Ultraviolet Divergences in Radiative Corrections to Weak Decays*

Giuliano Preparata† and William I. Weisberger‡
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

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The problem of second-order radiative corrections to general semileptonic weak decays is considered. Previous results on ultraviolet divergences in the calculations for zero-momentum-transfer β decays can be proven for the general process, provided that the models of the strong interactions are restricted to renormalizable theories of elementary fermion fields coupled to neutral massive vector bosons through a conserved vector current. A possible method for calculating finite radiative corrections to actual processes is also discussed.

I. INTRODUCTION

Within the past few years, we have achieved a compact expression of our knowledge of the basic low-energy properties of weak leptonic and semileptonic process through an effective current-current Lagrangian of what may be called the Gell-Mann[1]–Cabibbo[2] form. This theory embodies universal coupling of leptons and hadrons through the use of the $V-A$ currents constructed from the appropriate lepton field operators and the $V-A$ currents associated with the generators of the chiral $SU(3)\times SU(3)$ algebra of the strongly interacting particles.

Both lepton and hadron weak currents transform under equal-time commutation as raising and lowering operators of an appropriate angular-momentum algebra. The direction in the space of $SU(3)$ of the neutral component of the weak hadronic $SU(2)$ is determined by the well-known Cabibbo angle, θ_c.

Neglecting electromagnetic effects, the matrix elements of the conserved hadronic vector current are largely determined by the $SU(2)$ symmetry of the strong interactions, and many matrix elements of the nonconserved currents can be related to those of the conserved currents through sum rules[3] that involve, in principle, measurable forward scattering of high-energy neutrinos.

If one wishes to check precisely the universality postulate for this form of the Fermi interaction, it is required to verify the $\cos\theta_c$ factor, which relates the Fermi constant for μ decay to the decay constant for superallowed $0\rightarrow 0$ hadronic β decays, e.g., O^{16}. To do this, one must understand theoretical corrections to these transitions to an accuracy of about 1%. Assuming that the size of the nonrenormalizable higher-order weak interactions are fixed by the magnitude of the weak coupling constant, the relevant corrections are due to interactions of the charged particles participating in these decays with the electromagnetic radiation field.

It is the problem of developing a formalism which will allow us to calculate finite corrections of first order in the fine structure constant α that concerns us in this paper. We are, of course, aware that ultraviolet divergences arise in calculations of the effects of virtual electromagnetic interactions of charged particles for many processes beyond the restricted class considered here. However, we consider the calculation of finite radiative corrections to universal weak interactions, in the scheme just described, to be a sufficiently interesting and well-defined problem in its own right that it is worthwhile to ask what, if any, class of theories satisfying the Gell-Mann current algebra can give rise to finite second-order radiative corrections.[4]

A completely satisfactory theory would be one in which the radiative corrections to all weak processes were finite. A less satisfactory theory, but one in which finite corrections to all measurable effects were calculable, would be a theory in which the only divergence was a universal cutoff-dependent number multiplying the uncorrected matrix element for each transition. Such an infinite constant could be absorbed in a universal and unobservable renormalization of the Fermi constant, and we would still have finite corrections to the ratios of transition rates for various processes.

We consider that a theory with uncorrelated divergences for different processes and divergences depending on the values of kinematic variables of the problem to be unsatisfactory. In this case, it would not be possible to make a meaningful calculation of electromagnetic corrections to the originally universal theory. At the least, one would have to make separate and uncorrelated infinite renormalizations for different processes.

The first discussion of the implications of the current algebra for divergences in calculations of radiative corrections was due to Bjorken.[5] Assuming the entire quark-model algebra for the isospin current, he showed that the order-α corrections to the Fermi, i.e., polar vector, matrix element for leptonic β decay of members of an isomultiplet evaluated at zero four-momentum

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† Fulbright Scholar.

‡ Alfred P. Sloan Foundation Fellow 1967–1969.

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).

[2] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

[3] S. L. Adler, Phys. Rev. **140**, B736 (1965).

[4] One should ask whether electromagnetic corrections are finite to all orders in the fine structure constant. The restriction to second-order effects in semileptonic processes is due to technical complications in treating the strong and electromagnetic interactions simultaneously. Corrections to μ decay at all orders are discussed in Appendix C.

[5] J. D. Bjorken, Phys. Rev. **148**, 1467 (1967).

transfer from the leptons to the hadrons, diverged logarithmically. The coefficient of the divergent term was a universal number independent of the details of the strong interactions. This result answered certain questions raised by earlier, more conventional perturbative calculations.[6] It is well known that, in the point-coupling version of the weak interaction, the radiative corrections to μ decay are finite while the corrections to neutron β decay, calculated in the assumed absence of strong interactions, diverge logarithmically. It had been suggested that the strong interactions might supply the required convergence factors for these corrections.[7] If one truncated the sum over virtual intermediate states inherent in the second-order interaction with the radiation field and assumed convergent form factors for the hadron-photon vertices, it was possible to render the corrections finite. Bjorken showed that the existence of a local current algebra implies that completing the sum over intermediate states must restore the divergence of the original perturbation-theory calculation.

Abers, Norton, Dicus, and Quinn[8] generalized Bjorken's derivation, and showed that it followed from only the local $SU(2)$ algebra for the time components of the isospin current. Motivated by a desire to obtain finite radiative corrections in semileptonic processes analogous to those obtained for the four-point interaction in μ decay, several authors[9] introduced specific models of the hadron currents for which a further divergence in the axial-vector transition induced by second-order electromagnetic effects cancelled the infinite terms previously mentioned, and yielded an over-all finite result.

Sirlin[10] showed further that the introduction of charged vector mesons to mediate the weak interaction in a conventional way leads to a universal divergence in the radiative corrections for the Fermi matrix element at zero momentum transfer in μ decay and semileptonic processes. The result depends only on the model-independent $SU(2)$ algebra of the time components of the current. This is, of course, an example where corrections to relative rates can be taken as finite.

We emphasize that all the results summarized above were derived in the restricted case of spin-averaged matrix elements for decays between members of an isomultiplet at zero momentum transfer. It is to the problem of extending these results to general semileptonic processes that we address ourselves in this paper. What new ingredients enter the problem when one considers processes involving axial as well as polar vector currents, strangeness-changing as well as strangeness-conserving currents, and when one considers processes at arbitrary momentum transfer? The limit $q_\mu=0$ may be a good approximation if one considers decays among members of an isomultiplet, but not if he considers decays with large Q values, such as $K\rightarrow\pi+\mu+\nu$, or considers neutrino-hadron scattering in the crossed channel.

