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Daughters, Conspiracies, and Lorentz Symmetry*

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The Freedman-Wang prescription for the introduction of daughter trajectories is generalized to the arbitrary-spin case. The daughter problem is then investigated, within the framework of Lorentz symmetry, and a Regge-type representation for the asymptotic S matrix is obtained which satisfies all kinematic constraints. The representation is found to be equivalent to a Lorentz pole expansion in the case of elastic scattering. Some properties of the representations of the covering groups of the homogeneous Lorentz group and rotation group are also investigated.

1. INTRODUCTION

A DISTURBING feature of the usual Regge theory of the high-energy scattering, of two particles into two particles, is the number of kinematic constraints that have to be applied. These take many forms, from the removal of threshold and pseudothreshold factors of individual residue functions,[1] to the conspiracy of trajectories[2] in order to ensure the vanishing of helicity-flip elastic scattering amplitudes on the physical region boundary. Indeed it is the great variety of necessary constraints that makes the theory so unattractive.

In this paper, we examine in detail the possibility of restoring the usual asymptotic behavior of a Regge expansion for the scattering of particles with arbitrary nonzero masses and spins. First of all, while introducing our notation, we review the kinematics[3] of two-particle scattering processes, including Reggeization,[4] and collect various constraint equations for Regge residue functions.[5,6]

Our principal interest lies in the behavior of Regge expansions in the case of scattering of unequal-mass particles, with spin, at high energies $\sqrt{s}$, when the squared momentum transfer t is zero. It has been shown that the problem of obtaining asymptotic Regge behavior can be resolved, if each Regge pole conspires with a sequence of integrally spaced daughter poles,[7] according to some four-dimensional symmetry.[8] We eventually investigate this possibility in detail. Initially, we adopt a prescription similar to that used by Freedman and Wang[7] in the "spinless" case, and group Regge poles together into families. The members of each family conspire with correlated residues. They give contributions to s-channel helicity amplitudes $\mathcal{S}_{cd;ab}$ which are of the order of $s^{\alpha(0)}$, for large s, where $\alpha(0)$ is a constant. We discover, when particles have spin, that, in addition to the daughter poles of Freedman and Wang, one must introduce parity-doublet poles with the same signature. Only in this way may the usual analyticity properties of the S matrix be retained.

We then investigate the problem within a group-theoretical framework and show that it is possible to sum, into a closed form,[9] the contributions to the helicity amplitudes of families of Regge poles with specially correlated residues. The closed-form expression possesses "good" analytic structure and asymptotic behavior at high energies. Moreover, we show that the contributions of such families to the s-channel helicity amplitudes satisfy the equal-mass conspiracy constraints of Toller[10] and of Freedman and Wang.[6] When invariance under spatial inversion is considered, one finds that in order to have factorizable residues[11] one may be forced to consider Regge poles in parity doublets.

In the next section, we consider the general properties of our special forms for Regge residues that are correlated near $t=0$, and discover that they possess the usual threshold behavior,[1] for positive values of t. They also possess the singularities at $t=0$, which we should have expected from our earlier Freedman-Wang-type analysis.

We then investigate the leading asymptotic behavior of the s-channel helicity amplitudes at high energies, with $t=0$, which depends on both the external particle mass ratios[12,13] and a Regge family parameter j_0. We then proceed to apply our analysis to obtaining a representation for the scattering of spinless particles and compare it with that of Freedman and Wang.[7]

In Appendices A, B, C, and D, we state and derive some useful properties of representations of the covering

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groups of the rotation group and homogeneous Lorentz group.

2. KINEMATICS AND REGGEIZATION

In this section, we recall some well-known results concerning the kinematics and Reggeization of scattering amplitudes of particles with arbitrary masses and spins. We consider the interaction of four particles (r) with spins s_r, momenta P_r, masses m_r, and intrinsic parities η_r, where $r=a,b,c,d$. As usual, we shall be concerned with related physical processes:

$$\begin{aligned} s&: (a)+(b)\to(c)+(d)\,,\\ t&: (a)+(\bar{c})\to(\bar{b})+(d)\,,\\ \text{and}\quad u&: (a)+(\bar{d})\to(c)+(\bar{b})\,, \end{aligned} \tag{1}$$

where $(\bar{r})$ denotes the antiparticle of the particle (r) and s, t, and u are the squares of the c.m. energies in the s, t, and u channels, respectively,

$$s=(p_a+p_b)^2\,,\quad t=(p_a-p_c)^2\,,\quad u=(p_a-p_d)^2\,. \tag{2}$$

We shall be interested in the high-energy behavior of the s-channel c.m. helicity amplitudes,[14]

$$\mathcal{S}_{cd;ab}=\langle p_c,s_c,c;p_d,s_d,d|T|p_a,s_a,a;p_b,s_b,b\rangle\,,$$

which are simply related to the scattering amplitudes in an arbitrary Lorentz frame,[15]

$$\langle P_c,s_c,c;P_d,s_d,d|T|P_a,s_a,a;P_b,s_b,b\rangle\,,$$

by

$$\begin{aligned}\langle P_c,s_c,c;P_d,s_d,d|T|P_a,s_a,a;P_b,s_b,b\rangle&=D_{a'a}{}^{s_a}(R_a)\\ &\times D_{b'b}{}^{s_b}(R_b)D_{c'c}{}^{s_c*}(R_c)D_{d'd}{}^{s_d*}(R_d)\mathcal{S}_{cd;ab}\,,\end{aligned} \tag{3}$$

where the representation functions $D_{r'r}{}^{s_r}$ of the covering group of the rotation group $SU(2)$ correspond to Wigner rotations R_r, which transform the states $|P_r,s_r,r\rangle$ into the c.m. helicity states $|p_r,s_r,r\rangle$. The amplitudes $\mathcal{S}_{cd;ab}$ are functions of the total c.m. energy s and the c.m. scattering angle Θ_s, of particle (c) relative to particle (a), with

$$z_s=\cos\Theta_s=\frac{s(t-u)+(m_a^2-m_b^2)(m_c^2-m_d^2)}{\Delta(s;a,b)\Delta(s;c,d)} \tag{4}$$

and

$$\sin\Theta_s=\frac{2s^{1/2}\phi}{\Delta(s;a,b)\Delta(s;c,d)}\,. \tag{5}$$

The threshold function $\Delta(s;a,b)$ is given by

$$\Delta(s;a,b)=\Delta^+(s;a,b)\Delta^-(s;a,b)\,, \tag{6}$$

where

$$\Delta^\pm(s;a,b)=[s-(m_a\pm m_b)^2]^{1/2} \tag{7}$$

and ϕ is the Kibble[3] boundary function

$$\begin{aligned}\phi(s,t,u)=[stu&-s(m_a^2-m_c^2)(m_b^2-m_d^2)\\ &-t(m_a^2-m_b^2)(m_c^2-m_d^2)\\ &-(m_a^2m_d^2-m_b^2m_c^2)\Delta_{bc}{}^{ad}]^{1/2}\,,\end{aligned} \tag{8}$$

with

$$\Delta_{bc}{}^{ad}=m_a^2+m_d^2-m_c^2-m_b^2\,. \tag{9}$$

We choose our masses to satisfy $m_a\geq m_c\geq m_d\geq m_b$ so that the point $t=0$ lies within the s-channel physical region. For convenience we shall consider the c.m. four-momenta p_r in a standard Lorentz frame.

$$\begin{aligned}
p_a&=\left(\frac{s+m_a^2-m_b^2}{2s^{1/2}},0,0,\frac{\Delta(s;a,b)}{2s^{1/2}}\right),\\
p_b&=\left(\frac{s+m_b^2-m_a^2}{2s^{1/2}},0,0,\frac{-\Delta(s;a,b)}{2s^{1/2}}\right),\\
p_c&=\left(\frac{s+m_c^2-m_d^2}{2s^{1/2}},\frac{\Delta(s;c,d)\sin\Theta_s}{2s^{1/2}},0,\frac{\Delta(s;c,d)\cos\Theta_s}{2s^{1/2}}\right),\\
\text{and}\quad p_d&=\left(\frac{s+m_d^2-m_c^2}{2s^{1/2}},\frac{-\Delta(s;c,d)\sin\Theta_s}{2s^{1/2}},0,\frac{-\Delta(s;c,d)\cos\Theta_s}{2s^{1/2}}\right).
\end{aligned} \tag{10}$$

As is well known,[15] the s-channel c.m. helicity amplitudes $\mathcal{S}_{cd;ab}$ may be expressed as linear combinations of the t-channel c.m. helicity amplitudes $\mathcal{T}_{bd;ac}$ analytically continued to the region of negative t and positive s. We shall be interested in crossing a Regge-type expansion, from the t channel, to the s channel and must define precisely the continuations of our functions Δ and ϕ outside the relevant physical regions. This we do in Appendix

[14] M. Jacob and J. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).
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A. The crossing relation then takes the form[16]

$$S_{cd;ab}=(-1)^{\sigma(p)+2s_a+2s_c}e^{i\pi(a-d)}\sum_{a'b'c'd'} d_{a'a}{}^{s_a}(\chi_a{}^{st})d_{b'b}{}^{s_b}(\chi_b{}^{st})d_{c'c}{}^{s_c}(\chi_c{}^{st})d_{d'd}{}^{s_d}(\chi_d{}^{st})T_{b'd';a'c'}\,, \tag{11}$$

where $\sigma(p)$ vanishes unless both particles (b) and (c) are fermions, in which case $\sigma(p)=1$. The crossing angles $\chi_r{}^{st}$ are given by

$$\cos\chi_a{}^{st}=-\frac{(s+m_a^2-m_b^2)(t+m_a^2-m_c^2)-2m_a^2\Delta_{cb}{}^{ad}}{\Delta(s;a,b)\Delta(t;a,c)}\,,\quad \sin\chi_a{}^{st}=\frac{-2m_a\phi}{\Delta(s,a,b)\Delta(t,a,c)}\,,$$

$$\cos\chi_b{}^{st}=\frac{(s+m_b^2-m_a^2)(t+m_b^2-m_d^2)+2m_b^2\Delta_{cb}{}^{ad}}{\Delta(s;a,b)\Delta(t;b,d)}\,,\quad \sin\chi_b{}^{st}=\frac{-2m_b\phi}{\Delta(s,a,b)\Delta(t,b,d)}\,,$$

$$\cos\chi_c{}^{st}=\frac{(s+m_c^2-m_d^2)(t+m_c^2-m_a^2)+2m_c^2\Delta_{cb}{}^{ad}}{\Delta(s;c,d)\Delta(t;a,c)}\,,\quad \sin\chi_c{}^{st}=\frac{2m_c\phi}{\Delta(s,c,d)\Delta(t,a,c)}\,, \tag{12}$$

and

$$\cos\chi_d{}^{st}=-\frac{(s+m_d^2-m_c^2)(t+m_d^2-m_b^2)-2m_d^2\Delta_{cb}{}^{ad}}{\Delta(s;c,d)\Delta(t;b,d)}\,,\quad \sin\chi_d{}^{st}=\frac{2m_d\phi}{\Delta(s,c,d)\Delta(t,b,d)}\,,$$

where all square roots are taken to be positive in the physical region. The functions $d_{r'r}{}^{s_r}(\chi_r{}^{st})$ are representations of the group $SU(2)$ corresponding to rotations about the y axis.[17]

We now consider t-channel partial-wave analysis, where we use formulas (4)–(10) with the substitutions $t\leftrightarrow s$ and $b\leftrightarrow c$.

Following Jacob and Wick,[14] we define f amplitudes by

$$f_{bd;ac}=\frac{2t^{1/2}}{[\Delta(t;b,d)\Delta(t;a,c)]^{1/2}}T_{bd;ac}\,, \tag{13}$$

which are simply related to differential cross sections,

$$d\sigma=|f_{bd;ac}(t,\Theta_t)|^2d\Omega. \tag{14}$$

These may be expanded in partial waves,

$$f_{bd;ac}(t,z)=\left(\frac{p'}{p}\right)^{1/2}\sum_J(2J+1)F_{bd;ac}{}^J(t)d_{\lambda\lambda'}{}^J(z)\,, \tag{15}$$

where the initial and final t-channel c.m. three-momenta p and p' are given by

$$p=\Delta(t;a,c)/2t^{1/2}\,, \tag{16}$$

$$p'=\Delta(t;b,d)/2t^{1/2}. \tag{17}$$

In these expressions, λ and λ' are the total helicities, $\lambda=a-c$, $\lambda'=b-d$, and J is the total angular momentum.

The parity-conserving amplitudes[4] $f_{bd;ac}{}^{\pm}$ are linear combinations of the form

$$f_{bd;ac}{}^{\pm}(t,z)=\xi_{\lambda\lambda'}(z)f_{bd;ac}(t,z)\pm(-1)^{\lambda+\lambda_m}\eta_{bd}\xi_{\lambda-\lambda'}(z)f_{-b-d;ac}(t,z)\,, \tag{18}$$

where

$$\xi_{\lambda\mu}(z)=(1+z)^{-\frac{1}{2}|\lambda+\mu|}(1-z)^{-\frac{1}{2}|\lambda-\mu|}\,,\quad \lambda_m=\max(|\lambda|,|\lambda'|)\,, \tag{19}$$

and

$$\eta_{bd}=\eta_b\eta_d(-1)^{s_b+s_d-\nu}\,,$$

with

$$\nu=\tfrac{1}{2},J \text{ half-integral}$$
$$=0,J \text{ integral.}$$

Partial-wave expansions are of the form

$$f_{bd;ac}{}^{\pm}(t,z)=\left(\frac{p}{p'}\right)^{1/2}\sum_J(2J+1)[F_{bd;ac}{}^{J\pm}(t)e_{\lambda\lambda'}{}^{J+}(z)+F_{bd;ac}{}^{J\mp}(t)e_{\lambda\lambda'}{}^{J-}(z)]\,, \tag{20}$$

where the parity-conserving partial-wave amplitudes $F_{bd;ac}{}^{J\pm}$ are defined by

$$F_{bd;ac}{}^{J\pm}(t)=F_{bd;ac}{}^J(t)\pm\eta_{bd}F_{-b-d;ac}{}^J(t)\,, \tag{21}$$

and the representation functions $e_{\lambda\lambda'}{}^{J\pm}(z)$ are linear combinations of the usual representation functions $d_{\lambda\lambda'}{}^J(z)$,

$$e_{\lambda\mu}{}^{J\pm}(z)=\tfrac{1}{2}(\xi_{\lambda\mu}(z)d_{\lambda\mu}{}^J(z)\pm(-1)^{\lambda+\lambda_m}\xi_{\lambda-\mu}(z)d_{\lambda-\mu}{}^J(z))\,. \tag{22}$$

The Reggeization of the amplitudes $F_{bd;ac}{}^{J\pm}$ has been discussed in detail by Gell-Mann, Goldberger, Low, Marx, and Zachariasen.[4] They find

$$f_{bd;ac}{}^{\pm}(t,z)\approx\left(\frac{p}{p'}\right)^{1/2}\sum_\alpha\left(\beta_{bd;ac}{}^{\alpha\pm}(t)\frac{(2\alpha^{\pm}+1)}{\sin\pi\alpha^{\pm}}E_{\lambda\lambda'}{}^{(\alpha\pm)+}(-z)-\beta_{bd;ac}{}^{\alpha\pm}\frac{(2\alpha^{\mp}+1)}{\sin\pi\alpha^{\mp}}E_{\lambda\lambda'}{}^{(\alpha\mp)-}(-z)\right)\,, \tag{23}$$

[16] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 239 (1968). We use the same crossing angles, but the expressions appear different because of a difference in definition of kinematic threshold functions (see Appendix A).

