L are either vector or axial-vector currents (V or A, respectively); $K\cdot L=V$ when $K=L=A$, V; and $K\cdot L=A$ when $K\neq L=A$, V. The f_{abc} is an antisymmetric structure constant. The purpose of the present paper is to publicize the ETCR $[T^{00},K_a{}^0]$, as well as several related ETCR's between selected components of $T^{\mu\nu}$ and $K_a{}^\mu$, and to exhibit some interesting applications of these results.

The ETCR $[T^{00},K_a{}^0]$ has been derived previously by Gross and the present author[3] from Schwinger's action principle. Its value is[3a]

$$[T^{00}(\mathbf{x},t),K_a{}^0(\mathbf{y},t)]=-i\partial_\mu K_a{}^\mu(x)\delta(\mathbf{x}-\mathbf{y})-iK_a{}^k(x)\partial_k\delta(\mathbf{x}-\mathbf{y})\,. \quad (1.3)$$

In Sec. 2, the ETCR's $[T^{0i},K_a{}^0]$ and $[T^{00},K_a{}^i]$ are derived in a model-independent fashion. Section 3 is devoted to a study of the $[K_a{}^0,L_b{}^i]$ ETCR in the presence of weak and electromagnetic interactions. In Ref. 3 this ETCR was derived in the absence of such interactions and was found to be of the standard form

$$[K_a{}^0(\mathbf{x},t),L_b{}^i(\mathbf{y},t)]=if_{abc}(K\cdot L)_c{}^i(x)\delta(\mathbf{x}-\mathbf{y})+iR_{ab,KL}{}^{ji}(y)\partial_j\delta(\mathbf{x}-\mathbf{y})\,, \quad (1.4a)$$

$$R_{ab,KL}{}^{ji}=R_{ba,LK}{}^{ij}\,, \quad (1.4b)$$

if and only if the ETCR $[K_a{}^0,\partial_\mu L_b{}^\mu]$ contains no Schwinger terms (ST). In the latter eventuality (1.4a) must be modified by an additional term, proportional to a δ function, so that even the once integrated ETCR between the charge and a current space component is no longer consistent with (1.4a). Recently Lee and Zumino[4] have shown, on the basis of gauge invariance of electromagnetic interactions, that similar modifications of (1.4a) occur in the presence of electromagnetic interactions, when the ST $R_{ab,KL}{}^{ij}$ is a c number. In Sec. 3 we use the method of Ref. 3 to show the nature of this modification for electromagnetic interactions without reference to gauge principles or to the form of the ST. Furthermore, it is demonstrated that weak interactions similarly lead to a modification of (1.4a). Thus the space components of currents do not transform according to a definite representation of the group algebra in the presence of weak and electromagnetic effects.

In Sec. 4, we use our commutators to derive the second Weinberg sum rule in a fashion which has not been given before, and which follows closely Weinberg's

---

* Junior Fellow, Society of Fellows.

[1] J. Schwinger, Phys. Rev. 130, 406 (1963). It should be remembered that the ETCR (1.1) is valid as written only for systems with spins $\leqslant 1$. For higher spins, further model-dependent terms may be written. We shall not make use of this ETCR.

[2] M. Gell-Mann, Physics 1, 63 (1964).

[3] D. J. Gross and R. Jackiw, Phys. Rev. 163, 1688 (1967). The derivation introduces an external gravitational field, and makes assumptions about the dependence of various quantities on this field, in a fashion analogous to that of Ref. 1. These assumptions are most likely not satisfied for arbitrary systems with spin>1, and further model-dependent terms may then appear in the right-hand side of (1.3). (These must vanish upon integration over $\mathbf{x}$.) In the present paper we ignore such complications. In the note added in proof we present a canonical derivation of (1.3) which makes no reference to external gravitational fields.

[3a] In the noted added in proof a derivation of Eq. (1.3) is presented which differs from the one given in Ref. 3.

[4] T. D. Lee and B. Zumino, Phys. Rev. 163, 1667 (1967).

original derivation of the first sum rule.[5] Section 5 contains a derivation of a linear relation between the decay constants of the pion, kaon, and κ.

2. DERIVATION OF ETCR BETWEEN SELECTED COMPONENTS OF K_a^μ AND SELECTED COMPONENTS OF $T^{\mu\nu}$

In addition to the $[T^{00},K_a^0]$ ETCR, (1.3), we can give in a model-independent fashion the $[T^{0i},K_a^0]$ and $[T^{00},K_a^i]$ ETCR. We begin with a derivation of the first of these two. To this end we make use of the following commutators which hold for currents arising from linear transformations of the form

$$\Psi_i \to \Psi_i+\Lambda^a F_{ij}{}^a\Psi_j+O(\Lambda^2)\,, \quad (2.1)$$

$$[K_a{}^0(\mathbf{x},t),\Psi_i(\mathbf{y},t)]=-iF_{ij}{}^a\Psi_j(x)\delta(\mathbf{x}-\mathbf{y})\,, \quad (2.2a)$$

$$[K_a{}^0(\mathbf{x},t),\Pi_i(\mathbf{y},t)]=i\Pi_j(x)F_{ji}{}^a\delta(\mathbf{x}-\mathbf{y})\,. \quad (2.2b)$$

In (2.2b), Π_i is the momentum canonically conjugate to Ψ_i. Using the formula[6]

$$T^{0i}=\Pi_k\partial^i\Psi_k \quad (2.3)$$

and the commutators (2.2), we find

$$\begin{aligned}[T^{0i}(\mathbf{x},t),K_a{}^0(\mathbf{y},t)]&=-i\Pi_j(x)F_{jk}{}^a\partial^i\Psi_k(x)\delta(\mathbf{x}-\mathbf{y})\\&\quad+i\Pi_j(x)F_{jk}{}^a\partial_x{}^i[\Psi_k(x)\delta(\mathbf{x}-\mathbf{y})]\\&=i\Pi_j(x)F_{jk}{}^a\Psi_k(x)\partial^i\delta(\mathbf{x}-\mathbf{y})\,. \end{aligned}\quad (2.4)$$

Recognizing the coefficient of $\partial^i\delta(\mathbf{x}-\mathbf{y})$ to be $K_a{}^0(x)$, we obtain finally

$$[T^{0i}(\mathbf{x},t),K_a{}^0(\mathbf{y},t)]=iK_a{}^0(x)\partial^i\delta(\mathbf{x}-\mathbf{y})\,. \quad (2.5)$$

From (1.3) and (2.5) it is possible to determine $[T^{00},K_a{}^i]$ by Lorentz invariance. Setting M^{0i} to be the generator of Lorentz transformations

$$M^{0i}=tP^i-\int x^iT^{00}(x)d^3x \quad (2.6)$$

(P^i is the momentum operator), we have by virtue of Lorentz invariance

$$\begin{aligned}[M^{0i},K_a{}^\mu(x)]&=i[x^i\dot{K}_a{}^\mu(x)-t\partial^iK_a{}^\mu(x)]\\&\quad+i[g^{i\mu}K_a{}^0(x)-g^{0\mu}K_a{}^i(x)]\,, \end{aligned}\quad (2.7a)$$

$$\begin{aligned}[M^{0i},T^{0\mu}(x)]&=i[x^i\dot{T}^{0\mu}(x)-t\partial^iT^{0\mu}(x)]\\&\quad+i[g^{i\mu}T^{00}(x)-T^{i\mu}(x)-g^{0\mu}T^{0i}(x)]\,. \end{aligned}\quad (2.7b)$$