In order to formulate precise criteria for answering these questions, we find it advantageous to use the conventional renormalization techniques of the Feynman-Dyson formalism. In particular, to ensure that the strong-interaction theory exists, at least to every finite order of perturbation theory after renormalization, we assume a renormalizable theory of hadronic matter.

It then appears that to generalize the Bjorken *et al.* results on radiative corrections to β decay we must impose quite stringent requirements on the theory. The only renormalizable models of strong interactions for which second-order radiative corrections to relative rates of general semileptonic processes are finite are theories in which the $SU(3)\times SU(3)$ currents are constructed from bilinear combinations of spin-$\frac{1}{2}$ fields and the fermions interact via neutral vector mesons coupled to a conserved vector current.[11] Chiral $SU(3)\times SU(3)$ may be broken only by mass terms, so that the bosons, in fact, must couple to the baryon number current.

This seems necessary and sufficient to ensure that the divergence in semileptonic processes is a universal number. If we want to render the corrections finite, so that one can calculate corrections to hadron-lepton universality, we are required further to impose special relations for the equal-time commutators of the spatial components of the hadronic currents as proposed in the papers of Ref. 9.

One may ask whether the models thus described have any claim to physical reality. Since we are unable to calculate detailed dynamical effects in this or any other strong interaction theory, we can not answer this question. It is interesting that there is at least one model in which the results of Refs. 5, 8, and 9 can be generalized, but it seems that the simplicity and model independence of the results of the original papers are restricted to the special transitions considered there.

The outline of the paper is as follows. In Sec. II, we present a modified version of the derivation of Abers *et al.* for the general case. It is shown that a particular relation [Eq. (2.6)] is required in order to eliminate the ultraviolet divergences. Assuming the required relation to be valid, we complete the derivation.

The crucial relation, a sort of sum rule for the electro-

[6] See S. M. Berman and A. Sirlin, Ann. Phys. (N. Y.) **20**, 20 (1962), and references cited therein.

[7] See, e.g., G. Källén, Nucl. Phys. **B1**, 225 (1967). This calculation, however, is not gauge-invariant. We discuss a gauge-invariant prescription for calculating model-dependent effects in Sec. IV.

[8] E. S. Abers, R. E. Norton, and D. A. Dicus, Phys. Rev. Letters **18**, 676 (1967); H. Quinn, Ph.D. thesis, Stanford University, 1967 (unpublished); E. S. Abers, D. A. Dicus, R. E. Norton, and H. Quinn, Phys. Rev. **167**, 1461 (1968).

[9] K. Johnson, F. E. Low, and H. Suura, Phys. Rev. Letters **18**, 1224 (1967); N. Cabibbo, L. Maiani, and G. Preparata, Phys. Letters **25B**, 31; **25B**, 132 (1967).

[10] A. Sirlin, Phys. Rev. Letters **19**, 877 (1967).

[11] Radiative corrections to β decay in this model have also been discussed by C. G. Callan, Jr., Phys. Rev. **169**, 1175 (1968). Our conclusions agree with Callan's. We wish to acknowledge hearing a helpful seminar by Dr. Callan during the course of our investigations. Our discussion differs from his in the technical details of our derivations.

magnetic renormalization of the vertex of a general hadron current, is discussed in Sec. III. We derive the required result for the class of theories described above, and give simple counter-examples in low orders of perturbation theory for other models. Thus, we show that the restrictions imposed are both necessary and sufficient for the desired result. As a by-product, we prove the Ademollo-Gatto theorem directly in relativistic perturbation theory. In conclusion, we discuss our result and indicate how realistic calculations of radiative corrections might be carried out if ultraviolet divergences are presumed to be absent.

Some details of the derivations are elaborated in the Appendices. In Appendix A, we show that the particular kinematic limit that is useful for deriving Ward identities in the text yields the entire radiative correction for a vertex function, automatically including the correct contribution from the wave-function renormalization of the external lines. Appendix B contains some details associated with the results proved in Sec. III. In Appendix C, we prove that if a four-vector current is conserved except for mass terms in the Lagrangian, then the renormalization of the vertex functions for both the current and its divergence are finite. This result immediately yields as a corollary the known result[5] that radiative corrections to μ decay in the four-point Fermi interaction are finite to all orders of the fine structure constant.

II. OUTLINE OF THE CALCULATION

In this section, we essentially review the derivation of Ref. 8 and point out the relation that must hold if we are to achieve finite radiative corrections to general semileptonic processes. We consider a weak scattering $\nu+A\to l^-+B$, where A and B are the initial and final hadron states, respectively, l^- is a negatively charged lepton, and ν is its associated neutrino. The choice of charge label for the lepton is for convenience only and to avoid having to write extra symbols describing the charge degree of freedom in our equations.

Neglecting electromagnetic effects, the matrix element for this process is

$$M^0=(G_F/\sqrt{2})\bar{u}(l)\gamma_\lambda(1-\gamma_5)u(\nu)\times\langle B(p')|J_\lambda{}^w(0)|A(p)\rangle. \quad (2.1)$$

G_F is the Fermi constant, the u's are lepton spinors, and $J_\lambda{}^w$ is the Cabibbo current.

For the four-point interaction theory, there are three diagrams contributing to second-order electromagnetic effects (Fig. 1). In correspondence with Fig. 1, we write

$$M^a=\tfrac{1}{2}(Z_e-1)M^0, \quad (2.2a)$$

$$M^b=\frac{G_F}{\sqrt{2}}\frac{-ie^2}{(2\pi)^4}\int d^4k\, D_{\alpha\beta}(k)\times\left(\bar{u}(l)\gamma_\alpha\frac{1}{l-k-m_e}\gamma_\lambda(1-\gamma_5)u(\nu)\right)\times T_{\beta\lambda}{}^{ew}(k,-q-k), \quad (2.2b)$$

$$M^c=\frac{G_F}{\sqrt{2}}[\bar{u}(l)\gamma_\lambda(1-\gamma_5)u(\nu)]\frac{-ie^2}{2(2\pi)^4}\int d^4k\, D_{\alpha\beta}(k)\times T_{\alpha\beta\lambda}{}^{eew}(k,-k,-q), \quad (2.2c)$$

where Z_e is the second-order lepton wave-function renormalization and $D_{\alpha\beta}(k)$ is the photon propagator. The hadron amplitudes appearing in M^b and M^c are defined by

$$T_{\beta\lambda}{}^{ew}(k,-q)=-i\int dx\, e^{ik\cdot x}\langle B(p')|\times T^*(J_\beta{}^e(x)J_\lambda{}^w(0))|A(p)\rangle, \quad (2.3)$$

