Method of Pole Dominance and Applications*

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A method for calculating scattering amplitudes at low to moderate energies in terms of their bound-state and resonant poles is presented. By comparing expressions obtained for s-wave scattering lengths with corresponding predictions of current algebra, the widths of the ρ, K^*, Δ, and ω resonances are calculated in close agreement with experiment. Furthermore, without reference to current algebra, all the detailed features of πN elastic scattering (such as the behavior of the P_{11} phase shift) up to an energy of 400 MeV are obtained within an error of about 10 to 20%. The principal advantage of the present method over previous pole-dominance models is that the calculation of amplitudes has been reduced to the evaluation of a number of rapidly convergent integrals. In some cases, one or two undetermined subtraction constants must be introduced, but in the examples of πK and πN elastic scattering only one unknown constant arises, and it can be evaluated by means of the Adler self-consistency condition or an $M=1$ $O(4)$ assignment for the pion.

I. INTRODUCTION

THIS paper suggests a method for representing two-particle scattering amplitudes at low to moderate energies in terms of their most prominent bound-state and resonance poles. Perhaps the most novel feature of the present approach is that it succeeds in expressing the amplitude in terms of a number of rapidly convergent integrals and a small number of subtraction constants. Early work along these lines was done by Frazer, Dietz, and Höhler, Chew and Mandelstam, and Cini and Fubini.[1,2] The present work differs from these earlier efforts principally in the use of the Froissart-Gribov form of the partial-wave amplitude together with estimates of convergence based on Regge asymptotics. There is also a difference of emphasis inasmuch as we make no attempt to calculate the positions and residues of poles, but rather simply require them as input. Reduced in this way to asking relatively modest questions, there is still a remarkable amount that can be learned. Furthermore, comparison of our results for s-wave scattering lengths with those calculated from current algebra by Weinberg[3] gives relations involving the parameters of the poles.

In Sec. II we present the method of pole dominance by giving a detailed discussion of its application to $\pi\pi$ scattering. We find unique results without any undetermined constants if only the ρ poles are kept. Furthermore, the contribution of the f, although accompanied by one subtraction constant, is estimated to introduce a correction of about 1% near threshold. In Sec. III the same techniques are applied to the example of πK elastic scattering. Including contributions associated with the ρ and K^* poles only, one undetermined constant arises. In Sec. IV we discuss the difficulties associated with unequal-mass kinematics that arise if one insists on basing the analysis on partial waves in the s channel ($\pi K \to \pi K$) rather than the t channel ($\pi\pi \to K\bar{K}$). A method is given by which one can obtain the same results as in Sec. III. Sec. IV may be omitted without loss of continuity by the reader who is not interested in this rather technical problem. The most critical test of the method of pole dominance is given in Sec. V—application to πN elastic scattering. This provides the only direct confrontation with experimental data in this paper (i.e., not relying on current algebra as an intermediary). Close agreement is obtained, not only with s- and p-wave scattering lengths, but with detailed features of the phase shifts (most notably the P_{11}) through energies of several hundred MeV. As a final example, in Sec. VI the reaction $\pi\omega \to \pi\pi$ and the associated decay $\omega \to \pi\pi\pi$ are discussed. The results for the latter are exactly the same as were obtained by Gell-Mann, Sharp, and Wagner,[4] but it is suggested that the formulas may be appreciably more accurate than one would previously have had reason to believe. By also considering elastic $\pi\rho$ scattering, the width of the ω is successfully calculated.

In both πK and πN elastic scattering, one constant, undetermined by the method of pole dominance, appears. This constant, which may be taken to be the isospin-symmetric combination of s-wave scattering lengths, is known either from partial conservation of axial-vector current (PCAC) and the Adler self consistency condition[5] or from $O(4)$ analysis and an $M=1$ assignment for the pion[6] to be small in both cases. Thus the one constant that the present analysis does

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[1] W. R. Frazer, in *Proceedings of the Tenth Annual International Conference on High-Energy Physics, Rochester, 1960*, edited by E. C. G. Sudarshan *et al.*, (Interscience Publishers, Inc., New York, 1961), p. 282; K. Dietz and G. Höhler, in *Proceedings of the International Conference on High-Energy Physics, Geneva, 1962*, edited by J. Prentki (CERN, Scientific Information Service, Geneva, 1962), p. 138.

[2] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); Nuovo Cimento **19**, 752 (1961); M. Cini and S. Fubini, Ann. Phys. (N. Y.) **10**, 352 (1960).

[3] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966); Y. Tomozawa, Nuovo Cimento **46A**, 707 (1967).

[4] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962); W. R. Frazer, S. Patil, and H. L. Watson, *ibid.* **11**, 231 (1963).

[5] S. Adler, Phys. Rev. **137**, 1022 (1965).

[6] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967); S. Mandelstam, *ibid.* **168**, 1884 (1968).

not determine is provided by other theoretical considerations.

II. PION-PION SCATTERING

$\pi\pi$ elastic scattering is a particularly suitable reaction for introducing the method of pole dominance, as it is free from the complications of spin and unequal masses. Also, it is of considerable interest in its own right. The starting point is the Froissart-Gribov formula for the partial-wave amplitude, even though we shall only require physical values of angular momentum. Letting s, t, u be the usual Mandelstam variables, q the center-of-mass three-momentum in the s channel, and I the isotopic spin in the s channel, we have in units with $\hbar=c=m_\pi=1$,

$$a_J^{(I)}(s)=\frac{1}{\pi}\int_4^\infty \frac{dt}{2q^2}\,\mathrm{Im}A^{(I)}(s,t)Q_J\left(1+\frac{t}{2q^2}\right) -\frac{1}{\pi}\int_4^\infty \frac{du}{2q^2}\,\mathrm{Im}A^{(I)}(s,\,4-s-u)Q_J\left(-1-\frac{u}{2q^2}\right). \quad (2.1)$$

If $\alpha^{(I)}(s)$ denotes the rightmost singularity in the J plane, then the high-energy behavior of the integrand in (2.1) is $t^{\alpha^{(I)}(s)-J-1}$, and the integrals converge for $J>a^{(I)}(s)$.

Consider first the case $I=1$, for which the ρ trajectory is the leading J-plane singularity, and (2.1) is nonvanishing for odd values of J only. We shall attempt to derive expressions that are accurate even when the energy is comparable to the mass of the ρ. Hence, even though (2.1) converges for $J=1$ when $s<m_\rho^2$, we shall nevertheless use (2.1) only for the values $J=3,\,5,\,7,\,\cdots$. The $J=1$ wave shall be handled separately. For $J>1$, we shall suppose that (2.1) can be accurately evaluated by including only ρ poles in the t and u channels and using the narrow-width approximation. The evaluation of the $J=3$ wave becomes inaccurate as the energy approaches the mass of the first Regge recurrence of the ρ meson, believed to be the $\rho_V(1660)$. At a higher level of approximation than we are presently considering, the $J=3$ wave would be handled separately and the contribution of the $\rho_V(1660)$ in the crossed channels for $J=5,\,7,\,9,\,\cdots$ would also be included.

To evaluate the contribution from the crossed ρ poles, let us first define γ_ρ, the pole residue, by requiring that for $s\sim m_\rho^2$

$$\frac{s^{1/2}}{q^3}e^{i\delta_1}\sin\delta_1\sim\frac{\gamma_\rho}{m_\rho^2-s}. \quad (2.2)$$

The pole position is complex, of course, but we shall make the approximation of taking m_ρ^2 to be real. In this approximation γ_ρ is no longer completely well defined but a reasonable definition (since the resonance is elastic and not too broad) is

$$\gamma_\rho=m_\rho^2\Gamma_\rho/q_\rho^3\,, \quad (2.3)$$

where

$$q_\rho=(\tfrac{1}{4}m_\rho^2-1)^{1/2} \quad (2.4)$$

and Γ_ρ is the experimentally observed width of the ρ resonance.

Perhaps the most dubious aspect of evaluating (2.1) by ρ poles alone is that there may be significant contributions from an s-wave $I=0$ enhancement. In close analogy with Weinberg's current-algebra approach,[3] we shall proceed by *assuming* that this is not the case. Recalling the $s\leftrightarrow t$ isospin crossing matrix

$$\beta_{I'I}=\begin{pmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix} \quad (2.5)$$

and using amplitudes with normalization specified by

$$A^{(I)}(s,z)=\sum_J(2J+1)a_J^{(I)}(s)P_J(z)\,, \quad (2.6a)$$

$$a_J^{(I)}(s)=\frac{s^{1/2}}{q}e^{i\delta_J^{(I)}(s)}\sin\delta_J^{(I)}(s)\,, \quad (2.6b)$$

one can evaluate the contribution of the crossed ρ poles. The easiest way to get the total contribution to $J=3,\,5,\,7,\,\cdots$ is to sum the contributions to all waves and to remove the $J=1$ part. In this way one obtains

$$A^{(1)}(s,z)=\tfrac{3}{8}\gamma_\rho(2s+m_\rho^2-4)\left(\frac{1}{m_\rho^2-t}-\frac{1}{m_\rho^2-u}\right) -\frac{9\gamma_\rho q_\rho^2}{2q^2}Q_1\left(1+\frac{m_\rho^2}{2q^2}\right)P_1\left(1+\frac{s}{2q_\rho^2}\right)z+3a_1^{(I)}(s)z. \quad (2.7)$$

The p wave has been explicitly added in (2.7), since the first two terms have no p-wave part, but contain the other waves accurately. The next step is to evaluate the partial-wave amplitude $a_1^{(1)}(s)$. For this purpose we define

$$b_J^{(I)}(s)=a_J^{(I)}(s)/q^{2J} \quad (2.8)$$

and use the partial-wave dispersion relation

$$b_J^{(I)}(s)=\frac{1}{\pi}\int_{-\infty}^0\frac{\mathrm{Im}b_J^{(I)}(s')}{s'-s}ds' +\frac{1}{\pi}\int_4^\infty\frac{\mathrm{Im}b_J^{(I)}(s')}{s'-s}ds'. \quad (2.9)$$

For the case at hand ($I=J=1$), the right-hand integral is highly convergent as a consequence of the unitarity bound. It may be evaluated by simply including a ρ pole in the narrow-width approximation once again.