(The dot indicates time differentiation.) Next consider the Jacobi identity

$$\begin{aligned}&[M^{0i},[T^{00}(\mathbf{x},t),K_a{}^0(\mathbf{y},t)]]\\&\quad=[T^{00}(\mathbf{x},t),[M^{0i},K_a{}^0(\mathbf{y},t)]]\\&\qquad+[[M^{0i},T^{00}(\mathbf{x},t)],K_a{}^0(\mathbf{y},t)]\,. \end{aligned}\quad (2.8a)$$

Evaluation of the commutators proceeds with help of (1.3), (2.5), and (2.7) to the result

$$\begin{aligned}&[-t\partial^i\partial_\mu K_a{}^\mu(x)+x^i\partial^0\partial_\mu K_a{}^\mu(x)]\delta(\mathbf{x}-\mathbf{y})-[t\partial^iK_a{}^k(x)-x^i\dot{K}_a{}^k(x)]\partial_k\delta(\mathbf{x}-\mathbf{y})+K_a{}^0(x)\partial^i\delta(\mathbf{x}-\mathbf{y})\\&=[-t\partial^i\partial_\mu K_a{}^\mu(x)+x^i\partial^0\partial_\mu K_a{}^\mu(x)]\delta(\mathbf{x}-\mathbf{y})-[t\partial^iK_a{}^k(x)-x^i\dot{K}_a{}^k(x)]\partial_k\delta(\mathbf{x}-\mathbf{y})+\dot{K}_a{}^i(x)\delta(\mathbf{x}-\mathbf{y})\\&\quad+2K_a{}^0(x)\partial^i\delta(\mathbf{x}-\mathbf{y})+i(x^i-y^i)[\dot{T}^{00}(\mathbf{x},t),K_a{}^0(\mathbf{y},t)]-i[T^{00}(\mathbf{x},t),K_a{}^i(\mathbf{y},t)]\,. \end{aligned}\quad (2.8b)$$

We use the continuity equation for T^{00},

$$\dot{T}^{00}=-\partial_iT^{0i}\,, \quad (2.8c)$$

and (2.5) again to obtain, after some rearrangement,

$$\begin{aligned}[T^{00}(\mathbf{x},t),K_a{}^i(\mathbf{y},t)]&=-iK_a{}^0(x)\partial^i\delta(\mathbf{x}-\mathbf{y})\\&\quad-i(x^i-y^i)\partial_x{}^k(K_a{}^0(x)\partial_k\delta(\mathbf{x}-\mathbf{y}))\\&\quad-i\dot{K}_a{}^i(x)\delta(\mathbf{x}-\mathbf{y})\,. \end{aligned}\quad (2.8d)$$

The first two terms on the right side can be combined, and the final result is

$$\begin{aligned}[T^{00}(\mathbf{x},t),K_a{}^i(\mathbf{y},t)]&=-i\dot{K}_a{}^i(x)\delta(\mathbf{x}-\mathbf{y})\\&\quad+iK_a{}^0(y)\partial^i\delta(\mathbf{x}-\mathbf{y})\,. \end{aligned}\quad (2.9)$$

This formula was assumed in Ref. 3, and formed the basis of a calculation of the ST in the $[K_a{}^i,L_b{}^i]$ ETCR The present derivation justifies that assumption.

We are not able to derive in a model-independent fashion the $[T^{kl},K_a{}^\mu]$ and the $[T^{0k},K_a{}^i]$ ETCR.[7] These involve a knowledge of the commutation properties of $(\delta\mathfrak{L})/(\delta\partial^i\Psi)$ which are obviously model-dependent. ($\mathfrak{L}$ is the Lagrange density.) In summary therefore, the commutators that can be given in a model-independent fashion are[3,6]

$$\begin{aligned}[T^{00}(\mathbf{x},t),K_a{}^0(\mathbf{y},t)]&=-i\partial_\mu K_a{}^\mu(x)\delta(\mathbf{x}-\mathbf{y})\\&\quad-iK_a{}^k(x)\partial_k\delta(\mathbf{x}-\mathbf{y})\,, \end{aligned}\quad (2.10a)$$

$$\begin{aligned}[T^{00}(\mathbf{x},t),K_a{}^i(\mathbf{y},t)]&=-i\dot{K}_a{}^i(x)\delta(\mathbf{x}-\mathbf{y})\\&\quad+iK_a{}^0(y)\partial^i\delta(\mathbf{x}-\mathbf{y})\,, \end{aligned}\quad (2.10b)$$

$$[T^{0i}(\mathbf{x},t),K_a{}^0(\mathbf{y},t)]=iK_a{}^0(x)\partial^i\delta(\mathbf{x}-\mathbf{y})\,. \quad (2.10c)$$

The ETCR of T^{00} with $K_a{}^\mu$, Eqs. (2.10a) and (2.10b), is a local statement of the fact that $K_a{}^\mu$ transforms as a vector under Lorentz transformations. By this we mean that if we form M^{0i} as in (2.6), use (2.10) to evaluate

[5] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967); S. Glashow, H. J. Schnitzer, and S. Weinberg, *ibid.* **19**, 139 (1967).

[6] In offering formula (2.3) for T^{0i} we make use of the *canonical* energy-momentum tensor as given by Noether's theorem, $T^{\mu\nu}=(\delta\mathfrak{L}/\delta\partial_\mu\Psi_k)\partial^\nu\Psi_k-g^{\mu\nu}\mathfrak{L}$ ($\mathfrak{L}$ is the Lagrange density). We are able to derive model-independent ETCR only with this definition for T^{0i}. However, for arbitrary spin, additional gradient terms may be present which presumably contribute model-dependent terms to the right-hand side of (2.5). (These must vanish upon integration over $\mathbf{x}$.) See also Ref. 3. In the present paper we ignore these.

[7] By exploiting further Jacobi identities, for example, with T^{00}, T^{00}, $K_a{}^i$ or M^{0i}, T^{0i}, $K_a{}^0$, one can partially but not completely determine the $[T^{0i}, K_a{}^i]$ and $[T^{mn},K_a{}^0]$ ETCR.

commutators with T^{00}, then we arrive at (2.7a) by explicit calculation. Sugawara's recently proposed theory[8] presents a realization of the above commutators, with $T^{\mu\nu}$ chosen to be a bilinear functional of the moments.

Let us integrate (2.10a) and (2.10c) with respect to **y**. Introducing the charge

$$K_a(t)\equiv\int K_a{}^0(\mathbf{x},t)d^3x\,, \tag{2.11}$$

we have

$$[T^{00}(\mathbf{x},t),K_a(t)]=-i\partial_\mu K_a{}^\mu(x)\,, \tag{2.12a}$$

$$[T^{0k}(\mathbf{x},t),K_a(t)]=0\,. \tag{2.12b}$$

Thus it is seen that the divergence of a current is given by the commutator of the charge with the Hamiltonian density.[9] As we shall see below, this places restrictions which can be fruitfully exploited. Equation (2.12a) is to be compared to the more familiar formula for the divergence of a current,

$$\partial_\mu K_a{}^\mu=\delta\mathfrak{L}/\delta\Lambda^a|_{\Lambda=0}\,. \tag{2.13}$$

In conclusion, we wish to record here one more commutator which follows from (2.10), $\partial_\mu K_a{}^\mu$ with T^{00}. We find

$$[T^{00}(\mathbf{x},t),\partial_\mu K_a{}^\mu(\mathbf{y},t)]=-i\partial^0\partial_\mu K_a{}^\mu(x)\delta(\mathbf{x}-\mathbf{y})\,. \tag{2.14}$$

We are unable to derive the ETCR $[T^{0k},\partial_\mu K_a{}^\mu]$ in a model-independent fashion.