$$T_{\alpha\beta\lambda}{}^{eew}(k,-k,-q)=\int dxdy\, e^{ik\cdot x-ik\cdot y}\times\langle B(p')|T^*(J_\alpha{}^e(x)J_\beta{}^e(y)J_\lambda{}^w(0))|A(p')\rangle, \quad (2.4)$$

where T^* denotes the covariant ordered product. We have suppressed the variables p and p' in our symbols for these amplitudes since they do not enter explicitly into the relations we will derive. It is straightforward to verify that the sum of these corrections is gauge-invariant. That is, if we write

$$\delta M=M^a+M^b+M^c=\int dk\, D_{\alpha\beta}(k)M_{\alpha\beta}(k),$$

then

$$k_\alpha k_\beta M_{\alpha\beta}=0.$$

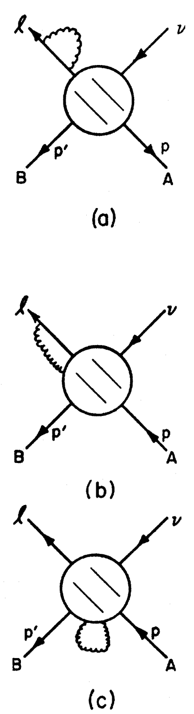

FIG. 1. Second-order electromagnetic corrections to a general weak semileptonic process.

We use this freedom to work in the Feynman gauge. If $D_{\alpha\beta}(k)=g_{\alpha\beta}/k^2$, then the divergent contribution from Fig. 1(b) is

$$M^{b(\mathrm{div})}=\frac{G}{\sqrt{2}}\frac{-ie^2}{(2\pi)^4}\int\frac{d^4k}{k^4}\bar{u}(l)(k\gamma_\alpha-2k_\alpha)\gamma_\lambda(1-\gamma_5)u(\nu)\times T_{\alpha\lambda}{}^{ew}(k,-k-q)$$

$$=\frac{G}{\sqrt{2}}\frac{-ie^2}{(2\pi)^4}\int\frac{d^4k}{k^4}\{2\bar{u}\gamma_\lambda(1-\gamma_5)u\langle B|J_\lambda{}^w|A\rangle+\bar{u}k\gamma_\lambda(1-\gamma_5)uT_{\alpha\alpha}{}^{ew}(k,-k-q)+\tfrac{1}{2}\bar{u}k[\gamma_\alpha,\gamma_\lambda](1-\gamma_5)u\times T_{\alpha\lambda}{}^{ew}(k,-k-q)\}. \quad (2.5)$$

Under restricted conditions to be discussed in the following section, the *divergent* contribution from M^c is given by a generalization of a result of Refs. 5 and 8.

$$\int\frac{d^4k}{k^2}T_{\alpha\alpha\lambda}{}^{eew}(k,-k,-q)$$

$$=\int\frac{d^4k}{k^2}\frac{\partial}{\partial k_\lambda}T_{\alpha\alpha}{}^{ew}(k,-k-q)+(\text{finite terms})$$

$$=2\int\frac{d^4k}{k^4}k_\lambda T_{\alpha\alpha}{}^{ew}(k,-k-q)+(\text{finite terms}). \quad (2.6a)$$

Then, we have

$$M^{c(\mathrm{div})}=-\frac{G}{\sqrt{2}}\frac{-ie^2}{(2\pi)^4}\int\frac{d^4k}{k^4}[\bar{u}k(1-\gamma_5)u]\times T_{\alpha\alpha}{}^{ew}(k,k-q). \quad (2.6b)$$

This cancels the corresponding term involving the trace $T_{\alpha\alpha}{}^{ew}$ in (2.5). The only remaining divergent term depending on the details of strong-interaction dynamics is the last term in (2.5) which involves the part of $T_{\alpha\lambda}{}^{ew}$ antisymmetric in the indices α and λ. To evaluate the divergent contribution from this term, we resort to the techniques suggested by Bjorken.[5]

The asymptotic behavior of the time-ordered product of two currents in the limit $|k_0|\to\infty$, $|\mathbf{k}|$ fixed is given by an elementary application of the Riemann-Lesbesgue lemma as

$$-i\int dx\, e^{ik\cdot x}\langle B|T(J_\alpha{}^e(x)J_\lambda{}^w(0))|A\rangle\xrightarrow[|k_0|\to\infty]{}\frac{1}{k_0}\int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\langle B|[J_\alpha{}^e(\mathbf{x},0),J_\lambda{}^w(0)]|A\rangle$$

$$=\frac{1}{k_0}\{g_{\alpha0}J_\lambda{}^w(q)+J_\alpha{}^w(q)g_{\lambda0}-g_{\alpha0}g_{\lambda0}J_0{}^w(q)+g_{\alpha i}g_{\lambda j}[\delta_{ij}\hat{J}_0{}^w(q)+\epsilon\langle Q\rangle\epsilon_{ijk}\hat{J}_k{}^w(q)]\}. \quad (2.7)$$

The last line of this equation is derived from various field-theoretic models of the currents. We have used the abbreviated notation

$$J_\lambda{}^w(q)=\langle B(p')|J_\lambda{}^w(0)|A(p)\rangle.$$

$\hat{J}_\lambda{}^w$ refers to the corresponding matrix element of the piece of the weak current constituted from the terms bilinear in the fundamental Fermi fields only. $\epsilon=\pm1$ and denotes the chirality of the weak current; $J_\lambda{}^w=V+\epsilon A$. $\langle Q\rangle$ denotes the average charge of the spin-$\frac{1}{2}$ particles forming the isospin current, which we suppose to transform as members of $SU(3)$ triplets. If there are no fundamental charged mesons in the theory, then $\hat{J}_\lambda{}^w=J_\lambda{}^w$.

To use the result (2.7) in our integrals, it is necessary to assume further that the time-ordered product is, in fact, the appropriate covariant matrix element. Then the result above has the unique covariant form

$$\lim_{|k_0|\to\infty,|\mathbf{k}|\,\mathrm{fixed}}T_{\alpha\lambda}{}^{ew}(k,-k-q)$$

$$=\frac{1}{k^2}\left[k_\alpha J_\lambda{}^w(q)+k_\lambda J_\alpha{}^w-\frac{k_\alpha k_\lambda}{k^2}k\cdot J^w(q)+\left(\frac{k_\alpha k_\lambda}{k^2}-g_{\alpha\lambda}\right)k\cdot\hat{J}^w(q)+\epsilon\langle Q\rangle\epsilon_{\alpha\lambda\sigma\beta}k_\sigma\hat{J}_\beta{}^w(q)\right]. \quad (2.8)$$

Under suitable smoothness assumptions,[12] the matrix element in the Bjorken limit gives the correct divergent terms in our photon integrations. With this result we can evaluate the remaining divergent term. Combining all the contributions, one has

$$\delta M^{\mathrm{div}}=\frac{-ie^2}{(2\pi)^4}\int\frac{d^4k}{k^4}\times\tfrac{3}{2}(M^0+2\epsilon\langle Q\rangle\hat{M}^0). \quad (2.9)$$

M^0 is defined in (2.1) and $\hat{M}^0$ is the corresponding matrix element with the weak current $J_\lambda{}^w$, replaced by its fermion part $\hat{J}_\lambda{}^w$. There are at this point various choices to get a more or less satisfactory result.

