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# Application of Hard-Pion Four-Point Functions to Pion-Pion Scattering*

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Application of hard-pion four-point functions is made to $\pi\pi$ scattering on the basis of the $SU(2)\times SU(2)$ current algebra, a conserved vector current, a partially conserved axial-vector current, and the hypothesis of single-meson dominance of intermediate states in $T$ products. The calculation uses techniques previously developed for exploiting the content of the current algebra and the subsidiary conditions for an $N$-point process. The $\pi\pi$ scattering amplitude is shown to include, besides the well-known pole diagrams, a set of seagull terms. The Weinberg scattering lengths and effective ranges are found to be accurate to within a few percent, since the hard-pion corrections at threshold are only of $O(m_\pi^2/m_\rho^2)$. In the low-energy region, the scattering-phase shifts are seen to be generally small and essentially model-independent, while at the $K$-meson mass we find $\delta_0^0-\delta_0^2\simeq 35°$. All existing data (up to 1 GeV) can be fitted by adjusting one model-dependent parameter.

## I. INTRODUCTION

THE $\pi\pi$ system plays an important role in the understanding of the strong interactions. Because of its inherent simplicity, it has proved to be a testing ground for dynamical theories such as $S$-matrix theory.[1] For a long time, the only well-known experimental feature of the $\pi\pi$ interaction has been the $\rho$ resonance which dominates the low-energy isospin 1, $P$-wave $\pi\pi$ amplitude. Information on other partial waves and isospin channels has been almost nonexistent. In particular, the $S$-wave $I=0$ scattering has been the topic of rather extensive speculations in the past. Fortunately, this situation appears to be changing rapidly.[2–4] The recent detailed analyses of pion-production data undertaken by Walker *et al.*[3] and by Schlein and Malamud[4] appear to support certain consistent features for low-energy scattering in the $I=0$, 2 channels. Indeed, there appears now to be sufficient ground to believe that the $S$-wave $I=0$ phase shift does in fact rise through 90° leading to a resonance (which we will call the $\sigma$ meson) somewhere between 700 MeV and 1 GeV. It also appears that the $I=2$ scattering is small and generally repulsive in the $S$ wave in this energy domain.

With the development of the soft-pion current-algebra methods, there has been a renewed interest in the $S$-wave $\pi\pi$ interaction since it appears in many soft-pion current-algebra calculations. The good results of the soft-pion analyses [which include the $K_{l4}$ form factors, low-energy single-pion production, and the evaluation of the pion-nucleon $I=\frac{1}{2}$, $\frac{3}{2}$ $S$-wave scattering lengths (in particular, the relation $a_{1/2}+2a_{3/2}=0$)] all require the assumption of a weak low-energy $\pi\pi$ interaction.[5] Similarly, the soft-pion current-algebra calculation produces the correct shape dependence in the $K\to 3\pi$ Dalitz plot[6] without the need of a $\pi\pi$ interaction. Using soft-pion current-algebra methods, Weinberg[5,7] has in fact obtained the remarkably small values for the $I=0$ and $I=2$ scattering lengths of[8] $a_0^0=0.15m_\pi^{-1}$ and $a_0^2=-0.043m_\pi^{-1}$. Subsequent analysis by Khuri,[9] based on additional assumptions involving commutation relations of the axial charges with the scalar and pseudoscalar densities, yields essentially the same result. On the other hand, Fulco and Wong,[10] using a dispersion-theoretic approach, have argued in favor of considerably larger scattering lengths to obtain phase shifts comparable to those in the data by Walker *et al.*[3] They have commented that because of the unphysical extrapolation involved in the soft-pion calculation, the physical scattering lengths may be considerably larger than the Weinberg values. In this paper, we will show that this is not necessarily the case, and if one generalizes the usual current-algebra analyses to the on-shell hard-pion calculations, the Weinberg scattering lengths are found to be valid within a few percent. Furthermore, agreement can also be achieved with all the data from threshold to 1 GeV for the $S$- and $P$-wave scattering amplitudes.[11]

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[1] See, e.g., G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); Nuovo Cimento **19**, 752 (1961); F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961); G. F. Chew, *ibid.* **16**, 60 (1966).

