n-Particle Kinematics and a Generalized Partial-Wave Analysis

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A generalized partial-wave analysis is derived for n-particle reactions.

I. INTRODUCTION

The assumption of an S matrix invariant under Poincaré transformations leads to a covariant scattering theory and results in a partial-wave expansion of the scattering amplitude characterized by a set of unknown Poincaré-invariant reduced amplitudes. The unknown amplitudes are specified by theoretical models or experimental data, and since a large quantity of data from multiparticle reactions is now available, it is reasonable to ask how much can be learned about multiparticle reactions through a model-independent approach, that is, assuming only that the scattering operator transforms as a scalar under the Poincaré group.

The basic mathematical problem involved in deriving the partial-wave analysis is the reduction of the tensor product of n single-particle states, that is, the Clebsch-Gordan problem. Macfarlane[1] has shown that it is possible to carry out the tensor-product reduction in a stepwise fashion. His scheme results in a partial-wave analysis in which the Poincaré-invariant amplitudes are labeled by the eigenvalues of diagonal operators of the two-particle subsystems specified in the coupling scheme. This analysis is useful because the eigenvalues of the diagonal operators can often be associated with the quantum numbers of intermediate resonances. On the other hand, the partial-wave expansion is not symmetric under particle interchange; symmetrization of the expansion results in unmanageable sums that are usually cut off arbitrarily.[2]

It is therefore of some interest to couple particles together in as highly symmetric a fashion as possible in order that the resulting partial-wave analysis treats all particles as symmetrically as possible. Goldberg[3] and Werle[4] have considered this problem, and Roffman[5] has derived a symmetric coupling that can be implemented provided that the two-particle coupling is known. In this paper, we shall derive a partial-wave analysis for the reaction

$$1+2\to 3+4+\cdots+n\,, \tag{1}$$

symmetric in particles 3 through n, using the results of a previous work dealing with the reduction of n-fold tensor-product representations of noncompact groups.[6] In Sec. II the salient features of the coupling scheme discussed in Ref. 6 will be given and the partial-wave analysis derived. It will be shown how the invariant parameters in the partial-wave expansion are determined and one set of parameters will be given. The partial-wave expansion will be carried out in the simplest frame of reference, namely, the c.m. system, although it is possible to derive the expansion in an arbitrary frame of reference. In Sec. III the results of Sec. II will be used to describe the reaction

$$1+2\to 3+4+5\,, \tag{2}$$

assuming that particles 3–5 are spinless.

II. KINEMATICS AND A GENERALIZED PARTIAL-WAVE ANALYSIS

In Ref. 6 we have shown how to decompose the n-fold tensor products of positive-mass representations of the Poincaré group in an arbitrary frame of reference. In the most convenient frame of reference, namely, the c.m. of particles 3 through n, it is possible to write the following reduction formula:

$$\begin{aligned}|[M_3,s_3]\mathbf{p}_3\sigma_3\rangle\cdots|[M_n,s_n]\mathbf{p}_n\sigma_n\rangle &= N\sum_{s}^{\infty}\sum_{K=-s}^{+s}\sum_{\sigma=-s}^{+s}\sum_{\alpha_3\cdots\alpha_n}\left(\frac{2s+1}{4\pi}\right)^{1/2}D_{K\sigma}{}^{s*}(R)\\ &\times\prod_{i=3}^{n}D_{\alpha_i\sigma_i}{}^{s_i}(R_i)\,|[Ms]\mathbf{p}=\mathbf{0},\sigma;K,\alpha_3,\cdots,\alpha_n,\mu\rangle.\end{aligned} \tag{3}$$