3. MODIFICATION OF CURRENT COMMUTATORS DUE TO ELECTROMAGNETIC AND WEAK INTERACTIONS

In the present section we demonstrate with the help of Eq. (2.10a) that current commutators are modified in the presence of weak and electromagnetic interactions, which couple to the hadronic currents. The Gell-Mann ETCR (1.2) of course does not change since it is a consequence of the group structure of the transformation (2.1). However, the $[K_a{}^0,L_b{}^i]$ ETCR is not directly related to this group algebra, and may be modified. To exhibit this modification, we consider as in Ref. 3 the Jacobi identity[10] between $T^{00}(\mathbf{x},t)$, $K_a{}^0(\mathbf{y},t)$, $L_b{}^0(\mathbf{z},t)$. With the help of (2.10a) it is found that[3]

$$\begin{aligned}&f_{abc}\partial_\mu(K\cdot L)_c{}^\mu(\mathbf{p}+\mathbf{q}+\mathbf{r})+if_{abc}(q+r)_k\\&\quad\times(K\cdot L)_c{}^k(\mathbf{p}+\mathbf{q}+\mathbf{r})+i[K_a{}^0(\mathbf{q}),\partial_\mu L_b{}^\mu(\mathbf{p}+\mathbf{r})]\\&\quad-i[L_b{}^0(\mathbf{r}),\partial_\mu K_a{}^\mu(\mathbf{p}+\mathbf{q})]-r_k[K_a{}^0(\mathbf{q}),L_b{}^k(\mathbf{p}+\mathbf{r})]\\&\qquad+q_k[L_b{}^0(\mathbf{r}),K_a{}^k(\mathbf{p}+\mathbf{q})]\,.\end{aligned} \tag{3.1}$$

Equation (3.1) has been written in momentum space where the Fourier transforms are by definition

$$K_a{}^\mu(\mathbf{p})=\int e^{i\mathbf{x}\cdot\mathbf{p}}K_a{}^\mu(\mathbf{x},t)d^3p\,, \tag{3.2a}$$

$$\partial_\mu K_a{}^\mu(\mathbf{p})=\int e^{i\mathbf{x}\cdot\mathbf{p}}\partial_\mu K_a{}^\mu(\mathbf{x},t)d^3p\,. \tag{3.2b}$$

The commutators with the charges $K_a(t)=K_a{}^0(\mathbf{0})$ can be taken to be, by definition,

$$[K_a{}^0(\mathbf{0}),\partial_\mu L_b{}^\mu(\mathbf{p})]=iC_{ab,KL}(\mathbf{p})\,, \tag{3.3a}$$

$$[K_a{}^0(\mathbf{0}),L_b{}^i(\mathbf{p})]=if_{abc}(K\cdot L)_c{}^i(\mathbf{p})+iB_{ab,KL}{}^i(\mathbf{p})\,. \tag{3.3b}$$

The term $B_{ab,KL}{}^i(\mathbf{p})$ which is inserted in (3.3b), allows for a possible breaking of the usual current-algebra assumption $B_{ab,KL}{}^i=0$. We shall show that in the presence of weak and electromagnetic interactions $B_{ab,KL}{}^i\neq0$.

The local version of (3.3) will in general contain ST. Without loss of generality, we have

$$[K_a{}^0(\mathbf{p}),\partial_\mu L_b{}^\mu(\mathbf{q})]=iC_{ab,KL}(\mathbf{p}+\mathbf{q})+p_iS_{ab,KL}{}^i(\mathbf{p},\mathbf{q})\,, \tag{3.4a}$$

$$[K_a{}^0(\mathbf{p}),L_b{}^i(\mathbf{q})]=if_{abc}(K\cdot L)_c{}^i(\mathbf{p}+\mathbf{q})+iB_{ab,KL}{}^i(\mathbf{p}+\mathbf{q})+p_jR_{ab,KL}{}^{ji}(\mathbf{p},\mathbf{q})\,. \tag{3.4b}$$

Equations (3.4) serve to define $S_{ab,KL}{}^i$ and $R_{ab,KL}{}^{ji}$. The ST are proportional to p_i since (3.3) holds. Because they depend on **p** and **q**, rather than just on **p**+**q**, we have allowed for an arbitrary number of derivatives of δ functions in position space.

Inserting (3.4) in (3.1) results in

$$\begin{aligned}&f_{abc}\partial_\mu(K\cdot L)_c{}^\mu(\mathbf{p}+\mathbf{q}+\mathbf{r})-C_{ab,KL}(\mathbf{p}+\mathbf{q}+\mathbf{r})\\&\quad+C_{ba,LK}(\mathbf{p}+\mathbf{q}+\mathbf{r})+iq_iS_{ab,KL}{}^i(\mathbf{q},\,\mathbf{p}+\mathbf{r})\\&\quad-ir_iS_{ba,LK}{}^i(\mathbf{r},\,\mathbf{p}+\mathbf{q})-ir_kB_{ab,KL}{}^k(\mathbf{p}+\mathbf{q}+\mathbf{r})\\&\quad+iq_kB_{ba,KL}{}^k(\mathbf{p}+\mathbf{q}+\mathbf{r})-r_kq_jR_{ab,KL}{}^{jk}(\mathbf{q},\,\mathbf{p}+\mathbf{r})\\&\qquad+q_kr_jR_{LK,ba}{}^{jk}(\mathbf{r},\,\mathbf{p}+\mathbf{q})=0\,.\end{aligned} \tag{3.5}$$

Now set **q** and **r** to zero. This leads to the following restriction on $C_{ab,KL}$:

$$C_{ab,KL}(\mathbf{p})-C_{ba,LK}(\mathbf{p})=f_{abc}\partial_\mu(K\cdot L)_c{}^\mu(\mathbf{p})\,. \tag{3.6}$$

This equation will be exploited in Sec. 5. Equation (3.6) simplifies (3.5) into

$$\begin{aligned}&iq_iS_{ab,KL}{}^i(\mathbf{q},\,\mathbf{p}+\mathbf{r})-ir_iS_{ba,LK}{}^i(\mathbf{r},\,\mathbf{p}+\mathbf{q})\\&\quad-ir_kB_{ab,KL}{}^k(\mathbf{p}+\mathbf{q}+\mathbf{r})+iq_kB_{ba,LK}{}^k(\mathbf{p}+\mathbf{q}+\mathbf{r})\\&\quad-r_kq_jR_{ab,KL}{}^{jk}(\mathbf{r}+\mathbf{p}+\mathbf{q})+q_kr_jR_{ba,LK}{}^{jk}(\mathbf{r},\,\mathbf{p}+\mathbf{q})=0\,.\end{aligned} \tag{3.7}$$

Next set **r** (or **q**) to zero. This gives

$$q_iS_{ab,KL}{}^i(\mathbf{q},\mathbf{p})=-q_iB_{ba,LK}{}^i(\mathbf{p}+\mathbf{q})\,. \tag{3.8}$$

Equation (3.8) shows that the ST in the ETCR between the zero component of a current and the divergence of

[8] H. Sugawara, Phys. Rev. **170**, 1659 (1968).

