define finite renormalized vertices and propagators by

$$\Gamma_\lambda{}^{ba}(p,p)=\frac{1}{Z_1}\tilde{\Gamma}_\lambda{}^{ba}(p',p),$$

$$D_\lambda{}^{ba}(p',p)=\frac{1}{Z_D}\tilde{D}^{ba}(p',p),$$

$$S_b(p')=Z_2{}^b\tilde{S}_b(p'),$$

$$S_a(p)=Z_2{}^a\tilde{S}_a(p).$$

All the renormalized quantities denoted by the tilde are finite and cutoff-independent. The over-all renormalization of the weak current and its divergence are finite if $Z_1/\sqrt{(Z_2{}^aZ_2{}^b)}$ and $Z_D/\sqrt{(Z_2{}^aZ_2{}^b)}$ are respectively finite, that is, independent of any cutoffs.

To prove this is so, we substitute for the renormalized quantities in the Ward identity and take a variation with respect to the cutoff. This yields

$$0=f^{ab}\left[\delta(Z_1/Z_2{}^b)\tilde{S}_b{}^{-1}(p')\begin{pmatrix}1\\ \gamma_5\end{pmatrix}-\delta(Z_1/Z_2{}^a)\begin{pmatrix}1\\ \gamma_5\end{pmatrix}\tilde{S}_a{}^{-1}(p)\right]+\delta(Z_1/Z_D)\tilde{D}^{ba}(p',p).$$

By evaluating this expression with b or a or both on the mass shell, one finds

$$\delta(Z_1/Z_2{}^b)=\delta(Z_1/Z_2{}^a)=\delta(Z_1/Z_D)=0.$$

Thus, the ratios of these renormalization constants are independent of any cutoffs and hence finite, and the combinations $Z_1/\sqrt{(Z_2{}^aZ_2{}^b)}$ and $Z_D/\sqrt{(Z_2{}^aZ_2{}^b)}$ are finite.

It is well known that the four-point interaction for μ decay can be rewritten by a Fierz transformation as a $V-A$ interaction between the charge-retaining currents. By the preceding theorem, the electromagnetic renormalization of such a μ-e vertex is finite to all orders. This is, of course, not a new result.[6]

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Eigenvectors for the Partial-Wave "Crossing Matrices"

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Let a, b, c, d be spinless particles of equal mass, and consider the process $a+b\to c+d$. It was shown elsewhere that the crossing symmetry of the scattering amplitude for such a process implies an infinite number of finite-dimensional "crossing relations" for the associated partial waves. In this paper, we derive explicit expressions for complete orthogonal and biorthogonal sets of eigenvectors of the partial-wave crossing matrices. The general form of a partial wave which is consistent with crossing symmetry is thus determined.

I. INTRODUCTION

IN a previous paper,[1a] we considered the process $a+b\to c+d$, where a, b, c, d were spinless particles of equal mass. The scattering amplitude F of such a process was expanded in terms of eigenfunctions which displayed its dependence on all the Mandelstam variables. It was shown that the crossing symmetry of F is equivalent to a sequence of finite-dimensional "crossing relations" for the partial waves.

Here we study the spectral properties of the partial-wave crossing matrices and construct their eigenvectors. With the aid of these eigenvectors, it is easy to state the general form of the partial waves which is consistent with the crossing symmetry of F. Section II summarizes the pertinent results from Ref. 1a. The eigenfunctions are tabulated in Sec. III together with their orthogonality and normalization properties. Section IV sketches the requisite derivations.

In a forthcoming paper,[1b] the eigenfunctions associated with the expansion of F (as well as the eigenvectors of the crossing matrices) will be identified with a subset of basis vectors of certain irreducible representations of the group $SU(3)$. The partial-wave crossing matrices that we discuss here are the matrix elements of the Weyl reflections between these vectors.