In Sec. III, we show that the simple formula for $M^{c(\mathrm{div})}$ is valid if, and only if, there are no fundamental charged bosons in the theory. Therefore we will have $\hat{J}_\lambda{}^w=J_\lambda{}^w$. The divergence for radiative corrections to semileptonic processes is a universal factor multiplying the uncorrected matrix element which can be absorbed in a universal infinite-charge renormalization of the Fermi constant for semileptonic processes. Finite relative corrections to such transitions can be calculated, but one cannot, in this approach, compare the corrections to μ decay with those to hadronic β decays.

[12] Examination shows that the integration region for the virtual photon loop does not coincide with the kinematic region included in the Bjorken limit. To show that the asymptotic form (2.8) gives the leading contribution in the region of interest, it is sufficient that the amplitude $T_{\alpha\beta}$ as an analytic function in the energy plane (a) satisfy a dispersion relation with a finite number of subtractions, and that (b) be polynomially bounded in k^2 on the cuts. S. B. Treiman and G. Tiktopoulos (private communication).

By inspection of (2.9), finite radiative corrections in models with currents composed of fermions are achieved if their charges are chosen so that

$$\epsilon\langle Q\rangle=-\tfrac{1}{2}.$$

If the weak interaction is taken to be mediated by an intermediate vector meson, then the result (2.6) generalizes Sirlin's result to an arbitrary current at $q_\alpha=0$. There is a universal multiplicative divergent factor which depends only on local extension of the equal-time commutation rules for the time components of the weak and electric currents. For nonzero momentum transfer, however, new model-dependent cutoff-dependent contributions arise for the boson-mediated interaction. These are related to the singular interactions of the intermediate bosons with photons.

III. ELECTROMAGNETIC RENORMALIZATION OF A WEAK HADRONIC CURRENT

We have now to deal with the relation (2.6) which is equivalent to discussing the electromagnetic renormalization of the hadronic weak-current vertex.

$$\delta J_\lambda{}^w=\frac{-ie^2}{2(2\pi)^4}\int d^4k\, D_{\alpha\beta}(k)T_{\alpha\beta\lambda}{}^{eew}(k,-k,-q). \quad (3.1)$$

We will define the amplitude $T_{\alpha\beta\lambda}$ in the expression above by the limiting procedure

$$T_{\alpha\beta\lambda}{}^{eew}(k,-k,-q) =\lim_{\kappa\to0} T_{\alpha\beta\lambda}{}^{eew}(k+\kappa,-k,-q-\kappa). \quad (3.2)$$

The reason for this artifice is that the amplitude thus defined includes the proper contribution to the vertex correction from the wave-function renormalization of the external lines and no extra contributions or cancellations need be put in by hand. It is clear that the limiting process can affect only those terms whose value is ambiguous at $\kappa=0$. This includes only electromagnetic self-energy insertions on the external lines. The virtues claimed for this procedure are proved in Appendix A.

With this definition we can derive the following Ward identity[13]:

$$\begin{aligned}(q+\kappa)^\lambda T_{\alpha\beta\lambda}{}^{eew}(k+\kappa,-k,-q-\kappa) &=M_{\alpha\beta}{}^{eew}(k+\kappa,-k,-q-\kappa)\\ &\quad+T_{\alpha\beta}{}^{ew}(k+\kappa,-k-q-\kappa)+T_{\alpha\beta}{}^{ew}(k+q,-k), \quad (3.3)\end{aligned}$$

where $M_{\alpha\beta}{}^{eew}$ is defined in analogy with $T_{\alpha\beta}{}^{eew}$ except that the weak current $J_\lambda{}^w$ is replaced by its divergence $\partial_\lambda J_\lambda{}^w$.

[13] A related approach is described in the last paper of Ref. 8.

A form which will prove useful is obtained by differentiating with respect to κ:

$$\begin{aligned}T_{\alpha\beta\lambda}{}^{eew}=&-(q+\kappa)^\delta\frac{\partial}{\partial\kappa_\lambda}T_{\alpha\beta\delta}{}^{eew}(k+\kappa,-k,-q-\kappa)|_{\kappa=0}\\ &+\frac{\partial}{\partial\kappa_\lambda}M_{\alpha\beta}{}^{eew}(k+\kappa,-k,-q-\kappa)|_{\kappa=0}\\ &+\frac{\partial}{\partial k_\lambda}T_{\alpha\beta}{}^{ew}(k,-k-q). \quad (3.3')\end{aligned}$$

In order to derive the generalized renormalization formula, it is required to show that the entire divergent contribution, which arises when this amplitude is contracted with the photon propagator and integrated with respect to the photon loop momentum, comes from the last term in Eq. (3.3′).

We note, incidentally, that the original result of Abers *et al.* follows immediately from the expression given above. Recall that we are referring to the case when $q=0$ and $\partial_\lambda J_\lambda=0$. Then the second term on the right is absent. The first term has an explicit factor of κ^δ. In the limit $\kappa\to0$ the only nonvanishing contribution comes from electromagnetic self-energy insertions on the external legs. A simple calculation shows that one obtains from this term precisely the second-order electromagnetic mass-shift insertions on the external lines.