For the left-hand-cut integral, the natural choice is to include the contribution due to ρ exchange. One reason is that it provides a good approximation to the upper end of the cut, and hence, if the integral is reasonably convergent, it should be adequate for evaluating the entire integral. Another reason is that the ρ-exchange left-hand cut is needed in order to cancel the singularity in the second term of (2.7), which by standard analyticity requirements should not be present in $A^{(1)}(s,z)$. The contribution of the ρ to the imaginary part on the left-hand cut is

$$\mathrm{Im}_L b_1^{(1)}(s)=\frac{3\pi\gamma_\rho q_\rho^2}{4q^2}P_1\left(1+\frac{m_\rho^2}{2q^2}\right)P_1\left(1+\frac{s}{2q_\rho^2}\right). \quad (2.10)$$

For large s (2.10) goes as $1/s$, so that its contribution to the left-hand cut integral converges quite rapidly. If one believes that the complete left-hand cut integral also converges rapidly, then no corrections to account for other left-hand cut contributions would appear to be required. The expression finally obtained for the left-hand cut contribution, when substituted into (2.7), precisely cancels the second term. This can be demonstrated by doing the integral. A more elegant method is to notice that the two contributions to $b_1^{(1)}(s)$ have cancelling cuts, vanish at infinity, and thus, by a standard theorem of complex analysis, must cancel. Assembling the results, we are left with

$$A^{(1)}(s,t)=\tfrac{3}{8}\gamma_\rho(2s+m_\rho^2-4)\left(\frac{1}{m_\rho^2-t}-\frac{1}{m_\rho^2-u}\right)+\tfrac{3}{4}\gamma_\rho\frac{t-u}{m_\rho^2-s}=\tfrac{3}{8}\gamma_\rho\left(\frac{s-u}{m_\rho^2-t}-\frac{s-t}{m_\rho^2-u}\right)+\tfrac{3}{4}\gamma_\rho\frac{t-u}{m_\rho^2-s}. \quad (2.11)$$

(2.11) is a formula that undoubtedly has been conjectured previously by many others. However, with the proviso of no strong s-wave enhancement, the reasoning by which it has been obtained here precludes the possibility of substantial additions such as $c(t-u)$. By use of crossing, one readily obtains from (2.11)

$$2A^{(0)}(s,t)-5A^{(2)}(s,t)=6A^{(1)}(t,s)-3A^{(1)}(s,t)=\frac{27\gamma_\rho}{8}\left(\frac{s-u}{m_\rho^2-t}+\frac{s-t}{m_\rho^2-u}\right). \quad (2.12)$$

Equation (2.12) implies, in particular, that the s-wave scattering lengths satisfy

$$2a_0-5a_2=(27/2)\gamma_\rho/m_\rho^2. \quad (2.13)$$

Unfortunately, there is a good deal of confusion about the experimental width of the ρ meson, values obtained ranging from 90 to 150 MeV.[7] If we use the value $\Gamma_\rho=120$ MeV, then $\gamma_\rho=1.53$ and

$$2a_0-5a_2\approx 0.70, \quad (2.14)$$

which is almost the same as Weinberg's value of 0.69. However, as both methods assume that there is no strong s-wave scattering, it is not implied that this close agreement proves (2.14) necessarily to be correct.

Another way of expressing the agreement with current algebra is to equate the two expressions, obtaining

$$\gamma_\rho=\frac{2m_\rho^2}{9m^2}g^2\left(\frac{G_V}{G_A}\right)^2, \quad (2.15)$$

where m is the nucleon mass and $g^2=14.6$. A similar formula has been previously deduced from current-algebra considerations alone[8]:

$$\gamma_\rho=\frac{m_\rho^2}{3m^2}g^2\left(\frac{G_V}{G_A}\right)^2K^2(0), \quad (2.16)$$

where $K(0)$ is a form factor associated with an off-mass-shell continuation of the πN coupling constant. Notice that (2.15) and (2.16) agree for $K^2(0)=\frac{2}{3}$. Using (2.3), (2.15) may be reexpressed in the form

$$\Gamma_\rho=\frac{2q_\rho^3}{9m_\rho^2}g^2\left(\frac{G_V}{G_A}\right)^2=120\text{ MeV}. \quad (2.17)$$

In the case of $A^{(0)}(s,t)$ the s wave should be treated separately while the higher waves are evaluated by saturating (2.1) with ρ-meson poles in the narrow-width approximation. This yields, in analogy with (2.7),

$$A^{(0)}(s,t)=\tfrac{3}{4}\gamma_\rho(2s+m_\rho^2-4)\left(\frac{1}{m_\rho^2-t}+\frac{1}{m_\rho^2-u}\right)-\frac{3\gamma_\rho q_\rho^2}{q^2}Q_0\left(1+\frac{m_\rho^2}{2q^2}\right)P_1\left(1+\frac{s}{2q_\rho^2}\right)+a_0^{(0)}(s). \quad (2.18)$$

The contribution of ρ exchange to the imaginary part of $b_0^{(0)}(s)$ on the left-hand cut is

$$\mathrm{Im}_L b_0^{(0)}(s)=\frac{3\pi\gamma_\rho q_\rho^2}{2q^2}P_1\left(1+\frac{s}{2q_\rho^2}\right). \quad (2.19)$$

(2.19) approaches a constant for large s, thus giving a logarithmically divergent integral. The divergence expresses the fact that higher-mass exchanges are also important for this wave. Their contribution at low energies can be approximated by making one subtraction in the partial-wave dispersion relation, thereby introducing one undetermined constant. Assuming that the contribution to $b_0^{(0)}(s)$ from the (subtracted) right-hand cut is vanishingly small, we then have

$$A^{(0)}(s,t)=\tfrac{3}{4}\gamma_\rho(2s+m_\rho^2-4)\left(\frac{1}{m_\rho^2-t}+\frac{1}{m_\rho^2-u}\right)+\text{const}. \quad (2.20)$$

[7] J. Pišút and M. Roos, CERN Report No. TH 885, 1968 (unpublished).

[8] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters, **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

It thus appears that the method of pole dominance leaves one undetermined constant for pion-pion scattering at the present level of approximation. For other reactions this will generally be the case, but for the present example there is a trick that enables us to determine the constant.

The key observation is that the combination $A^{(0)}+2A^{(2)}$ crosses into itself in all three channels (it corresponds to $\pi^0\pi^0$ scattering) and therefore has no ρ poles. With the same approximations as above it follows that

$$A^{(0)}(s,t)+2A^{(2)}(s,t)=0. \tag{2.21}$$

The most questionable point in this reasoning (aside from the possibility of an s-wave enhancement) is that heavy-mass exchanges may be non-negligible in the left-hand cut part of the s-wave partial-wave dispersion relation. If one assumes the integral to be reasonably convergent (as some Reggeized models would suggest), then it may be small enough for (2.21) to be a fairly good approximation, even if not as accurate as (2.11). Combining (2.11) and (2.21) gives

$$A^{(0)}(s,t)=-2A^{(2)}(s,t)=\tfrac{3}{4}\gamma_\rho\left(\frac{s-y}{m_\rho^2-t}+\frac{s-t}{m_\rho^2-u}\right). \tag{2.22}$$

For the s-wave scattering lengths, one has

$$a_0=0.155 \quad\text{and}\quad a_2=-0.078. \tag{2.23}$$

These results are numerically quite close to Weinberg's, although their algebraic structure is somewhat different. (He found $2a_0+7a_2=0$.)