[17] M. Andrews and J. Gunson, J. Math. Phys. **5**, 1391 (1964).

Our more conventional (Ref. 25) definition differs by a phase $(-1)^{\lambda-\lambda'}$.

where $\beta^{\alpha\pm}$ and $\alpha^{\pm}$ denote the residues and positions of Regge poles in the amplitudes $F_{bd;ac}{}^{J+}$ and $F_{bd;ac}{}^{J-}$, respectively. The functions $E_{\lambda\mu}{}^{\alpha\pm}(z)$ coincide with analytic continuations in z of the functions $e_{\lambda\mu}{}^{J\pm}(z)$, for integral or half-integral values of α (see Appendix B).

When we take signature $\mathbf{s}$ into account, we replace the functions $F_{bd;ac}{}^{\alpha\pm}$ by $F_{bd;ac}{}^{\mathbf{s}\alpha\pm}$, where

$$F_{bd;ac}{}^{\mathbf{s}\alpha\pm}=\tfrac{1}{2}(1+\mathbf{s}e^{-i\pi(\alpha-\nu)})F_{bd;ac}{}^{\alpha\pm} \tag{24}$$

and the trajectory corresponding to a physical particle with spin J has signature $\mathbf{s}=(-1)^{J-\nu}$. We shall later refer to several equivalent forms of the Regge representation and for convenience list them here. If we define a generalized residue function $\Gamma_{bd;ac}{}^{\mathbf{s}\alpha}(t)$ by

$$\Gamma_{bd;ac}{}^{\mathbf{s}\alpha\pm}(t)=\left(\frac{p}{p'}\right)^{1/2}\frac{(2\alpha+1)}{\sin\pi\alpha}\times(1+\mathbf{s}e^{-i\pi(\alpha-\nu)})\beta_{bd;ac}{}^{\alpha\pm}, \tag{25}$$

the t-channel amplitudes for large s are given by

$$f_{bd;ac}{}^{\pm}\approx\sum_{\alpha}\left[\Gamma_{bd;ac}{}^{\mathbf{s}\alpha\pm}E_{\lambda\lambda'}{}^{(\alpha\pm)+}(-z)-\Gamma_{bd;ac}{}^{\mathbf{s}\alpha\mp}E_{\lambda\lambda'}{}^{(\alpha\mp)-}(-z)\right] \tag{26}$$

$$=\sum_{\alpha}\xi_{\lambda\lambda'}(-z)\left[\Gamma_{bd;ac}{}^{\mathbf{s}\alpha\pm}D_{\lambda\lambda'}{}^{\alpha\pm}(-z)-\Gamma_{bd;ac}{}^{\mathbf{s}\alpha\mp}D_{\lambda\lambda'}{}^{\alpha\mp}(-z)\right]+(-1)^{\lambda+\lambda_m}\xi_{\lambda-\lambda'}(-z)\left[\Gamma_{bd;ac}{}^{\mathbf{s}\alpha\pm}D_{\lambda-\lambda'}{}^{\alpha\pm}(-z)+\Gamma_{bd;ac}{}^{\mathbf{s}\alpha\mp}D_{\lambda-\lambda'}{}^{\alpha\mp}(-z)\right] \tag{27}$$

$$=\sum_{\alpha}\Gamma_{bd;ac}{}^{\mathbf{s}\alpha\pm}\left[\xi_{\lambda\lambda'}{}^{+}(-z)D_{\lambda\lambda'}{}^{(\alpha\pm)+}(-z)+\xi_{\lambda\lambda'}{}^{-}(-z)D_{\lambda\lambda'}{}^{(\alpha\pm)-}(-z)\right]-\Gamma_{bd;ac}{}^{\mathbf{s}\alpha\mp}\left[\xi_{\lambda\lambda'}{}^{-}(-z)D_{\lambda\lambda'}{}^{(\alpha\mp)+}(-z)+\xi_{\lambda\lambda'}{}^{+}(-z)D_{\lambda\lambda'}{}^{(\alpha\mp)-}(-z)\right], \tag{28}$$

where the functions $D_{\lambda\lambda'}{}^{(\alpha\pm)\pm}(z)$ and $\xi_{\lambda\lambda'}{}^{\pm}(z)$ are defined in Appendix B [(B5), (B12)].

The behavior of s-channel helicity amplitudes, at high energies, may be obtained by using expression (13) and the crossing relation (11).

Let us now recall some properties of the reduced Regge residues,

$$\bar{\beta}_{bd;ac}{}^{\alpha}(t)=(pp')^{-\alpha}\beta_{bd;ac}{}^{\alpha}(t). \tag{29}$$

They are real, analytic functions of t with poles at normal and pseudothresholds. The behavior at these singular points follows directly from that of the amplitudes $F_{bd;ac}{}^{J\pm}(t)$,[1]

$$\bar{\beta}_{bd;ac}{}^{\alpha\pm}(t)\approx[\Delta^{+}(t;a,c)]^{N\pm}[\Delta^{-}(t;a,c)]^{P\pm}\times[\Delta^{+}(t;b,d)]^{N'\pm}[\Delta^{-}(t;b,d)]^{P'\pm}, \tag{30}$$

where

$$N^{\pm}=-(s_a+s_c)+\tfrac{1}{2}(1\mp\eta_{ac}) \tag{31}$$

and

$$P^{\pm}=-(s_a+s_c)+\tfrac{1}{2}[1\mp(-1)^{2s_c}\eta_{ac}],\quad m_a\geqq m_c, \tag{32}$$

with $\Delta^{\pm}(t;a,c)$ defined by Eq. (7).

This leads to threshold behavior of the amplitudes $f_{bd;ac}{}^{\pm}(t,z)$ of the form

$$f_{bd;ac}{}^{\pm}(t,z)\approx[\Delta(t;a,c)\Delta(t;b,d)]^{\lambda_m}\bar{\beta}_{bd;ac}{}^{\pm}. \tag{33}$$

The residue functions $\beta_{bd;ac}$ are not independent. Parity conservation alone implies[5]

$$\beta_{bd;ac}=\eta_{bd}\eta_{ac}\beta_{-b-d;-a-c}. \tag{34}$$

At the physical-region boundary of the s channel, we expect helicity flip amplitudes to vanish. If we use the crossing relation (11), we obtain more constraints on the t-channel amplitudes.[6] For arbitrary masses in the case of s-channel backward scattering or unequal masses, $m_a\neq m_c$, $m_b\neq m_d$, in the case of forward scattering, we find

$$0=\mathcal{S}_{cd;ab}\approx\mathcal{T}_{bd;ac},\quad b-d\neq a-c. \tag{35}$$

In the case of forward s-channel elastic scattering, we find

$$0=\mathcal{S}_{cd;ab}\approx\sum_{a'b'c'd'}d_{a'a}{}^{s_a}(-\tfrac{1}{2}\pi)d_{b'b}{}^{s_b}\times(-\tfrac{1}{2}\pi)d_{c'c}{}^{s_c}(\tfrac{1}{2}\pi)d_{d'd}{}^{s_d}(\tfrac{1}{2}\pi)\mathcal{T}_{b'd';a'c'},\quad a-c\neq b-d. \tag{36}$$

These constraints are known as equal-mass conspiracy relations, first obtained by studying $N\bar{N}$ scattering[2] and later investigated in detail by Toller and collaborators,[10] and Freedman and Wang.[6]

3. ANALYTICITY TROUBLES AND REGGE FAMILIES

Let us examine the Regge representation (26) more closely. All the s dependence of the amplitude $f_{bd;ac}{}^{\pm}(t,z)$ is contained in the functions $E_{\lambda\lambda'}{}^{\alpha\pm}(-z)$, where

$$z=\cos\Theta_t=\frac{t(s-u)+(m_a^2-m_c^2)(m_b^2-m_d^2)}{\Delta(t;a,c)\Delta(t;b,d)}. \tag{37}$$

We note that for zero s-channel momentum transfer t the energy dependence is lost unless both $m_a=m_c$ and $m_b=m_d$. When two masses are equal,[18] i.e., $m_a=m_c$ or $m_b=m_d$, the cosine of the scattering angle tends to zero with t, and when all masses are unequal, it tends to -1.

[18] If only two masses are equal $m_a=m_c$ or $m_b=m_d$, the point $t=0$ lies outside the s-channel physical region.

Moreover, the functions $E_{\lambda\lambda'}{}^{\alpha\pm}(z)$ have logarithmic branch points when $z=-1$, provided $\alpha-\lambda'$ is not an integer.[17] The Regge representation then conflicts with the usually assumed analyticity properties of the S matrix. One way of recovering suitable behavior for large s is to use a modification of the prescriptions which Freedman and Wang proposed,[7] for the case of the scattering of spinless particles.

We introduce a family of poles with related residue functions, enabling us to regain Regge asymptotic behavior in s, and simultaneously eliminate spurious t singularities from the helicity amplitudes. Since we are concerned with the behavior of Regge terms near $t=0$, we must be careful to remove correctly the t singularities artificially introduced into the amplitudes $F^{J\pm}$ by the half-angle factors $\xi_{\lambda\mu}(z)$, Eq. (19). If we inadvertently introduce families of poles directly into the amplitudes $F_{bd;ac}{}^{J+}$ and $F_{bd;ac}{}^{J-}$, to produce uniform Regge behavior, we shall encounter difficulties when trying to obtain the correct asymptotic form of the t-singularity-free $f_{bd;ac}$ amplitudes. Consequently, we introduce daughters by considering expansions (28) in terms of the functions $D_{\lambda\lambda'}{}^{(\alpha\pm)\pm}(-z)$ in preference to $E_{\lambda\lambda'}{}^{(\alpha\pm)\pm}(-z)$, which we define in Appendix B. The expansion is so constructed that if we introduce families of poles into amplitudes F^{J+} and F^{J-}, with residues $\Gamma^{s\alpha\pm}$, to cancel out singularities in the coefficients of the half-angle factors $\xi_{\lambda\lambda'}{}^{\pm}(z)$ *independently*, the analyticity of the amplitudes $f_{bd;ac}$ at $t=0$ is guaranteed.

A property of the functions $D_{\lambda\lambda'}{}^{(\alpha\pm)\pm}(z)$ which we use is the differing "odd" or "even" asymptotic behavior in z (B13). With the new half-angle factor combinations,

$$\xi_{\lambda\lambda'}{}^{(\pm)}(z)=\tfrac{1}{2}[\xi_{\lambda\lambda'}(z)(\pm)e^{\pm i\pi\mu_m}\xi_{\lambda\lambda'}(z)],\quad \operatorname{Im}z\gtrless 0,\quad (38)$$

where $\mu_m=\frac{1}{2}(|\lambda+\lambda'|-|\lambda-\lambda'|)$, we may express the functions $E_{\lambda\lambda'}{}^{\pm}(z)$ in the form

$$E_{\lambda\lambda'}{}^{(\alpha)\pm}(z)=\xi_{\lambda\lambda'}{}^{\pm}(z)D_{\lambda\lambda'}{}^{(\alpha)+}(z)+\xi_{\lambda\lambda'}{}^{\mp}(z)D_{\lambda\lambda'}{}^{(\alpha)-}(z).\quad (39)$$

Let us now consider the Regge expansion (28) incorporating signature $\mathbf{s}$ (24) with $\operatorname{Re}(-z)$ large and positive and $\operatorname{Im}(-z)>0$. We use expression (B13) for the asymptotic form of the functions $D_{\lambda\lambda'}{}^{\alpha\pm}(z)$, and define a reduced generalized residue (25)

$$\bar{\Gamma}_{bd;ac}{}^{s\alpha}=(pp')^{-\alpha}\Gamma_{bd;ac}{}^{s\alpha}.\quad (40)$$

Substitution into Eq. (28) gives

$$f_{bd;ac}{}^{\pm}\approx\sum_{\alpha}\bar{\Gamma}_{bd;ac}{}^{s\alpha\pm}(t)\{\xi_{\lambda\lambda'}{}^{+}(-z)h_{\lambda\lambda'}{}^{(\alpha\pm)+}[x^{\alpha\pm}+O(x^{(\alpha\pm)-2})]+pp'\xi_{\lambda\lambda'}{}^{-}(-z)h_{\lambda\lambda'}{}^{(\alpha\pm)-}[x^{(\alpha\pm)-1}+O(x^{(\alpha\pm)-3})]\}$$
$$-\bar{\Gamma}_{bd;ac}{}^{s\alpha\mp}(t)\{\xi_{\lambda\lambda'}{}^{-}(-z)h_{\lambda\lambda'}{}^{(\alpha\mp)+}[x^{\alpha\mp}+O(x^{(\alpha\mp)-2})]+pp'\xi_{\lambda\lambda'}{}^{+}(-z)h_{\lambda\lambda'}{}^{(\alpha\mp)-}[x^{(\alpha\mp)-1}+O(x^{(\alpha\mp)-3})]\},\quad (41)$$

where $x=pp'z$ and the functions $h_{\lambda\lambda'}{}^{(\alpha\pm)\pm}$ are asymptotic constants. We now consider in particular the contribution of a Regge pole in amplitude F^{J+} to the helicity amplitudes $f_{bd;ac}{}^{\pm}$, in the case where s is large and t is small but finite,

$$\begin{aligned} f_{bd;ac}{}^{+}&\approx\bar{\Gamma}^{s\alpha+}\{\xi^{+}[(\tfrac{1}{2}s)^{\alpha}-c_1pp'(\tfrac{1}{2}s)^{\alpha-1}]\\ &\qquad+c_2\xi^{-}[pp'(\tfrac{1}{2}s)^{\alpha-1}]\},\\ f_{bd;ac}{}^{-}&\approx-\bar{\Gamma}^{s\alpha+}\{\xi^{-}[(\tfrac{1}{2}s)^{\alpha}-c_1pp'(\tfrac{1}{2}s)^{\alpha-1}]\\ &\qquad+c_2\xi^{+}[pp'(\tfrac{1}{2}s)^{\alpha-1}]\},\end{aligned}\quad (42)$$

where c_1 and c_2 are constants. In the asymptotic region where s is large and t is finite the functions $\xi^{\pm}(z)$ behave like (B1):

$$\xi^{+}(z)\approx(st/\mathfrak{M})^{-\lambda_m},\quad \xi^{-}(z)\approx(st/\mathfrak{M})^{-\lambda_m-1},\quad (43)$$

$$\mathfrak{M}=\tfrac{1}{4}(m_a{}^2-m_c{}^2)(m_d{}^2-m_b{}^2),\quad (44)$$

and we see at once that this gives rise to the usual Regge behavior,[4]

$$f_{bd;ac}{}^{+}\approx s^{\alpha-\lambda_m},\quad f_{bd;ac}{}^{-}\approx s^{\alpha-\lambda_m-1}.\quad (45)$$