[9] The statement occasionally found in the literature that $\partial_\mu K_a{}^\mu=i[K_a,\mathfrak{L}]$ is evidently not generally true.

[10] Although the Jacobi identity can be discredited [see, e.g., K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppls. **37** and **38**, 74 (1966)], no contradictions have ever been found in commutators involving three time components. Furthermore, many of our present results can be obtained by use of the Jacobi identity with integrated operators (see Ref. 3) $H=\int T^{00}d^3x$, $K_a{}^0$, $L_b{}^0$, and M^{0i}, $K_a{}^0$, $L_b{}^0$.

a current contains at most one derivative of the δ function. Furthermore, it is seen that the presence of such a ST necessarily breaks the usual algebra of current commutators. Inserting (3.8) into (3.7) we have

$$r_k q_j R_{ab,KL}{}^{jk}(\mathbf{q},\,\mathbf{p}+\mathbf{r}) = q_k r_j R_{ba,LK}{}^{jk}(\mathbf{r},\,\mathbf{p}+\mathbf{q})\,. \quad (3.9)$$

Finally differentiating (3.9) by r_i and setting $\mathbf{r}$ to zero gives

$$q_j R_{ab,KL}{}^{ji}(\mathbf{q},\mathbf{p}) = q_k R_{ba,LK}{}^{ik}(\mathbf{0},\,\mathbf{p}+\mathbf{q}) \equiv q_j R_{ab,KL}{}^{ji}(\mathbf{p}+\mathbf{q})\,. \quad (3.10)$$

Inserting this back into (3.11) shows that the following symmetry is satisfied:

$$R_{ab,KL}{}^{ij} = R_{ba,LK}{}^{ji}\,. \quad (3.11)$$

It is seen therefore that the ST in $[K^0,L^i]$ ETCR contains at most one derivative of the δ function and fulfills the symmetry (3.11).

In summary, the form of the commutators as determined by local Lorentz invariance, Eq. (2.10a), is

$$[K_a{}^0(\mathbf{x},t),\partial_\mu L_b{}^\mu(\mathbf{y},t)] = iC_{ab,KL}(x)\delta(\mathbf{x}-\mathbf{y}) - iB_{ba,LK}{}^i(y)\partial_i\delta(\mathbf{x}-\mathbf{y})\,, \quad (3.12\text{a})$$

$$[K_a{}^0(\mathbf{x},t),L_b{}^i(\mathbf{y},t)] = if_{abc}(K\cdot L)_c{}^i(x)\delta(\mathbf{x}-\mathbf{y}) + iB_{ab,KL}{}^i(x)\delta(\mathbf{x}-\mathbf{y}) + iR_{ab,KL}{}^{ji}(y)\partial_j\delta(\mathbf{x}-\mathbf{y})\,, \quad (3.12\text{b})$$

$$C_{ab,KL}(x) - C_{ba,LK}(x) = f_{abc}\partial_\mu(K\cdot L)_c{}^\mu(x)\,, \quad (3.12\text{c})$$

$$R_{ab,KL}{}^{ji}(x) = R_{ba,LK}{}^{ij}(x)\,. \quad (3.12\text{d})$$

To evaluate the modification $B_{ab,KL}{}^i$ in the presence of weak and electromagnetic interactions we must specify the nature of these interactions, i.e., we must determine $\partial_\mu K_a{}^\mu$. We shall assume that, in the absence of weak and electromagnetic current, all divergences vanish. The consequences of partially conserved axial-vector and strangeness-changing vector currents have been studied in Ref. 3. These effects are irrelevant for our present purposes and may be suppressed for simplicity. We treat separately two cases: electromagnetic interactions and weak interactions with local current-current coupling. It is clear that a theory of weak interactions mediated by an intermediate boson is analogous to the electromagnetic interaction and we do not discuss this. In both instances we shall work to lowest order in these interactions. The divergence of the current (to lowest order) is easily evaluated from (2.12a), (1.2), and (3.12b).

For electromagnetic interactions we find

$$\partial_\mu K_a{}^\mu = ea_\mu f_{a3b}K_b{}^\mu + O(e^2)\,, \quad (3.13)$$

where a_μ is the electromagnetic field and the interaction is of the form

$$\mathcal{L}_{\text{em}} = -ea_\mu \mathcal{J}_{\text{em}}{}^\mu\,. \quad (3.14)$$

Inserting (3.13) in (3.12a) and using (1.2) and (3.12b) to evaluate (3.12a), we obtain an expression for $C_{ab,KL}$ and $B_{ba,LK}{}^i$. The formula for $C_{ab,KL}$ is a complicated expression which we do not present here. We remark only that the condition (3.12c) can be shown to hold when use is made of (3.12d). The form of $B_{ab,KL}{}^i$ is

$$B_{ab,KL}{}^i = -ef_{a3c}a_jR_{cb,KL}{}^{ji} + O(e^2)\,. \quad (3.15)$$

This cannot vanish identically. It is to be emphasized that these corrections persist even in the integrated ETCR.

Therefore in the presence of electromagnetic interactions, we find the model-dependent result

$$[K_a{}^0(\mathbf{x},t),L_b{}^i(\mathbf{y},t)] = if_{abc}(K\cdot L)_c{}^i(x)\delta(\mathbf{x}-\mathbf{y}) - ief_{a3c}a_j(x)R_{cb,KL}{}^{ji}(x)\delta(\mathbf{x}-\mathbf{y}) + iR_{ab,KL}{}^{ji}(y)\partial_j\delta(\mathbf{x}-\mathbf{y}) + O(e^2)\,. \quad (3.16)$$

If the ST is of the minimal form

$$R_{cb,KL}{}^{ji} = C\delta_{KL}\delta_{cb}\delta_{ij}\,, \quad C\neq 0 \quad (3.17)$$

then the current commutator is

$$[K_a{}^0(\mathbf{x},t),L_b{}^i(\mathbf{y},t)] = if_{abc}(K\cdot L)_c{}^i(x)\delta(\mathbf{x}-\mathbf{y}) + iC\delta_{KL}[\delta_{ab}\partial^i - ef_{a3b}a^i(x)]\delta(\mathbf{x}-\mathbf{y}) + O(e^2)\,. \quad (3.18)$$

This is the form given by Lee and Zumino[4] on the basis of gauge invariance.

Next we consider weak interactions with the current-current interaction

$$\mathcal{L}_{JJ} = \tfrac{1}{2}\sqrt{2}GJ_\mu J^{\mu\dagger}\,, \quad (3.19\text{a})$$

$$J^\mu = F_a(V_a{}^\mu + A_a{}^\mu) + j^\mu\,. \quad (3.19\text{b})$$

Here j^μ is the lepton current and F_a is an eight-vector

$$F_a = (\cos\theta, i\,\sin\theta, 0,\, \cos\theta, i\,\sin\theta, 0,0,0)\,. \quad (3.19\text{c})$$

With this interaction the divergence is

$$\partial_\mu K_a{}^\mu = -\tfrac{1}{2}\sqrt{2}Gf_{abc}F_b(V_c{}^\mu + A_c{}^\mu)J_\mu{}^\dagger + \text{H.c.} + O(G^2)\,. \quad (3.20)$$

Upon insertion of (3.20) into (3.12a), $B_{ba,LK}{}^i$ may be determined. For notational simplicity, we assume the ST to be of the minimal form (3.17). We then find that

$$B_{ab,KL}{}^i = -\tfrac{1}{2}\sqrt{2}CGF_c(V_d{}^i + A_d{}^i)(f_{abc}F_d{}^* + f_{adc}F_b{}^*) - \tfrac{1}{2}\sqrt{2}CGf_{abc}j^i + \text{H.c.} + O(G^2)\,. \quad (3.21)$$

This too does not vanish.