With the one possible exception discussed in the preceding paragraph, we have used highly convergent integrals for all key evaluations, so that contributions to the amplitude at low energy from higher-mass states should be greatly suppressed. To demonstrate this suppression quantitatively we include the f meson as well as the ρ. As the f has spin 2, the problem of the convergence of left-hand-cut integrals becomes more severe. This is one of the major difficulties in generalizing the method of pole dominance in its present form. The best we can do now is to consider

$$\begin{aligned}\tfrac{2}{3}[A^{(0)}(s,t)-A^{(2)}(s,t)]&=A^{(1)}(t,s)+A^{(2)}(t,s)\\&=A^{(1)}(u,s)+A^{(2)}(u,s).\end{aligned} \tag{2.24}$$

Working at fixed s, this combination has no crossed f poles. The s and d waves are treated separately by the same methods as before. Since there are crossed ρ poles, the left-hand cut integral in the s-wave partial-wave dispersion relation is logarithmically divergent, and one subtraction is required, giving rise to one undetermined constant. One finds

$$\begin{aligned}\tfrac{2}{3}[A^{(0)}(s,t)-A^{(2)}(s,t)]=\tfrac{3}{4}\gamma_\rho&\left(\frac{s-u}{m_\rho^2-t}+\frac{s-t}{m_\rho^2-u}\right)\\&+\frac{10}{3}\frac{\gamma_f q^4}{q_f^2}\frac{P_2(1+t/2q^2)}{m_f^2-s}+\text{const},\end{aligned} \tag{2.25}$$

where γ_f, defined analogously to γ_ρ, is

$$\gamma_f=m_f^2\Gamma_f/q_f^3=0.96. \tag{2.26}$$

Although there is one unknown constant in (2.25), it does not contribute to $2A^{(0)}-5A^{(2)}$, which can be constructed from (2.25) by use of crossing. For the f contribution to the scattering lengths one obtains

$$(2a_0-5a_2)_f=-5\gamma_f/q_f^2m_f^2=-0.003\,, \tag{2.27}$$

representing about a 0.5% correction to (2.13). Although we are unable to calculate the contributions of the f to the two scattering lengths separately, it is plausible that they are also of this order of magnitude.

Projecting the $J=0$, $I=0$ partial-wave from (2.22) gives

$$a_0^{(0)}(s)=\tfrac{3}{2}\gamma_\rho\left[\frac{2s+m_\rho^2-4}{4q^2}\ln\left(1+\frac{4q^2}{m_\rho^2}\right)-1\right]. \tag{2.28}$$

Since this expression is real, it only has a chance of making sense when the phase shift is small and takes the value

$$\delta_0^{(0)}(s)=\tfrac{3}{2}\gamma_\rho\frac{q}{s^{1/2}}\left[\frac{2s+m_\rho^2-4}{4q^2}\ln\left(1+\frac{4q^2}{m_\rho^2}\right)-1\right]. \tag{2.29}$$

This gives, for example, $\delta_0^{(0)}(m_K^2)=27°$ and

$$\delta_0^{(0)}(m_K^2)-\delta_0^{(2)}(m_K^2)\approx 40°\,, \tag{2.30}$$

a number of some interest in the analysis of CP violation.

Since $\delta_0^{(0)}(s)$ continues to rise with increasing s, the lack of unitarity begins to become disturbing for $s\geqslant 15$ or so. This may be an indication of a need for a resonance with mass of about 700–800 MeV. If there is such a resonance, then it should be included in the pole-dominance calculations, of course. For reasonable parameters, such as $m_\sigma^2\approx 30$ and $\Gamma_\sigma\approx 1$ (still in units of pion masses), one obtains corrections to the s-wave scattering lengths $a_0^\sigma=0.33$ and $a_2^\sigma=0.13$, to be added to the values in (2.23). The addition of such a resonance is consistent with the fact that $\delta_0^{(2)}(m_\rho^2)$ is known experimentally to be negative. The σ correction would raise $\delta_0^{(2)}(m_\rho^2)$ from $-33°$ to $-28°$ only. It is also conceivable that there is an $I=J=0$ resonance with mass 400–500 MeV. These important and interesting questions shall not be pursued further as they are somewhat peripheral to the central issue of this paper.

III. PION-KAON SCATTERING

As a second application of the method of pole dominance we consider πK elastic scattering. The poles that should be included in the first approximation are the $K^*(893)$ in the s and u channels and the ρ in the t channel. Once again we *assume* that there is no strong s-wave enhancement in any of the three channels. We shall furthermore (as a matter of convenience) assume universality of the ρ-meson coupling[9] to be valid. Then $a_1^{(1)}(t)$ has the pole and residue given by

$$\frac{1}{q_t q_t'} a_1^{(1)}(t) \sim \frac{\gamma_\rho}{m_\rho^2 - t}, \tag{3.1}$$

with γ_ρ as defined in (2.3) and

$$q_t = (\tfrac{1}{4}t - 1)^{1/2}, \tag{3.2a}$$

$$q_t' = (\tfrac{1}{4}t - m_K^2)^{1/2}. \tag{3.2b}$$

The K^* pole and residue are given by

$$(1/q_s^2) a_1^{(1/2)}(s) \sim \gamma_{K^*}/m_{K^*}^2 - s, \tag{3.3}$$

with

$$q_s^2 = [s - (m_K+1)^2][s - (m_K-1)^2]/4s, \tag{3.4}$$

$$\gamma_{K^*} = m_{K^*}^2 \Gamma_{K^*}/q_{K^*}^3 = 1.68, \tag{3.5}$$

and

$$q_{K^*}^2 = [m_{K^*}^2 - (m_K+1)^2] \times [m_{K^*}^2 - (m_K-1)^2]/4m_{K^*}^2. \tag{3.6}$$

Starting from

$$a_J^{(I)}(t) = \frac{1}{\pi}\int_{(m_K+1)^2}^{\infty} \frac{ds}{2q_t q_t'} \operatorname{Im} A^{(I)}(t,s) Q_J\left(\frac{2s+t-\Sigma}{4q_t q_t'}\right) - \frac{1}{\pi}\int_{(m_K+1)^2}^{\infty} \frac{du}{2q_t q_t'} \operatorname{Im} A^{(I)}(t, \Sigma - t - u) \times Q_J\left(\frac{-2u-t+\Sigma}{4q_t q_t'}\right), \tag{3.7}$$

where

$$\Sigma = 2(m_K^2 + 1), \tag{3.8}$$

and using the isospin crossing relations

$$A^{(0)}(t,s) = \tfrac{1}{3}(\sqrt{6})[A^{(1/2)}(s,t) + 2A^{(3/2)}(s,t)], \tag{3.9a}$$

$$A^{(1)}(t,s) = \tfrac{2}{3}[A^{(1/2)}(s,t) - A^{(3/2)}(s,t)], \tag{3.9b}$$

we obtain, by the same reasoning as in Sec. II,

$$A^{(1)}(t,s) = \tfrac{1}{2}\gamma_{K^*}(4q_{K^*}^2 + 2t) \times \left(\frac{1}{m_{K^*}^2 - s} - \frac{1}{m_{K^*}^2 - u}\right) + 3z_t\left[a_1^{(1)}(t) - \frac{\gamma_{K^*}(4q_{K^*}^2 + 2t)}{2q_t q_t'} Q_1\left(\frac{2m_{K^*}^2 + t - \Sigma}{4q_t q_t'}\right)\right]$$
$$= \gamma_{K^*}(t + 2q_{K^*}^2)\left(\frac{1}{m_{K^*}^2 - s} - \frac{1}{m_{K^*}^2 - u}\right) + \tfrac{3}{4}\gamma_\rho\left(\frac{s-\mu}{m_\rho^2 - t}\right). \tag{3.10}$$

Evaluation of (3.10) at threshold [i.e., $s = (m_K+1)^2$, $u = (m_K-1)^2$, $t=0$] gives the following relation for the s-wave scattering lengths:

$$a_{1/2} - a_{3/2} = \left(1 + \frac{1}{m_K}\right)^{-1}\left(\frac{3\gamma_{K^*}}{m_{K^*}^2} + \frac{9\gamma_\rho}{2m_\rho^2}\right) = \left(1 + \frac{1}{m_K}\right)^{-1}(0.123 + 0.233) = 0.28. \tag{3.11}$$

Equation (3.11) is very close to the current-algebra prediction of 0.27. Comparing (3.11) with the expression given by current algebra, and also using (2.15) and (3.5), gives in analogy with (2.17),

$$\Gamma_{K^*} = \frac{q_{K^*}^3}{6m^2} g^2 \left(\frac{G_V}{G_A}\right)^2 = 47 \text{ MeV}. \tag{3.12}$$

Taking the ratio of (2.17) and (3.12),

$$\Gamma_{K^*}/\Gamma_\rho = 3q_{K^*}^3/4q_\rho^3, \tag{3.13}$$

a relation that has been obtained previously from current-algebra considerations alone by Riazuddin and Fayyazuddin.[8] [Exact $SU(3)$ gives a ratio of $\frac{3}{4}$.]

In similar fashion,

$$A^{(0)}(t,s) = \tfrac{1}{2}(\sqrt{6})(t + 2q_{K^*}^2) \times \left(\frac{1}{m_{K^*}^2 - s} + \frac{1}{m_{K^*}^2 - u}\right) + \text{const}. \tag{3.14}$$

Equation (3.14) contains one undetermined constant because the partial-wave dispersion relation for the s-wave requires one subtraction. This constant cannot be determined by the methods of this paper. (In Feynman-diagram language, one would say that this constant depends on the choice of a coupling scheme.)

The results of this section would be significantly altered by a low-mass scalar resonance in either of the channels. The correction from the 2^+ mesons $f(1260)$ and $K_V(1419)$ is probably of the order of 1% near threshold, as was shown to be the case in the $\pi\pi$ example.

IV. PROBLEM OF UNEQUAL MASSES

In the preceding section results were obtained for πK elastic scattering by focusing attention on t-channel partial waves. If the method of pole dominance is to be generally applicable, the same results should be obtainable from s-channel considerations as well. Accomplishing this is not trivial, however, because there are kinematical complications associated with unequal masses.