If we let t become zero, we have singular terms in expression (42). We are interested in introducing additional poles with singular residues into F^{J+} and F^{J-} in order to obtain an expression in which the coefficients of $\xi^{+}(z)$ and $\xi^{-}(z)$ are manifestly analytic at $t=0$. In order to remove the second term in the coefficient of ξ^{+}, from the amplitude $f_{bd;ac}{}^{+}$, we propose to add a pole, with opposite signature, at the point $(\alpha-1)$ in amplitude $F_{bd;ac}{}^{J+}$, such that the generalized reduced residue $\bar{\Gamma}^{-s(\alpha-1)+}(t)$ satisfies the condition

$$\bar{\Gamma}^{-s(\alpha-1)+}(t)\to c_1pp'\bar{\Gamma}^{s\alpha+}(t)\quad\text{as}\quad t\to 0.\quad (46)$$

This is a *daughter* pole of the type introduced by Freedman and Wang. We note that in the nonzero-spin case there is an additional singularity in $f_{bd;ac}{}^{-}$ with coefficient $\xi_{\lambda\lambda'}{}^{+}(-z)$. This can only be removed by the introduction of a pole, with the opposite parity and the same signature as the first daughter, and residue $\bar{\Gamma}^{-s(\alpha-1)-}$, at the point $(\alpha-1)$ in $F_{bd;ac}{}^{J-}$, such that

$$\bar{\Gamma}^{-s(\alpha-1)-}(t)\to c_2pp'\bar{\Gamma}^{s\alpha+}(t)\quad\text{as}\quad t\to 0.\quad (47)$$

Note that the effect of the introduction of the first parity-doublet daughter poles is to remove the leading pp' singularities from *both* $f_{bd;ac}{}^{+}$ *and* $f_{bd;ac}{}^{-}$. In general, we shall need an infinite sequence of integrally spaced parity doublets to remove all the poles at $t=0$ from the expansion (42). We refer to this process of cancellation as a Regge-pole conspiracy. In the equal-mass case, $m_a=m_c$, $m_b=m_d$, the momentum product pp' is not singular, and we have no reason to introduce an infinite number of daughter poles at $t=0$ to restore analyticity. On the other hand, it seems unreasonable to have a Regge *spectrum* that changes abruptly when one considers a slight variation in the mass of an external particle. As we shall see later, it is possible to obtain a daughter representation which automatically satisfies the conspiracy conditions (36) in the equal-mass limit.

In the previous discussion, we have tacitly assumed that the Regge residue of the parent pole remains finite at $t=0$. We now define a k-evasive solution to the analyticity problem to be one in which the parent residue $\bar{\Gamma}^{s\alpha}(t)$ (40) has the property $\bar{\Gamma}^{s\alpha}(t)\approx t^k$ as $t\to 0$. In this case, we have many ways of introducing daughter doublet trajectories. These lead to different asymptotic behaviors of the helicity amplitudes. To see this, let us write $\bar{\Gamma}^{s\alpha}$ in the form $\bar{\Gamma}^{s\alpha}=t^r t^{k-r}\gamma^{s\alpha}(t)$, where $\gamma^{s\alpha}(0)$ is finite. We may then consider the parent contribution to the helicity amplitude $f_{bd;ac}{}^+$,

$$f^+\approx t^r\gamma^{s\alpha}(t)[\xi^+(a_0t^{k-r}s^{\alpha}+\cdots+a_{r-k}s^{\alpha-k+r}+a_{r-k+1}t^{-1}s^{\alpha-k-r-1}+\cdots)+\xi^-(t^{k-r-1}s^{\alpha-1}+\cdots)] \quad (48)$$

and introduce daughter-doublet poles to remove terms which are singular at $t=0$. The leading asymptotic behavior for large s, in the limit as t tends to zero, is then given by

$$f_{bd;ac}{}^+\approx\xi^+t^rs^{\alpha-k+r}. \quad (49)$$

We refer to such a solution of the daughter problem as a *k-evasive solution of order r*. It is to be noted that a k-evasive solution of order zero may be referred to as a k-evasive conspiracy and an ordinary conspiracy is a 0-evasive solution of order zero.

We conclude this section with some remarks concerning the 0-evasive asymptotic and threshold behavior of the various helicity amplitudes. We rewrite Eq. (41) in the form

$$f_{bd;ac}{}^\pm\approx\bar{\Gamma}^\pm[\xi_{\lambda\lambda'}{}^+(-z)s^{\alpha\pm}+pp'\xi_{\lambda\lambda'}{}^-(-z)s^{(\alpha\pm)-1}]-\bar{\Gamma}^\mp[\xi_{\lambda\lambda'}{}^-(-z)s^{\alpha\mp}+pp'\xi_{\lambda\lambda'}{}^+(-z)s^{(\alpha\mp)-1}]. \quad (50)$$

The helicity amplitude $f_{bd;ac}$ (18) is then found to have the asymptotic form (B5):

$$f_{bd;ac}\approx\Gamma^+(s^{\alpha+}+pp's^{(\alpha+)-1})+\Gamma^-(s^{\alpha-}+pp's^{(\alpha-)-1}). \quad (51)$$

As is well known,[1] the functions $\xi_{\lambda\lambda'}{}^\pm(-z)$ behave like $|1+z|^{-\frac{1}{2}(|\lambda|+|\lambda'|)}$ as $z\to -1$ and the final asymptotic expressions are[19]

$$f_{bd;ac}{}^\pm(t,z)\approx(st)^{-\frac{1}{2}(|\lambda|+|\lambda'|)}s^{\alpha_m}, \quad (52)$$

$$f_{bd;ac}(t,z)\approx s^{\alpha_m}, \quad (53)$$

for large s and small t, where $\alpha_m=\max(\alpha_+,\alpha_-)$. For future reference, we recall the threshold behavior of the parity conserving amplitudes (20) and note that near the normal and pseudothresholds the amplitude $f_{bd;ac}$ has the following behavior[19] with parity doubling:

$$f_{bd;ac}\approx[\Delta(t;a,c)]^{-(s_a+s_c)}[\Delta(t;b,d)]^{-(s_b+s_d)}. \quad (54)$$

4. LORENTZ SYMMETRY APPROACH AT $t=0$

In the previous section, we obtained a solution to the problem of obtaining Regge asymptotic behavior and the usual S-matrix analyticity, in the case of the scattering of unequal-mass particles with spin. It has been shown that, by using the theory of group contractions, one may obtain solutions to the daughter problem if the Regge-pole *spectrum* has Lorentz symmetry at $t=0$, even in the case of unequal-mass scattering.[8] The scattering amplitude does not possess this symmetry, however, since the homogeneous Lorentz group is not in general a little group of the Poincaré group. Such a restriction, on the Regge-pole spectrum, has also been suggested by considerations of properties of the complex homogeneous Lorentz group.[20]

We now propose to construct[9] a representation for the asymptotic S matrix which automatically satisfies, not only the daughter-doublet conditions at $t=0$, but also the kinematic constraints (35) and (36) discussed in Sec. 2. Our representations will differ in detail from those presented before[8,13] and depend upon a daughter-doublet addition formula which we derive in Appendix D:

$$\sum_{\kappa=0}^{\infty}d_{s\sigma-\kappa m}{}^{j_0\sigma+1}(\delta)d_{s'\sigma-\kappa m'}{}^{j_0\sigma+1}(\delta')D_{mm'}{}^{\sigma-\kappa}(\theta)=\sum_{\mu}d_{m\mu}{}^{s}(\psi)d_{\mu m'}{}^{s}(\psi')D_{ss'\mu}{}^{j_0\sigma+1}(\gamma), \quad (55)$$

where

$$\sin\psi=\frac{\sinh\delta'\sin\theta}{\sinh\gamma},\quad \cos\psi=\frac{\cosh\delta\sinh\delta'\cos\theta+\sinh\delta\cosh\delta'}{\sinh\gamma},$$
$$\sin\psi'=\frac{\sinh\delta\sin\theta}{\sinh\gamma},\quad \cos\psi'=\frac{\cosh\delta'\sinh\delta\cos\theta+\sinh\delta'\cosh\delta}{\sinh\gamma}, \quad (56)$$

with

$$\cosh\gamma=\cosh\delta'\cosh\delta+\sinh\delta'\sinh\delta\cos\theta. \quad (57)$$

The functions $d_{s\sigma-\kappa m}{}^{j_0\sigma+1}(\delta)$ and $D_{ss'\mu}{}^{j_0\sigma+1}(\gamma)$ are analytic continuations in σ of the discrete nonunitary representations of the group $SL(2C)$ discussed in Appendix C.

Let us now consider the Regge expansion in the form (27). The trajectories $\alpha(t)$ are normally independent

[19] These results appear to differ from those obtained by J. D. Jackson and G. E. Hite, Ref. 1, Appendix D.

[20] G. Cosenja, A. Sciarrino, and M. Toller, CERN report, Geneva, 1968 (unpublished).

but at $t=0$ they group together in such a way that to each parent trajectory, with $\alpha=\sigma$, there correspond daughter-doublet trajectories, with $\alpha=\sigma\text{-}\kappa$, where $\kappa=1,2,3,\cdots$. We shall refer to such a collection of poles in the unequal-mass case at $t=0$ as *Lorentz families*, which must not be confused with *Lorentz poles* which only have physical significance in the case of elastic scattering.

We are especially interested in the region near $t=0$, and shall assume our daughter and parent residues to be correlated in a particular way for infinitesimal values of t. Let us compare the Regge expansion (27) with the left-hand side of the addition formula (55). We identify α with $\sigma\text{-}\kappa$, m and m' with λ and $\pm\lambda'$ and θ with $\pi-\Theta_t$. Our object is to find what choice of boost angles $\delta(t)$ and $\delta'(t)$ will lead to functions $\gamma(s,t)$, $\psi(s,t)$, and $\psi'(s,t)$ such that for large s at $t=0$ the functions $d_{m\mu}{}^s(\psi)$ and $d_{\mu m'}{}^s(\psi)$ are bounded and $\cosh\gamma(0,s)\approx s$. If this be the case the property [(C11) and (C12)]

$$D_{ss'\mu}{}^{j_0\sigma+1}(\gamma)\approx(e^\gamma)^{\sigma-|j_0\pm\mu|}\quad\text{as}\quad|\gamma|\to\infty \tag{58}$$

leads to "good" asymptotic power behavior.

Consider two four-vectors

$$A=e^{-iK_3\delta}(1,0,0,0)=(\cosh\delta,\,0,\,0,\,\sinh\delta)$$

and

$$B=e^{-iJ_2\Theta}e^{-iK_3\delta'}(1,0,0,0)$$
$$=(\cosh\delta',\,\sinh\delta',\,\sin\Theta,\,0,\,\sinh\delta'\cos\Theta). \tag{59}$$

The angle γ corresponding to the boost $e^{-iK_3\gamma}$ of the vector B in the Lorentz frame where A is at rest, i.e., $A=0$, is given by

$$A_\mu B_\mu=\cosh\gamma$$
$$=\cosh\delta\cosh\delta'+\sinh\delta\sinh\delta'\cos(\pi-\Theta). \tag{60}$$

We see that, provided we take the parameters δ and δ' to be the boost angles of four-vectors q and q' with

$$\frac{\mathbf{q}\cdot\mathbf{q}'}{|\mathbf{q}||\mathbf{q}'|}=\cos(\pi-\theta)=\cos\Theta, \tag{61}$$

the boost angle γ is obtained by taking the scalar product

$$\cos\gamma=\frac{q_\mu q_\mu'}{|q_\mu||q_\mu'|}. \tag{62}$$

By looking at the t-channel four-momenta in the standard c.m. frame (10), we see that the most general candidates for the vectors q and q' are

$$q=p_a+\zeta p_c$$

and

$$q'=p_b+\zeta' p_d, \tag{63}$$

where ζ and ζ' are free parameters. The boost angle γ is now given by

$$\cosh\gamma=(1/4t|q||q'|)\{[t(1+\zeta)+(m_a{}^2-m_c{}^2)(1-\zeta)]$$
$$\times[t(1+\zeta')+(m_b{}^2-m_d{}^2)(1-\zeta')]-(1-\zeta)(1-\zeta')$$
$$\times[t(s-u)+(m_a{}^2-m_c{}^2)(m_b{}^2-m_d{}^2)]\} \tag{64}$$

with

$$|q|=\zeta t+(1-\zeta)(m_a{}^2-\zeta m_c{}^2)$$

and

$$|q'|=\zeta' t+(1-\zeta')(m_b{}^2-\zeta' m_d{}^2). \tag{65}$$

In order to have Regge behavior at $t=0$, the constant term in the square brackets must vanish. Indeed, we see that it does so automatically, for all values of ζ and ζ'.

In the Bethe-Salpeter approaches of Domokos[21] and Freedman and Wang,[7] and in the S-matrix approach of Domokos and Tindle[8] and Toller *et al.*,[20] the daughter problem has been considered with $\zeta=\zeta'=-1$ when q and q' coincide with the initial and final t-channel four-momentum transfers p_a-p_c and $p_b\text{-}p_d$, respectively. In this special case the expression for $\cosh\gamma$ takes a particularly simple form,

$$\cosh\gamma=\frac{-(s-u)}{[t-2(m_a{}^2+m_c{}^2)]^{1/2}[t-2(m_b{}^2+m_d{}^2)]^{1/2}}. \tag{66}$$

Let us now examine the expression for the hyperbolic sine of γ at $t=0$,

$$\sinh\gamma=\left[\frac{(2s-\Sigma m^2)^2}{4(m_a{}^2+m_c{}^2)(m_b{}^2+m_d{}^2)}-1\right]^{1/2}. \tag{67}$$

This function has spurious square-root branch points in s which have no simple physical interpretation. Moreover, on crossing from the t channel, where we define our Regge expansion, to the s channel, these may contribute factors which interfere with the asymptotic phases of the function $D_{ss'\mu}{}^{j_0\sigma+1}(\gamma)$. This problem does not arise in the equal-mass case, when $m_a=m_c$ and $m_b=m_d$, since for arbitrary ζ and ζ' the function $\sinh\zeta$ reduces to the s-channel threshold form $\Delta(s;a,b)/2m_am_b$. The spurious kinematic branch points in s in $\sinh\gamma$ are likely to occur as long as the momenta $|q|$ and $|q'|$ (65) contain spurious kinematic branch points in t. We may, however, obtain expressions with acceptable kinematic singularities, i.e., singularities at threshold, pseudo-threshold, or on the physical boundary, if, and only if, we have $\zeta=\zeta'=0$ or $\approx\infty$. This limits our choice of boost angles δ and δ' to those associated with the individual particle four-momenta:

$$\cosh\delta_a=\frac{t+m_a{}^2-m_c{}^2}{2t^{1/2}m_a},\quad\sinh\delta_a=\frac{\Delta(t;\,a,c)}{2t^{1/2}m_a},$$
$$\cosh\delta_b=\frac{t+m_b{}^2-m_d{}^2}{2t^{1/2}m_b},\quad\sinh\delta_b=\frac{\Delta(t;\,b,d)}{2t^{1/2}m_b},\ \text{etc.} \tag{68}$$

[21] G. Domokos, Phys. Rev. **159**, 1387 (1967).