4. DERIVATION OF WEINBERG'S SECOND SUM RULE

In the present section,[11] we make use of the commutators derived in Sec. 2, as well as additional assumptions, to derive Weinberg's second sum rule.[5] We restrict ourselves to $SU(2)\times SU(2)$, and shall argue that the extension to $SU(3)\times SU(3)$, may in general have corrections

[11] A brief summary of this derivation has been presented in R. Jackiw, Phys. Letters **27B**, 96 (1968).

of the order of the kaon mass. (Naturally we ignore weak and electromagnetic effects.)

Weinberg's two sum rules are[5]

$$\int \rho_V(a^2)\frac{da^2}{a^2}=\int \rho_A(a^2)\frac{da^2}{a^2}+\int \rho_A{}^0(a^2)da^2, \quad (4.1a)$$

$$\int \rho_V(a^2)da^2=\int \rho_A(a^2)da^2. \quad (4.1b)$$

The spectral functions are defined by

$$\langle\Omega|V_a{}^\mu(x)V_b{}^\nu(0)|\Omega\rangle =\delta_{ab}\int e^{ikx}\theta(k^0)\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)\rho_V(k^2)d^4k, \quad (4.2a)$$

$$\langle\Omega|A_a{}^\mu(x)A_b{}^\nu(0)|\Omega\rangle=\delta_{ab}\int e^{ikx}\theta(k^0) \times\left[\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)\rho_A(k^2)-k^\mu k^\nu\rho_A{}^0(k^2)\right]d^4k. \quad (4.2b)$$

The second sum rule, (4.1b), has been frequently considered to be on less firm footing than the first sum rule (4.1a) because the latter has been proved only from assumptions about asymptotic behavior,[5,12] or within specific Lagrangian models.[13] The first sum rule, on the other hand, can be established by methods, additional to the ones used in deriving (4.1b), which make the assumption that the $I=1$ components in the ST in the current commutators of space components with time components are absent. Specifically, in the notation (3.1b),

$$[K_a{}^0(\mathbf{x},t),L_b{}^i(\mathbf{y},t)]=i\epsilon_{abc}(K\cdot L)_c{}^i(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}) -iR_{ab,KL}{}^{ji}(y)\partial_j\delta(\mathbf{x}-\mathbf{y}), \quad (4.3)$$

one assumes that

$$\epsilon_{cab}R_{ab,KL}{}^{ji}=0. \quad (4.4)$$

[The ϵ_{abc} are the $SU(2)$ structure constants.] It then follows that

$$R_{aa,AA}{}^{ij}=R_{aa,VV}{}^{ij}, \quad (4.5a)$$

which implies (4.1). Alternatively, one may assume that the ST is a c number.[5] It then follows that it is a scalar in group space

$$R_{ab,KL}{}^{ij}=\delta_{ab}\delta_{KL}\delta_{ij}C,\quad C\neq 0. \quad (4.5b)$$

This latter derivation is attractive in that the assumptions concern themselves with the isotopic spin properties of certain commutators, and no reference need be made to questionable Lagrangian models or to untestable asymptotic behavior. We now show that the ETCR, derived in Sec. 1, can be used to give a proof of the second sum rule (4.1b), from assumptions very analogous to those given above in the derivation of (4.1a). Thus we tie the validity of the second sum rule to the algebraic structure of commutators, and not directly to field-theoretic models or to asymptotic behavior.

It is easy to verify from (4.2) that (4.1b) is equivalent to

$$\langle\Omega|[\dot{V}_a{}^i(\mathbf{x},0)-\partial^i V_a{}^0(\mathbf{x},0),V_b{}^j(0)]|\Omega\rangle =\langle\Omega|[\dot{A}_a{}^i(\mathbf{x},0)-\partial^i A_a{}^0(\mathbf{x},0),A_b{}^j(0)]|\Omega\rangle. \quad (4.6)$$

Thus it is sufficient to derive (4.6). The following three model-dependent assumptions are made. (1) The assumptions made in the original derivation of Weinberg's first sum rule are made: The Jacobi identity is utilized, and is therefore assumed to be valid. The ETCR of the time component of a current with a current four vector is of the standard form and all ST are c numbers. (We operate in a theory without electromagnetic or weak interactions.) All the currents are conserved. It will be seen that the latter assumption can be relaxed somewhat to take into account partial conservation of axial-vector current (PCAC). (2) Equal-time commutators of space components of currents are c numbers. From the above two assumptions we can prove that

$$[\dot{K}_a{}^i(\mathbf{x},t)-\partial^i K_a{}^0(\mathbf{x},t),L_b{}^j(\mathbf{y},t)]=iC_{ab,KL}{}^{ij}(x)\delta(\mathbf{x}-\mathbf{y}) +i\epsilon_{abc}(K\cdot L)_c{}^i\partial^i\delta(\mathbf{x}-\mathbf{y}), \quad (4.7a)$$

$$C_{ab,KL}{}^{ij}=C_{ba,LK}{}^{ji}. \quad (4.7b)$$

The third assumption which we formulate in the last restrictive fashion is (3) the commutator of the $I=1$ part of $C_{ab,VA}{}^{ij}$ with $\int A_c{}^0d^3x$ has no vacuum expectation. This implies that

$$\langle\Omega|C_{aa,VV}{}^{ij}|\Omega\rangle=\langle\Omega|C_{aa,AA}{}^{ij}|\Omega\rangle, \quad (4.8)$$

which then leads to (4.6) and is the desired result.

To prove (4.7) we begin with Jacobi identity for $T^{00}(z)$, $K_a{}^i(x)$, $L_b{}^j(y)$.

$$[[T^{00}(\mathbf{z},t),K_a{}^i(\mathbf{x},t)],L_b{}^j(\mathbf{y},t)] +[[K_a{}^i(\mathbf{x},t),L_b{}^j(\mathbf{y},t)],T^{00}(\mathbf{z},t)] +[[L_b{}^j(\mathbf{y},t),T^{00}(\mathbf{z},t)],K_a{}^i(\mathbf{x},t)]=0. \quad (4.9)$$

The second term is zero according to the assumption (2). The remaining internal commutators are evaluated with the help of (2.10b), (4.3), and (4.5b). We find

$$[\dot{K}_a{}^i(\mathbf{x},t)-\partial^i K_a{}^0(\mathbf{x},t),L_b{}^j(\mathbf{y},t)]\delta(\mathbf{z}-\mathbf{x})-i\epsilon_{abc}(K\cdot L)_c{}^j(z) \times\delta(\mathbf{z}-\mathbf{y})\partial^i\delta(\mathbf{z}-\mathbf{x})=[\dot{L}_b{}^j(\mathbf{y},t)-\partial^i L_b{}^0(\mathbf{y},t),K_a{}^i(\mathbf{x},t)] \times\delta(\mathbf{z}-\mathbf{y})-i\epsilon_{bac}(K\cdot L)_c{}^i(z)\delta(\mathbf{z}-\mathbf{x})\partial^j\delta(\mathbf{z}-\mathbf{y}). \quad (4.10)$$

---
[12] T. Das. V. S. Mathur, and S. Okubo, Phys. Rev. Letters 18, 761 (1967).