Having isolated particular s-channel partial waves for special consideration, it is necessary to properly take account of threshold behaviors before performing dispersion relations. We believe the following rules to be correct: (1) $a_J(s)$ vanishes at the normal threshold proportionally to $[s-(m_K+1)^2]^J$; (2) $a_J(s)$ is finite

[9] J. J. Sakurai, Ann. Phys. (N. Y.) 11, 1 (1960).

at $s=(m_K-1)^2$ for $J=0$, but vanishes there like $[s-(m_K-1)^2]^{1/2}$ for $J\geq 1$; (3) $a_J(s)$ has contributions which at $s=0$ behave like s, $s\ln s$, and $s^{-\alpha(0)}$, where $\alpha(0)$ is the intercept of the leading s-channel Regge singularity[10]—this rule holding for all J.

Carrying out the calculations of Sec. III by use of the above rules, or slightly modified ones, leads to expressions having spurious poles at either $s=(m_K-1)^2$ or $s=0$ (or having the wrong equal-mass limit) and consequently differing from (3.10) and (3.14). Suitable analyticity necessarily requires admixtures of lower partial waves to cancel off these spurious poles. This automatically happens in writing down a Feynman diagram for the K^* pole in $K\pi$ scattering, for example, with different couplings giving somewhat different prescriptions.

A convenient method of circumventing the difficulties, within the method of pole dominance, is to replace the partial-wave expansion by another expansion for which the analyticity of the individual terms is more favorable. As a specific possibility let us consider

$$A(s,t)=\sum_{n=0}^{\infty}(2n+1)c_n(s)P_n(\tilde{z}), \quad (4.1)$$

with

$$\tilde{z}=1+t/2\tilde{q}^2 \quad (4.2)$$

and

$$\tilde{q}^2=\tfrac{1}{4}[s-(m_K+1)^2]. \quad (4.3)$$

This expansion coincides with partial waves in the limit of equal masses, while for unequal masses it mixes the various waves. Inverting (4.1) in a Froissart-Gribov form,

$$c_n(s)=\frac{1}{\pi}\int_4^{\infty}\frac{dt}{2\tilde{q}^2}\,\mathrm{Im}A(s,t)Q_n\left(1+\frac{t}{2\tilde{q}^2}\right) -\frac{1}{\pi}\int_{(m_K+1)^2}^{\infty}\frac{du}{2\tilde{q}^2}\,\mathrm{Im}A(s,\Sigma-s-u) \times Q_n\left(1+\frac{\Sigma-s-u}{2\tilde{q}^2}\right). \quad (4.4)$$

From (4.4) and standard analyticity assumptions for $A(s,t)$, one can easily see that these amplitudes have the threshold behavior

$$c_n(s)\sim(\tilde{q})^{2n}. \quad (4.5)$$

At $s=0$ and $s=(m_K-1)^2$, $c_n(s)$ is analytic, except for possible left-hand-cut singularities which are logarithmic at worst.

The method of pole dominance described in the preceding sections may now be carried out using (4.4) to replace the standard Froissart-Gribov formula. A resonance of spin J will contribute to all the terms $n=0, 1, \cdots, J$, in general, since a Jth-order polynomial in t is required. For application of this expansion to the example of πK scattering it is convenient (but not essential) to consider the isospin combination that is free of crossed-channel K^* poles:

$$\tfrac{2}{3}A^{(1/2)}(s,t)+\tfrac{1}{3}A^{(3/2)}(s,t)=A^{(3/2)}(u,t) =(\sqrt{\tfrac{1}{6}})A^{(0)}(t,s)+\tfrac{1}{2}A^{(1)}(t,s). \quad (4.6)$$

Then, by the same methods as before,

$$\tfrac{2}{3}A^{(1/2)}(s,t)+\tfrac{1}{3}A^{(3/2)}(s,t)=\tfrac{3}{8}\gamma_\rho\left(\frac{2s+m_\rho^2-\Sigma}{m_\rho^2-t}\right) -(n=0\text{ and }n=1\text{ projections})+\tfrac{2}{3}c_0^{(1/2)}(s) +\tfrac{1}{3}c_1^{(3/2)}(s)+[2c_1^{(1/2)}(s)+c_1^{(3/2)}(s)]\tilde{z}. \quad (4.7)$$

The c_0's and c_1's are once again given by "partial-wave" dispersion relations (after removing the threshold factor $\tilde{q}^2$ from c_1). The convergence properties of these integrals are the same as for ordinary partial waves in the equal-mass case. The K^* resonance contributes to the right-hand cut of both $c_0^{(1/2)}$ and $c_1^{(1/2)}$, its total contribution to (4.7) in the narrow-width approximation being

$$\gamma_{K^*}\left(\frac{t+2q_{K^*}^2}{m_{K^*}^2-s}-\frac{1}{2}\right). \quad (4.8)$$

For the left-hand-cut integrals there is again a cancellation against the subtracted $n=0$ and $n=1$ projections, one unknown constant remaining because the dispersion relations for the c_0's require a subtraction. In this way one finds

$$\tfrac{2}{3}A^{(1/2)}(s,t)+\tfrac{1}{3}A^{(3/2)}(s,t) =\frac{3\gamma_\rho}{8}\left(\frac{s-u}{m_\rho^2-t}\right)+\gamma_{K^*}\frac{t+2q_{K^*}^2}{m_{K^*}^2-s}+\text{const}. \quad (4.9)$$

By use of crossing, both (3.10) and (3.14) can be deduced from (4.9).

Lest the reader be misled, it should be emphasized that the prescription suggested in (4.1)–(4.4) is not unique and, in fact, there is an ambiguity associated with its non-uniqueness. An example of another possible expansion is

$$A(s,t)=\sum_{n=0}^{\infty}(2n+1)d_n(s)P_n\left(1+\frac{\tilde{q}_R^2}{q_R^2}\frac{t}{2\tilde{q}^2}\right), \quad (4.10)$$

which has the convenient feature that a resonance of mass m_R and spin J will only contribute to the Jth term. The contribution of such a resonance as it arises from a right-hand cut integral would be of the form

$$\frac{\gamma}{m_R^2-s}(\tilde{q})^{2J}P_J\left(1+\frac{\tilde{q}_R^2}{q_R^2}\frac{t}{2\tilde{q}^2}\right). \quad (4.11)$$

[10] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).

Equation (4.11) differs from what one would get from expanding (4.1) by a polynomial in s of degree $J-1$. It is easy to show that such a polynomial carries all the ambiguity in the choice of an expansion having suitable analyticity of the individual terms. In the example of πK scattering, an unknown constant should therefore be added to (4.8). As it happens, this constant can be absorbed in the constant already appearing in (4.9), leaving the final results unaffected.

V. PION-NUCLEON SCATTERING

The obvious advantage of studying πN elastic scattering over the other reactions discussed so far is that it allows a direct comparison of calculated and experimental quantities. In the following we shall calculate the four $J=\frac{1}{2}$ phase shifts as well as the s- and p-wave scattering lengths. The accuracy of the techniques shall be estimated by finding the sensitivity of the calculations to various possible modifications. Our purpose is to determine the reliability of the approximations rather than to attempt achieving the best possible solution. The degree of success that shall be reached, however, does suggest that the present formalism may be usefully incorporated in a thorough analysis of scattering data.

Some standard formulas given by Frautschi and Walecka,[11] for example, that shall be required are

$$f_1(s,z)=\sum_{l=0}^{\infty} f_{l+}(s)P_{l+1}'(z)-\sum_{l=2}^{\infty} f_{l-}(s)P_{l-1}'(z)\,, \quad (5.1\text{a})$$

$$f_2(s,z)=\sum_{l=1}^{\infty} [f_{l-}(s)-f_{l+}(s)]P_l'(z)\,, \quad (5.1\text{b})$$

where $f_{l\pm}(s)$ is the partial wave having orbital angular momentum l and total angular momentum $J=l\pm\frac{1}{2}$. The inverse formulas are

$$f_{l\pm}(s)=(1/q)e^{i\delta_{l\pm}(s)}\sin\delta_{l\pm}(s)$$
$$=\tfrac{1}{2}\int_{-1}^{1}[f_1(s,z)P_l(z)+f_2(s,z)P_{l\pm1}(z)]dz\,. \quad (5.2)$$

Invariant amplitudes, free of kinematical singularities, are defined by

$$A(s,t)=8\pi W\left[\frac{W+m}{(W+m)^2-1}f_1(s,t)-\frac{W-m}{(W-m)^2-1}f_2(s,t)\right], \quad (5.3\text{a})$$

$$B(s,t)=8\pi W\left[\frac{1}{(W+m)^2-1}f_1(s,t)+\frac{1}{(W-m)^2-1}f_2(s,t)\right], \quad (5.3\text{b})$$

[11] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

where $W=s^{1/2}$ and m is the nucleon mass. The amplitude appropriate to unpolarized nucleons is

$$F(s,t)=A(s,t)+\left(\frac{s-u}{4m}\right)B(s,t)\,. \quad (5.4)$$

The most prominent poles to be included in a first approximation are the ρ, N, and Δ. We continue to assume that there is no strong $\pi\pi$ s-wave enhancement. The contribution of each of these poles to $F(s,t)$ is as follows. Near the nucleon pole one has

$$F^{(1/2)}(s,t)\sim\frac{3\pi g^2}{m}\frac{t-2}{m^2-s}\,, \quad (5.5)$$

where $g^2=14.6$ measures the pion-nucleon coupling. In the vicinity of the ρ pole

$$F^{(1)}(t,s)\sim\frac{3\pi\gamma_\rho}{m}\frac{2s+m_\rho^2-\Sigma}{m_\rho^2-t}\,. \quad (5.6)$$