In this case, we obtain simple expressions for γ,

$$\cosh\gamma_{ab}{}^{s}=(m_a{}^2+m_b{}^2-s)/2m_am_b\,,$$

and (69)

$$\sinh\gamma_{ab}{}^{s}=\Delta(s;a,b)/2m_am_b\,.$$

We note that $\cosh\gamma_{ab}{}^{s}$ is independent of t and has the required (58) behavior for large s. This analysis suggests that the most general form for a generalized residue function ${}^{\kappa}\Gamma_{bd;ac}{}^{\sigma}(t)$ (25), corresponding to a parent Regge pole ($\kappa=0$) or daughter pole at $J=\sigma-\kappa$ in the amplitude $F_{-b-d;ac}{}^{J}(t)$, which "conspires" at $t=0$ is

$$^{\kappa}\Gamma_{bd;ac}{}^{\sigma}={}^{\kappa}G_{bd;ac}{}^{j_0\sigma}(t)\;{}^{\kappa}\boldsymbol{\Delta}_{bd;ac}{}^{j_0\sigma}(t)\,, \tag{70}$$

where

$$\begin{aligned}{}^{\kappa}\boldsymbol{\Delta}_{bd;ac}{}^{j_0\sigma}(t) &=\{C_{a-c\lambda}{}^{s_as_cs}[d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(\delta_a)+\epsilon_{ac}d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(-\delta_c)]\}\\ &\quad\times\{C_{b-d\lambda'}{}^{s_bs_ds'}[d_{s'\sigma-\kappa\lambda'}{}^{j_0\sigma+1}(\delta_b)\\ &\qquad+\epsilon_{bd}d_{s'\sigma-\kappa\lambda'}{}^{j_0\sigma+1}(-\delta_d)]\}+j_0\leftrightarrow -j_0\,, \end{aligned}\tag{71}$$

and $\epsilon_{ac},\epsilon_{bd}$ are free parameters. We have introduced the Clebsch-Gordan coefficients $C_{ab\lambda}{}^{s_as_bs}$ so that, as we shall see later, one has consistency with the equal-mass conspiracy relations (36). This leads to a natural interpretation of s and s' as the channel spins. The parameter j_0 is yet to be identified although it is restricted by the inequality $|j_0|\leq\min(s,s')$ (C4).

The function ${}^{\kappa}G_{bd;ac}{}^{j_0\sigma}(t)$ which becomes independent of the daughter-son parameter κ at $t=0$ shall be referred to as a *dynamical residue*. Similarly, for reasons which will become apparent later, we refer to the function ${}^{\kappa}\boldsymbol{\Delta}_{bd;ac}{}^{j_0\sigma}$ as the *kinematic residue*. For simplicity, unless otherwise stated, we shall take ${}^{\kappa}\boldsymbol{\Delta}_{bd;ac}{}^{j_0\sigma}$ to be of the form

$$^{\kappa}\Delta_{bd;ac}{}^{j_0\sigma}={}^{\kappa}\Delta_{bd}{}^{j_0\sigma}\;{}^{\kappa}\Delta_{ac}{}^{j_0\sigma}\,, \tag{72}$$

where

$$\begin{aligned}{}^{\kappa}\Delta_{bd}{}^{j_0\sigma}&=C_{b-d\lambda'}{}^{s_bs_ds'}d_{s'\sigma-\kappa\lambda'}{}^{j_0\sigma+1}(\delta_b)\,,\\ {}^{\kappa}\Delta_{ac}{}^{j_0\sigma}&=C_{a-c\lambda}{}^{s_as_cs}d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(\delta_a)\,,\end{aligned}\tag{73}$$

and we shall omit indices when it is not likely to lead to confusion. In this case, the rotation angles corresponding to ψ and ψ' in the addition formula (55) are given by (56):

$$\begin{aligned}\cos\psi&=-\frac{(s+m_a{}^2-m_b{}^2)(t+m_a{}^2-m_c{}^2)-2m_a{}^2\Delta_{cb}{}^{ad}}{\Delta(s;a,b)\Delta(t;a,c)}\,,\\ \sin\psi&=\frac{2m_a\phi}{\Delta(s;a,b)\Delta(t;a,c)}\,,\\ \cos\psi'&=-\frac{(s+m_b{}^2-m_a{}^2)(t+m_b{}^2-m_d{}^2)+2m_b{}^2\Delta_{cb}{}^{ad}}{\Delta(s;a,b)\Delta(t;b,d)}\,,\\ \sin\psi'&=\frac{2m_b\phi}{\Delta(s;a,b)\Delta(t;b,d)}\,,\end{aligned}\tag{74}$$

and comparison with the expressions (12) for the crossing angles indicates

$$\psi=-\chi_a{}^{st}\,,\quad \psi'=\pi+\chi_b{}^{st}\,. \tag{75}$$

These angles have the required property that the functions $d_{\lambda\mu}{}^{s}(\psi)$ and $d_{\mu\lambda'}{}^{s'}(\psi')$ remain bounded in the s-channel physical region.

We have found that a parent Regge pole with $\alpha=\sigma(t)$ and daughters with kinematic residues ${}^{\kappa}\Delta_{bd}{}^{j_0\sigma}(t)\;{}^{\kappa}\Delta_{ac}{}^{j_0\sigma}(t)$ may conspire near $t=0$, to give a contribution to t-channel helicity amplitudes of the form

$$\begin{aligned}f_{-b-d;ac}(t,z)&\approx\Sigma_{bd;ac}{}^{j_0\sigma}(s,t)\\ &=\sum_{\kappa}{}^{\kappa}\Delta_{bd}{}^{j_0\sigma}(t)\;{}^{\kappa}\Delta_{ac}{}^{j_0\sigma}(t)D_{\lambda\lambda'}{}^{\sigma-\kappa}(\pi-\Theta_t) &(76)\\ &=\sum_{\mu}d_{\lambda\mu}{}^{s}(-\chi_a{}^{ts})d_{\mu\lambda'}{}^{s'}(\pi-\chi_b{}^{ts})D_{ss'\mu}{}^{j_0\sigma+1}(\gamma_{ab}{}^{s})\,. &(77)\end{aligned}$$

It is to be noted that if we replace the crossing angles (12) in the expression (77) by those obtained in the limit $s\to\infty$,

$$\cos\chi_a\to-\frac{t+m_a{}^2-m_c{}^2}{\Delta(t;a,c)}\,,\quad \sin\chi_a\to\frac{-2m_a(-t)^{1/2}}{\Delta(t;a,c)}\,,$$

we obtain the representation suggested by Klein[13] for the behavior of helicity amplitudes near $t=0$. He was, however, unable to demonstrate the correspondence with a Regge-pole daughter sum in the unequal-mass case. For finite values of s, the major qualitative difference between these representations is that our t-channel helicity-flip amplitudes only vanish ($\sin\chi_r=0$) on the s-channel (or t- or u-channel) physical boundary, whereas those of Klein only vanish at $t=0$ for all s.

Since our expression for $\Sigma_{bd;ac}{}^{j_0\sigma}(s,t)$, Eq. (77), varies smoothly as a function of the external mass ratios, provided we stay in the s-channel physical region, it should be possible to satisfy both the kinematic constraints (35) and (36) at the boundary of the physical region. Indeed, we find that, provided the dynamical residue $G_{bd;ac}{}^{j_0\sigma}(t)$ (70) be *helicity-independent*,[22] the kinematic constraints (35) and (36) are satisfied automatically. When we consider the equal-mass limit, we find $\cosh\delta\to0$ and $\delta\to i\pi/2$. The expression for the helicity amplitudes after continuation to the bound-state region is then of the form

$$\begin{aligned}f_{-b-d;ac}\approx\sum_{\sigma,\kappa}G^{\sigma}(t)C_{a-c\lambda}{}^{s_as_cs}C_{b-d\lambda'}{}^{s_bs_ds'}d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(i\pi/2)\\ \times d_{s'\sigma-\kappa\lambda'}{}^{j_0\sigma+1}(i\pi/2)d_{\lambda\lambda'}{}^{\sigma-\kappa}(\pi-\Theta_t)\,,\end{aligned}\tag{78}$$

which is consistent with that obtained by Freedman and Wang.[6] In other words, the Clebsch-Gordan-coefficient helicity dependence is such that our *Lorentz*

[22] Helicity independent up to a phase $(-1)^{\lambda,\lambda'}$ which can be determined by comparison with a Lorentz-pole expansion in the equal-mass case. We shall in general neglect over-all spin-dependent phase factors.

families may be described as *Lorentz poles* in the case of elastic scattering. We now see the reason for introducing the Clebsch-Gordan coefficients into the expression for $\mathbf{\Delta}_{bd;ac}$ (71). We may identify j_0 with the equal-mass Lorentz-pole classification parameter M, and observe that $\sigma=n$ corresponds to the four-dimensional angular momentum.

This indicates that our solution to the unequal-mass problem is consistent with the equal-mass conspiracy theory developed by Domokos,[21] Freedman and Wang,[6] and Toller.[10]

Let us now consider properties of the kinematic residue functions $\Delta_{bd}\Delta_{ac}$ under spatial inversion. The constraint (34) suggests we construct parity-conserving functions

$$\Delta_{bd;ac}=\Delta_{bd}\Delta_{ac}+\eta_{bd}\eta_{ac}\Delta_{-b-d}\Delta_{-a-c}\,, \tag{79}$$

such that

$$\Delta_{bd;ac}=\eta_b\eta_d\Delta_{-b-d;-a-c}. \tag{80}$$

Factorization of the parity-conserving residues $\beta_{bd;ac}{}^{s\alpha\pm}(t)$ (25) suggests that we should express them in the form $\beta_{bd}{}^{\alpha\pm}\beta_{ac}{}^{\alpha\pm}$. Now, from Eq. (21), we see that

$$^{\kappa}\Gamma_{bd;ac}{}^{\sigma\pm}={}^{\kappa}G_1{}^{\sigma}(t)\;{}^{\kappa}\Delta_{bd;ac}{}^{\sigma}\pm\eta_{bd}{}^{\kappa}G_2{}^{\sigma}(t)\;{}^{\kappa}\Delta_{-b-d;ac}{}^{\sigma}\,, \tag{81}$$

and it only factorizes and satisfies conditions (21) and (34) if $G_1(t)=G_2(t)=G(t)$. Since the functions $\bar{\Delta}_{bd;ac}$ and $\bar{\Delta}_{-b-d;ac}$ (73) are not linearly dependent in general, we are obliged to introduce parity-doublet Regge poles with $\sigma^+=\sigma^-=\sigma$ into amplitudes F^{J+} and F^{J-} with correlated residues at $t=0$. The factorized residues of such parity doublets are of the form

$$\pm\Gamma_{bd;ac}{}^{\pm}=G(t)(\Delta_{bd}\pm\eta_{bd}\Delta_{-b-d})(\Delta_{ac}\pm\eta_{ac}\Delta_{-a-c}) \tag{82}$$

$$\equiv G(t)\Delta_{bd}{}^{\pm}\Delta_{ac}{}^{\pm}\equiv G(t)\Delta_{bd;ac}{}^{\pm}. \tag{83}$$

In the special case where $j_0=0$ we find $d_{sj\mu}{}^{0\sigma+1}(\beta)=d_{sj-\mu}{}^{0\sigma+1}(\beta)$ (C22) and from the defining equations [(73) and (83)] we have

$$\Delta_{bd;ac}{}^{0\sigma\pm}=\Delta_{bd}{}^{0\sigma}\Delta_{ac}{}^{0\sigma}[1\pm\eta_b\eta_d(-1)^{s'-\nu}]\times[1\pm\eta_a\eta_c(-1)^{s-\nu}]. \tag{84}$$

This means that for a particular choice of spins s and s' the contribution to the helicity amplitudes comes either from a pole in F^{J+} or a pole in F^{J-}.

If we denote the positions of parity-doublet Regge poles by α, and parent doublet positions by σ, we have, near $t=0$, the helicity amplitude expansion

$$f_{bd;ac}{}^{\pm}\approx\sum_{\alpha}\pm G^{\alpha}[\mathbf{\Delta}_{bd;ac}{}^{j_0\alpha\pm}E_{\lambda\lambda'}{}^{\alpha+}(-z)+\mathbf{\Delta}_{bd;ac}{}^{j_0\alpha\mp}E_{\lambda\lambda'}{}^{\alpha-}(-z)] \tag{85}$$

$$=\sum_{\sigma}\pm G^{\sigma}[\mathbf{\Sigma}_{bd;ac}{}^{j_0\sigma}\xi_{\lambda\lambda'}(-z)\pm(-1)^{\lambda+\lambda_m}\times\eta_{bd}\mathbf{\Sigma}_{-b-d;ac}{}^{j_0\sigma}\xi_{\lambda-\lambda'}(-z)], \tag{86}$$

where the general residues ${}^{\kappa}\mathbf{\Delta}_{bd;ac}{}^{j_0\sigma\pm}(t)$ and family sums $\mathbf{\Sigma}_{bd;ac}{}^{j_0\sigma}(t,z)$ are given by

$$\mathbf{\Delta}_{bd;ac}{}^{\pm}=(\mathbf{\Delta}_{bd}\pm\eta_{bd}\mathbf{\Delta}_{-b-d})(\mathbf{\Delta}_{ac}\pm\eta_{ac}\mathbf{\Delta}_{-a-c}) \tag{87}$$

and

$${}^{\kappa}\mathbf{\Delta}_{ac}{}^{j_0\sigma}(t)=C_{a-c\lambda}{}^{s_as_cs}\times[d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(\delta_a)+\epsilon_{ac}d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(-\delta_c)] \tag{88}$$

with $\alpha=\sigma-\kappa$, and

$$\mathbf{\Sigma}_{bd;ac}=\Sigma_{bd;ac}{}^{*}+\eta_{bd}\eta_{ac}\Sigma_{-b-d;-a-c}{}^{*}\,, \tag{89}$$

where

$$\begin{aligned}\Sigma_{bd;ac}{}^{*}=C_{a-c\lambda}{}^{s_as_cs}C_{b-d\lambda'}{}^{s_bs_ds'}&\\ \times[d_{\lambda\mu}{}^{s}(-\chi_a{}^{st})d_{\mu\lambda'}{}^{s'}(\pi+\chi_b{}^{st})D_{ss'\mu}{}^{j_0\sigma+1}(\gamma_{ab}{}^{s})&\\ +\epsilon_{ac}\epsilon_{bd}d_{\lambda\mu}{}^{s}(-\chi_c{}^{st})d_{\mu\lambda'}{}^{s'}(\pi+\chi_d{}^{st})D_{ss'\mu}{}^{j_0\sigma+1}(\gamma_{cd}{}^{s})&\\ +\epsilon_{ac}d_{\lambda\mu}{}^{s}(\chi_c{}^{ut})d_{\mu\lambda'}{}^{s'}(\pi-\chi_b{}^{ut})D_{ss'\mu}{}^{j_0\sigma+1}(\gamma_{bc}{}^{u})&\\ +\epsilon_{bd}d_{\lambda\mu}{}^{s}(\chi_a{}^{ut})d_{\mu\lambda'}{}^{s'}(\pi-\chi_d{}^{ut})D_{ss'\mu}{}^{j_0\sigma+1}(\gamma_{ad}{}^{u})].&\end{aligned} \tag{90}$$

It is to be noted that a change in the sign of j_0 is, apart from phase factors, equivalent to changing the signs of helicities. In this way for each Regge family we have obtained a sum of terms of the form (76).