[13] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).

Upon integration over $\mathbf{x}$, (4.10) becomes

$$[\dot{K}_a{}^i(\mathbf{z},t)-\partial^i K_a{}^0(\mathbf{z},t),\ L_b{}^j(\mathbf{y},t)] =\left[\dot{L}_b{}^j(\mathbf{y},t)-\partial^j L_b{}^0(\mathbf{y},t),\ \int K_a{}^i(\mathbf{x},t)d^3x\right]\delta(\mathbf{z}-\mathbf{y}) +i\epsilon_{abc}(K\cdot L)_c{}^i(z)\partial^j\delta(\mathbf{z}-\mathbf{y}). \quad (4.11)$$

This proves (4.7a) when the following definition is made:

$$C_{ab,KL}{}^{ij}(y) \equiv\left[L_b{}^j(\mathbf{y},t)-\partial^j L_b{}^0(\mathbf{y},t),\ \int K_a{}^i(\mathbf{x},t)d^3x\right]. \quad (4.12)$$

Note that the conservation of the currents plays no role at this stage.

To prove (4.7b), we insert (4.7a) into (4.10) and obtain immediately (4.7b).

Formula (4.7) is rather remarkable in that it predicts that the ST in the ETCR is model-independent, as long as assumptions (1) and (2) hold. Only the term proportional to the δ function is model-dependent. We note that in the algebra of fields[12] in which assumptions (1) and (2) are satisfied, the ETCR (4.7) can be evaluated explicitly and does indeed satsify (4.7). Furthermore Sugawara's model[8] also satisfies these assumptions.

Next we proceed to derive (4.8). We consider the Jacobi identity between $\dot{K}_a{}^i(x)-\partial^i K_a{}^0(x)$, $L_b{}^j(y)$, and $M_c{}^0(z)$.

$$[[\dot{K}_a{}^i(\mathbf{x},t)-\partial^i K_a{}^0(\mathbf{x},t),L_b{}^j(\mathbf{y},t)],M_c{}^0(\mathbf{z},t)] +[[L_b{}^j(\mathbf{y},t),M_c{}^0(\mathbf{z},t)],\ \dot{K}_a{}^i(\mathbf{x},t)-\partial^i K_a{}^0(\mathbf{x},t)] +[[M_c{}^0(\mathbf{z},t),\dot{K}_a{}^i(\mathbf{x},t)-\partial^i K_a{}^0(\mathbf{x},t)],L_b{}^j(\mathbf{y},t)]=0. \quad (4.13)$$

The commutators that we have given before are sufficient to evaluate the first two internal commutators. To evaluate the third internal commutator we use

$$[M_c{}^0(\mathbf{z},t),\dot{K}_a{}^i(\mathbf{x},t)]=\partial^0[M_c{}^0(\mathbf{z},t),K_a{}^i(\mathbf{x},t)] -[\dot{M}_c{}^0(\mathbf{z},t),K_a{}^i(\mathbf{x},t)]. \quad (4.14a)$$

The first ETCR on the right side is readily evaluated. The second can be rewritten as

$$[\dot{M}_c{}^0(\mathbf{z},t),K_a{}^i(\mathbf{x},t)]=[\partial_\mu M_c{}^\mu(\mathbf{z},t),K_a{}^i(\mathbf{x},t)] -\partial_j{}^z[M_c{}^j(\mathbf{z},t),K_a{}^i(\mathbf{x},t)]. \quad (4.14b)$$

The second term on the right side is zero according to assumption (2). The first term is zero since by assumption the currents are conserved. We shall prove below that it may be zero also when a special form PCAC obtains. Hence (4.14b) is zero. We find therefore from (4.13) the result that

$$[M_c{}^0(\mathbf{z},t),C_{ab,KL}{}^{ij}(\mathbf{x},t)]=i\epsilon_{cad}C_{db,M\cdot L\,K}{}^{ij}(x)\delta(\mathbf{z}-\mathbf{x}) +i\epsilon_{cbd}C_{ad,M\cdot L\,K}{}^{ij}\delta(\mathbf{z}-\mathbf{x})-i\epsilon_{cbd}\epsilon_{dae}(K\cdot L\cdot M)_e{}^i(x)\partial^j \times\delta(\mathbf{x}-\mathbf{z})-i\epsilon_{cad}\epsilon_{dbe}(K\cdot L\cdot M)_e{}^j(x)\partial^i\delta(\mathbf{x}-\mathbf{z}). \quad (4.15)$$

This equation shows that $C_{ab,KL}{}^{ij}$ cannot be a c number, for if it were the left side would vanish, while the right side would contain the q-number operators $(K\cdot L\cdot M)_e{}^i$. From (4.15) our result now follows. Set $M=L=A$, $K=V$ and contract with ϵ_{fab} to give

$$[A_c{}^0(\mathbf{z},t),\epsilon_{fab}C_{ab,VA}{}^{ij}(\mathbf{x},t)]=i\epsilon_{fab}\epsilon_{cad}C_{db,AA}{}^{ij}(x)\delta(\mathbf{z}-\mathbf{x}) +i\epsilon_{fab}\epsilon_{cbd}C_{ad,VV}{}^{ij}(x)\delta(\mathbf{z}-\mathbf{x})-i\epsilon_{fab}\epsilon_{cbd}\epsilon_{dae}V_e{}^i(x) \times\partial^j\delta(\mathbf{x}-\mathbf{z})-i\epsilon_{fab}\epsilon_{cad}\epsilon_{dbe}V_e{}^j(x)\partial^i\delta(\mathbf{x}-\mathbf{z}). \quad (4.16a)$$

Upon taking vacuum expectation values, the left side vanishes according to assumption (3). ($\epsilon_{fab}C_{ab,AV}{}^{ij}$ is just the $I=1$ part of $C_{ab,AV}{}^{ij}$.) On the right side $V_e{}^i$ has no vacuum expectation value, and we are left with

$$\epsilon_{fab}\epsilon_{cad}\langle\Omega|C_{db,VV}{}^{ij}(x)|\Omega\rangle=\epsilon_{fab}\epsilon_{cdb} \times\langle\Omega|C_{ad,AA}{}^{ij}(x)|\Omega\rangle. \quad (4.16b)$$

Contracting f with c yields finally

$$\langle\Omega|C_{aa,VV}{}^{ij}|\Omega\rangle=\langle\Omega|C_{aa,AA}{}^{ij}|\Omega\rangle, \quad (4.16c)$$

which is the desired result.