In the general case, some care is needed in defining what we mean by the radiative correction to the vertex. The expression (3.1) includes such effects as electromagnetic corrections to the masses of internal and external hadron lines and to coupling constants for the strongly interacting particles. In limiting our attention to ultraviolet divergences for purely weak processes, we wish to exclude possible divergent contributions from internal mass and coupling-constant shifts. In order to do this, we must renormalize these electromagnetic-mass and strong-coupling-constant corrections. Since we work only to second order in the interaction with the radiation field, both the self-masses and the coupling constants can be renormalized by *additive* counter-terms which are local polynomials in the field operators of the fundamental hadrons. We denote these local counter-terms by the collective symbol $\delta g(x)$. Then the correction to the vertex function from the counter terms (CT) can be written as

$$\delta J_\lambda{}^{w(\mathrm{CT})}(\kappa,-q-\kappa)=-i\int dx\, e^{-i(q+\kappa)\cdot x}\times\langle B|T(J_\lambda{}^w(x)\delta g(0))|A\rangle. \quad (3.4)$$

If we assume that

$$[J_0{}^w(\mathbf{x},0),\delta g(\mathbf{y},0)]=\delta^{(3)}(\mathbf{x}-\mathbf{y})\mathcal{O}(\mathbf{x},0),$$

where $\mathcal{O}(x)$ is some local operator, then we can derive from (3.4) an expression analogous to (3.3):

$$\delta J_\lambda{}^{w(\mathrm{CT})}(q,0) = -(q+\kappa)^\delta \frac{\partial}{\partial\kappa_\lambda}\delta J_\delta{}^{w(\mathrm{CT})}(\kappa, -q-\kappa)|_{\kappa=0} + \frac{\partial}{\partial\kappa_\lambda}M^{w(\mathrm{CT})}(\kappa, -q-\kappa)|_{\kappa=0}. \quad (3.5)$$

$M^{w(\mathrm{CT})}$ is the counter-term correction to the vertex function of the divergence of the weak current. The radiative correction to the weak vertex itself is defined as

$$\delta\bar{J}_\lambda{}^w = A_\lambda + B_\lambda + \frac{-ie^2}{(2\pi)^4}\int d^4k \times D_{\alpha\beta}(k)\frac{\partial}{\partial k_\lambda}T_{\alpha\beta}{}^{ew}(k, -k-q), \quad (3.6)$$

with

$$A_\lambda = \lim_{\kappa\to 0}\frac{-ie^2}{2(2\pi)^4}\int d^4k \times D_{\alpha\beta}(k)\left[-(q+\kappa)^\delta\frac{\partial}{\partial\kappa_\lambda}T_{\alpha\beta\delta}{}^{eew}(k+\kappa, -k, -q-\kappa)\right] - (q+\kappa)^\delta\frac{\partial}{\partial\kappa_\lambda}J_\delta{}^{w(\mathrm{CT})}(\kappa, -q-\kappa), \quad (3.7\mathrm{a})$$

$$B_\lambda = \lim_{\kappa\to 0}\frac{-ie^2}{2(2\pi)^4}\int d^4k \times D_{\alpha\beta}(k)\frac{\partial}{\partial\kappa_\lambda}M_{\alpha\beta}{}^{eew}(k+\kappa, -k, -q-\kappa) - \frac{\partial}{\partial\kappa_\lambda}M^{w(\mathrm{CT})}(\kappa, -q-\kappa). \quad (3.7\mathrm{b})$$

It is our task to find the conditions under which A_λ and B_λ are finite.

We first note the contribution of the counter-term corrections to the zero-momentum-transfer vertex for a current conserved by the strong interactions. The second term on the right of (3.5) is absent. At $q_\lambda=0$, the only surviving contribution comes from possible insertions of the interaction vertex δg on external legs. These are just mass shifts on external lines and cancel the corresponding terms found in (3.3) in the same limit.

Note that this is a direct proof in relativistic perturbation theory that the first-order contribution of a local symmetry-breaking interaction $\delta g(x)$ does not contribute to the charge renormalization of an otherwise conserved current. This is the well-known theorem of Ademollo and Gatto.[14]

We further note that theories with fundamental charged spin-zero bosons will not yield the desired result. This is most easily seen from simple field-theory examples. Consider the decay $\pi^+\to\pi^0+e^++\nu$ in a world of non-strongly-interacting pions. The relevant decay matrix element including radiative corrections has the form

$$M = \frac{G}{\sqrt{2}}\bar{u}(l)\gamma_\lambda(1-\gamma_5)u(\nu) \times[(\pi^++\pi^0)_\lambda f_+(q^2) + (\pi^+-\pi^0)_\lambda f_-(q^2)],$$

where we have used the names of the pions for their momenta and f_+, f_- are invariant form factors. A simple perturbation calculation shows that the second-order radiative corrections to $f_+(0)$ give just the result of Bjorken *et al.* However, it is also found that $f_-(0)$, which is induced only when we introduce electromagnetic effects, is also logarithmically divergent. This term is not seen in previous calculations where the kinematic factor $(\pi^+-\pi^0)_\lambda$ is taken as zero. It therefore appears that theories with fundamental charged spinless particles will not permit a simple generalization of the $q_\lambda=0$ result to general momentum transfer.

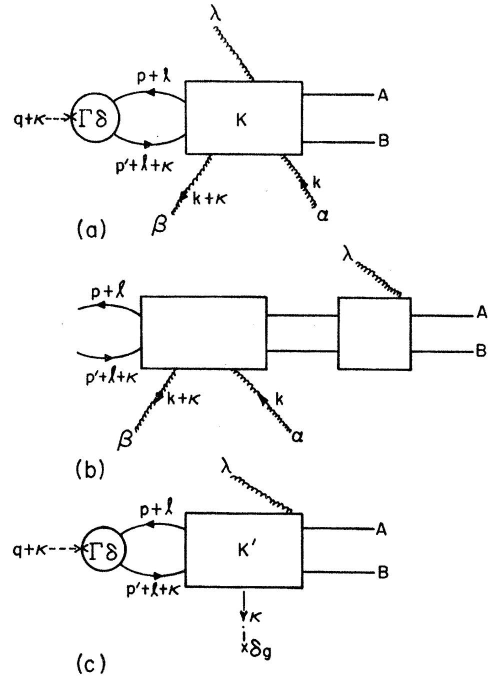

FIG. 2. (a) General form of the two-photon contribution to A_λ. (b) Type of diagram *absent* in the kernel K. (c) Counter-term contributions to A_λ.

Thus, we are restricted at this point to theories of charged spin-$\frac{1}{2}$ particles interacting with neutral bosons. In this case the weak current is bilinear in the Fermi fields. This leads to the conclusion that A_λ is finite.