For convenience, we now list in Table I the corresponding angles appearing in the addition formula (55) where [(68) and (69)]

TABLE I. Angles in the addition formula.

δ	δ'	θ	ψ	ψ'	γ
δ_a	δ_b	$\pi-\Theta$	$-\chi_a{}^{st}$	$\pi+\chi_b{}^{st}$	$\gamma_{ab}{}^{s}$
δ_a	$-\delta_d$	$\pi-\Theta$	$\chi_a{}^{ut}$	$\pi-\chi_d{}^{ut}$	$\gamma_{ad}{}^{u}$
$-\delta_c$	δ_b	$\pi-\Theta$	$\chi_c{}^{ut}$	$\pi-\chi_b{}^{ut}$	$\gamma_{bc}{}^{u}$
$-\delta_c$	$-\delta_d$	$\pi-\Theta$	$-\chi_c{}^{st}$	$\pi+\chi_d{}^{st}$	$\gamma_{cd}{}^{s}$

$$\cosh\delta_c=\frac{t+m_c^2-m_a^2}{2t^{1/2}}\,,\qquad \sinh\delta_c=\frac{\Delta(t;\,a,c)}{2t^{1/2}}\,,$$

$$\cosh\delta_d=\frac{t+m_d^2-m_b^2}{2t^{1/2}}\,,\qquad \sinh\delta_c=\frac{\Delta(t;\,b,d)}{2t^{1/2}}\,, \tag{91}$$

and

$$\cosh\gamma_{ad}{}^{u}=\frac{m_a^2+m_d^2-u}{2m_am_d}\,,\quad \sinh\gamma_{ad}{}^{u}=\frac{\Delta(u;\,a,d)}{2m_am_d}. \tag{92}$$

For different angle combinations, one may use the symmetry relations of the functions $d_{sj\lambda}{}^{j_0\sigma}(\delta)$ and $d_{\lambda\lambda'}{}^{j}(\theta)$ discussed in Appendices B and C.

5. ANALYTICITY AND ASYMPTOTIC BEHAVIOR OF REGGE FAMILIES

In this section we shall examine the Regge-pole and Lorentz-family expansions (85) and (86) for small values of t, and compare them in the light of the daughter-doublet formalism developed in Sec. 3. We also present a discussion of the behavior of kinematic residues $\Delta_{bd;ac}{}^{j_0\sigma}(t)$ for arbitrary values of t. This may prove useful in developing models for Regge-residue functions.

Let us consider the equation

$$\sum_{\kappa} [(pp')^{-\sigma+\kappa} d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(\delta_a) d_{s'\sigma-\kappa\lambda'}{}^{j_0\sigma+1}(\delta_b)] \times (pp')^{\sigma-\kappa} D_{\lambda\lambda'}{}^{\sigma-\kappa}(\pi-\Theta_t) = \sum_{\mu} [d_{\lambda\mu}{}^{s}(-\chi_a) d_{\mu\lambda'}{}^{s'}(\pi+\chi_b)] D_{ss'\mu}{}^{j_0\sigma+1}(\gamma_{ab}{}^{s}), \quad (93)$$

where the parameters δ, Θ, χ, and γ are defined by Eqs. (12), (37), (68), and (69). We recall that $\sigma-\kappa$ denotes the position of a daughter pole with parent trajectory $\alpha=\sigma(t)$, and that the term in square brackets on the *left*-hand side is proportional to the reduced Regge residue ${}^{\kappa}\bar{\beta}_{bd;ac}{}^{\sigma}$ [Eqs. (29) and (97)]. We shall refer to the term on the *right*-hand side as a *Lorentz-family* sum.

In order to determine the behavior of our amplitudes for large positive s, we first require some properties of the group-representation functions derived in Appendices B and C:

$$D_{ss'\mu}{}^{j_0\sigma}(\gamma) \approx |e^{\pm\gamma}|^{\sigma-1-|j_0\mp\mu|} \quad \text{as} \quad |e^{\pm\gamma}| \to \infty, \quad (94)$$

$$(\sinh\delta)^{-\sigma+\kappa} d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(\delta) \approx |e^{\pm\delta}|^{-|j_0\mp\lambda|+\kappa} \quad \text{as} \quad |e^{\pm\delta}| \to \infty, \quad (95)$$

and

$$d_{\lambda\mu}{}^{s}(\chi) \approx (\sin\chi)^{|\lambda\pm\mu|} \quad \text{as} \quad \sin\chi\to 0,\ \cos\chi \lesseqgtr 0. \quad (96)$$

In order to simplify the discussion, we shall first consider a Regge family with kinematic residues of the form ${}^{\kappa}\Delta_{bd;ac}{}^{j_0\sigma}(t)$ (72), and denote the contributions to amplitudes by $f_{bd;ac}{}^{\pm(1)}$ etc. It is then easy to obtain the full contribution from poles with kinematic residues ${}^{\kappa}\mathbf{\Delta}_{bd;ac}{}^{j_0\sigma}$ (87).

The reduced Regge residue ${}^{\kappa}\bar{\beta}_{bd;ac}{}^{\sigma}$ (29) is given in terms of the kinematic residue by

$${}^{\kappa}\bar{\beta}_{-b-d;ac}{}^{\sigma}(t) = (-1)^{\kappa} \frac{\sin\pi\sigma}{2\sigma-2\kappa+1} p'(pp')^{-(\sigma-\kappa+\frac{1}{2})} \times {}^{\kappa}G^{j_0\sigma}(t) \quad {}^{\kappa}\Delta_{bd;ac}{}^{j_0\sigma}(t). \quad (97)$$

As t approaches zero, the hyperbolic sines and cosines of the boost angles δ_r (68) become singular, and Eq. (94) implies

$$\bar{\beta}_{bd;ac}{}^{(1)} \approx t^{\frac{1}{2}(|j_0+\lambda|+|j_0+\lambda'|-2\kappa)}. \quad (98)$$

Note that the sign of λ or λ' is the same as that of the hyperbolic tangent of the corresponding boost angle δ appearing in the addition formula (55), as we approach $t=0$. At $t=0$, the $\frac{1}{2}(|j_0+\lambda|+|j_0+\lambda'|)$-evasive parent residue[23] with $\kappa=0$, may generate an evasive Regge asymptotic behavior of the type discussed in Sec. 3 (49),

$$f_{bd;ac}{}^{(1)} \approx t^{r} s^{\sigma-\frac{1}{2}(|j_0+\lambda|+|j_0+\lambda'|)+r}. \quad (99)$$

[23] See note preceding Eq. (48).

At this point, we should like to discuss in detail the prescription for obtaining the behavior at $t=0$. The real analytic function ${}^{\kappa}\bar{\beta}_{-b-d;ac}{}^{\sigma}$ which multiplies $D_{\lambda\lambda'}{}^{\sigma}(-z)$ is not necessarily analytic at $t=0$. It must have a square-root branch point at $t=0$ for $|\lambda-\lambda'|$ an odd integer to cancel a singularity in the half-angle factor $\xi_{\lambda\lambda'}(-z)$ which is present in the function $D_{\lambda\lambda'}{}^{\sigma}(-z)$. The total contribution to the amplitude $f_{bd;ac}$ will then be *analytic* in t at $t=0$. The asymptotic form of the parent contribution is then given by expanding the function $D_{\lambda\lambda'}{}^{\sigma}(-z)$, *with the half-angle factors removed*, in a power series in (st) and then letting t become small,

$$f_{bd;ac}{}^{(1)} \approx [t^{\frac{1}{2}(|j_0+\lambda|+|j_0+\lambda'|)}] \times t^{-\sigma}[(st)^{\frac{1}{2}|\lambda+\lambda'|}(4\mathfrak{M}-st)^{\frac{1}{2}|\lambda-\lambda'|}] \times [(st)^{\sigma-\lambda_m} + a_1(st)^{\sigma-\lambda_m-1} + \cdots]$$
$$\approx t^{\frac{1}{2}(|j_0+\lambda|+|j_0+\lambda'|-|\lambda-\lambda'|)} \times [s^{\sigma-\frac{1}{2}|\lambda-\lambda'|} + a_1 t^{-1} s^{\sigma+\frac{1}{2}|\lambda-\lambda'|-1} + \cdots].$$

Now $|j_0+\lambda|+|j_0+\lambda'|-|\lambda-\lambda'|$ is an *even* integer so we may adopt the daughter cancellation procedure discussed in Sec. 2 (49) to obtain the asymptotic form for *zero t*:

$$f^{(1)} \approx s^{\sigma-\frac{1}{2}(|j_0+\lambda|+|j_0+\lambda'|)}.$$

However, for any finite t, no matter how small, for large enough s, we still find

$$f_{bd;ac}{}^{(1)} \approx s^{\sigma}. \quad (100)$$

Let us now consider the Lorentz-family sum at $t=0$. The crossing angles χ [Eq. (12)] then have the property $\sin\chi \approx s^{-1/2}$ for s large. If we use the asymptotic relations (96) and (94) in conjunction with Eq. (93), we see that the leading power behavior at $t=0$ is of the form (99), and the Lorentz symmetry solution is a $\frac{1}{2}(|j_0+\lambda| + |j_0+\lambda'|)$-evasive solution *of order zero*.[23] For finite t, the crossing angles do not tend to zero as s becomes large, and the power behavior is again given by expression (100).

We have shown that the four-dimensional symmetry solution with a particular value of j_0 gives rise to evasive residues $\beta_{\lambda\lambda'}$ at $t=0$ unless $j_0=-\lambda=-\lambda'$. Moreover, whether we consider the Lorentz-family expansion at $t=0$ or take the leading terms in a Regge-pole expansion for large s and small finite t and then let t become zero, we may obtain the same behavior for the helicity amplitudes $f_{bd;ac}{}^{(1)}$. After crossing to the s channel (11), the asymptotic behavior in s remains the same,

$$\mathcal{S}_{cd;ab}{}^{(1)} \approx s^{\sigma-\frac{1}{2}(|\lambda+j_0|+|\lambda'+j_0|)}, \quad t=0,$$
$$\approx s^{\sigma}, \quad t\neq 0.$$
$$(\lambda=a-c,\ \lambda'=b-d) \quad (101)$$

The change for infinitesimal values of t can be most easily followed by using a Lorentz-family expansion (86) in place of the usual Regge expansion (85). The asymptotic behavior of parity-conserving amplitudes $f_{bd;ac}{}^{\pm(1)}$ may be obtained directly from Eqs. (99) and (100) by using the definition (18) and properties of the half-angle factor functions $\xi_{\lambda\lambda'}(z)$ (B2). It should be noted that if at $t=0$ we have $m_a=m_c$ or $m_b=m_d$ or both, the asymptotic s behaviors of the amplitudes $\mathcal{S}_{cd;ab}^{(1)}$ are of the form $s^{\sigma-|\lambda'+j_0|}$, $s^{\sigma-|\lambda+j_0|}$ and s^{σ}, respectively.

We now use the most general form for the kinematic residue functions and obtain the following Regge asymptotic behavior for helicity amplitudes. We assume that they are dominated by the contributions of a single Lorentz-family parity doublet at $t=0$, with conspiracy parameter j_0 and kinematic factors ${}^{\kappa}\boldsymbol{\Delta}_{bd;ac}{}^{j_0\sigma\pm}(t)$ (87)

$$f_{bd;ac}(t,z)\approx s^{\sigma-\frac{1}{2}(|\lambda'+j_0|+|\lambda+j_0|)}+s^{\sigma-\frac{1}{2}(|\lambda-j_0|+|\lambda'-j_0|)}. \quad (102),$$

$$f_{bd;ac}{}^{\pm}\approx(st)^{-\frac{1}{2}(|\lambda|+|\lambda'|)}\left(s^{\sigma-\frac{1}{2}(|\lambda\pm j_0|+|\lambda'+j_0|)}+s^{\sigma-\frac{1}{2}(|\lambda\mp j_0|+|\lambda'-j_0|)}\right),$$
$$\text{according as}\quad |\lambda\pm\lambda'|=|\lambda|+|\lambda'|, \quad (103)$$

and

$$\mathcal{S}_{cd;ab}\approx s^{\sigma-\frac{1}{2}(|\lambda+j_0|+|\lambda'+j_0|)}+s^{\sigma-\frac{1}{2}(|\lambda-j_0|+|\lambda'-j_0|)}. \quad (104)$$

As noted by Jackson and Hite,[1] the singular behavior of the amplitudes $f^{\pm}$ at $t=0$ is a reflection of the most singular behavior of $\xi_{\lambda\lambda'}(z)$ and $\xi_{\lambda-\lambda'}(z)$ together (B2).

It is interesting to note that the form of $\gamma_{ab}{}^{s}$ suggests that the region of "pure" power behavior is $s\gg m_1{}^2+m_2{}^2$, where m_1 and m_2 are the masses of the two lightest particles.