Since universality[9] for the electric coupling of the ρ has been assumed in (5.6), γ_ρ is the same as in (2.3). Near the Δ pole,

$$F^{(3/2)}(s,t)\sim\frac{8\pi q_\Delta^2\gamma_\Delta}{m_\Delta^2-s}h(t)\,, \quad (5.7)$$

where $m_\Delta=1236$ MeV is the mass of the Δ,

$$q_\Delta^2=[m_\Delta^2-(m+1)^2][m_\Delta^2-(m-1)^2]/4m_\Delta^2\,, \quad (5.8)$$

$$\gamma_\Delta=m_\Delta^2\Gamma_\Delta/q_\Delta^3\approx14.6\,, \quad (5.9)$$

$$h(t)=3\frac{m_\Delta+m}{(m_\Delta+m)^2-1}\left(1+\frac{t}{2q^2_\Delta}\right)+\frac{m_\Delta-m}{(m_\Delta-m)^2-1}+\frac{2m_\Delta^2+t-\Sigma}{4m}\left[\frac{3}{(m_\Delta+m)^2-1}\times\left(1+\frac{t}{2q_\Delta^2}\right)-\frac{1}{(m_\Delta-m)^2-1}\right], \quad (5.10)$$

and

$$\Sigma=2(m^2+1)\,. \quad (5.11)$$

The method of pole dominance can now be applied to the calculation of $F^{(I)}(s,t)$. For the isospin-symmetric amplitude, one finds

$$F^{(1/2)}(s,t)+2F^{(3/2)}(s,t)=\frac{3\pi g^2}{m}(t-2)\left(\frac{1}{m^2-s}+\frac{1}{m^2-u}\right)+16\pi q_\Delta^2\gamma_\Delta h(t)\left(\frac{1}{m_\Delta^2-s}+\frac{1}{m_\Delta^2-u}\right)+c_0+c_1t\,. \quad (5.12)$$

The undetermined constants c_0 and c_1 arise as subtraction constants in the dispersion relation for the $J=0$ t-channel partial wave. Two subtractions are required because the Δ-exchange contribution to the left-hand cut integral is linearly divergent. Similarly, the isospin-antisymmetric amplitude is

$$F^{(1/2)}(s,t)-F^{(3/2)}(s,t)=\frac{9\pi\gamma_\rho}{2m}\left(\frac{s-u}{m_\rho^2-t}\right)+\frac{3\pi g^2}{m}(t-2)\left(\frac{1}{m^2-s}-\frac{1}{m^2-u}\right)-8\pi q_\Delta^2\gamma_\Delta h(t)\times\left(\frac{1}{m_\Delta^2-s}-\frac{1}{m_\Delta^2-u}\right)+c_2(s-u). \quad (5.13)$$

The constant c_2 arises as a subtraction in the $J=1$ t-channel partial-wave dispersion relation, which otherwise would be logarithmically divergent. The constants c_1 and c_2 cannot be determined by present methods from a consideration of $F^{(I)}(s,t)$ alone. However, in the subsequent discussion when the A's and B's are treated separately, only one undetermined constant (corresponding to c_0) will appear. The difference is attributable to the energy factor multiplying $B(s,t)$ in (5.4). A comparison of formulas allows us to deduce that

$$c_1=c_2=0. \quad (5.14)$$

One combination of s-wave scattering lengths can now be obtained. The normalization conventions are such that

$$F^{(I)}(s=(m+1)^2,\, t=0)=4\pi[1+(1/m)]a_{2I}, \quad (5.15)$$

where a_{2I} denotes the s-wave scattering length of isospin I. From (5.13)–(5.15) and $h(0)=1/m$,

$$a_1-a_3=\left(1+\frac{1}{m}\right)^{-1}\left(\frac{9\gamma_\rho}{2m_\rho^2}+\frac{3g^2}{2m^2}-\frac{2\gamma_\Delta}{m_\Delta^2}\right)=[1+1/m]^{-1}(0.233+0.486-0.375)=0.299. \quad (5.16)$$

This result is very close to the prediction of current algebra. More importantly, it is quite close to the experimental value

$$a_1-a_3=0.271\pm0.007 \quad (5.17)$$

determined by Hamilton[12] from data on low-energy scattering. Equating (5.16) to the current-algebra prediction gives, in analogy with (2.17) and (3.12),

$$\Gamma_\Delta=(3q_\Delta^3/4m^2)g^2[1-\tfrac{1}{3}(G_V/G_A)^2]=115\text{ MeV}, \quad (5.18)$$

which is to be compared with the experimental value of 120 MeV.

Because of the presence of c_0 in (5.12), the isospin-symmetric combination a_1+2a_3 cannot be calculated. However, the Adler self-consistency condition[5] or the assignment of the π trajectory to an $M=1$ representation of $O(4)$[6] predicts it to be very small.

$$a_1+2a_3=-0.002\pm0.008 \quad (5.19)$$

is the value found by Hamilton. The specification of $F^{(I)}(s,t)$ can be completed by simply requiring that this combination vanish.

Formulas for the A's and B's have already been alluded to in justification of (5.14). A more important reason to work out separate formulas for the A's and B's is that a complete specification of pion-nucleon scattering requires them. Labeling amplitudes by their t-channel isospins,

$$A^{(0)}(t,s)=\frac{16\sqrt{6}\pi q_\Delta^2\gamma_\Delta}{3}g_1(t)\left(\frac{1}{m_\Delta^2-s}+\frac{1}{m_\Delta^2-u}\right)+(16\sqrt{6})\pi c, \quad (5.20a)$$

$$A^{(1)}(t,s)=-\frac{16\pi q_\Delta^2\gamma_\Delta}{3}g_1(t)\left(\frac{1}{m_\Delta^2-s}-\frac{1}{m_\Delta^2-u}\right)-\frac{6\pi\gamma_\rho\mu_\rho}{m}\left(\frac{s-u}{m_\rho^2-t}\right), \quad (5.20b)$$

$$B^{(0)}(t,s)=\frac{16\sqrt{6}\pi q_\Delta^2\gamma_\Delta}{3}g_2(t)\left(\frac{1}{m_\Delta^2-s}-\frac{1}{m_\Delta^2-u}\right)+4\sqrt{6}\pi g^2\left(\frac{1}{m^2-s}-\frac{1}{m^2-u}\right), \quad (5.20c)$$

$$B^{(1)}(t,s)=-\frac{16\pi q_\Delta^2\gamma_\Delta}{3}g_2(t)\left(\frac{1}{m_\Delta^2-s}+\frac{1}{m_\Delta^2-u}\right)+8\pi g^2\left(\frac{1}{m^2-s}+\frac{1}{m^2-u}\right)+12\pi\gamma_\rho(1+2\mu_\rho)\frac{1}{m_\rho^2-t}, \quad (5.20d)$$

where

$$g_1(t)=3\frac{m_\Delta+m}{(m_\Delta+m)^2-1}\left(1+\frac{t}{2q_\Delta^2}\right)-\frac{m_\Delta-m}{(m_\Delta-m)^2-1}, \quad (5.21a)$$

$$g_2(t)=\frac{3}{(m_\Delta+m)^2-1}\left(1+\frac{t}{2q_\Delta^2}\right)-\frac{1}{(m_\Delta-m)^2-1}. \quad (5.21b)$$

$\gamma_\rho\mu_\rho$ describes the magnetic coupling of the ρ to the nucleon. A prediction, based on universality, is that μ_ρ should be the anomalous isovector magnetic moment of the nucleon, i.e.,

$$\mu_\rho=1.85. \quad (5.22)$$

[12] J. Hamilton, Phys. Letters **20**, 687 (1966).

The subtraction constant c in (5.20a) can be determined by requiring the isospin-symmetric combination of s-wave scattering lengths to vanish, in which case

$$c=\frac{mg^2}{4m^2-1}-\gamma_\Delta\frac{2m_\Delta-m}{3m_\Delta^2}. \tag{5.23}$$

To obtain the $J=\frac{1}{2}$ partial waves, we use (5.20a)–(5.20d) and

$$f_{1-}^{(I)}(W)=\frac{(W+m)^2-1}{16\pi s}[A_1^{(I)}+(W-m)B_1^{(I)}]-\frac{(W-m)^2-1}{16\pi s}[A_0^{(I)}-(W+m)B_0^{(I)}], \tag{5.24}$$

where

$$A_l^{(I)}=\tfrac{1}{2}\int_{-1}^{1}A^{(I)}(s,z)P_l(z)dz, \tag{5.25a}$$

$$B_l^{(I)}=\tfrac{1}{2}\int_{-1}^{1}B^{(I)}(s,z)P_l(z)dz. \tag{5.25b}$$

$f_{1-}(W)$ equals $q^{-1}e^{i\delta}\sin\delta$ for the p wave when $W>m+1$, while for $W<-m-1$ it describes the s wave (in accordance with the MacDowell symmetry).