[2] V. Hagopian, W. Selove, J. Alitti, J. P. Baton, M. Neveu-Reue, R. Gessarolli, and A. Romano, Phys. Rev. Letters **14**, 1077 (1965).

[3] W. D. Walker, J. Carroll, A. Garfinkel, and B. Y. Oh, Phys. Rev. Letters **18**, 630 (1967).

[4] P. E. Schlein, Phys. Rev. Letters **19**, 1052 (1967); E. Malamud and P. E. Schlein, *ibid.* **19**, 1056 (1967).

[5] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[6] H. D. I. Abarbanel, Phys. Rev. **153**, 1547 (1967).

[7] Different but also quite small scattering lengths have been obtained by J. Schwinger, Phys. Letters **24B**, 473 (1967). This calculation uses a different assumption for chiral breakdown based on a nonlinear representation of the chiral group.

[8] We use the experimental value of 94 MeV for the pion decay constant $F_\pi$.

[9] N. N. Khuri, Phys. Rev. **153**, 1477 (1967).

[10] J. R. Fulco and D. Y. Wong, Phys. Rev. Letters **19**, 1399 (1967). These results are, however, contradicted by other analyses; see, e.g., H. Goldberg, Northeastern University Report (unpublished).

[11] A brief description of some of the results of this paper was given in R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. Letters **20**, 475 (1968).

The $\pi\pi$ calculation discussed in this paper represents an application of current-algebra techniques recently developed[12] for calculating hard-pion $N$-point functions involving $\pi$, $\rho$, $A_1$, and $\sigma$ mesons. We find that the assumptions of single-meson dominance, the chiral $SU(2)\times SU(2)$ current commutation relations, conserved vector current (CVC), and partially conserved axial-vector current (PCAC) require the inclusion of a specific set of pole and seagull diagrams in the calculation of a scattering amplitude. In Sec. II, we review the general formalism for calculating vertex functions and scattering amplitudes. Section III is devoted to the calculation to the total $\pi\pi$ scattering amplitude. In Sec. IV, we carry out the partial-wave analysis for $S$ and $P$ waves. These results are then compared with the experimental data in Sec. V.

## II. REVIEW OF THE FORMALISM

The basic framework of the present calculation is provided by the following set of equal-time commutation relations involving $SU(2)$ triplets of vector and axial-vector currents, $V^\mu{}_a$ and $A^\mu{}_a$:

$$\delta(x^0-y^0)[V^0{}_a(x),V^\mu{}_b(y)]=i\epsilon_{abc}V^\mu{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}, \quad (2.1a)$$

$$\delta(x^0-y^0)[A^0{}_a(x),V^\mu{}_b(y)]=i\epsilon_{abc}A^\mu{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}, \quad (2.1b)$$

$$\delta(x^0-y^0)[V^0{}_a(x),A^\mu{}_b(y)]=i\epsilon_{abc}A^\mu{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}, \quad (2.1c)$$

$$\delta(x^0-y^0)[A^0{}_a(x),A^\mu{}_b(y)]=i\epsilon_{abc}V^\mu{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}, \quad (2.1d)$$

where "$c$-No. S.T." stands for $c$-number Schwinger terms. The vector and axial-vector currents are assumed to obey the conservation laws

$$\partial_\mu V^\mu{}_a=0, \quad (2.2)$$

$$\partial_\mu A^\mu{}_a=m_\pi{}^2F_\pi\varphi_a, \quad (2.3)$$

where $F_\pi$ is the pion decay constant[8] and $m_\pi$ the pion mass. Besides these hypotheses, two other dynamical assumptions were made in the analysis given in I and III.[12] The first is the assumption that one may saturate intermediate sums in the $T$ products of current operators by single-meson states. This hypothesis is essentially the hard-pion analog of the "gentleness" assumption in soft-pion calculations and is a generalized version of $\rho$ dominance for $N$-point functions. It was seen in I and III that single-meson saturation leads to the following result. A $T$-matrix amplitude for a given process is to be calculated by retaining all "tree" and "seagull" diagrams from an appropriately constructed "effective" Lagrangian.[13] More precisely, let $\varphi_a$, $v^\mu{}_a$, $a^\mu{}_a$, and $\sigma$ be a set of interpolating Heisenberg fields for the $\pi$, $\rho$, $A_1$, and $\sigma$ mesons ($a=1, 2, 3$ is the isotopic index). Then the effective Lagrangian was seen to have the form