We should now like to point out that, if $m_a=m_c$ and both particles (a) and (c) have either integral or half-integral spins when j_0 is an integer, alternate daughter residue functions ${}^{\kappa}\beta_{bd;ac}{}^{\sigma\pm}$ will vanish as we approach $t=0$. To see this, we use the relation (C19):

$$d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(\pm i\pi/2)=(-1)^{s+\kappa-\lambda+j_0}d_{s\sigma-\kappa-\lambda}{}^{j_0\sigma+1}(\pm i\pi/2). \quad (105)$$

The Regge residue functions $\beta_{bd;ac}{}^{\pm}$ contain a factor

$$\Delta_{ac}{}^{\pm}=C_{a-c\lambda}{}^{s_a s_c s}[d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(\beta_a)\pm(-1)^{s-\nu}\eta_a\eta_c d_{s\sigma-\kappa-\lambda}{}^{j_0\sigma+1}(\beta_a)] \quad (106)$$

and, in the equal-mass limit as $t\approx-|t|(1+i\epsilon)\to 0$, we find $\beta_a\approx i\pi/2$. Relation (106) then implies that

$$\Delta_{ac}{}^{\pm}=0 \quad\text{for}\quad \pm\eta_a\eta_c(-1)^{j_0+\kappa+\lambda}\neq 1, \quad (107)$$

and alternate daughter residues vanish. A similar result holds in the case of particles b and d. This phenomenon was observed by Domokos[21] in a Bethe-Salpeter model with zero-spin particles and by Freedman and Wang[6] in the case of $N\bar{N}$ scattering.

For nonzero values of t, we have no reason to believe that there is any connection between the Regge poles in a Lorentz family. In particular, there is no longer any need to associate a single value of the parameter j_0 with a given trajectory residue. We may still, however, take a model with residues $\bar{\beta}_{bd;ac}$ expressed as linear combinations of zero-t kinematic residues. If we suppress the sum over spins s,s', this could be of the form

$$\bar{\beta}_{bd;ac}{}^{\alpha\pm}\approx\sum_{j_0 j_0'}G^{j_0 j_0'\alpha\pm}(t)(pp')^{-\alpha\pm}\Delta_{bd}{}^{j_0\alpha\pm}\Delta_{ac}{}^{j_0'\alpha\pm}, \quad (108)$$

where $G^{j_0 j_0'\alpha\pm}$ becomes diagonal in j_0 when $t=0$. Let us now consider the analytic properties of the reduced kinematic residues,

$${}^{\kappa}\bar{\Delta}_{bd;ac}{}^{\sigma\pm}(t)=(pp')^{-\sigma+\kappa}\,{}^{\kappa}\Delta_{bd;ac}{}^{\sigma\pm}(t). \quad (109)$$

We show in Appendix C that the function $(\sinh\delta)^{-\sigma+\kappa}\times d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(\delta)$ (C14) may be represented as a terminating series of real analytic functions of t. Consequently, both the kinematic reduced residue ${}^{\kappa}\bar{\Delta}_{bd;ac}{}^{\sigma}(t)$, and the dynamical residue $G(t)$, are manifestly real analytic when κ is an integer.

In order to investigate the t-channel threshold behavior of the residue function ${}^{\kappa}\Delta_{bd}{}^{\sigma\pm}$, when t approaches $(m_b+m_d)^2$, we need the relation (C20):

$$(\sinh\delta)^{-\sigma+\kappa}d_{s\sigma-\kappa\pm\lambda}{}^{j_0\sigma+1}(\delta)\approx(\sinh\delta)^{-s'}\pm c(\sinh\delta)^{-s'+1}\quad\text{as}\quad e^{-\delta}\to\pm 1, \quad (110)$$

where c is a constant. As t approaches the normal threshold $(m_b+m_d)^2$, the boost angle δ tends to zero, and we find on substituting (110) into the expression for $\bar{\Delta}_{bd}{}^{\pm}(t)$ [(108) and (106)]:

$$\bar{\Delta}_{bd}{}^{\pm}(t)\approx[\Delta(t;b,d)]^{-s'}[1\pm\eta_b\eta_d(-1)^{s'-\nu}]+c[\Delta(t;b,d)]^{-s'+1}[1\mp\eta_b\eta_d(-1)^{s'-\nu}]. \quad (111)$$

In general, there will occur in the expression for the residues $\beta_{bd;ac}{}^{\pm}$ (108) a term with $s'=s_b+s_d$. The "most singular" threshold behavior then coincides with that described in Sec. 2[1] (30). It is interesting to note that, in the case of a $j_0=0$ conspiracy, although there is no parity doubling, the threshold behavior is still given by Eq. (111) because of the restriction (84).

We have shown that each Regge residue in a Lorentz family has the correct t-channel threshold behavior. If we now continue the expression for the Lorentz-family sum (86) into the positive-t region, we find that it also possesses analytic properties which generate the usual threshold behavior in the helicity amplitudes.

In order to see this we need expressions for the behavior of the functions $d_{\lambda\mu}{}^{s}(\chi)$ and $\xi(z)$ for large $|\cos\chi|$ and large $|z|$, derived in Appendix B:

$$d_{\lambda\mu}{}^{s}(\chi_a)\approx(z_a)^{s}e^{\pm i(\frac{1}{2}\pi)(\lambda-\mu)},\quad \mathrm{Im}(z_a)\gtrless 0 \quad (112)$$

where $z_a=\cos(\chi_a)$, and

$$\xi_{\lambda\lambda'}(z)\approx(z)^{-\lambda_m}e^{\pm i(\frac{1}{2}\pi)|\lambda-\lambda'|},\quad \mathrm{Im}(z)\gtrless 0,\ \mathrm{Re}(z)>0. \quad (113)$$

Near the normal threshold, the cosines of the crossing angles χ (12) and c.m. scattering angle Θ_t (37) become infinite. We use expressions (112) and (113) in conjunction with the Lorentz-family form (93), with large $s\approx-|s|(1+i\epsilon)$, to obtain the result

$$f_{bd;ac}{}^{\pm}\approx\sum_{\sigma}G(t)[\Delta(t;b,d)]^{-s'+\lambda_m}e^{-i(\frac{1}{2}\pi)(|\lambda+\lambda'|+\mu+\lambda')}\times(1\pm\eta_{bd})+O([\Delta(t;b,d)]^{-s'+\lambda_m+1}). \quad (114)$$

We see that this behavior coincides with that stated in Sec. 2 (30). From the definitions of the amplitudes $f_{bd,ac}^{\pm}$ in terms of $f_{bd;ac}$ (18) we find that the threshold behavior of amplitudes $f_{bd;ac}$ coincides with that given in Sec. 3 (54).

If one is prepared to retain the correlated Regge-family trajectories and parity doubling away from $t=0$, one may use the Lorentz-family decomposition, which satisfies all kinematic constraints. Alternatively, one could choose particular values of j_0 and construct residue functions with specific kinematic factors of the form (108).

6. AN EXAMPLE—THE SPINLESS CASE

As an example of the behavior of the Regge-pole and Lorentz-pole expansions near $t=0$, we consider the scattering of spinless particles. In this case, the functions $d_{mm'}{}^{j}(z)$ and $d_{ss'\lambda}{}^{j_0\sigma}(\delta)$ are particularly simple. The effect of parity conservation is trivial (84):

$$\mathbf{\Delta}_{;}{}^{\pm}=\mathbf{\Delta}_{;}(1\pm\eta_b\eta_d)(1\pm\eta_a\eta_c)$$

and either the f^+ or the f^- amplitude vanishes. The Regge expansion is of the form (27):

$$f=\sum_\alpha \Gamma^\alpha D_{00}{}^\alpha(-z)$$
$$=\left(\frac{p}{p'}\right)^{1/2}\sum_\alpha\frac{(2\alpha+1)}{\sin\pi\alpha}\beta^\alpha(t)(1+\mathrm{s}e^{i\pi\alpha})\mathcal{P}^\alpha(-z). \quad (115)$$

Let us now consider the group-representation functions which occur in the daughter and parent kinematic residues (23),

$$d_{0\sigma-\kappa 0}{}^{0\sigma+1}(\delta)$$
$$={}^\kappa W^\sigma(\delta)=(\tfrac{1}{4})^\kappa\left[\frac{(2\sigma-2\kappa+1)\Gamma(2\sigma-2-\kappa)\Gamma(\kappa+1)}{(\sigma+1)}\right]^{1/2}$$
$$\times\frac{\Gamma(\sigma-\kappa+1)\Gamma(\sigma+\frac{3}{2}-\kappa)}{\Gamma(2\sigma-2\kappa+2)\Gamma(\sigma+\frac{3}{2})}$$
$$\times(2\sinh\delta)^\sigma C_\kappa{}^{-(\sigma+1/2)}(-\coth\delta), \quad (116)$$

where $C_\kappa{}^\sigma(x)$ denotes the Gegenbauer polynomial of degree κ in x. We recall that σ denotes the position of the parent pole and $(\sigma-\kappa)$ the position of daughter poles for $\kappa=1,2,3\cdots$. The most general form for the kinematic residue is (87):

$$\mathbf{\Delta}_{;}{}^{\sigma-\kappa}=[{}^\kappa W^\sigma(\delta_a)+\epsilon_{ac}\,{}^\kappa W^\sigma(-\delta_c)]$$
$$\times[{}^\kappa W^\sigma(\delta_b)+\epsilon_{bd}\,{}^\kappa W^\sigma(-\delta_d)], \quad (117)$$

where ϵ_{ac} and ϵ_{bd} are arbitrary parameters. We recall (68):

$$\sinh\delta_a=\frac{\Delta(t;a,c)}{2t^{1/2}m_a},\quad -\coth\delta_a=\frac{m_c{}^2-m_a{}^2-t}{\Delta(t;a,c)}. \quad (118)$$

The proportionality between the reduced residue ${}^\kappa\tilde{\beta}^\sigma$ and the kinematic residue ${}^\kappa\mathbf{\Delta}_{;}{}^\sigma$, is demonstrated in Eq. (97). Let us, first of all, examine the behavior of ${}^\kappa\mathbf{\Delta}_{;}{}^\sigma$ for small values of t. The argument of the function $C_\kappa{}^{-(\sigma+\frac{1}{2})}(x)$ tends to -1, and we have a singularity of the form $t^{-\sigma}$ for all κ. Consequently, the reduced residue ${}^\kappa\tilde{\beta}^\sigma$ will have a pole of order κ at $t=0$.

We now consider the first two terms in the daughter expansion (85) for large s and small t with ${}^\kappa\mathbf{\Delta}_{;}{}^\sigma$ of the form ${}^\kappa\Delta_{;}{}^\sigma={}^\kappa W^\sigma(\delta_a)\,{}^\kappa W^\sigma(\delta_b)$,

$$f\approx G(t)\sum_\kappa {}^\kappa\Delta^\sigma(t)\mathcal{P}_{\sigma-\kappa}(-z). \quad (119)$$

From expressions (116) and (B9) for the functions ${}^\kappa W^\sigma(\delta)$ and $\mathcal{P}_\sigma(z)$, we find, keeping terms less singular than t^{-2},

$${}^0\Delta^\sigma(t)\mathcal{P}_\sigma(-z)\approx N_\sigma\left(\frac{-s}{m_am_b}\right)^\sigma\left(1-\frac{2\mathfrak{M}\sigma}{st}\right), \quad (120)$$
$${}^1\Delta^\sigma(t)\mathcal{P}_{\sigma-1}(-z)\approx -N_\sigma\frac{2\mathfrak{M}\sigma}{tm_am_b}\left(\frac{-s}{m_am_b}\right)^{\sigma-1}, \quad (121)$$

where $N_\sigma=1/(\sigma+1)$ and $\mathfrak{M}=\frac{1}{4}(m_a{}^2-m_c{}^2)(m_d{}^2-m_b{}^2)$. We see immediately that the daughter contribution (121) cancels out the t^{-1} singularity in the parent expansion (120):

$${}^0\Delta^\sigma(t)D_{00}{}^\sigma(z)+{}^1\Delta^\sigma(t)D_{00}{}^{\sigma-1}(z)\approx\frac{1}{\sigma+1}\left(\frac{-s}{m_am_b}\right)^\sigma. \quad (122)$$

Moreover, the daughter contribution will vanish in the equal-mass limit when $\mathfrak{M}=0$. This is a property possessed by all the odd daughter residues (107) because the Gegenbauer function is an odd function of the argument, $-\coth\delta$ [Eq. (118)], for κ odd and in the equal-mass case $\coth\delta$ tends to zero with t. We also see that in this limit the $t^{-\kappa}$ singularities in the residue functions are no longer present, owing to the appearance of $\sqrt{t}$ zero in the functions $\Delta(t;a,c)$ and $\Delta(t;b,d)$.

Near the threshold and pseudothreshold the kinematic threshold functions $\Delta(t;a,c)$, $\Delta(t;b,d)$ tend to zero. The Gegenbauer polynomial develops a pole of order κ, and the reduced kinematic residue ${}^\kappa\bar{\mathbf{\Delta}}^\sigma(t)=(pp')^{-\sigma+\kappa}\,{}^\kappa\mathbf{\Delta}^\sigma$ remains finite.

Before proceeding to examine the Lorentz-family expansion, we first note that the boost functions are given by

$$D_{000}{}^{0\sigma+1}(\gamma^s)=V^\sigma(\gamma^s)=\frac{-e^{-(\sigma+1)\gamma s}}{2(\sigma+1)\sinh\gamma^s},$$
$$D_{000}{}^{0\sigma+1}(\gamma^u)=V^\sigma(\gamma^u)=\frac{e^{(\sigma+1)\gamma u}}{2(\sigma+1)\sinh\gamma^u}. \quad (123)$$

The addition formula (55) gives, for the contribution of a single Lorentz family (89),

$$f=G^\sigma(t)[V^\sigma(\gamma_{ab}{}^s)+\epsilon_{ac}V^\sigma(\gamma_{bc}{}^u)$$
$$+\epsilon_{bd}V^\sigma(\gamma_{ad}{}^u)+\epsilon_{ac}\epsilon_{bd}V^\sigma(\gamma_{cd}{}^s)]. \quad (124)$$

Now,

$$\sinh\gamma_{ab}{}^s=\frac{\Delta(s;a,b)}{2m_am_b}\quad\text{and}\quad\cosh\gamma_{ab}{}^s=\frac{m_a{}^2+m_b{}^2-s}{2m_am_b},$$

so

$$2V^\sigma(\gamma_{ab}{}^s)=-\frac{1}{\sigma+1}\left[\frac{\Delta(s;a,b)}{2m_am_b}\right]^\sigma\left[\frac{m_a{}^2+m_b{}^2-s}{\Delta(s;a,b)}-1\right]^{\sigma+1}$$

$$\approx\frac{2}{\sigma+1}e^{-i\pi\sigma}\left(\frac{s}{m_am_b}\right)^\sigma\quad\text{as}\quad s\to\infty\,.$$

We note that this is of precisely the same form as the one we have obtained using the Freedman-Wang prescription for Regge-pole cancellation (122). We see that the leading term in (125) will be a good approximation to the function $V^\sigma(\gamma_{ab}{}^s)$, provided $s\gg m_a{}^2+m_b{}^2$. Moreover, a natural scaling factor for asymptotic behavior has appeared—the product of the masses of the two lightest particles.