Here $[M_i,s_i]$, $\mathbf{p}_i$, and σ_i are the mass, spin, momentum, and z component of the spin of the ith particle. Denoting the 4-momentum of the ith particle by p_i, $i=3$, 4, $\cdots$, n, gives $M^2=(p_3+p_4+\cdots+p_n)^2$ and $\mathbf{p}_3+\mathbf{p}_4+\cdots+\mathbf{p}_n=0$ by energy-momentum conservation. μ is a set of continuous degeneracy parameters of the form $p_i\cdot p_j$ and $\epsilon_{\alpha\beta\gamma\delta}p_n{}^\alpha p_{n-1}{}^\beta p_{n-2}{}^\gamma p_i{}^\delta$; a convenient set will be given in Eq. (14). R is a rigid-body rotation that rotates all $n-2$ outgoing momenta $\mathbf{p}_i$ through the same angle. It contains the effect of coupling all of the particles together into one "over-all particle." $D_{K\sigma}{}^s(R)$ is a Wigner function that compares the spin of the "over-all particle" $|[Ms]\mathbf{p}=\mathbf{0},\ \sigma;K,\ \alpha_i,\ \mu\rangle$ along the z axis of the c.m. coordinate system with the spin along a coordinate system rotated by R. The rotation R induces rotations

[1] A. J. Macfarlane, Rev. Mod. Phys. **34**, 41 (1962).
[2] H. Goldberg, J. Math. Phys. **7**, 434 (1966).
[3] H. Goldberg, Nuovo Cimento **47**, 495 (1967).
[4] J. Werle, *Relativistic Theory of Reactions* (John Wiley & Sons, Inc., New York, 1966).
[5] E. Roffman, J. Math. Phys. **9**, 62 (1968).
[6] W. H. Klink and G. Smith, University of Iowa Report No. 68-8, 1968 (to be published in Commun. Math. Phys.).

R_i for all i outgoing particles, changing their spin component from σ_i to α_i in the Wigner function $D_{\alpha_i\sigma_i}{}^{s_i}(R_i)$. N is a normalization constant depending on how the states $|[M_i,s_i]\mathbf{p}_i\sigma_i\rangle$ are normalized.

Implicit in the construction of a single-particle state $|[M_i,s_i]\mathbf{p}_i,\sigma_i\rangle$ is an operation that boosts or orbits the ith particle from its rest frame to the c.m. frame. A Lorentz transformation is involved in carrying the particle from its rest frame to the c.m. frame, and since Lorentz transformations can always be decomposed into right cosets with respect to the rotation subgroup, it is always possible to orbit the particle with right-coset representatives; the rotation has no effect on the rest-frame four-vector $\hat{p}=(\mathbf{0},M)$. Thus, a boost operation can always be thought of as a right-coset decomposition of the homogeneous Lorentz group; equivalently, the covering group $SL(2,C)$ of the Lorentz transformations can be decomposed into right cosets with respect to the little group $SU(2)$. That is,

$$SL(2,C)=\bigcup_c SU(2)\Lambda_c\,, \tag{4}$$

so that to boost a particle at rest to an arbitrary momentum one simply writes[7]

$$\begin{aligned} H(p)&=\Lambda_c^{-1}H(\hat{p})\Lambda_c^{\dagger-1}\\ &=M\Lambda_c^{-1}\Lambda_c^{\dagger-1}\,, \end{aligned} \tag{5}$$

where Λ_c is a right coset representative, the inverse of which boosts the particle to a momentum p and

$$H(p)\equiv\begin{pmatrix} E+p_z & p_x-ip_y\\ p_x+ip_y & E-p_z\end{pmatrix}. \tag{6}$$

M is the mass of the particle. The notation $\Lambda_c^{-1}(p)$ will thus mean a boost from $\hat{p}$ to p.

There are many choices of Λ_c possible. One choice, the choice we shall make in this article, is the set of rotationless Lorentz transformations, which in covering group form are Hermitian matrices and can conveniently be written

$$\Lambda_c^{-1}(p)=R(\Omega\mathbf{p})L_z(|\mathbf{p}|)R^{-1}(\Omega\mathbf{p})\,, \tag{7}$$

where $R(\Omega\mathbf{p})$ is a rotation dependent on the polar angles $\Omega\mathbf{p}$ of $\mathbf{p}$ and $L_z(|\mathbf{p}|)$ is a Lorentz transformation along the z axis.