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[1] (a) A. P. Balachandran and J. Nuyts, Phys. Rev. 172, 1821 (1968); (b) A. P. Balachandran, W. J. Meggs, J. Nuyts, and P. Ramond, International Center for Theoretical Physics ICTP, Report No. IC/68/46 (unpublished); and Phys. Rev. (to be published); (c) see also A. P. Balachandran and J. Nuyts (to be published).

II. SUMMARY OF PREVIOUS RESULTS

The common mass of the particles is taken to be $\frac{1}{2}$. The Mandelstam variables s, t, u therefore fulfill the identity $s+t+u=1$. The cosines of the scattering angles in the three channels are defined by the equations

$$z_s=1+2t/(s-1)\,,\quad z_t=1+2u/(t-1)\,,\qquad z_u=1+2s/(u-1)\,. \tag{2.1}$$

The scattering amplitude F is expanded in the form

$$F(s,t)=\sum_{n,\,l=0}^{\infty}2(n+l+1)(2l+1)a_n{}^lS_n{}^l(s,t)\,, \tag{2.2}$$

where

$$S_n{}^l(s,t)=R_n{}^l(s)P_l(z_s)=(1-s)^lP_n{}^{(2l+1,0)}(2s-1)P_l(z_s) \tag{2.3}$$

and $P_n{}^{(2l+1,0)}$ and P_l are Jacobi and Legendre polynomials. Let

$$(f,g)=\int_0^1 ds\int_0^1 dt\,\theta(1-s-t)f^*(s,t)g(s,t) \tag{2.4}$$

define the scalar product for functions of s and t. Because of the orthogonality relation

$$(S_n{}^l,S_N{}^L)=\frac{\delta_{lL}\delta_{nN}}{2(n+l+1)(2l+1)}\,, \tag{2.5}$$

it follows that

$$a_n{}^l=(S_n{}^l,F)=\int_0^1 ds(1-s)R_n{}^l(s)f_l(s)\,, \tag{2.6}$$

where f_l is the lth partial wave:

$$f_l(s)=\frac{1}{2}\int_{-1}^1 dz_sP_l(z_s)F(s,t)\,. \tag{2.7}$$

The integration in (2.4) is over the Mandelstam triangle which is enclosed by the boundaries $s=0$, $t=0$, $u=0$. The corresponding measure is invariant under s, t, u permutations. The series (2.2) converges to F in the norm induced by the scalar product (2.4).

The Weyl reflection W_{st} is the transposition of s and t [1b]:

$$W_{st}F(s,t)=F(t,s)\equiv G(s,t)\,. \tag{2.8}$$

The other two transpositions W_{tu} and W_{us} are defined similarly. As $F+G$ and $F-G$ are even and odd under W_{st}, we regard F as having a definite parity $\epsilon(=\pm1)$ under W_{st}, W_{tu}, W_{us} hereafter. This entails no loss of generality. (The combinations of the scattering amplitudes which are eigenfunctions of the different transpositions are, in general, different. They depend also on the eigenvalue ϵ. But, for notational simplicity, we use the same symbols F, $a_n{}^l$ for these functions and their Fourier coefficients and suppress their dependence on the indices of the transpositions and on ϵ.)

As a consequence of [1a]

$$(S_n{}^l,W_{\alpha\beta}S_N{}^L)=0\,,\quad \alpha,\beta=s,t,u \quad \text{if} \quad n+l\neq N+L\,, \tag{2.9}$$

the following crossing relations for $a_n{}^l$ [and hence for f_l through (2.6)] are readily inferred from (2.2):

$$2(\sigma+1)\sum_{L=0}^{\sigma}(S_{\sigma-l}{}^l,W_{\alpha\beta}S_{\sigma-L}{}^L)(2L+1)a_{\sigma-L}{}^L=\epsilon a_{\sigma-l}{}^l\,,\qquad l=0,1,2,\cdots,\sigma\,,\qquad \sigma=0,1,2,\cdots. \tag{2.10}$$