Since the formulas (5.20a)–(5.20d) are written in a purely real form, the partial waves projected from them obviously cannot be unitary. In evaluating phase shifts we shall simply replace $e^{i\delta}\sin\delta$ by δ, an approximation that should make sense as long as phase shifts are less than 30° or so. In the vicinity of a resonance pole, unitarity can be restored, at least approximately, by taking the pole position to be complex. Otherwise, taking complex positions for the resonance poles would generally introduce only a small correction. Thus we can now calculate the $J=\frac{1}{2}$ waves, which are nonresonant at low energy, using (5.20a)–(5.20d), and (5.24). Defining

$$z_\rho=1+m_\rho^2/2q^2, \tag{5.26a}$$

$$z_N=1+(\Sigma-s-m^2)/2q^2, \tag{5.26b}$$

$$z_\Delta=1+(\Sigma-s-m_\Delta^2)/2q^2, \tag{5.26c}$$

and

$$\alpha_1=3\frac{m_\Delta+m}{(m_\Delta+m)^2-1}, \tag{5.27a}$$

$$\alpha_2=\frac{3}{(m_\Delta+m)^2-1}, \tag{5.27b}$$

$$\beta_1=3\left(\frac{q_\Delta^2}{q^2}-1\right)\frac{m_\Delta+m}{(m_\Delta+m)^2-1}+\frac{q_\Delta^2}{q^2}\frac{m_\Delta-m}{(m_\Delta-m)^2-1}, \tag{5.27c}$$

$$\beta_2=3\left(\frac{q_\Delta^2}{q^2}-1\right)\frac{1}{(m_\Delta+m)^2-1}-\frac{q_\Delta^2}{q^2}\frac{1}{(m_\Delta-m)^2-1}, \tag{5.27d}$$

the $I=\frac{1}{2}$, $J=\frac{1}{2}$ partial waves are given by

$$\begin{aligned}f_{1-}^{(1/2)}(W)=&\frac{(W+m)^2-1}{s}\left(\frac{3\gamma_\rho\mu_\rho}{16mq^2}(\Sigma-2s-m_\rho^2)Q_1(z_\rho)-\tfrac{1}{9}\gamma_\Delta[2\alpha_1Q_2(z_\Delta)+3\beta_1Q_1(z_\Delta)+\alpha_1Q_0(z_\Delta)]\right)\\&+\frac{(W+m)^2-1}{s}(W-m)\left(\frac{3\gamma_\rho}{8q^2}(1+2\mu_\rho)Q_1(z_\rho)-\frac{g^2}{8q^2}Q_1(z_N)+\tfrac{1}{9}\gamma_\Delta[2\alpha_2Q_2(z_\Delta)+3\beta_2Q_1(z_\Delta)+\alpha_2Q_0(z_\Delta)]\right)\\&+\frac{(W-m)^2-1}{s}\left(-\frac{3\gamma_\rho\mu_\rho}{8m}-\frac{3\gamma_\rho\mu_\rho}{16mq^2}(\Sigma-2s-m_\rho^2)Q_0(z_\rho)+\tfrac{1}{3}\gamma_\Delta[\alpha_1Q_1(z_\Delta)+\beta_1Q_0(z_\Delta)]-c\right)\\&+\frac{(W-m)^2-1}{s}(W+m)\left(\frac{3}{4}\frac{g^2}{m^2-s}-\frac{g^2}{8q^2}Q_0(z_N)+\frac{3\gamma_\rho}{8q^2}(1+2\mu_\rho)Q_0(z_\rho)+\tfrac{1}{3}\gamma_\Delta[\alpha_2Q_1(z_\Delta)+\beta_2Q_0(z_\Delta)]\right).\end{aligned} \tag{5.28}$$

Similarly, for the $I=\frac{3}{2}$, $J=\frac{1}{2}$ partial waves,

$$\begin{aligned}f_{1-}^{(3/2)}(W)=&\frac{(W+m)^2-1}{s}\left(\frac{3\gamma_\rho\mu_\rho}{32mq^2}(2s+m_\rho^2-\Sigma)Q_1(z_\rho)+\frac{1}{6}\frac{q^2\gamma_\Delta\alpha_1}{m_\Delta^2-s}-\frac{\gamma_\Delta}{36}[2\alpha_1Q_2(z_\Delta)+3\beta_1Q_1(z_\Delta)+\alpha_1Q_0(z_\Delta)]\right)\\&+\frac{(W+m)^2-1}{s}(W-m)\left(\frac{g^2}{4q^2}Q_1(z_N)-\frac{3\gamma_\rho(1+2\mu_\rho)}{16q^2}Q_1(z_\rho)+\frac{1}{6}\frac{q^2\gamma_\Delta\alpha_2}{m_\Delta^2-s}+\frac{\gamma_\Delta}{36}[2\alpha_2Q_2(z_\Delta)+3\beta_2Q_1(z_\Delta)+\alpha_2Q_0(z_\Delta)]\right)\\&+\frac{(W-m)^2-1}{s}\left(\frac{3\gamma_\rho\mu_\rho}{16m}-\frac{3\gamma_\rho\mu_\rho}{32mq^2}(2s+m_\rho^2-\Sigma)Q_0(z_\rho)-\frac{1}{2}\frac{q^2\gamma_\Delta\beta_1}{m_\Delta^2-s}+\tfrac{1}{12}\gamma_\Delta[\beta_1Q_0(z_\Delta)+\alpha_1Q_1(z_\Delta)]-c\right)\\&+\frac{(W-m)^2-1}{s}(W+m)\left(\frac{g^2}{4q^2}Q_0(z_N)-\frac{3\gamma_\rho(1+2\mu_\rho)}{16q^2}Q_0(z_\rho)+\frac{1}{2}\frac{q^2\gamma_\Delta\beta_2}{m_\Delta^2-s}+\tfrac{1}{12}\gamma_\Delta[\beta_2Q_0(z_\Delta)+\alpha_2Q_1(z_\Delta)]\right).\end{aligned} \tag{5.29}$$

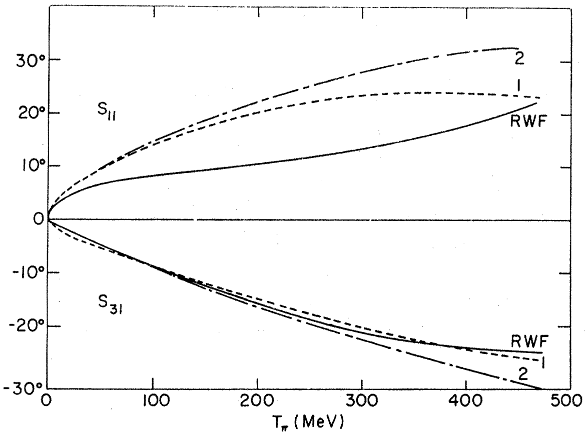

FIG. 1. The s-wave πN phase shifts of solutions 1 and 2 compared with the phenomenological analysis of Roper, Wright, and Feld (RWF) (see Ref. 13).

The phase shifts deduced from (5.28) and (5.29) are referred to as "solution 1." They are plotted in Figs. 1 and 2 together with the results of the phase-shift analysis of Roper, Wright, and Feld.[13] In particular, the upturning of the P_{11} phase shift is encouraging. Approximately equal contributions to this behavior are made by the ρ and Δ poles. It should be emphasized, however, that as we are not performing dynamical calculations in the conventional sense it does not necessarily follow that the ρ and Δ forces should be responsible for this behavior in a one-channel N/D calculation. Indeed, the suggestion has been made that the P_{11} phase shift should not come out positive in a one-channel N/D calculation without the explicit inclusion of a Castillejo-Dalitz-Dyson (CDD) pole.[14] In Ref. 14 it was also suggested that the turning positive of the P_{11} phase shift at low energy and the presence of a resonance in this wave at high energy are probably independent phenomena, a point of view that is supported by solution 1 inasmuch as it does not have a resonance pole in it. Other remarks to be made about solution 1 are that it gives excellent agreement for S_{31} and P_{31} waves, but only fair agreement for the S_{11} phase shift.

In solution 2 the sensitivity of the phase shifts to the magnetic coupling of the ρ is tested by taking $\mu_\rho=0$, with all other quantities as in solution 1. As can be seen in Figs. 1 and 2, this change causes the p-wave phase shifts to deviate significantly from their experimental values, while it has very little effect for the s waves. This reinforces the belief that (5.22) is a good approximation.

Solution 3 is the same as solution 1 except for the explicit addition of the Roper resonance with nucleon quantum numbers, mass $m_{11}=1470$ MeV, width $\Gamma_{11}=210$ MeV, and elasticity factor $x_{11}=0.65$. Thus, defining the residue

$$\gamma_{11}=\frac{m_{11}^2\Gamma_{11}x_{11}}{q_{11}^3}=3.6\,, \tag{5.30}$$

the contributions to be added to the A's and B's in (5.20a)–(5.20d) are

$$A_{11}^{(0)}=-\frac{(\sqrt{6})}{3}\frac{8\pi(m_{11}-m)}{(m_{11}-m)^2-1}q_{11}^2\gamma_{11}\times\left(\frac{1}{m_{11}^2-s}+\frac{1}{m_{11}^2-u}\right)+c_{11}\,, \tag{5.31a}$$

$$A_{11}^{(1)}=-\frac{2}{3}\frac{8\pi(m_{11}-m)}{(m_{11}-m)^2-1}q_{11}^2\gamma_{11}\times\left(\frac{1}{m_{11}^2-s}-\frac{1}{m_{11}^2-u}\right), \tag{5.31b}$$

$$B_{11}^{(0)}=\frac{\sqrt{6}}{3}\frac{8\pi q_{11}^2}{(m_{11}-m)^2-1}\gamma_{11}\times\left(\frac{1}{m_{11}^2-s}-\frac{1}{m_{11}^2-u}\right), \tag{5.31c}$$

$$B_{11}^{(1)}=\frac{2}{3}\frac{8\pi q_{11}^2}{(m_{11}-m)^2-1}\gamma_{11}\times\left(\frac{1}{m_{11}^2-s}+\frac{1}{m_{11}^2-u}\right). \tag{5.31d}$$

The phase shifts corresponding to solution 3 are shown in Figs. 3 and 4. One change is that the s-wave scattering lengths have been increased by about 10%. Other changes are generally of this same order or less, except for the P_{11} phase shift which becomes much more strongly positive for energies above 200 MeV, in close agreement with experiment. Fortunately, the change in this phase shift below 200 MeV is not very large.