Consider the first term on the right-hand side of (3.7a). The factor $(q+\kappa)_\lambda$ may be neglected here. In $T_{\alpha\beta\delta}(k+\kappa, -k, -q-\kappa)$ the momentum κ is brought in by the weak current $J_\lambda{}^w$ and carried out by the electric

[14] M. Ademollo and R. Gatto, Phys. Rev. Letters 13, 264 (1965).

current $J_\alpha{}^e$. For every diagram, some route for the momentum to flow along lines from the weak vertex to the charge vertex can be found which will satisfy the following properties: The derivative $\partial/\partial\kappa_\lambda$ acting on $T_{\alpha\beta\delta}$ effectively inserts a zero-momentum-transfer vector vertex on lines along this path. The resulting diagrams sum to give a contribution which is indicated symbolically in Fig. 2(a). Here $\Gamma_\delta{}^w$ is the proper weak vertex to all orders in the strong interaction; the full propagators on the fundamental fermion lines with loop momenta $p+l$, $p'+l+\kappa$ are included in the kernel. The crucial fact about the scattering kernel K is that it can *not* be divided as shown in Fig. 2(b) with the two photons to the left and the new vector vertex to the right of a two-particle intermediate state. Therefore, by standard power-counting arguments, all loop integrals over l enclosing the weak vertex are superficially convergent.

Any divergences that arise when we integrate over the virtual photon momenta must be electromagnetic shifts to masses or strong-coupling constants in the kernel K. The strong and electromagnetic corrections to the new zero-momentum vector vertex insertions are shown in Appendix B to give rise to no new infinities.

By the same argument the counter-term contributions to A_λ are symbolized by the diagram of Fig. 2(c). The kernel K' has *no* two-particle intermediate state to the left of the zero-momentum vector vertex. The self-mass and coupling-constant corrections of Fig. 2(a) are in one-to-one correspondence with the counter-term insertions of Fig. 2(c). Therefore, all ultraviolet divergences associated with the photon loop integration are cancelled by the counter terms. There might remain divergent fine-structure-dependent contributions to A_λ if the strong renormalization of the weak vertex was formally divergent in our theory. We show shortly, however, that the elimination of unwanted divergences in photon loop contributions to the terms we call B_λ further restricts the theory to exclude such a possibility.

The singularities of B_λ may be analyzed in the same way as those of A_λ if the divergence operator $\partial_\sigma J_\sigma{}^w$ is bilinear in the Fermi fields. This happens if the only terms in the Lagrangian which break the chiral $SU(3)\times SU(3)$ invariance of the theory are mass terms. Non-invariant interaction terms yield contributions to $\partial_\sigma J_\sigma{}^w$ which are trilinear in the field operators. The corresponding vertex corrections are superficially linearly divergent, and the single differentiation with respect to κ is not sufficient to remove divergences in photon loop integrals encircling the weak vertex. A simple example is given in Fig. 3 for a theory with a $g\bar{\psi}\psi\phi$ interaction. ψ is a spin-$\frac{1}{2}$ field; ϕ is a scalar field. The requirement of a chiral, $SU(3)\times SU(3)$-invariant, renormalizable interaction restricts us to the case of neutral vector bosons coupled to the baryon number current. For such a theory, the strong renormalizations of the vertices for the weak currents and their divergences are finite, as shown in Appendix C. Thus, we find ourselves able to prove the validity of the formula

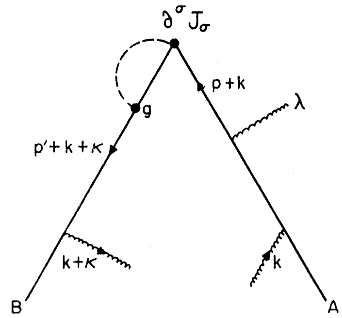

FIG. 3. Divergent contribution to B_λ in theory with scalar bosons.

$$\delta \bar{J}_\lambda{}^{w(\mathrm{div})}=\frac{-ie^2}{2(2\pi)^4}\int d^4k\, D_{\alpha\beta}(k)\frac{\partial}{\partial k_\lambda}T_{\alpha\beta}{}^{ew}(k,-k-q)$$

for all currents only in the restricted class of models described above.

IV. CONCLUSIONS

We have achieved an extension of previous results on general behavior of ultraviolet divergences in radiative corrections to weak processes, but only at the cost of a severe restriction on the field-theory models for the strong interactions we may consider. The original result of Abers *et al.* imposed no such restrictions, not even that of renormalizability, which we found necessary in order to have a well-defined framework in which to discuss the problem. The generality of the original result must appear as rather an accident.

The question arises as to what lessons for the actual calculation of physically meaningful quantities can be extracted from the preceding theoretical discussion. We would suggest that a tenable optimistic point of view is the following.

We have shown that even in the context of the four-point Fermi interaction it is possible to achieve finite second-order radiative corrections to β decay. If we have a theory without ultraviolet divergences, we might then assume that contributions from photons of large virtual momenta are unimportant.

Sirlin has shown how to define and isolate the infrared-divergent terms in a gauge-invariant and model-independent manner. These terms are collectively ultraviolet-convergent and include wave-function renormalization on the external lepton line. One could add model-dependent corrections to this calculation by keeping a finite number of virtual states in the hadronic contribution to the decay amplitude. The photons are coupled to these states in a gauge-invariant manner and the electromagnetic vertices of the hadrons are supplied with form factors to render these contributions finite. Since these corrections will be dominated by virtual photons of fairly low mass (the form factors give an effective cutoff of a few GeV), it is reasonable to treat the four-point Fermi interaction as exact even though it is known that it must acquire intrinsic structure at momentum transfers of a few hundred GeV. Some applications of this approach are presently under investigation.

ACKNOWLEDGMENTS

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APPENDIX A

We wish to justify the definition of the vertex renormalization given by the κ-limiting process in Sec. III. It is clear that the definition

$$T_{\alpha\beta\lambda}{}^{eew}(k,-k,-q)=\lim_{\kappa\to 0} T_{\alpha\beta\lambda}{}^{eew}(k+\kappa,-k,-q-\kappa)$$

can affect only terms whose definition is ambiguous when $\kappa=0$. These are precisely the electromagnetic insertions on external legs.

The first term is, for example, defined by

$$T_{\alpha\beta\lambda}{}^{\mathrm{B(orn)}}=\Sigma_{\alpha\beta}{}^{b}(p',-p'-\kappa;k+\kappa,-k)\times\tilde{S}_p(p+\kappa)\Gamma_\lambda{}^{w}(p'+\kappa,-p),$$

where $\tilde{S}_F$ is the renormalized Feynman propagator.