This equation shows that F will have the right symmetry under $W_{\alpha\beta}$ if $a_{\sigma-l}{}^l$ is a linear combination of the eigenvectors of the matrix

$$2(\sigma+1)(S_{\sigma-l}{}^l,W_{\alpha\beta}S_{\sigma-L}{}^L)(2L+1) \tag{2.11}$$

for a suitable eigenvalue ϵ. [$a_{\sigma-l}{}^l$ must of course be real if f_l is real when s lies between 0 and 1. There are also constraints on the form of this expansion from the analyticity of F and the L^2 convergence of the series (2.2) which we do not discuss in this paper.] It may be emphasized that *the problem is to find a complete, linearly independent set of eigenvectors with simple orthogonality or biorthogonality properties*. If such were not the case, any symmetric or antisymmetric polynomial in α, β is an eigenfunction of $W_{\alpha\beta}$ and the projections of these polynomials onto the space of $S_{\sigma-L}{}^L$ solves the problem (2.10) (cf. Sec. IV).

III. EIGENVECTORS

(a) We first take up W_{tu}. From (2.3) and (2.1), $S_n{}^l$ is seen to be either even or odd under W_{tu} according to whether l is even or odd. The corresponding crossing matrices in (2.11) are diagonal. Thus we need only restrict the sum in (2.2) to either even or odd l to ensure the right parity of F under W_{tu}. This is a well-known result.

(b) The construction of the eigenvectors for s-t crossing is a bit more involved. We state here three distinct sets of eigenvectors for these matrices, the first two of which form a biorthogonal system and the last one an orthogonal system. The derivations are postponed to the next section.[2]

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[2] The crossing matrices are related to ${}_4F_3$ functions (Ref. 1a). It is amusing that the considerations which follow yield identities involving ${}_4F_3$'s which do not seem to be available in the mathematical literature. Further identities of this sort can be derived from the properties of the group $SU(3)$. See, in this connection, K. J. Lezuo [J. Math. Phys. **8**, 1163 (1967)] who also expresses Weyl reflections in terms of ${}_4F_3$'s. The Weyl group in $SU(3)$ has been discussed by A. J. Macfarlane, E. C. G. Sudarshan, and C. Dullemond, Nuovo Cimento **30**, 845 (1963); N. Mukunda and L. K. Pandit, J. Math. Phys. **8**, 746 (1963); K. J. Lezuo (quoted above) and further references cited therein.

Biorthogonal System

$$(1)\quad \xi_{\sigma-L}{}^{L}(\rho,\epsilon)=(-1)^{L+\rho}\frac{(-L)_{\sigma-\rho}(\sigma+L-\rho)!\rho!}{(\sigma-\rho)!(\sigma+L+1)!(\sigma-L)!L!}+\epsilon(-1)^{L+\sigma+\rho}\frac{(-L)_{\rho}(L+\rho)!(\sigma-\rho)!}{\rho!(\sigma+L+1)!(\sigma-L)!L!},$$

$$\rho=0,1,2,\cdots,[\tfrac{1}{2}\sigma]. \quad (3.1)$$

Much of the notation is explained by the equation

$$2(\sigma+1)\sum_{L=0}^{\sigma}(S_{\sigma-l}{}^{l},W_{st}S_{\sigma-L}{}^{L})\times(2L+1)\xi_{\sigma-L}{}^{L}(\rho,\epsilon)=\epsilon\xi_{\sigma-l}{}^{l}(\rho,\epsilon).$$