Another possible choice of $\Lambda_c(p)$, called the helicity choice, is

$$\Lambda_c^{-1}(p)=R(\Omega\mathbf{p})L_z(|\mathbf{p}|)\,. \tag{8}$$

This choice is particularly convenient since it leads to rotationally invariant states so that

$$\begin{aligned} U(R)|[ms]\mathbf{p}\lambda\rangle&=U(R)U(R(\Omega\mathbf{p})L_z(|\mathbf{p}|))|[ms]\mathbf{0}\lambda\rangle\\ &=U(RR(\Omega\mathbf{p})L_z(|\mathbf{p}|))|[ms]\mathbf{0}\lambda\rangle\\ &=|[ms]\mathbf{p}'\lambda\rangle\,, \end{aligned} \tag{9}$$

with $U(R)$ an arbitrary rotation operator and $\mathbf{p}'\equiv RR(\Omega\mathbf{p})L_z(|\mathbf{p}|)\mathbf{0}$.

Lorentz transformations carrying a particle from one frame to another also lead to Wigner rotations. In the coset notation, the Wigner rotation can be written

$$(p,\Lambda)=\Lambda_c(\Lambda p)\Lambda\Lambda_c^{-1}(p)\,. \tag{10}$$

An unusual coset choice has been given by Rideau[8] in which the representatives Λ_c are

$$\Lambda_c=\begin{pmatrix}\lambda & 0\\ z & 1/\lambda\end{pmatrix}, \tag{11}$$

with $\lambda>0$, z complex. These coset labels have the advantage of forming a group so that the Wigner rotation (p,Λ_c) is always the identity and thus no Wigner D function appears when Λ_c acts on a general state $|\mathbf{p},\sigma\rangle$.[8]

In order to derive the decomposition (3), it is necessary to choose a boost [e.g., (7) ,(8), or (11)]. In particular, it is necessary to specify the boost $\Lambda_{D_i}{}^{-1}$ which carries the ith particle to the nth particle rest frame in order to calculate the rotation R and the $n-2$ rotations R_i. In Ref. 6, we have shown that the parameters in Λ_{D_i} not only specify the rotations,[9] but also serve to distinguish equivalent representations of the Poincaré group which result from the n-fold tensor product decomposition. If we choose Λ_{D_i} to have the same form as Λ_c, Eq. (8),

$$\begin{aligned} \Lambda_{D_i}&=RL_zR^{-1}\\ &=\begin{pmatrix}|U_i| & V_i\\ -V_i^* & |U_i|\end{pmatrix}\begin{pmatrix}D_i & 0\\ 0 & 1/D_i\end{pmatrix}\begin{pmatrix}|U_i| & -V_i\\ V_i^* & |U_i|\end{pmatrix}\\ &=\begin{pmatrix}|U_i|^2D_i+|V_i|^2/D_i & |U_i|V_i(1/D_i-D_i)\\ |U_i|V_i^*(1/D_i-D_i) & |U_i|^2/D_i+|V_i|^2D_i\end{pmatrix}, \end{aligned} \tag{12}$$

then the parameters $\{|U_i|,V_i,D_i\}$ can all be written in

[7] It is customary to say that an element of the Lorentz group carries the four-momentum p to p', which in the covering-group form is written $H(p')=\Lambda H(p)\Lambda^+$. However, in following the notation of Mackey [*Theory of Group Representation* (Dept. of Mathematics, Univ. of Chicago, Chicago, Illinois, 1955)], it turns out that the inverse element is used, so that $H(p')=\Lambda^{-1}H(p)\Lambda^{-1\dagger}$. This has as a consequence, when $SL(2,C)$ is decomposed into right cosets with respect to $SU(2)$, that $H(p)=\Lambda_c^{-1}H(\hat{p})\Lambda_c^{-1\dagger}$ and thus a boost is the inverse of the usual definition. In order to agree with the conventional choices of boosts, all coset representatives are specified in terms of inverse elements. For a further discussion of this point, see W. Klink, in *Proceedings of the Summer Institute of Theoretical Physics, University of Colorado, Boulder, Colorado, 1968* (to be published).

[8] G. Rideau, Ann. Inst. Henri Poincaré **3**, 339 (1965).