The two photon lines are to be contracted with the propagator $D_{\alpha\beta}$ and integrated over the loop momentum. That is, the contribution to the weak vertex is

$$\delta J_\lambda{}^{\mathrm{B}}(q)=\lim_{\kappa\to 0}\int d^4k\, D_{\alpha\beta}(k)T_{\alpha\beta\lambda}{}^{B}(k+\kappa,-k,-q-\kappa).$$

The answer requires an expansion of Σ up to terms of first order in κ. By TP invariance, it follows that

$$\Sigma_{\alpha\beta}(p',-p'-\kappa;k+\kappa,-k)=\Sigma_{\beta\alpha}(p'+\kappa,-p';k,-k-\kappa).$$

Since the photon propagator is symmetric in the indices α and β, and since we need only first-order terms in κ, we can write, *inside the k integral*,

$$\begin{aligned}\Sigma_{\alpha\beta}(p',-p'-\kappa;k+\kappa,-k)&=\tfrac{1}{2}[\Sigma_{\beta\alpha}(p',-p'-\kappa;k+\kappa,-k)+\Sigma_{\alpha\beta}(p'+\kappa,-p';k,-k-\kappa)]\\&=\tfrac{1}{2}[\Sigma_{\alpha\beta}(p',-p';k,-k)+\Sigma_{\beta\alpha}(p'+\kappa,-p'-\kappa;k+\kappa,-k-\kappa)].\end{aligned}$$

Using the translation invariance and the spherical symmetry of the k integration, we obtain

$$\begin{aligned}\Sigma^b(p,\kappa)&=\int d^4k\, D_{\alpha\beta}(k)\Sigma_{\alpha\beta}(p',-p'-\kappa;k+\kappa,-k)\\&=\frac{1}{2}\int d^4k\, D_{\alpha\beta}(k)[\Sigma_{\alpha\beta}{}^{b}(p,-p;k,-k)+\Sigma_{\alpha\beta}{}^{b}(p+\kappa,-p-\kappa;k,-k)]\\&=\tfrac{1}{2}[2\delta m_b+(Z_b-1)\tilde{S}_F{}^{-1}(p'+\kappa)].\end{aligned}$$

The last step follows from the standard Feynman-Dyson definition of the proper self-energy part. A similar result obviously holds for the self-energy insertion on the other external line.

Thus we see that our limiting procedure gives the full self-mass insertion on the external line, but only $\frac{1}{2}$ of the wave-function renormalization correction, which is the desired contribution to the vertex correction.

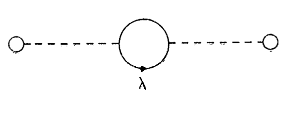

FIG. 4. Vector-vertex insertion on a boson line.

APPENDIX B

We show that the zero-momentum vector vertex insertions introduced by the derivation with respect to κ suffer only finite renormalization from both the strong and electromagnetic interactions, and, hence, introduce no additional divergences into the calculation. The only primitively divergent diagrams with the vector insertion are those with either two external boson lines or two external fermion lines.

Consider first the insertion on a meson propagator as shown in Fig. 4. The strong renormalization of this diagram is finite. In fact, the contribution of such a diagram is

$$\bar{A}_\lambda(q)=\frac{\partial}{\partial q_\lambda}D'(q),$$

where $D'(q)$ is the full propagator for a spin-0 meson; in the case of a vector meson both A and D' have two additional four-vector indices and the argument is unchanged.

From renormalization theory, we know that we can define a finite renormalized propagator by

$$D'=Z_m\tilde{D}(q),$$

where Z_m is a cutoff-dependent number and $\tilde{D}$ is the finite renormalized propagator. Thus, the derivative acts only on the finite renormalized propagator and the factor Z_m is absorbed in the charge renormalization of the vertices at either end of the boson line.

The electromagnetic renormalization of this insertion is given by the derivative of the diagram shown in Fig. 5(a) as well as in the corresponding counter-term diagrams of Figs. 5(b) and 5(c). Assuming strong renormalizations to have been carried out, we differentiate the first of these terms to obtain

$$\frac{\partial}{\partial\kappa_\lambda}\tilde{D}(q)\delta\pi_{\alpha\beta}(q,-q-\kappa;k+\kappa,-k)\tilde{D}(q+\kappa)$$
$$=\tilde{D}(q)\left\{\frac{\partial}{\partial\kappa_\lambda}\delta\pi_{\alpha\beta}(q,-q-\kappa;k+\kappa,-k)\right\}\tilde{D}(q)$$
$$+\tilde{D}(q)\delta\pi_{\alpha\beta}\frac{\partial}{\partial q_\lambda}\tilde{D}(q).$$

The virtual photons are contracted with a photon propagator $D_{\alpha\beta}(k)$ and integrated. Then arguments analogous to those of Appendix A give the resulting self-energy parts as

$$\delta\pi(q,\kappa)|_{\kappa=0}=\int d^4k\, D_{\alpha\beta}(k)\delta\pi_{\alpha\beta}|_{\kappa=0}$$
$$=\delta m^2-Z_m{}^{\mathrm{elec}}\tilde{D}^{-1}(q)+\text{finite terms},$$
$$\frac{\partial}{\partial\kappa_\lambda}\delta\pi(q,\kappa)|_{\kappa=0}=-\tfrac{1}{2}Z_m{}^{\mathrm{elec}}\frac{\partial}{\partial q_\lambda}\tilde{D}^{-1}(q)+\text{finite terms}.$$

$-\frac{1}{2}Z_m{}^{\mathrm{elec}}$ is just the multiplicative factor which gives the meson line contribution to the electromagnetic renormalization of the meson-fermion coupling constant. Thus Fig. 5(a) gives a net divergent contribution

$$\tilde{D}(q)\left[-\tfrac{1}{2}Z_m{}^{\mathrm{elec}}\frac{\partial}{\partial q_\lambda}D^{-1}(q)\right]\tilde{D}(q)$$
$$+\tilde{D}[\delta m^2-Z_m{}^{\mathrm{elec}}D^{-1}(q)]\frac{\partial}{\partial q_\lambda}\tilde{D}(q)$$
$$=\tilde{D}(q)\left(\delta m^2\frac{\partial}{\partial q_\lambda}D(q)+(-\tfrac{1}{2}Z_m{}^{\mathrm{elec}})\right)\frac{\partial}{\partial q_\lambda}D(q).$$

It is clear that these insertions will be cancelled by the differentiation of the corresponding counter-terms of Figs. 5(b) and 5(c).

The vector vertex insertions on a fermion line can be discussed in a similar manner. The strong renormalization problem is trivial since the differentiation is equivalent to inserting a low-energy vertex for a conserved current which suffers no renormalization by the strong interactions.

As for electromagnetic corrections, it can be shown that fermion diagrams corresponding the meson line insertions of Fig. 5 combine in a similar way to cancel their divergent parts.

APPENDIX C

We prove here that if a current is conserved, except for mass terms in the Lagrangian,[15] then the renormalizations of both the current and its divergence are finite. The former result is well known and can also be proved order by order in perturbation theory. This result is sufficient to show that the electromagnetic corrections to μ decay in the four-point Fermi theory are finite to all orders of the fine structure constant.