$$\mathcal{L}=\mathcal{L}_{(0)}+\mathcal{L}_I, \quad (2.4)$$

where $\mathcal{L}_{(0)}$ is the free-meson Lagrangian and the interaction Lagrangian is a polynomial in the meson fields:

$$\mathcal{L}_I=\mathcal{L}_{(3)}+\mathcal{L}_{(4)}+\cdots. \quad (2.5)$$

Here, $\mathcal{L}_{(3)}$ is cubic in the fields, $\mathcal{L}_{(4)}$ quartic, etc. If one then adopts the convention that the coupling constants of $\mathcal{L}_{(3)}$ are of first order, those in $\mathcal{L}_{(4)}$ are of second order, etc., then the prescription of single-meson saturation implies that one calculates a scattering amplitude involving $N$ "in" and "out" mesons using $\mathcal{L}$ to $(N-2)$nd-order perturbation theory. (This guarantees that only tree and seagull diagrams appear for a given process.) Thus, to calculate a vertex function ($N=3$), one need only use the effective Lagrangian to first-order perturbation theory; a two-body scattering amplitude is obtained by using $\mathcal{L}$ to second order, etc. In addition, the above assumptions imply that the currents are related to the fields by[14]

$$V^\mu{}_a=g_\rho v^\mu{}_a, \quad (2.6a)$$

$$A^\mu{}_a=g_A a^\mu{}_a+F_\pi\partial^\mu\varphi_a. \quad (2.6b)$$

Thus, the field-current identity[15] arises naturally here.

The free-meson Lagrangian $\mathcal{L}_{(0)}$ has the form

$$\mathcal{L}_{(0)}=\mathcal{L}_{(0)\pi}+\mathcal{L}_{(0)\sigma}+\mathcal{L}_{(0)\rho}+\mathcal{L}_{0A}, \quad (2.7)$$

where

$$\mathcal{L}_{(0)\pi}=-\varphi^\mu{}_a(\partial_\mu\varphi_a)+\tfrac{1}{2}(\varphi^\mu{}_a\varphi_{\mu a}-m_\pi{}^2\varphi_a{}^2), \quad (2.8a)$$

$$\mathcal{L}_{(0)\sigma}=-\sigma^\mu(\partial_\mu\sigma)+\tfrac{1}{2}(\sigma^\mu\sigma_\mu-m_\sigma{}^2\sigma^2), \quad (2.8b)$$

$$\mathcal{L}_{(0)\rho}=-\tfrac{1}{2}G^{\mu\nu}{}_a(\partial_\mu v_{\nu a}-\partial_\nu v_{\mu a})+\tfrac{1}{2}(\tfrac{1}{2}G^{\mu\nu}{}_aG_{\mu\nu a}-m_\rho{}^2v_{\mu a}v^\mu{}_a), \quad (2.8c)$$

$$\mathcal{L}_{(0)A}=-\tfrac{1}{2}H^{\mu\nu}{}_a(\partial_\mu a_{\nu a}-\partial_\nu a_{\mu a})+\tfrac{1}{2}(\tfrac{1}{2}H^{\mu\nu}{}_aH_{\mu\nu a}-m_A{}^2a_{\mu a}a^\mu{}_a). \quad (2.8d)$$

The masses appearing in Eqs. (2.8) are the physical $\pi$, $\sigma$, $\rho$, and $A_1$ masses. The cubic part of the Lagrangian was seen in I to have the form

$$\mathcal{L}_{(3)}=\mathcal{L}_{(3)\pi\rho A}+\mathcal{L}_{(3)\sigma}, \quad (2.9)$$

[12] R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters **19**, 1085 (1967); Phys. Rev. **174**, 1999 (1968) (we will refer to this paper as I). R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, preceding paper, Phys. Rev. **175**, 1802 (1968) (we will refer to this paper as III).