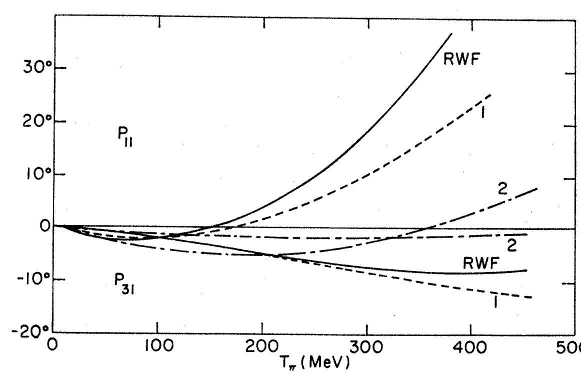

FIG. 2. The $J=\frac{1}{2}$ p-wave phase shifts of solutions 1 and 2 compared with RWF.

[13] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965).

[14] J. H. Schwarz, Phys. Rev. **152**, 1325 (1966); J. S. Ball, G. L. Shaw, and D. Y. Wong, *ibid.* **155**, 1725 (1967).

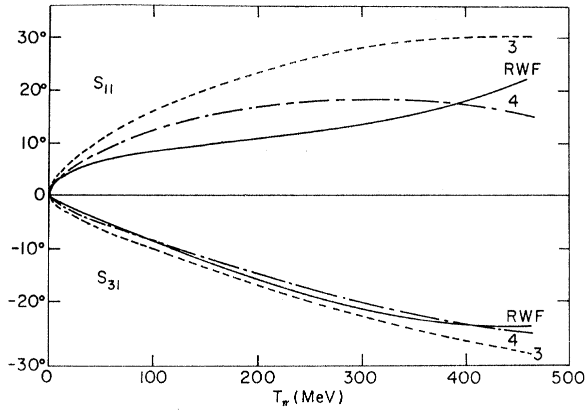

FIG. 3. The s-wave phase shifts of solutions 3 and 4 compared with RWF.

In solution 3 the s-wave scattering lengths are about 20% larger than the experimental values. Furthermore, inclusion of the D_{13} resonance would increase them by another 10% or so. A likely cause for this discrepancy involves the narrow-width approximation for the $\Delta(1236)$ resonance. Its width is of magnitude comparable to its distance from threshold. Since we have taken this resonance to be concentrated at 1236 MeV inside integrals in which it multiplies rapidly decreasing functions, finite-width effects may be estimated by taking its "effective mass" to be somewhat lower than 1236 MeV. The principal effect of reducing the mass by 10 MeV, say, is a decrease in q_Δ and hence an increase in γ_Δ to 16.6 from the value of 14.6 given in (5.9). Solution 4 shows the effect of making this change in solution 3. The agreement of the phase shifts, especially the S_{11}, is generally improved by this modification.

In Table I we compare the s- and p-wave scattering lengths for each of the four solutions with the experimental values.[12,15] The $J=\frac{1}{2}$ scattering lengths are determined from (5.28) and (5.29), with the modifications indicated for solutions 2, 3, and 4. For the $J=\frac{3}{2}$ p-wave scattering lengths (in the notation $a_{2I,2J}$), we use

$$a_{2I,3}=\frac{1}{4m^2}a_{2I}+a_{2I,1}-\frac{1}{8\pi m}B^{(I)}(\text{threshold}). \qquad (5.32)$$

To summarize the discussion of πN scattering, we find it proper to conclude that using N, ρ, and Δ poles, as well as the isospin-symmetric s-wave scattering length, we have succeeded in representing πN scattering up to about 400-MeV pion lab energy within an error of 10 to 20%. To achieve better accuracy would require including additional resonances, determining couplings more accurately, and correcting for finite-width effects. Furthermore, at an improved level of accuracy it may be necessary to include corrections for $\pi\pi$ scattering in the s wave.

[15] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

VI. ω DECAY

The techniques developed in the preceding sections may be used to calculate the amplitude for the process $\pi\omega\to\pi\pi$ in terms of ρ-meson poles. This amplitude also describes the decay of the ω into three pions, providing, in particular, an accurate expression for the distribution over the Dalitz plot. The formula to be obtained for the integrated partial width is exactly the same as was suggested by Gell-Mann, Sharp, and Wagner[4] some time ago. Nevertheless the derivation is given in some detail, as the method of pole dominance provides reason for believing the amplitude to be accurately represented in a larger region than just in the immediate vicinity of the poles.

Invoking parity conservation, the process $\pi\omega\to\pi\pi$ has just one independent helicity amplitude

$$f(s,z)=\sum_{J=1}^{\infty}(2J+1)f_J(s)d_{0-1}{}^J(\theta). \qquad (6.1)$$

All kinematical singularities are removed from (6.1) by constructing

$$\tilde{f}(s,z)=f(s,z)[stu-(m_\omega{}^2-1)^2]^{-1/2}. \qquad (6.2)$$

The following kinematical relations are required:

$$\sin\theta=[stu-(m_\omega{}^2-1)^2]^{1/2}/k(s)\,, \qquad (6.3)$$

$$k(s)=\tfrac{1}{2}\{(s-4)[s-(m_\omega+1)^2][s-(m_\omega-1)^2]\}^{1/2}=2qq's^{1/2}, \qquad (6.4)$$

$$q=(\tfrac{1}{4}s-1)^{1/2}, \qquad (6.5a)$$

$$q'=\{[s-(m_\omega+1)^2]\times[s-(m_\omega-1)^2]\}^{1/2}/2s^{1/2}, \qquad (6.5b)$$

$$d_{0-1}{}^J(\theta)=-[J(J+1)]^{-1/2}\sin\theta P_J'(\cos\theta). \qquad (6.6)$$

From (6.1)–(6.6) and the orthogonality of the d functions one obtains

$$f_J(s)=-\tfrac{1}{2}[J(J+1)]^{-1/2}k(s)\times\int_{1}^{1}\tilde{f}(s,z)(1-z^2)P_J'(z)dz. \qquad (6.7)$$

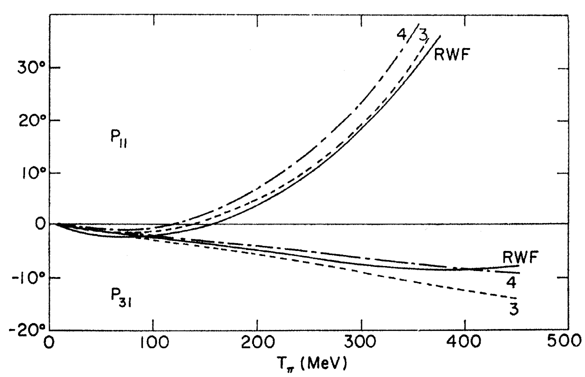

FIG. 4. The $J=\frac{1}{2}$ p-wave phase shifts of solutions 3 and 4 compared with RWF.

TABLE I. πN s- and p-wave scattering lengths.

	Experiment[a]	Solution 1	2	3	4
a_1	0.180	0.200	0.200	0.220	0.185
a_3	−0.091	−0.100	−0.100	−0.110	−0.093
a_{11}	−0.101	−0.091	−0.120	−0.083	−0.071
a_{31}	−0.039	−0.040	−0.025	−0.042	−0.034
a_{13}	−0.027	−0.053	−0.039	−0.036	−0.018
a_{33}	0.216	0.239	0.233	0.240	0.265

[a] See Refs. 12 and 15.

The remaining procedure is now the same as before. Substituting a fixed-s dispersion relation for $\tilde{f}(s,z)$ in (6.7) and performing the z integration yields an expression of the Froissart-Gribov form. The remaining integrals are highly convergent for $J>1$ (only odd J occurs) and can be evaluated by the substitution of ρ poles. The $J=1$ wave is considered separately. To calculate it, one first removes threshold factors by forming

$$\tilde{f}_1(s)=f_1(s)/k(s). \tag{6.8}$$

The partial-wave dispersion relation for $\tilde{f}_1(s)$ then has a highly convergent right-hand cut, which is once again evaluated by substitution of a ρ pole in the narrow-width approximation. The contribution of ρ exchange to the left-hand discontinuity of $f_1(s)$ vanishes at large s like $1/s^2$, so that the cancellation discussed in the previous sections occurs without a need for any unknown subtraction constants.