[15] It has been suggested by K. Wilson (private communication) that such a current be called in general a partially conserved current. We adopt his terminology here.

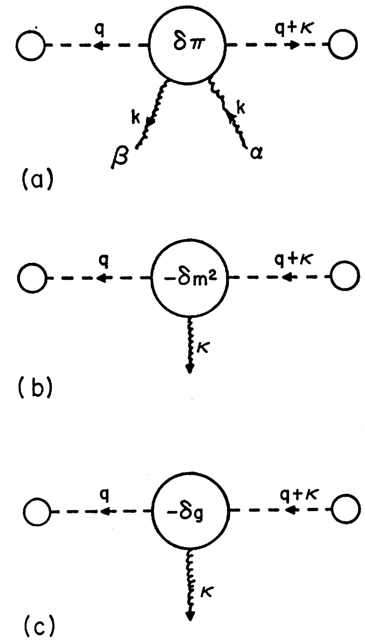

FIG. 5. Two-photon and counter-term corrections to boson lines on which zero-momentum vector vertices may be inserted.

We consider the vertex of a current between two states a and b with quantum numbers of the fundamental fields in our theory. The generalized partial-conservation assumption implies that these are spin-$\frac{1}{2}$ fields. One derives straightforwardly the following Ward identity:

$$(p'-p)^\lambda\Gamma_\lambda{}^{ba}(p',p)=f_{ab}\left[S_b{}^{-1}(p')\binom{1}{\gamma_5}-\binom{1}{\gamma_5}S_a{}^{-1}(p)\right]$$
$$+D^{ba}(p',p),$$

where $\Gamma_\lambda{}^{ba}$ is the proper vertex of the current, D^{ba} is the proper vertex of the divergence, S_b and S_a are the unrenormalized propagators of the particles b and a. The multiplicative factors 1 or γ_5 refer to the two cases of a polar-vector or axial-vector current, respectively.

The generalized partial-conservation assumption implies that $\partial_\lambda J_\lambda(x)$ as well as J_λ itself is a bilinear local product of Fermi fields. Therefore, the proper vertex functions for both the current and its divergence are rendered finite by a single multiplicative renormalization. Following the standard Dyson procedure, we define finite renormalized vertices and propagators by

$$\Gamma_\lambda{}^{ba}(p,p)=\frac{1}{Z_1}\tilde{\Gamma}_\lambda{}^{ba}(p',p),$$

$$D_\lambda{}^{ba}(p',p)=\frac{1}{Z_D}\tilde{D}^{ba}(p',p),$$

$$S_b(p')=Z_2{}^b\tilde{S}_b(p'),$$

$$S_a(p)=Z_2{}^a\tilde{S}_a(p).$$

All the renormalized quantities denoted by the tilde are finite and cutoff-independent. The over-all renormalization of the weak current and its divergence are finite if $Z_1/\sqrt{(Z_2{}^aZ_2{}^b)}$ and $Z_D/\sqrt{(Z_2{}^aZ_2{}^b)}$ are respectively finite, that is, independent of any cutoffs.

To prove this is so, we substitute for the renormalized quantities in the Ward identity and take a variation with respect to the cutoff. This yields

$$0=f^{ab}\left[\delta(Z_1/Z_2{}^b)\tilde{S}_b{}^{-1}(p')\binom{1}{\gamma_5}-\delta(Z_1/Z_2{}^a)\binom{1}{\gamma_5}\tilde{S}_a{}^{-1}(p)\right]+\delta(Z_1/Z_D)\tilde{D}^{ba}(p',p).$$

By evaluating this expression with b or a or both on the mass shell, one finds

$$\delta(Z_1/Z_2{}^b)=\delta(Z_1/Z_2{}^a)=\delta(Z_1/Z_D)=0.$$

Thus, the ratios of these renormalization constants are independent of any cutoffs and hence finite, and the combinations $Z_1/\sqrt{(Z_2{}^aZ_2{}^b)}$ and $Z_D/\sqrt{(Z_2{}^aZ_2{}^b)}$ are finite.

It is well known that the four-point interaction for μ decay can be rewritten by a Fierz transformation as a $V-A$ interaction between the charge-retaining currents. By the preceding theorem, the electromagnetic renormalization of such a μ-e vertex is finite to all orders. This is, of course, not a new result.[6]

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Eigenvectors for the Partial-Wave "Crossing Matrices"

A. P. BALACHANDRAN*
International Center for Theoretical Physics and Physics Department, Syracuse University, Syracuse, New York 13210

AND

W. J. MEGGS* AND P. RAMOND†
Physics Department, Syracuse University, Syracuse, New York 13210

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Let a, b, c, d be spinless particles of equal mass, and consider the process $a+b\rightarrow c+d$. It was shown elsewhere that the crossing symmetry of the scattering amplitude for such a process implies an infinite number of finite-dimensional "crossing relations" for the associated partial waves. In this paper, we derive explicit expressions for complete orthogonal and biorthogonal sets of eigenvectors of the partial-wave crossing matrices. The general form of a partial wave which is consistent with crossing symmetry is thus determined.

I. INTRODUCTION

In a previous paper,[1a] we considered the process $a+b\rightarrow c+d$, where a, b, c, d were spinless particles of equal mass. The scattering amplitude F of such a process was expanded in terms of eigenfunctions which displayed its dependence on all the Mandelstam variables. It was shown that the crossing symmetry of F is equivalent to a sequence of finite-dimensional "crossing relations" for the partial waves.

Here we study the spectral properties of the partial-wave crossing matrices and construct their eigenvectors. With the aid of these eigenvectors, it is easy to state the general form of the partial waves which is consistent with the crossing symmetry of F. Section II summarizes the pertinent results from Ref. 1a. The eigenfunctions are tabulated in Sec. III together with their orthogonality and normalization properties. Section IV sketches the requisite derivations.

In a forthcoming paper,[1b] the eigenfunctions associated with the expansion of F (as well as the eigenvectors of the crossing matrices) will be identified with a subset of basis vectors of certain irreducible representations of the group $SU(3)$. The partial-wave crossing matrices that we discuss here are the matrix elements of the Weyl reflections between these vectors.

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[1] (a) A. P. Balachandran and J. Nuyts, Phys. Rev. 172, 1821 (1968); (b) A. P. Balachandran, W. J. Meggs, J. Nuyts, and P. Ramond, International Center for Theoretical Physics ICTP, Report No. IC/68/46 (unpublished); and Phys. Rev. (to be published); (c) see also A. P. Balachandran and J. Nuyts (to be published).