[9] We have not been able to give R a physical interpretation and, in fact, it is not clear that a physical interpretation exists, even for only three outgoing particles. R is defined mathematically in terms of the boosts Λ_{D_i} and the boost $\Lambda_c(P_D)$, where $P_D\equiv\Sigma\Lambda_{D_i}{}^{-1}p_i$, in the following way: $Rp_i=\Lambda_c(P_D)\Lambda_{D_i}{}^{-1}p_i$. Then $R_i=\Lambda_{D_i}\times\Lambda_c^{-1}(P_D)R\Lambda_c^{-1}(p_i)$.

terms of scalar products of the momenta:

$$p_i\cdot p_j=\det H(p_i+p_j)-m_i^2-m_j^2$$
$$=\det[\Lambda^{-1}(\Lambda_{D_i}^{-1}H(\hat{p}_i)\Lambda_{D_i}^{-1\dagger}+\Lambda_{D_i}^{-1}H(\hat{p}_j)\Lambda_{D_j}^{-1\dagger})\Lambda]-m_i^2-m_j^2,$$

$$p_i\cdot p_j=\tfrac{1}{2}M_iM_j\left\{(V_i|U_j|-|U_i|V_j^*)^2\left(\frac{1}{D_i^2D_j^2}+D_i^2D_j^2\right)+|V_iV_j^*+|U_i||U_j||^2\left(\frac{D_i^2}{D_j^2}+\frac{D_j^2}{D_i^2}\right)\right\};$$

$$\epsilon_{\alpha\beta\gamma\delta}p_n{}^\alpha p_{n-1}{}^\beta p_{n-2}{}^\gamma p_i{}^\delta=\tfrac{1}{2}M_nM_{n-1}M_{n-2}M_i\times\left(\frac{1}{D_{n-1}^2}-D_{n-1}^2\right)\left(\frac{1}{D_{n-2}^2}-D_{n-2}^2\right)\left(\frac{1}{D_i^2}-D_i^2\right)\times|U_{n-2}||V_{n-2}||U_i|(\mathrm{Im}V_i-\mathrm{Re}V_i),\quad(13)$$

with $D_i>0$, U_i, V_i complex, $i=3,4,\cdots,n$, and $D_n=U_n=U_{n-1}=1$, $V_n=V_{n-1}=0$, $V_{n-2}>0$. In order to specify uniquely the continuous invariants μ in Eq. (3), it is necessary to solve for $\{|U_i|,V_i,D_i\}$ in terms of an independent set of scalars taken from Eq. (13). A convenient independent set is

$$\begin{aligned}\mu&=p_n\cdot p_i, & i&=3,\cdots,n-1;\\ &=p_{n-1}\cdot p_i, & i&=3,\cdots,n-2;\\ &=p_{n-2}\cdot p_i, & i&=3,\cdots,n-3;\\ &=\epsilon_{\alpha\beta\gamma\delta}p_n{}^\alpha p_{n-1}{}^\beta p_{n-2}{}^\gamma p_i{}^\delta, & i&=3,\cdots,n-3.\end{aligned}\quad(14)$$

To obtain a generalized partial-wave analysis, it is necessary to couple the two incoming particles together.[10] Assuming that the two incoming particles define the z axis of our coordinate system and that the S matrix commutes with the generators of the Poincaré group, we get, using (3),

$$\begin{aligned}&\langle[M_3s_3]\mathbf{p}_3\sigma_3|\cdots\langle[M_ns_n]\mathbf{p}_n\sigma_n|S|[M_1s_1]\mathbf{p}_1\sigma_1\rangle|[M_2s_2]\mathbf{p}_2\sigma_2\rangle\\ &=N\sum_{s,s',K,\sigma,\alpha_i}^{\infty}\left(\frac{2s+1}{4\pi}\right)^{1/2}\left(\frac{2s'+1}{4\pi}\right)^{1/2}D_{K\sigma}{}^s(R)\prod_{i=3}^{n}D_{\alpha_i\sigma_i}{}^{s_i*}(R_i)\langle[Ms]\mathbf{p}\sigma;K,\alpha_i,\mu|S|[M's']\mathbf{p}',\sigma_1+\sigma_2\rangle\\ &\qquad\delta(M'-M)\delta^3(\mathbf{p}'-\mathbf{p})\delta_{ss'}\delta_{\sigma,\sigma_1+\sigma_2}\\ &=N\sum_{s=|\sigma_1+\sigma_2|}^{\infty}\sum_{K=-s}^{+s}\sum_{\alpha_3\cdots\alpha_n}\frac{2s+1}{4\pi}D_{K,\sigma_1+\sigma_2}{}^s(R)\prod_{i=3}^{n}D_{\alpha_i\sigma_i}{}^{s_i*}(R_i)A_{[\sigma_1,\sigma_2,\alpha_3,\cdots,\alpha_n;\,K,\mu]}(M,s),\end{aligned}\quad(15)$$