From the above discussion, it may be concluded that *to high accuracy*,

$$\tilde{f}(s,t)=\text{const}\times\left(\frac{1}{m_\rho^2-s}+\frac{1}{m_\rho^2-t}+\frac{1}{m_\rho^2-u}\right). \tag{6.9}$$

The coupling constant $\gamma_{\rho\omega\pi}$ is defined by[16]

$$s^{1/2}t_1(s)/(qq')^{3/2}\sim(\gamma_{\rho\omega\pi}\gamma_\rho)^{1/2}/(m_\rho^2-s), \tag{6.10}$$

where (6.10) refers only to the immediate vicinity of the pole and $t_1(s)$ denotes the off-diagonal matrix element of $(S-1)/2i$ for $J=1$. ($\pi\omega\to\pi\pi$ is not an elastic process.) By combining (6.9), (6.10), employing standard rules for three-body decays, and including a finite-width correction for the ρ poles, we obtain

$$\Gamma_{\omega\to3\pi}=\frac{3}{64\pi}\frac{\gamma_\rho\gamma_{\rho\omega\pi}}{m_\rho^2m_\omega^3}\int_R\delta(\Sigma-s-t-u)[stu-(m_\omega^2-1)^2]\times\left|\frac{1}{m_\rho^2-im_\rho\Gamma_\rho'-s}+\frac{1}{m_\rho^2-im_\rho\Gamma_\rho'-t}+\frac{1}{m_\rho^2-im_\rho\Gamma_\rho'-u}\right|^2dsdtdu, \tag{6.11}$$

where $\Sigma=m_\omega^2+3$ and R is the region for which the second factor in the integrand is positive. The integral in (6.11) is increased by about 10% if the imaginary parts of the pole positions are omitted. The effect is not greater because the boundary of the Dalitz region is kinematically suppressed, and therefore regions close to the poles do not contribute strongly to the integral. It should be emphasized that previously there was no good reason to believe (6.11) to be an accurate formula for the contributions near the center of the Dalitz plot, the region from which the bulk of the integral arises.

Evaluating (6.11) numerically, we find that for $\gamma_\rho=1.53$, $\Gamma_{\omega\to3\pi}=11$ MeV, and $m_\omega=5.68$ (corresponding to an ω mass of 783 MeV and a π mass of 138 MeV), we have

$$\gamma_{\rho\omega\pi}=10.5. \tag{6.12}$$

This result is extremely sensitive to phase space. Repeating the calculation with $m_\omega=5.60$ (corresponding to an ω mass of 783 MeV and a π mass of 140 MeV), one obtains $\gamma_{\rho\omega\pi}=12.6$. The value $m_\pi=138$ MeV is preferred because that is the average for the three pion charges (each of which always occurs in the decay). This serves as a warning, however, that electromagnetic corrections could alter (6.12) by as much as 10%.

In order to obtain a "theoretical value" for the decay width $\Gamma_{\omega\to3\pi}$, an independent calculation of $\gamma_{\rho\omega\pi}$ is required. For this purpose we consider $\pi\rho$ elastic scattering, with the inclusion of only the π and ω poles in the s and u channels and the ρ pole in the t channel. The A_2 meson (or mesons) probably represents a small correction near threshold. An A_1 meson, having $J^P=1^+$ and hence contributing to the $\pi\rho$ s wave, might not be negligible. However, for purposes of comparison with the current-algebra formulas for the s-wave scattering lengths, the A_1 contribution should probably be omitted. The reason for this is that the current-algebra calculations assume the absence of strong s-wave scattering. This reasoning has been successfully applied in other contexts by von Hipple and Kim.[17]

Once again assuming universality of the ρ electric couplings (this time to the ρ), the method of pole dominance gives for the unpolarized isospin-antisymmetric amplitude (the only one not requiring a subtraction constant)

$$F^{(1)}(t,s)=\tfrac{3}{4}\gamma_\rho\frac{s-u}{m_\rho^2-t}+\tfrac{1}{8}\gamma_\rho(2t-1-2m_\rho^2)\left(\frac{1}{1-s}-\frac{1}{1-u}\right)+\tfrac{1}{2}\gamma_{\rho\omega\pi}(t+2q_\omega^2)\left(\frac{1}{m_\omega^2-s}-\frac{1}{m_\omega^2-u}\right), \tag{6.13}$$

where

$$q_\omega^2=[m_\omega^2-(m_\rho+1)^2][m_\omega^2-(m_\rho-1)^2]/4m_\omega^2. \tag{6.14}$$

[16] The relationship between our constants γ_ρ and $\gamma_{\rho\omega\pi}$ and the couplings $\gamma_{\rho\pi\pi}$ and $f_{\rho\omega\pi}$ used in Ref. 4 is $\gamma_{\rho\pi\pi}^2/4\pi=\frac{3}{8}\gamma_\rho$ and $f_{\rho\omega\pi}^2/4\pi=\frac{3}{2}(\gamma_{\rho\omega\pi}/m_\rho^2)$.

[17] F. von Hippel and J. K. Kim, Phys. Rev. Letters **20**, 1303 (1968).

Evaluation of (6.13) at the s-channel threshold gives a relation among the s-wave scattering lengths:

$$\tfrac{1}{3}a_0+\tfrac{1}{2}a_1-\tfrac{5}{6}a_2=\left(1+\frac{1}{m_\rho}\right)^{-1}\times\left[\frac{3\gamma_\rho}{m_\rho^2}-\frac{(2m_\rho^2+1)\gamma_\rho}{2m_\rho^2(m_\rho^2-4)}+\frac{\gamma_{\rho\omega\pi}}{m_\omega^2}\right]. \quad (6.15)$$

Equating (6.15) with the current-algebra prediction for the scattering lengths,

$$\gamma_{\rho\omega\pi}=1.60\frac{m_\omega^2}{m^2}g^2\left(\frac{G_V}{G_A}\right)^2=11.6. \quad (6.16)$$

Substitution of (6.16) into (6.11) gives the theoretical value

$$\Gamma_{\omega\to3\pi}=12.1\text{ MeV}, \quad (6.17)$$

in satisfactory agreement with the experimental value of 11 MeV, especially in view of the sensitivity of the calculation to small corrections.

VII. CONCLUSION

In summary, the basic steps in the method of pole dominance are the following. The first step is to express the scattering amplitude in the form

$$A^{(I)}(t,s)=\frac{1}{\pi}\int_{s_0}^{\infty}ds'\,\mathrm{Im}A^{(I)}(t,s')\times\left[\frac{1}{s'-s}-\frac{1}{2q_tq_t'}\sum_{J=0}^{J_0}(2J+1)Q_J\left(\frac{2s'+t-\Sigma}{4q_tq_t'}\right)\times P_J\left(\frac{2s+t-\Sigma}{4q_tq_t'}\right)\right]+(u\text{ integral})+\sum_{J=0}^{J_0}(2J+1)a_J^{(I)}(t)P_J\left(\frac{2s+t-\Sigma}{4q_tq_t'}\right). \quad (7.1)$$

For a specified range of t, there is then a value of J_0 such that the integrals are highly convergent and may be evaluated by substituting for $\mathrm{Im}A^{(I)}(t,s')$ pole contributions in the narrow-resonance approximation. It is convenient to include only low-mass resonances. The second step is to calculate $a_J^{(I)}(t)$ for $J=0$, 1, $\cdots$, J_0 by means of a dispersion relation for $b_J^{(I)}(t)=a_J^{(I)}(t)/(q_tq_t')^J$. The right-hand cut integrals are convergent (except possibly for $J=0$) and may also be evaluated by substitution of pole contributions. The left-hand cut integral is evaluated by substituting the contributions from the *same* poles as were used in the evaluation of the s- and u-cut integrals in step one. These are the most important contributions whenever the left-hand cut integrals converge. In examples having no external spins, a spin-S contribution to the left-hand cut of the Jth partial wave converges for $S\leq J$. If $S>J$, higher-mass exchanges are necessary to cancel the divergence. They may be represented by making $S-J$ subtractions. The third step is to observe that except for terms corresponding to the previous subtractions, there is an exact cancellation between the left-hand cut integrals and the portion of the t- and u-cut integrals involving the sum of Q functions.

Of the various approximations involved in the method of pole dominance, the most difficult to justify quantitatively is the neglect of other contributions to the left-hand cut integrals. The procedure given seems plausible to us, but perhaps further justification should rest on comparison with experiment.

As a final (and somewhat gratuitous) remark, we suggest that if a Reggeized version of the method of pole dominance should be found, it would quite likely be free from the need for subtraction constants. Furthermore, it might provide a suitable formalism for implementing the bootstrap conditions. At that stage one would be performing true dynamical calculations, representing a substantial advance over what has been accomplished in this paper.

Note added in proof. It has been pointed out to me by Dr. H. Harari that a more careful calculation of the Δ finite-width effects increases the discrepancy in the πN s-wave scattering lengths, rather than decreasing it as claimed in the text. We have also noticed that a subtraction constant in $B^{(1)}$ [see (5.20d)] cannot be excluded. This constant can be chosen to fix up the isospin-antisymmetric scattering length. Thus, the success of Eqs. (5.16) and (5.18) is due, at least in part, to the cancellation of errors. The phase shifts would be only very slightly altered by reducing the Δ term by 30% and adding an appropriate constant to $B^{(1)}$.