7. GENERAL DISCUSSION AND CONCLUSIONS

We have shown that it is possible, within the framework of Regge-pole theory, to obtain asymptotic expansions of the s-channel helicity amplitudes for particles with arbitrary masses and spins, which manifestly satisfy all kinematic constraints. The Regge poles group together, with correlated residues, into Lorentz families when the momentum transfer vanishes, in order to preserve the usual Regge-type behavior at high energies. In the case of elastic scattering, these families may be identified with Lorentz poles, since our helicity-amplitude expansion then coincides with those proposed by Domokos,[21] Freedman and Wang,[6] and Toller.[10]

We have also demonstrated that asymptotic behavior may be determined either by adopting a prescription similar to that used by Freedman and Wang in the spinless case or by considering the expression for a Lorentz-family sum. The latter procedure is to be preferred when examining forward scattering, since the changes in power behavior which occur are then simply expressed in terms of crossing-angle products. Moreover, the Lorentz-family expansion in the physical region varies smoothly as a function of the external mass ratios and momentum transfer. We find that s-channel helicity amplitudes behave in the following way at $t=0$, for large s,

$$S_{cd;ab}\approx s^{\sigma-\frac{1}{2}(|a-c-j_0|+|d-b+j_0|)}+s^{\sigma-\frac{1}{2}(|c-a-j_0|+|b-d+j_0|)},$$

where the contribution arises from a Lorentz family with a parent Regge pole at $\alpha=\sigma(0)$ and conspiracy parameter j_0.

The group-theoretic approach suggests that we can write the ordinary Regge residue as a linear combination of kinematic residues Δ^{j_0} and a dynamical residue $G^{j_0j_0'}(t)$, depending upon the Lorentz-family parameter j_0. Provided that the function $G^{j_0j_0'}(t)$ becomes diagonal in j_0 at $t=0$, and poles group into parity doublets for nonzero j_0, all kinematic constraints are automatically satisfied. This suggests immediately the possibility of systematically constructing models for Regge residue functions.

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APPENDIX A: KINEMATIC FUNCTIONS

We consider the kinematic functions $\Delta(s;a,b)$ and ϕ, defined by Eqs. (6) and (8), to be positive in the s-channel physical region. We define the *principal sheets* of these functions to be such that they are *real analytic*, i.e., $[\phi(s)]^*=\phi(s^*)$ and $[\Delta(s;a,b)]^*=\Delta(s^*;a,b)$. This implies that, for $\Delta(s;a,b)$, we must cut the s plane from $s=(m_a+m_b)^2$ to $s=(m_a-m_b)^2$ and that for ϕ, the s plane is cut wherever s lies outside the physical regions of the s,t,u plane.

The process of analytic continuation in crossing from the s channel to the t channel takes us from a region where $s\approx|s|(1+i\epsilon)$ to a region where $s\approx-|s|(1+i\epsilon)$ and for consistency the imaginary part of s must vanish *between the normal and pseudothresholds*. Thus for $\Delta(s;a,b)$ and $\Delta(s;c,d)$ we pass onto the second sheet and $\Delta(s;a,b)=|\Delta(s;a,b)|$, $\Delta(s;c,d)=|\Delta(s;c,d)|$ in the t-channel physical region.

Similarly, the function ϕ is positive in each physical region.

In general, when we consider functions of the parameters s,t,u all ambiguities are removed by giving these parameters small "positive" imaginary parts above corresponding thresholds and small "negative" imaginary parts below the pseudothresholds. Confusion may occur in the case where $m_a>m_b+m_c+m_d$ when, for example, the function $\Delta(s;a,b)\Delta(s;c,d)$ has disjoint cuts, but we note that formally particle (a) is unstable. A small, imaginary part added to the mass m_a separates the different threshold cuts "vertically" and permits us to follow the trajectory of s through each cut from above.

It is to be noted that with this crossing path one does not "pass over" *dynamical* cuts in the s plane. This convention does not affect the expressions for crossing angles given by Cohen-Tannoudji *et al.*[16] The expressions only appear different because the threshold functions which they define change sign on crossing.

APPENDIX B: ANALYTIC CONTINUATIONS OF ROTATION GROUP REPRESENTATIONS

Let us first of all recall some properties of the half-angle factors $\xi_{\lambda\lambda'}(z)$ [24]:

$$\xi_{\lambda\lambda'}(z)=(1+z)^{-\frac{1}{2}|\lambda+\lambda'|}(1-z)^{-\frac{1}{2}|\lambda-\lambda'|},$$

where we always, in defining the principal sheet of the function $(1+z)^\alpha=[1-(-z)]^\alpha$ cut the z plane from -1 to $-\infty$ and take $|(1+z)^\alpha|=(1+z)^\alpha$ for $z>-1$. In this case $\xi_{\lambda\lambda'}(z)$ is a real analytic function for λ,λ' integral or half integral. For large $|z|$, we find

$$\xi_{\lambda\lambda'}(z)\approx|z|^{-\lambda_m}\begin{Bmatrix} e^{\pm(i\pi/2)|\lambda-\lambda'|}, & \mathrm{Re}(z)>0 \\ e^{\mp(i\pi/2)|\lambda+\lambda'|}, & \mathrm{Re}(z)<0\end{Bmatrix},\quad \mathrm{Im}(z)\gtrless0, \tag{B1}$$

where $\lambda_m=\max(|\lambda|,|\lambda'|)$. In addition we find as $|z|\to1$,

$$\xi_{\lambda\lambda'}(z)\approx(1-|z|)^{-\frac{1}{2}|\lambda\pm\lambda'|},\quad z\to\pm1, \tag{B2}$$

and we have the symmetry relation

$$\xi_{\lambda\lambda'}(z)=\xi_{\lambda-\lambda'}(-z). \tag{B3}$$

We also define new functions:

$$\xi_{\lambda\lambda'}{}^{\pm}(z)=\tfrac{1}{2}[\xi_{\lambda\lambda'}(z)\pm e^{\pm i\pi\mu_m}\xi_{\lambda-\lambda'}(z)],\quad \mathrm{Im}(z)\gtrless0 \tag{B4}$$

where $\mu_m=\frac{1}{2}(|\lambda+\lambda'|-|\lambda-\lambda'|)$, which have the properties

$$\xi_{\lambda\lambda'}{}^{+}(z)+\xi_{\lambda\lambda'}{}^{-}(z)=\xi_{\lambda\lambda'}(z)$$

and

$$\xi_{\lambda\lambda'}{}^{+}(z)-\xi_{\lambda\lambda'}{}^{-}(z)=e^{\pm i\pi\mu_m}\xi_{\lambda-\lambda'}(z). \tag{B5}$$

For angles θ in the range $-\pi<\theta<\pi$ and integral or half-integral parameters j, m, and m', we define[25,17]

$$\begin{aligned}\langle\lambda|e^{-iJ_2\theta}|\lambda'\rangle=d_{\lambda\lambda'}{}^j(\theta)&=d_{\lambda\lambda'}{}^j(z), && 0\leqq\theta<\pi\\ &=(-1)^{\lambda-\lambda'}d_{\lambda\lambda'}{}^j(z), && -\pi<\theta\leqq0\end{aligned} \tag{B6}$$

where $z=\cos\theta$.

The function $d_{\lambda\lambda'}{}^j(z)$ is defined for $|1-z|<2$ by

$$\begin{aligned}d_{\lambda\lambda'}{}^j(z)=&\left[\frac{\Gamma(j+\lambda_m+1)\Gamma(j-\mu_m+1)}{\Gamma(j+\mu_m+1)\Gamma(j-\lambda_m+1)}\right]^{1/2}\\ &\times\left(\frac{1+z}{2}\right)^{\frac{1}{2}|\lambda+\lambda'|}\left(\frac{1-z}{2}\right)^{\frac{1}{2}|\lambda-\lambda'|}(-1)^{\frac{1}{2}(\lambda-\lambda'+|\lambda-\lambda'|)}\\ &\times\frac{F(-j+\lambda_m,j+\lambda_m+1;1+|\lambda-\lambda'|:(1-z)/2)}{\Gamma(1+|\lambda-\lambda'|)},\end{aligned} \tag{B7}$$

where $F(a,b;c:z)$ is the ordinary hypergeometric function [HTF 2.8(1)]. There exists a unique analytic continuation of this function into the whole z plane cut from -1 to $-\infty$. When $|z|$ is large, we find [HTF 2.10(3)]

$$d_{\lambda\lambda'}{}^j(z)=D_{\lambda\lambda'}{}^j(z)+D_{\lambda'\lambda}{}^{-(j+1)}(z),$$

where

$$\begin{aligned}D_{\lambda\lambda'}{}^j(z)=&\frac{\Gamma(2j+1)(-1)^{\frac{1}{2}(\lambda-\lambda'+|\lambda-\lambda'|)}}{[\Gamma(j+\mu_m+1)\Gamma(j+\lambda_m+1)\Gamma(j-\mu_m+1)\Gamma(j-\lambda_m+1)]^{1/2}}\left(\frac{1-z}{1+z}\right)^{\frac{1}{2}|\lambda-\lambda'|}\\ &\times\left(\frac{1+z}{2}\right)^j F\left(-j+\lambda_m,\ -j-\mu_m,\ -2j:\frac{2}{1+z}\right).\end{aligned} \tag{B8}$$

In the special case $\lambda=\lambda'=0$, we find

$$D_{00}{}^j(z)=\mathcal{P}_j(z)=\frac{\Gamma(2j+1)}{[\Gamma(j+1)]^2}\left(\frac{1+z}{2}\right)^j F\left(-j,\ -j,\ -2j:\frac{2}{1+z}\right). \tag{B9}$$

We now continue this function in j away from half-integer or integer values, and define the principal sheet to have no cut for j large and positive.[17] This function possesses symmetry properties for complex j similar to those of the functions $d_{\lambda\lambda'}{}^j$ for integral or half-integral j:

$$D_{\lambda\lambda'}{}^j(z)=e^{\pm i\pi(j+\lambda)}D_{\lambda-\lambda'}{}^j(-z),\quad \mathrm{Im}(z)\gtrless0 \tag{B10}$$

$$D_{\lambda\lambda'}{}^j(z^*)=e^{\pm i\pi|\lambda-\lambda'|}D_{\lambda\lambda'}{}^j(z)\times\begin{Bmatrix}1, & \mathrm{Re}(z)>0\\ e^{\mp2i\pi j}, & \mathrm{Re}(z)<0\end{Bmatrix},\quad \mathrm{Im}(z)\gtrless0. \tag{B11}$$

We now define special linear combinations of these functions which occur in Regge theory:

$$e_{\lambda\lambda'}{}^{j\pm}(z)=\xi_{\lambda\lambda'}(z)d_{\lambda\lambda'}{}^j(z)\pm(-1)^{\lambda+\lambda_m}\xi_{\lambda-\lambda'}(z)d_{\lambda-\lambda'}{}^j(z),$$

$$E_{\lambda\lambda'}{}^{j\pm}(z)=\xi_{\lambda\lambda'}(z)D_{\lambda\lambda'}{}^j(z)\pm(-1)^{\lambda+\lambda_m}\xi_{\lambda-\lambda'}(z)D_{\lambda-\lambda'}{}^j(z),$$

[24] In this section we shall use (HTF) to denote *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953).

[25] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

and

$$D_{\lambda\lambda'}{}^{j\pm}(z)=D_{\lambda\lambda'}{}^{j}(z)\pm e^{\zeta i\pi\lambda'}D_{\lambda-\lambda'}{}^{j}(z)\,,\quad \text{where}\quad \zeta=\pm1,\quad \text{Im}z\gtrless0\,. \tag{B12}$$

The functions E^+ and D^+, and E^- and D^-, are so constructed that they contain "even" or "odd" powers of z, respectively, in an expansion for large $|z|$;

$$\begin{aligned}D^{j\pm}(z)&\approx z^{j-\frac{1}{2}\pm\frac{1}{2}}(a_0+a_1z^{-2}+a_2z^{-4}+\cdots)\,,\\ E^{j\pm}(z)&\approx z^{j-\lambda_m-\frac{1}{2}\pm\frac{1}{2}}(b_0+b_1z^{-2}+b_2z^{-4}+\cdots)\,,\end{aligned} \tag{B13}$$

where a_i and b_i are constants. Moreover, the $E^\pm$ functions may be simply expressed in terms of the $D^\pm$ functions:

$$E_{\lambda\lambda'}{}^{j\pm}(z)=\xi_{\lambda\lambda'}{}^{\pm}(z)D_{\lambda\lambda'}{}^{j+}(z)+\xi_{\lambda\lambda'}{}^{\mp}(z)D_{\lambda\lambda'}{}^{j-}(z)\,. \tag{B14}$$

The asymptotic behavior may be determined from that of the functions $D_{\lambda\lambda'}{}^{j}(z)$:

$$D_{\lambda\lambda'}{}^{j}(z)\approx\frac{|\tfrac{1}{2}z|^j\Gamma(2j+1)e^{\mp i(\frac{1}{2}\pi)(\lambda'-\lambda)}}{[\Gamma(j+\lambda_m+1)\Gamma(j+\mu_m+1)\Gamma(j-\lambda_m+1)\Gamma(j-\mu_m+1)]^{1/2}}\left\{\begin{matrix}1, & \text{Re}(z)>0\\ e^{\pm i\pi j}, & \text{Re}(z)<0\end{matrix}\right\},\quad \text{Im}(z)\gtrless0. \tag{B15}$$

In conclusion we state a property of the function $d_{\lambda\lambda'}{}^{j}(\theta)$ for small θ;

$$d_{\lambda\lambda'}{}^{j}(\theta)\approx(\sin\theta)^{|\lambda\pm\lambda'|},\quad \cos\theta\to\mp1\,. \tag{B16}$$

This relation may be derived directly from the definition (B6).