We have derived Weinberg's second sum rule for *conserved* axial-vector currents. When the axial-vector current is not conserved, then the first equation on the right side of (4.14b) is in our application ($M=A$, $K=V$)

$$[\partial_\mu A_c{}^\mu(\mathbf{z},t),V_a{}^i(\mathbf{x},t)]. \quad (4.17)$$

This becomes a model-dependent quantity, proportional to the pion mass, since the axial-vector current is presumably conserved for massless pions. However, we can still show that (4.17) is zero if a special form of PCAC holds. The form of PCAC that we require is $\partial_\mu A_c{}^\mu=\mu_\pi{}^2F_\pi\varphi_c$, where $\mu_\pi{}^2$ is the pion mass, F_π is the pion decay constant, and φ_c is the canonical pion field, canonically conjugate to $\dot{\varphi}_c$. To prove the desired result we consider the Jacobi identity between T^{00}, $V_a{}^0$, and $\partial_\mu A_c{}^\mu$;

$$[[T^{00}(\mathbf{x},t),V_a{}^0(\mathbf{y},t)],\partial_\mu A_c{}^\mu(\mathbf{z},t)] +[[V_a{}^0(\mathbf{y},t),\partial_\mu A_c{}^\mu(\mathbf{z},t)],T^{00}(\mathbf{x},t)] +[[\partial_\mu A_c{}^\mu(\mathbf{z},t),T^{00}(\mathbf{x},t)],V_a{}^0(\mathbf{y},t)]=0. \quad (4.18)$$

The first internal commutator is evaluated from (2.10a), the third from (2.14). The second commutator is

$$[V_a{}^0(\mathbf{y},t),\partial_\mu A_c{}^\mu(\mathbf{z},t)]=i\epsilon_{acd}\partial_\mu A_d{}^\mu(\mathbf{y},t)\delta(\mathbf{y}-\mathbf{z}). \quad (4.19)$$

This has been derived in Ref. 3, and can be shown to follow from (3.16). Therefore, we have

$$[\varphi_c(\mathbf{z},t),V_a{}^i(\mathbf{x},t)]\partial_i\delta(\mathbf{x}-\mathbf{y})-i\epsilon_{acd}\dot{\varphi}_d(x)\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{y}-\mathbf{z}) +[V_a{}^0(\mathbf{y},t),\dot{\varphi}_c(\mathbf{x},t)]\delta(\mathbf{x}-\mathbf{y})=0. \quad (4.20)$$

Since it is true that

$$[V_a{}^0(\mathbf{y},t),\varphi_c(\mathbf{x},t)]=i\epsilon_{acd}\varphi_d(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y}) \quad (4.21a)$$

and by assumption $\dot{\varphi}_c(\mathbf{x},t)$ is canonically conjugate to φ_c, it must also be true that

$$[V_a{}^0(\mathbf{y},t),\dot{\varphi}_c(\mathbf{x},t)]=i\epsilon_{acd}\dot{\varphi}_a(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y}). \quad (4.21b)$$

Therefore the last two terms in (4.20) cancel, leaving

$$[\varphi_c(\mathbf{z},t),V_a^i(\mathbf{x},t)]=0\,, \qquad (4.22)$$

which is the desired result.

We now discuss the assumptions that we have made in this derivation. The assumption about the validity of the Jacobi identity is technically necessary in view of our repeated use of that identity. Since it involves two space components and one time component, it may in fact be unreliable.[14] However, it can easily be verified that (4.4) may be integrated over **x** and (4.13) may be integrated over **z** and **y**, without losing any other information necessary to derive (4.16c). The Jacobi identity, when applied to such nonlocal operators as space integrals over local operators, should be reliable. The statement that ST are c numbers assures that the first sum rule (4.1a) is valid. It is clear that if q number ST's are present then many of our relations would have additional terms about which we could not say anything. It is also evident from arguments based on asymptotic chiral symmetry that the second sum rule cannot be valid if the first sum rule is invalid (at least for zero-mass pions). Therefore, the first assumption seems necessary.

The second assumption, if violated, would also produce terms about which we could not say anything, e.g., in (4.9). Thus it too seems necessary. This assumption is somewhat related to the first assumption. It has been shown[3] that in general the space-space ETCR can contain only terms proportional to a δ function and to its first derivative. Furthermore if the ST in the space-time ETCR are c numbers, then the gradient terms in the space-space ETCR necessarily vanish. (The derivation of Ref. 3 depends on certain assumptions, which are explicitly stated there and which are briefly summarized in Refs. 1, 3, and 6 of the present paper.) Therefore, the effective content of the second assumption is that the term proportional to the δ function is a c number. If a specific form for the space-space ETCR is known, our results may of course still be valid. For example, it can be verified that if the space-space ETCR is given by the quark-model formulas, then the second Weinberg sum rule still holds when assumptions (1) and (3) are made.

The third assumption is necessary and sufficient within our argument, as is seen from (4.16a). It cannot be replaced by the stronger statement that the commutator of A_c^0 with the $I=1$ part of $C_{ab,AV}{}^{ij}$ vanishes identically. For if this happened, then the terms on the right side of (4.16a) proportional to the δ function and to the gradient of the δ function would have to vanish separately. However, it is manifestly true that the gradient terms cannot vanish identically.

Finally, we take up the matter of the commutator (4.17) which may not vanish when $\partial_\mu A_a{}^\mu\neq 0$. The special form of PCAC, which allowed us to conclude that the ETCR $[\varphi_a,V_b{}^i]=0$, is rather restrictive. However, we note that the only model which incorporates PCAC and current algebra, viz., the σ model, does satisfy the assumption. Nevertheless, in general we must admit $O(\mu_\pi{}^2)$ corrections to (4.1b). Such corrections are not serious in $SU(2)\times SU(2)$. But if the present argument is extended to $SU(3)\times SU(3)$, the corrections become of the order of the kaon mass squared, which would make (4.1b) unreliable. This may be the explanation why the second Weinberg sum rule is not well satisfied for $SU(3)\times SU(3)$.

5. RELATION BETWEEN DECAY CONSTANTS

In this section we obtain a relation between the pion, kaon, and κ decay constants, by using (3.8) and partial conservation of currents. According to (3.2a) and (3.2b), we have

$$[K_a(t),\partial_\mu L_b{}^\mu(x)]=iC_{ab,KL}(x)\,, \qquad (5.1a)$$

$$C_{ab,KL}(x)-C_{ba,LK}(x)=f_{abc}\partial_\mu(K\cdot L)_c{}^\mu\,, \qquad (5.1b)$$

where

$$K_a(t)\equiv\int d^3y\,K_a{}^0(\mathbf{y},t)\,. \qquad (5.1c)$$

The consequences of (5.1) for $SU(2)\times SU(2)$ have been explored in Ref. 3. We now consider the implication for $SU(3)\times SU(3)$. Of particular interest is the case $K=L=A$, $a=1$, $b=4$. We have therefore

$$[A_1(t),\partial_\mu A_4{}^\mu(\mathbf{x},t)]-[A_4(t),\partial_\mu A_1{}^\mu(\mathbf{x},t)] =if_{14c}\partial_\mu V_c{}^\mu(x)=\tfrac{1}{2}i\partial_\mu V_7{}^\mu\,. \qquad (5.2)$$

We impose the PCAC condition on $\partial_\mu A_4{}^\mu$, $\partial_\mu A_2{}^\mu$, and $\partial_\mu V_7{}^\mu$, i.e., we identify these divergences with the kaon, pion, and κ fields.

$$\begin{aligned}\partial_\mu A_4{}^\mu&=F_K\mu_K{}^2a_4\,,\\ \partial_\mu A_1{}^\mu&=F_\pi\mu_\pi{}^2a_1\,,\\ \partial_\mu V_7{}^\mu&=-F_\kappa\mu_\kappa{}^2S_\kappa\,.\end{aligned} \qquad (5.3)$$