ρ is a degeneracy index and $[\frac{1}{2}\sigma]$ is the largest integer which does not exceed $\frac{1}{2}\sigma$. Also,

$$(a)_r\equiv a(a+1)\cdots(a+r-1),\quad (a)_0\equiv 1.$$

$$(2)\quad \eta_{\sigma-L}{}^{L}(\rho,\epsilon)=(L-\sigma)_{\rho}\frac{(\sigma+L+1)!(\sigma-\rho)!}{\rho!(\sigma+L+1-\rho)!}+\epsilon(L-\sigma)_{\sigma-\rho}\frac{(\sigma+L+1)!\rho!}{(\sigma-\rho)!(\rho+L+1)!},$$

$$\rho=0,1,2,\cdots,[\tfrac{1}{2}\sigma]. \quad (3.2)$$

The biorthogonality of the ξ's with the η's is expressed by

$$\sum_{L}\xi_{\sigma-L}{}^{L}(\rho,\epsilon)\eta_{\sigma-L}{}^{L}(\tau,\epsilon')(2L+1)=2\delta_{\epsilon\epsilon'}[\delta_{\rho\tau}+\epsilon\delta_{\sigma,\rho+\tau}],$$

$$\rho,\tau=0,1,2,\cdots,[\tfrac{1}{2}\sigma]. \quad (3.3)$$

Note that the last term never contributes if σ is odd.

Orthogonal System

$$\zeta_{\sigma-L}{}^{L}(\rho)=(-1)^{\sigma+\rho+L}\times(2\sigma+2)\times(S_{\sigma-L}{}^{L},W_{st}S_{\sigma-\rho}{}^{\rho})$$

$$=(-1)^{L}\frac{[\sigma!]^2}{(\sigma-\rho)!(\sigma+\rho+1)!}\times{}_4F_3(-\rho,\rho+1,L-\sigma,-\sigma-L-1;-\sigma,-\sigma,1;1),\quad \rho=0,1,2,\cdots,\sigma. \quad (3.4)$$

The indices σ,L have the usual meanings and ρ denotes the eigenfunction with the eigenvalue $(-1)^{\rho}$. The definition of ${}_4F_3$ is standard[3]:

$${}_4F_3(-\rho,\rho+1,L-\sigma,-\sigma-L-1;-\sigma,-\sigma,1;1)=\sum_{\nu=0}^{\nu_m}\frac{(-\rho)_\nu(\rho+1)_\nu(L-\sigma)_\nu(-\sigma-L-1)_\nu}{(-\sigma)_\nu(-\sigma)_\nu(1)_\nu\nu!},$$

$$\nu_m=\min(\rho,\sigma-L).$$

[3] *Bateman Manuscript Project*, edited by A. Erderlyi (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 183.

The ρ's satisfy the orthogonality relations

$$\sum_{L}\zeta_{\sigma-L}{}^{L}(\rho)\zeta_{\sigma-L}{}^{L}(\tau)(2L+1)=\frac{\delta_{\rho\tau}}{(2\rho+1)}. \quad (3.5)$$

(c) It remains to discuss W_{us}. The three sets of eigenvectors pertinent to these matrices are

$$\tilde{\xi}_{\sigma-L}{}^{L}(\rho,\epsilon)=(-1)^{L}\xi_{\sigma-L}{}^{L}(\rho,\epsilon),$$
$$\tilde{\eta}_{\sigma-L}{}^{L}(\rho,\epsilon)=(-1)^{L}\eta_{\sigma-L}{}^{L}(\rho,\epsilon), \quad (3.6)$$
$$\tilde{\zeta}_{\sigma-L}{}^{L}(\rho)=(-1)^{L}\zeta_{\sigma-L}{}^{L}(\rho).$$

Their orthogonality properties are trivially inferred from (3.3) and (3.5).

It may sometimes be necessary to know the transformation coefficients between the different sets of eigenvectors. The requisite formulas are implicit in the results of the next section and those of Ref. 1a.