[13] One must also assume that the *particle* vertex functions can be approximated by a polynomial in the momenta (which is presumably good for sufficiently low momentum transfers). Thus the Lagrangian of Eqs. (2.10) allows for at most cubic structures in momentum.

[14] The constant $g_\rho$ is defined by the vacuum–$\rho$-meson matrix element: $\langle 0|V^\mu{}_a(0)|\rho,kb\rangle=g_\rho\delta_{ab}[N\epsilon^\mu(k)]$, where $N\equiv[2\omega_k(2\pi)^3]^{-1/2}$ and $\epsilon^\mu$ is the $\rho$ polarization vector. Similarly, $g_A$ is defined by the vacuum–$A_1$-meson matrix element:

$$\langle 0|A^\mu{}_a(0)|A_1,k,b\rangle=g_A\delta_{ab}[N\epsilon^\mu(k)].$$

[15] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

where

$$\mathfrak{L}_{(3)\pi\rho A}=\tfrac{1}{2}\epsilon_{abc}(2g_{\pi\pi\rho}\varphi^{\mu}{}_{b}\varphi_{c}v_{\mu a}+\lambda_{\pi\pi\rho}\varphi_{\mu a}\varphi_{\nu b}G^{\nu\mu}{}_{c}+2g_{\pi\rho A}v_{\mu a}\varphi_{b}a^{\mu}{}_{c}+2\mu_{\pi\rho A}\varphi_{a}G^{\mu\nu}{}_{b}H_{\mu\nu c}+2\lambda_{\pi\rho A}v_{\mu a}\varphi_{\nu b}H^{\mu\nu}{}_{c}$$
$$+2\tilde{\lambda}_{\pi\rho A}a_{\mu a}\varphi_{\nu b}G^{\mu\nu}{}_{c}+g_{\rho\rho\rho}v_{\mu a}v_{\nu b}G^{\nu\mu}{}_{c}+2g_{\rho AA}v_{\mu a}a_{\nu b}H^{\nu\mu}{}_{c}+\lambda_{\rho AA}a_{\mu a}a_{\nu b}G^{\nu\mu}{}_{c}+\mu_{\rho\rho\rho}G_{\mu\nu a}G^{\nu\lambda}{}_{b}G_{\lambda}{}^{\mu}{}_{c}$$
$$+\mu_{\rho AA}G_{\mu\nu a}H^{\nu\lambda}{}_{b}H_{\lambda}{}^{\mu}{}_{c})\,, \quad (2.10a)$$

and the $\sigma$ couplings are

$$\mathfrak{L}_{(3)\sigma}=\tfrac{1}{2}g_{\sigma\pi\pi}\varphi_{a}\varphi_{a}\sigma+\tfrac{1}{2}\lambda_{\sigma\pi\pi}\varphi^{\mu}{}_{a}\varphi_{\mu a}\sigma+\tfrac{1}{2}g_{\sigma\rho\rho}v^{\mu}{}_{a}v_{\mu a}\sigma+\tfrac{1}{4}\lambda_{\sigma\rho\rho}G^{\mu\nu}{}_{a}G_{\mu\nu a}\sigma+\tfrac{1}{2}g_{\sigma AA}a^{\mu}{}_{a}a_{\mu a}\sigma+\tfrac{1}{4}\lambda_{\sigma AA}H^{\mu\nu}{}_{a}H_{\mu\nu a}\sigma$$
$$+\tilde{\lambda}_{\sigma\pi A}a^{\mu}{}_{a}\varphi_{\mu a}\sigma+\mu_{\sigma\rho\rho}v_{\mu a}G^{\mu\nu}{}_{a}\sigma_{\nu}+\mu_{\sigma AA}a_{\mu a}H^{\mu\nu}{}_{a}\sigma_{\nu}+\mu_{\sigma\pi A}\varphi_{\mu a}H^{\mu\nu}{}_{a}\sigma_{\nu}+\mu_{\sigma\pi\pi}\varphi_{a}\varphi^{\nu}{}_{a}\sigma_{\nu}+\lambda_{\sigma\pi A}\varphi_{a}a^{\nu}{}_{a}\sigma_{\nu}$$
$$+g_{\sigma\sigma\sigma}\sigma^{3}+\lambda_{\sigma\sigma\sigma}\sigma\sigma_{\mu}\sigma^{\mu}\,. \quad (2.10b)$$