where $A_{[\sigma_1\sigma_2\alpha_3\cdots\alpha_nK\mu]}(Ms)$ is the reduced amplitude and contains the dynamics

$$A_{[\sigma_1\sigma_2\alpha_3\cdots\alpha_nK\mu]}(Ms)\equiv\langle[Ms]\mathbf{p}=0,\sigma;\alpha_iK\mu|S\times|[Ms]\mathbf{p}=0,\sigma_1+\sigma_2\rangle.\quad(16)$$

Rohrlich[11] has pointed out that n vectors p_j in a four-dimensional space are fixed by $3n-10$ invariants $p_i\cdot p_j$ when three of the vectors (noncoplanar) are chosen as a basis and the relation $\sum_{j=1}^n p_j=0$ exists. He also points out that each vector p_j is only specified up to a sign by $p_1\cdot p_j$, $p_2\cdot p_j$, and $p_3\cdot p_j$, with p_1, p_2, and p_3 the basis vectors. The sign is fixed by the invariant $\mathrm{sgn}\epsilon_{\alpha\beta\gamma\delta}\times p_1{}^\alpha p_2{}^\beta p_3{}^\gamma p_j{}^\delta$. There are only $n-4$ such sign ambiguities because the basis vectors are considered known and one vector is uniquely fixed by the four-momentum conservation relation.

In our analysis there are $3n-12$ invariants of the form $p_i\cdot p_j$, Eq. (14). However, the nth particle is chosen to define the x-z plane, and the invariant mass is fixed by $M^2=(p_1+p_2)^2$, for a total of $3n-10$ parameters in agreement with Rohrlich. We also find it necessary to fix $n-5$ signs using the $n-5$ invariants $\epsilon_{\alpha\beta\gamma\delta}p_n{}^\alpha\times p_{n-1}{}^\beta p_{n-2}{}^\gamma p_i{}^\delta$; in addition, one sign must be fixed in order to specify the handedness of the coordinate system. It is easy to see that this sign can be fixed by $\epsilon_{\alpha\beta\gamma\delta}P^\alpha p_1{}^\beta p_n{}^\gamma p_l{}^\delta=M\epsilon_{ijk}p_1{}^ip_n{}^jp_l{}^k$, where $\mathbf{p}_l$ is any vector not in the x-z plane. Thus, having specified $3n-10$ $p_i\cdot p_j$ invariants and $n-4$ sign ambiguities, the scattering amplitude is uniquely determined.

III. SIMPLE EXAMPLE

If we consider for a moment the stepwise coupling scheme for a three-body final state, there are three possible couplings usually denoted (34)5, (35)4, and (45)3, with 3, 4, and 5 labeling the final-state particles. The coupling schemes are related by recoupling coefficients, the relativistic analog of Racah coefficients. In order to symmetrize the coupling, a sum over all possible couplings is taken and then all couplings are rewritten in terms of one coupling using the recoupling coefficients. This process results in a very complicated partial-wave analysis even if all final-state particles are spinless.

On the other hand, if all final-state particles are spinless, Eq. (16) reduces to a single Wigner function:

$$\langle[M_30]\mathbf{p}_3;[M_40]\mathbf{p}_4;[M_40]\mathbf{p}_5|S|[M_1s_1]\mathbf{p}_1\sigma_1;[M_2s_2]\mathbf{p}_2\sigma_2\rangle=N\sum_{s=\sigma_1+\sigma_2}^{\infty}\sum_{K=-s}^{+s}\frac{2s+1}{4\pi}D_{K,\sigma_1+\sigma_2}{}^s(R)A_{[\sigma_1\sigma_2;\,K,p_3\cdot p_4,p_3\cdot p_5,p_4\cdot p_5]}.\quad(17)$$

[10] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).
[11] F. Rohrlich, Nuovo Cimento 38, 673 (1965).