IV. DERIVATIONS

(a) The derivation of the biorthogonal system (3.1), (3.2) will be outlined first. It is known that the Appell polynomials[4]

$$\mathfrak{F}_{\rho,\sigma-\rho}(2,1,1,s,t)\equiv F_{\rho,\sigma-\rho}(s,t)=\frac{1}{\rho!(\sigma-\rho)!}(\partial_s)^{\rho}(\partial_t)^{\sigma-\rho}[s^{\rho}t^{\sigma-\rho}(1-s-t)^{\sigma}],$$

$$\rho=0,1,2,\cdots,\sigma,\quad \sigma=0,1,2,\cdots \quad (4.1)$$

as well as $S_{\sigma-L}{}^{L}$ are eigenfunctions of a partial differential operator $\mathcal{O}$ for the same eigenvalue $\sigma(\sigma+2)$.[5,6] Further, $\mathcal{O}$ is self-adjoint in the scalar product (2.4).[1a] So

$$(S_{\sigma-L}{}^{L},F_{\rho,\sigma'-\rho})=0\quad\text{if}\quad\sigma\neq\sigma'. \quad (4.2)$$

Next we note that

$$W_{st}[F_{\rho,\sigma-\rho}(s,t)+\epsilon F_{\sigma-\rho,\rho}(s,t)]=\epsilon[F_{\rho,\sigma-\rho}(s,t)+\epsilon F_{\sigma-\rho,\rho}(s,t)], \quad (4.3)$$

$$\epsilon=\pm1,\quad \rho=0,1,2,\cdots,[\tfrac{1}{2}\sigma].$$

Take the scalar product of this equation with $S_{\sigma-l}{}^{l}$ and use the resolution of the identity

$$\delta(s-s')\delta(t-t')=2\sum_{\sigma=0}^{\infty}\sum_{L=0}^{\sigma}(\sigma+1)(2L+1)\times S_{\sigma-L}{}^{L}(s,t)S_{\sigma-L}{}^{L}(s',t') \quad (4.4)$$

[4] Reference 3, Vol. II, p. 269. See also P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques, Polynomes d'Hermite* (Gauthier-Villars, Paris, 1926).

[5] The operator $\mathcal{O}$ is defined in Ref. 1a. After u is set equal to $1-s-t$ and the ∂_u terms are dropped, it reduces to the differential operator in Eq. (15), p. 272 of Ref. 4. See, in this connection, footnote 4 of Ref. 1a.

[6] The expansion of scattering amplitudes in a series of Appell polynomials has been investigated by J. Charap and B. M. Minton [Queen Mary College Report, 1968 (unpublished); and (to be published)].

and (2.9) or (4.2) to learn that

$$\xi_{\sigma-L}{}^{L}(\rho,\epsilon)=\frac{(2\sigma+2)}{\sigma!}(S_{\sigma-L}{}^{L},\,F_{\rho,\sigma-\rho}+\epsilon F_{\sigma-\rho,\rho})\quad(4.5)$$

are the requisite eigenvectors. The normalization has been arranged conveniently. The integrations in (4.5) can be performed by expanding $S_{\sigma-L}{}^{L}$ in a power series in s and t[7] and doing the partial integrations indicated by (4.1). This leads to

$$(S_{\sigma-L}{}^{L},F_{\rho,\sigma-\rho})=(-1)^{L+\rho}\frac{(-L)_{\sigma-\rho}(2\sigma+1)!(\sigma+L-\rho)!}{[(\sigma-\rho)!]^{2}(\sigma+L+1)!(\sigma-L)!L!}\times\int_0^1 ds\int_0^1 dt\,\theta(1-s-t)s^{\rho}t^{\sigma-\rho}(1-s-t)^{\sigma}.\quad(4.6)$$

Now use[8]

$$\int_0^1 ds\int_0^1 dt\,\theta(1-s-t)s^{\rho}t^{\sigma-\rho}(1-s-t)^{\sigma}=\frac{\sigma!\rho!(\sigma-\rho)!}{(2\sigma+2)!}\quad(4.7)$$

to obtain (3.1).