A first-order Lagrangian formalism has been used in Eqs. (2.8) and (2.10). Thus, $\varphi^{\mu}{}_{a}$, $\sigma^{\mu}$, $G^{\mu\nu}{}_{a}$, and $H^{\mu\nu}{}_{a}$ are to be varied independently of the meson fields $\varphi_a$, $\sigma$, $v^{\mu}{}_{a}$, and $a^{\mu}{}_{a}$. The Lagrange equations thus determine $\varphi^{\mu}{}_{a}$ in terms of the meson fields by

$$\varphi^{\mu}{}_{a}=\partial_{\mu}\varphi_{a}-\delta\mathfrak{L}_{I}/\delta\varphi^{\mu}{}_{a}\,. \quad (2.11)$$

Similarly, one has

$$G_{\mu\nu a}=\partial_{\mu}v_{\nu a}-\partial_{\nu}v_{\mu a}-2\delta\mathfrak{L}_{I}/\delta G^{\mu\nu}{}_{a}\,, \quad (2.12)$$

etc.[16]

Equations (2.6) allow one to make use of the field equations and the field canonical commutation relations to impose straightforwardly the current-algebra conditions (2.1)–(2.3). These constraints control the dynamics by determining some of the coupling constants. Thus, the $\mathfrak{L}_{(3)\pi\rho A}$ coupling constants are determined in I to obey the relations

$$g_{\pi\pi\rho}=g_{\rho AA}=g_{\rho\rho\rho}=m_{\rho}{}^{2}g_{\rho}{}^{-1}\,, \quad (2.13a)$$

$$g_{\rho}\lambda_{\pi\pi\rho}=[1-g_{A}{}^{2}(F_{\pi}{}^{2}m_{A}{}^{2})^{-1}]+g_{A}{}^{2}m_{\rho}{}^{2}(F_{\pi}{}^{2}m_{A}{}^{4})^{-1}\lambda_{A}\,, \quad (2.13b)$$

$$\lambda_{\pi\rho A}=-m_{A}{}^{-2}g_{\pi\rho A}=-F_{\pi}m_{\rho}{}^{2}(g_{\rho}g_{A})^{-1}\,, \quad (2.13c)$$

$$g_{\rho}m_{A}{}^{2}F_{\pi}\tilde{\lambda}_{\pi\rho A}=g_{A}m_{\rho}{}^{2}(\lambda_{A}-m_{A}{}^{2}m_{\rho}{}^{-2})\,, \quad (2.13d)$$

$$2F_{\pi}\mu_{\pi\rho A}=g_{A}g_{\rho}{}^{-1}-g_{\rho}g_{A}{}^{-1}\,, \quad (2.13e)$$

$$g_{\rho}{}^{2}m_{\rho}{}^{-2}+g_{A}{}^{2}m_{A}{}^{-2}=F_{\pi}{}^{2}\,, \quad (2.13f)$$

where $\lambda_{A}\equiv g_{\rho}m_{\rho}{}^{-2}\lambda_{\rho AA}$ is the anomalous moment of the $A_1$ meson. Equation (2.13f) is the first Weinberg[17] sum rule. Neither the second Weinberg sum rule ($g_A=g_\rho$) nor the KSRF relation[18] $g_{\rho}{}^{2}=2m_{\rho}{}^{2}F_{\pi}{}^{2}$ emerges as a consequence of the current-algebra requirements. Thus the coupling constants $\lambda_A$, $\mu_{\rho\rho\rho}$, and $\mu_{\rho AA}$ and two of the three parameters $x$, $y$, and $z$, where