APPENDIX C: REPRESENTATIONS OF THE HOMOGENEOUS LORENTZ GROUP

Any element of the Lorentz group may be decomposed into a product of rotations, and boosts in the z direction.[24] We are concerned here with representations of these boosts with a spin basis,[26,27]

$$\langle s\lambda|e^{-iK_3\gamma}|s'\lambda'\rangle=\delta_{\lambda\lambda'}d_{ss'\lambda}{}^{j_0\sigma}(\gamma)\,. \tag{C1}$$

The parameters j_0 and σ are related to the eigenvalues of the Casimir operators of the homogeneous Lorentz group by

$$\tfrac{1}{2}J_{\mu\nu}J_{\mu\nu}=j_0^2+\sigma^2-1\quad\text{and}\quad \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}J_{\mu\nu}J_{\lambda\rho}=-2i\sigma j_0\,, \tag{C2}$$

where the usual boost-rotation operators are

$$J_i=\tfrac{1}{2}\epsilon_{ijk}J_{jk}\quad\text{and}\quad K_i=J_{0i}\,. \tag{C3}$$

We now define $x=e^{-2\gamma}$ and obtain a series expansion for half-integral or integral j_0,s,s',λ and pure imaginary σ from the integral representation[28] in terms of rotation-group representations, for $|1-x|<1$:

$$d_{ss'\lambda}{}^{j_0\sigma}(x)=N_{ss'}\sum_{r,r'=0}^{\infty}(-1)^{r+r'}\Xi_{sr;\lambda j_0}\Xi_{s'r';\lambda j_0}\Omega_{rr';ss',\lambda}{}^{j_0\sigma}(x)\,, \tag{C4}$$

where

$$\Xi_{sr;\lambda j_0}=\frac{[\Gamma(s+\lambda+1)\Gamma(s-\lambda+1)\Gamma(s+j_0+1)\Gamma(s-j_0+1)]^{1/2}}{\Gamma(s-\lambda-r+1)\Gamma(s+j_0-r+1)\Gamma(r+\lambda-j_0+1)\Gamma(r+1)}\,, \tag{C5}$$

$$\Omega_{rr';ss';\lambda}{}^{j_0\sigma}(x)=\frac{\Gamma(s+s'-r-r'+j_0-\lambda+1)\Gamma(r+r'+\lambda-j_0+1)F(s'-\sigma+1,\,r+r'+\lambda-j_0+1;\,s+s'+2:1-x)}{\Gamma(s+s'+2)x^{\frac{1}{2}(\sigma-1-2r'-\lambda+j_0)}} \tag{C6}$$

and

$$N_{ss'}=\left[\frac{(2s+1)(2s'+1)\Gamma(1+s+\sigma)\Gamma(1+s'-\sigma)\sin\pi(\sigma-s')}{\Gamma(1+s-\sigma)\Gamma(1+s'+\sigma)\sin\pi(\sigma-s)}\right]^{1/2}.$$

The Γ functions in $\Xi_{sr;\lambda j_0}$ restrict the range of summation,

$$\begin{aligned}r_{\min}&=\max(0,\,j_0-\lambda)\,,\\ r_{\max}&=\min(s-\lambda,\,j_0+s)\,.\end{aligned} \tag{C7}$$

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[26] J. F. Boyce, R. Delbourgo, A. Salam, and J. Strathdee, International Centre for Theoretical Physics, Trieste, Report No. IC/67/9 (unpublished). There are several misprints in this paper.

[27] H. Joos, Fortschr. Physik 10, 65 (1962).

[28] See, for example, Toller's work (Ref. 10). His normalization $N_{ss'}$ is given by $[(2s+1)(2s'+1)]^{1/2}$, and his functions are less symmetric than the ones we use.

We now decompose $d_{s's\lambda}{}^{j_0\sigma}(x)$ into two parts,

$$d_{s's\lambda}{}^{j_0\sigma}(x)=D_{s's\lambda}{}^{j_0\sigma}(x)+(-1)^{s-s'}D_{s's\lambda}{}^{-j_0-\sigma}(x)\,,$$

in such a way that the function has simple asymptotic properties for large $|x|$. We continue the expression for $d_{ss'\lambda}{}^{j_0\sigma}(x)$ into the whole x plane cut along the real axis to $\pm\infty$ and use formulas (HTF 2.10) to obtain suitable forms for $\mathbf{\Omega}_{rr';ss';\lambda}{}^{j_0\sigma}(x)$ corresponding to $D_{ss'\lambda}{}^{j_0\sigma}(x)$. We find

$$\Omega_{rr';ss';\lambda}{}^{j_0\sigma}(x)=\frac{\Gamma(s+s'-r-r'+j_0-\lambda+1)\Gamma(\sigma-s'+r+r'+\lambda-j_0)}{x^{\frac{1}{2}(\sigma-1-2r'-\lambda+j_0)}(x-1)^{s'+1-\sigma}\Gamma(\sigma+s+1)}\times F\left(s'-\sigma+1,\,-s-\sigma,\,s'-\sigma+1-r-r'-\lambda+j_0:\frac{1}{1-x}\right),\quad |1-x|>1 \tag{C8}$$

and

$$\mathbf{\Omega}_{rr';ss';\lambda}{}^{j_0\sigma}(x)=\frac{\Gamma(r+r'+\lambda-j_0+1)\Gamma(s+\sigma+j_0-\lambda-r-r')}{x^{\frac{1}{2}(\sigma-1-2r'-\lambda+j_0)}(1-x)^{s'+1-\sigma}\Gamma(\sigma+s+1)}\times F\left(s'-\sigma+1,\,-s-\sigma,\,1-s-\sigma-j_0+\lambda+r+r':\frac{x}{x-1}\right).\quad \left|1-\frac{1}{x}\right|>1. \tag{C9}$$

The function $D_{ss'\lambda}{}^{j_0\sigma}(x)$ for complex σ is defined to be the continuation with $\mathbf{\Omega}_{rr';ss';\lambda}{}^{j_0\sigma}(x)$ of the form (C8) for $|1-x|>1$ and (C9) for $|1-1/x|>1$, with the additional factor $(-1)^{s-s'}$.

By using properties of the hypergeometric series (HTF 2.10) and the integral representation, it is possible to obtain many symmetry properties of these functions:

$$D_{ss'\lambda}{}^{j_0\sigma}(x)=D_{s's\lambda}{}^{j_0\sigma}(x)=D_{ss'-\lambda}{}^{-j_0\sigma}(x)=(-1)^{s-s'}D_{ss'-\lambda}{}^{j_0\sigma}(1/x)\,. \tag{C10}$$

If we take into account the restrictions (C7), we find that $D_{ss'\lambda}{}^{j_0\sigma}(x)$ has the asymptotic behavior

$$D_{ss'\lambda}{}^{j_0\sigma}(x)\approx|x|^{\frac{1}{2}(\sigma-1-|\lambda+j_0|)}\,,\quad |x|\to\infty\,. \tag{C11}$$

We then use relation (C10) to find

$$D_{ss'\lambda}{}^{j_0\sigma}(x)\approx|x|^{-\frac{1}{2}(\sigma-1-|\lambda-j_0|)}\,,\quad |1/x|\to\infty\,. \tag{C12}$$

In the case where $j_0=s=s'=\lambda=0$, we have

$$D_{000}{}^{0\sigma+1}(x)=\frac{x^{\frac{1}{2}(\sigma+1)}}{\sigma(x-1)},\quad |1-x|>1$$
$$=\frac{x^{-\frac{1}{2}(\sigma+1)}}{\sigma(x^{-1}-1)},\quad \left|1-\frac{1}{x}\right|>1. \tag{C13}$$

We consider the functions $d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(\delta)$, which occur in the reduced residue functions. The representation (C4) is not suitable since the sum over r' is not finite for complex s'. We obtain a differential equation[25] for the representation functions and find that it is possible to express them in the form

$$d_{s\sigma-\kappa s}{}^{j_0\sigma+1}(x)=M_{s\sigma-\kappa}\frac{(1-x)^{\sigma-\kappa-s}F(-\kappa,\,\sigma-\kappa+1-j_0,\,2\sigma-2\kappa+2:1-x)}{x^{\frac{1}{2}(\sigma+j_0-s)}\Gamma(2\sigma-2\kappa+2)}, \tag{C14}$$

where

$$M_{s\sigma-\kappa}=\left[(2s+1)(2\sigma-2\kappa+1)\frac{\Gamma(\sigma+s+1-\kappa)\Gamma(\sigma+j_0+1-\kappa)\Gamma(\sigma-j_0+1-\kappa)\Gamma(2\sigma+2-\kappa)\Gamma(\sigma+1-s)}{\Gamma(\sigma-s+1-\kappa)\Gamma(s+j_0+1)\Gamma(s-j_0+1)\Gamma(s+\sigma+2)\Gamma(1+\kappa)}\right]^{1/2}. \tag{C15}$$

For different values of λ, we may use the relation

$$D^\lambda(x)d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(x)=[(s-\lambda)(s+\lambda+1)(\sigma-\lambda-\kappa)(\sigma+\lambda+1-\kappa)]^{1/2}\times d_{s\sigma-\kappa\lambda+1}{}^{j_0\sigma+1}(x)-[(s+\lambda)(s-\lambda+1)(\sigma-\kappa+\lambda)\times(\sigma-\lambda+1-\kappa)]^{1/2}d_{s\sigma-\kappa\lambda-1}{}^{j_0\sigma+1}(x)\,, \tag{C16}$$

where

$$D^\lambda(x)=-2\lambda(1-x)x^{1/2}\frac{d}{dx}+[j_0(\sigma+1)+\lambda]x^{1/2}+[\lambda-j_0(\sigma+1)]x^{-1/2}. \tag{C17}$$

We note that since $-\kappa$ is a negative integer the hypergeometric function in the expression (C14) reduces to a polynomial in $(1-x)$ of degree κ. We shall be using these functions in the kinematic residues and it proves convenient to define a "reduced" function

$$\bar{d}_{sj\lambda}{}^{j_0\sigma}(x)=[(1-x)/x^{1/2}]^{-\sigma}d_{sj\lambda}{}^{j_0\sigma}(x)\,, \qquad (C18)$$

which is real analytic in the cut $x^{1/2}$ plane.

From the properties of the hypergeometric series (HTF 2.10), we find

$$\bar{d}_{sj\lambda}{}^{j_0\sigma}(x)=e^{\pm\pi i(\kappa+s)}\bar{d}_{sj\lambda}{}^{-j_0\sigma}(1/x)\,.$$

Moreover,

$$\bar{d}_{sj\lambda}{}^{j_0\sigma}(e^{\pm 2\pi i}x)=(-1)^{j_0-\lambda}\bar{d}_{sj\lambda}{}^{j_0\sigma}(x)$$

and

$$\bar{d}_{sj\lambda}{}^{j_0\sigma}(e^{\pm 2\pi i}x)=e^{\pm\pi i(\kappa+s+j_0-\lambda)}\bar{d}_{sj-\lambda}{}^{j_0\sigma}(1/x)\,. \qquad (C19)$$

By considering properties of the operator D^λ, we find for all λ

$$(1-x)^\sigma d_{s\sigma-\kappa\pm\lambda}{}^{j_0\sigma+1}(x)\approx(1-x)^{-\kappa-s} \pm c(1-x)^{-\kappa-s+1} \quad \text{as} \quad x\to 1. \qquad (C20)$$

As $x\to\infty,0$ we have similar asymptotic behavior to that of the functions $D_{ss'\lambda}{}^{j_0\sigma}(x)$:

$$\begin{aligned}\bar{d}_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(x)&\approx|x|^{-\frac{1}{2}|j_0+\lambda|}\,, & |x|&\to\infty\\ \bar{d}_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(x)&\approx|x|^{\frac{1}{2}|j_0-\lambda|}\,. & |x|&\to 0\end{aligned} \qquad (C21)$$

In addition, these functions possess similar symmetry properties,

$$d_{s\sigma-\kappa\lambda}{}^{j_0\sigma+1}(x)=d_{s\sigma-\kappa-\lambda}{}^{-j_0\sigma+1}(x)\,. \qquad (C22)$$

In conclusion we remark that in the case $j_0=s=\lambda=0$ the hypergeometric function (C14) may be simply expressed in terms of Gegenbauer polynomials $C_\kappa{}^\sigma$; We find, using [HTF 10.9(20)],

$$d_{0\sigma-\kappa0}{}^{0\sigma+1}(x)=(\tfrac{1}{4})^\kappa\left[\frac{(2\sigma-2\kappa+1)\Gamma(2\sigma-2-\kappa)\Gamma(\kappa+1)}{(\sigma+1)}\right]^{1/2} \times\frac{\Gamma(\sigma-\kappa+1)\Gamma(\sigma+\frac{3}{2}-\kappa)}{\Gamma(2\sigma-2\kappa+2)\Gamma(\sigma+\frac{3}{2})} \times\left(\frac{1-x}{x^{1/2}}\right)^\sigma C_\kappa{}^{-(\sigma+\frac{1}{2})}\left(\frac{x+1}{x-1}\right). \qquad (C23)$$

APPENDIX D: ADDITION FORMULA

We use the following equation connecting Lorentz transformations in the x-z plane:

$$e^{-iK_3\delta}e^{-iJ_2\theta}e^{-iK_3\delta'}=e^{-iJ_2\psi}e^{-iK_3\gamma}e^{-iJ_2\psi'}\,. \qquad (D1)$$

By considering these group elements represented as 2×2 matrices,

$$e^{-iJ_2\theta}\leftrightarrow\begin{bmatrix}\cos\frac{1}{2}\theta & -\sin\frac{1}{2}\theta\\ \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta\end{bmatrix},\quad e^{-iK_3\gamma}\leftrightarrow\begin{bmatrix}e^{\frac{1}{2}\gamma} & 0\\ 0 & e^{-\frac{1}{2}\gamma}\end{bmatrix}$$

one obtains the relations between the angles stated in Sec. 2 [(56) and (57)]. We now take $|j_0\sigma jm\rangle$ matrix elements of the Eq. (D1), where σ is an integer:

$$\sum_j d_{sjm}{}^{j_0\sigma}(\delta)d_{mm'}{}^j(\theta)d_{js'm'}{}^{j_0\sigma}(\delta') =\sum_\mu d_{m\mu}{}^s(\psi)d_{ss'\mu}{}^{j_0\sigma}(\gamma)d_{\mu m'}{}^{s'}(\psi')\,. \qquad (D2)$$

For δ, δ', and γ pure imaginary, we have essentially just invoked the "group property" for representations of the covering group of the four-dimensional rotation group. We now note that the functions $d_{sjm}{}^{j_0\sigma}(\delta)$ (C4) are nonzero provided that $j_0<j,s<\sigma-1$. For this reason, we rewrite the left-hand side of Eq. (D2) in the form

$$\sum_\kappa d_{s\sigma-1-\kappa m}{}^{j_0\sigma}(\delta)d_{mm'}{}^j(\theta)d_{\sigma-1-\kappa s'm'}{}^{j_0\sigma}(\delta)\,.$$

Then, by using the functions $D_{ss'\lambda}{}^{j_0\sigma}(x)$ and $D_{\lambda\lambda'}{}^j(x)$ defined in Appendixes B and C, we obtain a modified expression for arbitrary σ. In particular, in the s-channel physical region the addition formula takes the form[29] (55). It should be noted that the functions $D^{j_0\sigma}$ and D^j are determined uniquely by demanding that they have maximum symmetry and lead to similar asymptotic behavior of the left- and right-hand sides of Eq. (55) for large values of s.

We should like to remark that this addition formula for the $O(4)$ group has been given by Domokos[21] in the special case $m'=s'=j_0=s=m=0$ when it reduces to

$$\sum_j d_{0j0}{}^{0\sigma}(\delta)d_{00}{}^j(\theta)d_{j00}{}^{0\sigma}(\delta')=d_{000}{}^{0\sigma}(\gamma)\,.$$

[29] The addition formula as stated here holds up to spin-dependent phase factors $(-1)^{2s,s'}$ since for simplicity we only specify angles up to a rotation through 2π. That is all we need for our present analysis.