Here a_4 (a_1) is the kaon (pion) pseudoscalar field, while S_κ is the κ scalar field. The F's are the decay constants and the μ's are the masses of these mesons. The minus sign has been inserted for convention. Combining (5.2) with (5.3) yields

$$F_K\mu_K{}^2[A_1(t),a_4(\mathbf{x},t)]-F_\pi\mu_\pi{}^2[A_4(t),a_1(\mathbf{x},t)] =-\tfrac{1}{2}iF_\kappa\mu_\kappa{}^2S_\kappa(x)\,. \qquad (5.4)$$

Next we assume that the pseudoscalar mesons transform like members of a $(3,\bar{3})+(\bar{3},3)$ representation of $SU(3)\times SU(3)$. This means that

$$[A_a(t),a_b(x)]=id_{abc}S_c(x)\,. \qquad (5.5)$$

Inserting (4.5) in (4.4), we obtain

$$(F_K\mu_K{}^2-F_\pi\mu_\pi{}^2)d_{14c}S_c(x)=(F_K\mu_K{}^2-F_\pi\mu_\pi{}^2)S_6(x)/2 =-F_\kappa\mu_\kappa{}^2S_\kappa(x)/2\,. \qquad (5.6)$$

[14] K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppls. **37** and **38**, 74 (1966).

It therefore follows that

$$S_\kappa(x)=S_6(x) \tag{5.7}$$

and

$$F_K\mu_K{}^2+F_\kappa\mu_\kappa{}^2=F_\pi\mu_\pi{}^2. \tag{5.8}$$

This is our desired result.[15]

6. CONCLUSION

We have shown that the ETCR between selected components of the stress energy tensor and selected components of a current can be given in a model-independent fashion (for low spin systems), when the current arises from an internal transformation of the Lagrangian. Starting with the ETCR of time components of these components, we showed how weak and electromagnetic interactions affect the ETCR of space components with time components. A derivation of Weinberg's second sum rule was given from further assumptions about the isospin structure of certain commutators Finally, a linear relation between the decay constants of broken $SU(3)\times SU(3)$ has been found.

Note added in proof. We present a canonical proof of (1.3). Use is made of functional derivatives $\delta/\delta\Psi(x)$, with canonical momentum $\Pi(x)$ kept fixed, and $\delta/\delta\Psi(x)$, with the canonical field kept fixed. In addition, variational derivatives are used: $\delta'/\delta'\Psi$, $\delta'/\delta'\partial_i\Psi$, and $\delta'/\delta'\Psi$. For any functional $\mathcal{L}$, depending only on Ψ and $\partial_\mu\Psi$, we have

$$\frac{\delta\mathcal{L}(x)}{\delta\Psi(y)}=\frac{\delta'\mathcal{L}(x)}{\delta'\dot{\Psi}}\frac{\delta\dot{\Psi}(x)}{\delta\Psi(y)}+\frac{\delta'\mathcal{L}(x)}{\delta'\partial_i\Psi}\partial_i\delta(\mathbf{x}-\mathbf{y})+\frac{\delta'\mathcal{L}(x)}{\delta'\Psi}\delta(\mathbf{x}-\mathbf{y}), \tag{N1}$$

$$\frac{\delta\mathcal{L}(x)}{\delta\Pi(y)}=\frac{\delta'\mathcal{L}(x)}{\delta'\dot{\Psi}}\frac{\delta\dot{\Psi}(x)}{\delta\Psi(y)}.$$

When $\mathcal{L}$ is the Lagrange density, the equations of motion and the definition of Π can be used to rewrite the above as

$$\frac{\delta\mathcal{L}(x)}{\delta\Psi(y)}=\Pi(x)\frac{\delta\dot{\Psi}(x)}{\delta\Psi(y)}+\frac{\delta'\mathcal{L}(x)}{\delta'\partial_i\Psi}\partial_i\delta(\mathbf{x}-\mathbf{y})+\dot{\Pi}(x)\delta(\mathbf{x}-\mathbf{y})+\left(\partial_i\frac{\delta'\mathcal{L}(x)}{\delta'\partial_i\Psi}\right)\delta(\mathbf{x}-\mathbf{y})\frac{\delta\mathcal{L}(x)}{\delta\Pi(y)}=\Pi(x)\frac{\delta\Psi(x)}{\delta\Psi(y)}. \tag{N2}$$

The commutator in question is

$$C(\mathbf{x},\mathbf{y},t)=[T^{00}(\mathbf{x},t),j^0(\mathbf{y},t)]. \tag{N3}$$

(We suppress internal degrees of freedom.) Using the canonical formulas

$$T^{00}=\Pi\Psi-\mathcal{L},\quad j^0=\Pi\Psi, \tag{N4}$$

we have

$$C(\mathbf{x},\mathbf{y},t)=i\Pi(\mathbf{x},t)\frac{\delta\dot{\Psi}(\mathbf{x},t)}{\delta\Psi(\mathbf{y},t)}\Psi(\mathbf{y},t)-i\Pi(\mathbf{x},t)\Pi(\mathbf{y},t)\frac{\delta\dot{\Psi}(\mathbf{x},t)}{\delta\Psi(\mathbf{y},t)}-i\Pi(\mathbf{x},t)\Psi(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})-i\frac{\delta\mathcal{L}(\mathbf{x},t)}{\delta\Psi(\mathbf{y},t)}\Psi(\mathbf{y},t)+i\Pi(\mathbf{y},t)\frac{\delta\mathcal{L}(\mathbf{x},t)}{\delta\Pi(\mathbf{y},t)}. \tag{N5}$$

Equations (N2) simplify the above to

$$\begin{aligned}C(\mathbf{x},\mathbf{y},t)&=-i\Pi(\mathbf{x},t)\dot{\Psi}(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})-i\dot{\Pi}(\mathbf{x},t)\Psi(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})\\&\quad-i\frac{\delta'\mathcal{L}(\mathbf{x},t)}{\delta'\partial_i\Psi}\Psi(\mathbf{y},t)\partial_i\delta(\mathbf{x}-\mathbf{y})\\&\quad-i\left(\partial_i\frac{\delta'\mathcal{L}(\mathbf{x},t)}{\delta'\partial_i\Psi}\right)\Psi(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})\\&=-i\partial_\mu\left[\frac{\delta'\mathcal{L}(\mathbf{x},t)}{\delta'\partial_\mu\Psi}\Psi(\mathbf{x},t)\right]\delta(\mathbf{x}-\mathbf{y})\\&\quad-i\frac{\delta'\mathcal{L}(\mathbf{x},t)}{\delta'\partial_i\Psi}\Psi(\mathbf{x},t)\partial_i\delta(\mathbf{x}-\mathbf{y})\\&=-i\partial_\mu j^\mu(x)\delta(\mathbf{x}-\mathbf{y})-ij^i(x)\partial_i\delta(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{N6}$$

This is the desired result.

ACKNOWLEDGMENTS

I wish to acknowledge the hospitality at CERN, where this work was performed.

[15] A similar relation has been obtained by S. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968), who made use of a Lagrangian model for symmetry breaking and smoothness assumptions about various functions. I am indebted to Professor S. Glashow for pointing out the relevance of the methods developed in this paper to this problem.