$$x\equiv\sqrt{2}m_{\rho}/m_{A}\,,\quad y\equiv g_{A}/g_{\rho}\,,\quad z\equiv g_{\rho}/(F_{\pi}\sqrt{2}m_{\rho})\,, \quad (2.14)$$

are undetermined by the current algebra and must be obtained from experiment. One finds[12,19] $x\cong1$, $y\cong z\cong1$, and $\lambda_A\simeq1/2$. There are no data presently available for determining $\mu_{\rho\rho\rho}$ or $\mu_{\rho AA}$. However, these constants will not enter into any of the considerations of this paper.[20]

A similar analysis is applied in I to the $\sigma$ couplings. One finds

$$F_{\pi}g_{\sigma\pi\pi}=m_{\sigma}{}^{2}\lambda_{3}-m_{\pi}{}^{2}\lambda_{1}\,,\quad F_{\pi}\lambda_{\sigma\pi\pi}=-(\lambda_{1}+\lambda_{2})\,,$$
$$F_{\pi}g_{\sigma AA}=(x^{2}yz)^{-2}2m_{\rho}{}^{2}(\lambda_{1}-\lambda_{2})\,,$$
$$\sqrt{2}m_{\rho}\mu_{\sigma\pi A}=-x^{2}yz\mu_{\sigma AA}\,,$$
$$g_{\sigma\rho\rho}=0=\mu_{\sigma\rho\rho}\,, \quad (2.15)$$

where

$$\lambda_{1}\equiv(g_{A}m_{A}{}^{-2})\lambda_{\sigma\pi A}\,,\quad \lambda_{2}\equiv(g_{A}m_{A}{}^{-2})\tilde{\lambda}_{\sigma\pi A}\,,$$
$$\lambda_{3}\equiv\lambda_{1}+F_{\pi}\mu_{\sigma\pi\pi}\,. \quad (2.16)$$

Thus, $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_{\sigma\rho\rho}$, $\lambda_{\sigma AA}$, $g_{\sigma\sigma\sigma}$, $\lambda_{\sigma\sigma\sigma}$ are unconstrained by the current algebra. Only the constants $\lambda_1$, $\lambda_2$, and $\lambda_3$ will enter into the considerations of this paper. The coupling constants $\mu_{\rho\rho\rho}$, $\mu_{\rho AA}$ and $\lambda_{\sigma\rho\rho}$, $\lambda_{\sigma AA}$, $g_{\sigma\sigma\sigma}$, $\lambda_{\sigma\sigma\sigma}$ may be considered to be "orthogonal" to the current-algebra conditions. For, not only are they not determined by these constraints (as is also true of $\lambda_A$, $\lambda_1$, $\lambda_2$, $\lambda_3$), but the current algebra does not produce any relations at all involving these couplings.

The extension of the above discussion to quartic and higher parts of $\mathfrak{L}_I$ is given in III. In order to carry out this analysis, it is necessary to make a second dynamical assumption, beyond the single-meson saturation assumption. For, to determine scattering amplitudes it is necessary to evaluate the so-called $\sigma$ commutator, defined by[21]

$$\delta(x^{0}-y^{0})[\varphi_{a}(x),A^{0}{}_{b}(y)]=i\delta^{4}(x-y)\sigma_{ab}(x)\,. \quad (2.17)$$

As pointed out by Weinberg,[22] $\sigma_{ab}$ has the important physical significance of governing in part the breakdown of chiral invariance. The current algebra, which allows

[16] The canonically conjugate pairs of variables are $(\sigma_0,\sigma)$, $(\varphi_{0a},\varphi_a)$, $(G_{0ia},v_{ia})$, and $(H_{0ia},a_{ia})$.

[17] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

[18] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966). These calculations have been shown to be incomplete and when the full content of the current algebra is exploited, the analyses yield identities rather than a determination of $g_\rho$. See R. Arnowitt, M. H. Friedman, and P. Nath, Nucl. Phys. **B5**, 115 (1968).

[19] R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. **174**, 2008 (1968).