The eigenvectors $\eta_{\sigma-L}{}^{L}(\rho,\epsilon)$ are associated with the second class of Appell polynomials[9]

$$E_{\rho,\sigma-\rho}(1,1,s,t)\equiv E_{\rho,\sigma-\rho}(s,t)=\sum_{m=0}^{\rho}\sum_{n=0}^{\sigma-\rho}\frac{(\sigma+2)_{m+n}(-\rho)_m(\rho-\sigma)_n}{(m!)^2(n!)^2}s^m t^n,\quad \rho=0,1,2,\cdots,\sigma,\quad(4.8)$$

which are also eigenfunctions of $\mathcal{O}$ for the eigenvalue $\sigma(\sigma+2)$. The vectors $\eta_{\sigma-L}{}^{L}(\rho,\epsilon)$ are defined by

$$\eta_{\sigma-L}{}^{L}(\rho,\epsilon)=(2\sigma+2)(\sigma+1)!\times(S_{\sigma-L}{}^{L},\,E_{\rho,\sigma-\rho}+\epsilon E_{\sigma-\rho,\rho}).\quad(4.9)$$

The integral can be evaluated by the methods of Appendix A, Ref. 1a, after first setting

$$t^n=[(1-s)^n/2^n](1-z_s)^n$$

in (4.8).

[7] Cf. Appendix A of Ref. 1a.
[8] Cf. Ref. 4, p. 270, Eq. (7).
[9] Reference 4, p. 271.

It remains to discuss (3.3). But this is a simple consequence of[10]

$$(F_{\rho,\sigma-\rho},E_{\tau,\sigma-\tau})=\delta_{\rho\tau}/2(\sigma+1)^2\quad(4.10)$$

and (4.2) and (4.4).

(b) To obtain the orthogonal system (3.4), we start from

$$U_{\sigma-\rho}{}^{\rho}(s,t)=R_{\sigma-\rho}{}^{\rho}(u)P_\rho(z_u),\quad \rho=0,1,2,\cdots,\sigma,\quad \sigma=0,1,2,\cdots\quad(4.11)$$

[see (2.3) and (2.1) for definitions] which are eigenfunctions of W_{st} for eigenvalues $(-1)^\rho$ and eigenfunctions of $\mathcal{O}$ for eigenvalues $\sigma(\sigma+2)$. ($\mathcal{O}$ is invariant under all permutations of s, t, u.[1a]) As before, we see that

$$\zeta_{\sigma-L}{}^{L}(\rho)=(-1)^{\sigma+\rho}(2\sigma+2)(S_{\sigma-L}{}^{L},U_{\sigma-\rho}{}^{\rho})\quad(4.12)$$

are eigenvectors of the s-t crossing matrices. Since

$$(S_{\sigma-L}{}^{L},U_{\sigma-\rho}{}^{\rho})=(-1)^{L}(S_{\sigma-L}{}^{L},W_{st}S_{\sigma-\rho}{}^{\rho})=(-1)^{L+\rho}(S_{\sigma-L}{}^{L},T_{\sigma-\rho}{}^{\rho}),\quad(4.13)$$

where

$$T_{\sigma-\rho}{}^{\rho}(s,t)=R_{\sigma-\rho}{}^{\rho}(t)P_\rho(z_t)\quad(4.14)$$

and the last scalar product was evaluated in Appendix A, Ref. 1a, the result (3.4) follows. Finally, (3.5) can be derived from[1a]

$$(U_{\sigma-\rho}{}^{\rho},U_{\sigma-\tau}{}^{\tau})=\delta_{\rho\tau}/(2\sigma+2)(2\rho+1).\quad(4.15)$$

(c) The diagonalization of the u-s crossing matrices is now easy. To obtain $\xi_{\sigma-L}{}^{L}(\rho,\epsilon)$, for example, we start from $F_{\rho,\sigma-\rho}(s,u)$ which is also an eigenfunction of $\mathcal{O}$ for the eigenvalue $\sigma(\sigma+2)$. A simple change of variables and the symmetry property $P_L(-z_s)=(-1)^L P_L(z_s)$ yields the answer.

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[10] Reference 4, p. 272.