But, as one might guess, the complications that manifest themselves in the recoupling coefficients in the stepwise coupling scheme are contained in the rotation R. For the most general three-particle final state, R is a complex function of the $p_i\cdot p_j$ invariants.[9] However, if we consider only those reactions in which the final particles lie in the x-z plane, then R involves only a rotation about the y axis (since $\mathbf{p}_5$ defines the x-z plane):

$$\cos 2\theta=\frac{\dfrac{p_{5x}}{M}\left(\dfrac{4p_3\cdot p_5p_4\cdot p_5-4(p_4\cdot p_5)^2}{M_4M_5}-\dfrac{M_5}{M_4}p_3\cdot p_4-M_4M_5\right)}{(p_{5x}^2+p_{5z}^2)\left[\left(\dfrac{2p_4\cdot p_5}{M_4M_5}\right)^2-1\right]^{1/2}}$$
$$-\frac{\dfrac{2p_{5x}}{M}\left\{\dfrac{4(p_3\cdot p_5)^2-M_3^2M_5^2}{M_3M_5}\left[\left(\dfrac{2p_4\cdot p_5}{M_4M_5}\right)^2-1\right]-\dfrac{(4p_3\cdot p_5p_4\cdot p_5-M_5^2p_3\cdot p_4)^2}{M_3M_4^2M_5^3}\right\}^{1/2}}{(p_{5x}^2+p_{5z}^2)\left[\left(\dfrac{2p_4\cdot p_5}{M_4M_5}\right)^2-\mathrm{P}\right]^{1/2}}, \quad (18)$$

with

$$R=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

Thus the D function appearing in Eq. (17) is specified explicitly by the angle given in Eq. (18).

IV. CONCLUSION

Several conclusions can be drawn on the basis of Eq. (15). First, the number of multiparticle reactions being investigated is increasing; thus it is important to understand n-particle kinematics so that meaningful multiparticle experiments can be defined and so that multiparticle reactions can be systematically analyzed. As other authors[12] have pointed out, it is not at all clear which variables should be used to specify multiparticle reactions, and as our analysis shows, any set of invariants that uniquely specifies the $3n-6$ parameters in Eq. (13) is satisfactory.

Second, it is obviously necessary to have a symmetric analysis if there are several identical particles in the final state. Even if all final-state particles are different, the analysis becomes obscured by kinematical effects unless each final-state particle is given an equivalent role. Our analysis avoids this situation because it depends only upon scalar products of final-state momenta and the spin variables $(\alpha_3,\alpha_4,\cdots,\alpha_n)$ in a symmetric fashion. We believe that our analysis makes a clear distinction between kinematics and dynamics; the dynamics will be completely contained in the reduced amplitude (16) and in any assumption that allows the summation over s to be terminated.

Finally, even though the kinematic difficulties arising in multiparticle reactions are contained in the rotations R and R_i, so that explicit expressions for R and R_i as functions of $p_i\cdot p_j$ are quite complicated, nevertheless the relationships between R and R_i and the invariants $p_i\cdot p_j$ are straightforward and unambiguous. Thus, using the definitions given in Ref. 6, it would be possible to write a computer program that would read out the angles involved in the rotations for the various values of $p_i\cdot p_j$ occurring in a given experiment. Given such a computer program, we had hoped to find a systematic manner of handling multiparticle reactions by parametrizing experimental data in some convenient manner, the hope being that a special parametrization might lead to a multiparticle phase-shift analysis in which resonances would be predicted from experimental data. However, we have not been able to carry out such a phase-shift analysis and it is not even clear that it is possible to do so in a model-independent way.

[12] N. F. Bali, G. F. Chew, and Alberto Pignotti, Phys. Rev. **163**, 1572 (1967).