[20] We note, however, that they would contribute to the $\rho$ and $A_1$ electromagnetic mass shifts and could conceivably be used to cancel logarithmic infinities and make these shifts finite.

[21] The $\sigma$ commutator does not enter into three-point functions due to isospin invariance. We note, however, that in $SU(3)$ extensions, analogous $\sigma$ terms will contribute even to the three-point functions.

[22] S. Weinberg, Phys. Rev. **166**, 1568 (1968).

for chiral breakdown, also does not determine $\sigma_{ab}$, and some additional assumption concerning it must be made. Since, in other aspects of the analysis, single-meson intermediate states, i.e., pole dominance has been postulated, it is natural to assume that the $\sigma$ commutator is also governed by a resonance pole (much as one assumes $\partial_\mu A^\mu{}_a$ is dominated by the pion pole). The only meson for this purpose with the right quantum-numbers is the $\sigma$ meson itself. Thus in III, the assumption was made that $\sigma_{ab}(x)\sim\delta_{ab}\sigma(x)$. This postulate has the appeal of relating chiral breakdown to a physical agency, the $\sigma$ meson. In the soft-pion approximation, it leads directly to the Weinberg $\pi\pi$ $S$-wave scattering lengths,[5] and we will see below that the hard-pion analysis changes this result by only a few percent. Thus the $\sigma$ assumption appears to be in agreement with existing data.[23]

The $\sigma$-commutator assumption leads to one more condition on the $\mathcal{L}_{(3)\sigma}$ couplings, i.e.,

$$\lambda_3=1/\lambda_1 \qquad (2.18)$$

and allows one to construct the higher parts of $\mathcal{L}_I$ consistent with the current algebra restrictions. The explicit form of $\mathcal{L}_{(4)}$ is given in III. There exist a large number of coupling constants "orthogonal" to the current algebra in the same sense that $\mu_{\rho\rho\rho}$, $\mu_{\rho AA}$, etc., of $\mathcal{L}_{(3)}$ are. It is natural as a working hypothesis to set these constants to zero (two of which would contribute to $\pi\pi$ scattering) since they appear to be unrelated to the current-algebra principles.[24] This "minimal" Lagrangian is restated in the Appendix. The $\pi\pi$ amplitude calculated from it will then depend only on two constants, $\lambda_1$ and $\lambda_2$. Actually, at low energies, the amplitude is insensitive to both $\lambda_1$ and $\lambda_2$. At intermediate and high energies, $\lambda_1$ and $\lambda_2$ enter only in the combination $\lambda\equiv\lambda_1-\lambda_2+2(m_\pi^2/m_\sigma^2)\lambda_2$. Thus the amplitude will depend essentially on only one undetermined coupling constant $\lambda$, which will be seen to determine the width of the $\sigma$ meson. We now turn to the evaluation of the $\pi\pi$ scattering amplitude.

## III. $\pi\pi$ SCATTERING AMPLITUDE

As discussed in Sec. II, the effective Lagrangian technique requires that the Lagrangian be used only to second order in calculating a scattering amplitude. One need retain only the $\mathcal{L}_{(3)}+\mathcal{L}_{(4)}$ part of the $\mathcal{L}_I$ (since by convention the coupling constants of $\mathcal{L}_{(n)}$ are of order $n-2$) and one is to use $\mathcal{L}_{(4)}$ to first-order perturbation theory and $\mathcal{L}_{(3)}$ to second order. Thus, the only parts of $\mathcal{L}_{(4)}$ which will contribute to $\pi\pi$ scattering are the terms quartic in the pion fields. These structures are given in Eq. (A1) of the Appendix.

A convenient way of calculating the $S$-matrix element $\langle p_1c,p_2d\text{ out}|q_1a,q_2b\text{ in}\rangle$ is to contract one of the outgoing pions (e.g., the one of momentum $p_2{}^\mu$ and isotopic index $d$):

$$\langle p_1c,p_2d|q_1a,q_2b\rangle=i[2\omega_{p2}(2\pi)^3]^{-1/2}\times\int d^4x\,e^{-ip_2x}\langle p_1c|j_d(x)|q_1a,q_2b\rangle, \qquad (3.1)$$

where $j_d(x)$ is the source term in the pion field equations, i.e.,

$$(-\Box^2+m_\pi^2)\varphi_d(x)=j_d(x). \qquad (3.2)$$

Contributions to $j_d(x)$ arise from both $\mathcal{L}_{(3)}$ and $\mathcal{L}_{(4)}$. Thus, we write

$$j_d(x)=j_{d(3)}(x)+j_{d(4)}(x). \qquad (3.3)$$

$\mathcal{L}_{(4)}$ produces contributions to $j_d(x)$ cubic in the field variables and hence $j_{d(4)}$ contributes only seagull diagrams to Eq. (3.1). The use of a first-order Lagrangian formalism, however, implies the existence of additional seagull contributions arising from $j_{d(3)}$. For, from Eqs. (2.7)–(2.10), one has that $j_{d(3)}$ is a quadratic function of the various meson field variables. If one eliminates, by the first-order field equations, the variables $\varphi_{\mu a}$, $\sigma_{\mu a}$, and $G_{\mu\nu a}$ from $j_{d(3)}$, then cubic structures local in the fields (and second order in the coupling constants) will arise by Eqs. (2.11), (2.12), etc. We write the pion source $j_d(x)$ then as a sum of two terms, the "pole" part and the seagull part:

$$j_d(x)=j_d{}^{(\mathrm{P})}(x)+j_d{}^{(\mathrm{SG})}(x), \qquad (3.4a)$$

where by Eqs. (2.7)–(2.10) the pole part is

$$j_d{}^{(\mathrm{P})}=(2g_{\pi\pi\rho}-m_\rho^2\lambda_{\pi\pi\rho})\epsilon_{def}v_{e\mu}\partial^\mu\varphi_f+(g_{\sigma\pi\pi}-m_\pi^2\lambda_{\sigma\pi\pi}-m_\sigma^2\mu_{\sigma\pi\pi})\varphi_d\sigma \qquad (3.4b)$$

and (keeping terms only up to second order) the total seagull part (from both $j_{d(3)}$ and $j_{d(4)}$) is

$$j_d{}^{(\mathrm{SG})}=g_{\pi\pi\rho}\lambda_{\pi\pi\rho}\epsilon_{dfe}\epsilon_{egh}\varphi_g\partial_\mu\varphi_h\partial^\mu\varphi_f+\{\tfrac{1}{2}\mu_{\sigma\pi\pi}g_{\sigma\pi\pi}+(F_\pi)^{-2}m_\pi^2-\tfrac{1}{2}(F_\pi)^{-2}[m_A{}^4(g_A\lambda_{\sigma\pi A})^{-2}m_\sigma{}^2+2(g_Am_\pi\lambda_{\sigma\pi A}m_A{}^{-2})^2+g_A(m_A)^{-2}m_\pi{}^2\tilde{\lambda}_{\sigma\pi A}]\}\varphi_d(\varphi_a)^2+\{\tfrac{1}{2}\mu_{\sigma\pi\pi}\lambda_{\sigma\pi\pi}+\tfrac{1}{2}(F_\pi)^{-2}[(\lambda_{\sigma\pi A})^{-1}\tilde{\lambda}_{\sigma\pi A}+1-2(g_A\lambda_{\sigma\pi A}m_A{}^{-2})^2]\}\varphi_d\partial^\mu\varphi_a\partial_\mu\varphi_a-(F_\pi)^{-2}(\lambda_{\sigma\pi A})^{-1}\tilde{\lambda}_{\sigma\pi A}(\varphi_a\partial^\mu\varphi_a)\partial_\mu\varphi_d. \qquad (3.4c)$$

[23] As pointed out by M. G. Olsson and L. Turner, Phys. Rev. Letters **20**, 1121 (1968), the $\pi\pi$ scattering lengths enter importantly in the process $\pi+N\rightarrow 2\pi+N$. The present data appear to be consistent with the Weinberg value but not with the Schwinger value (Ref. 7). See also the analysis of L. N. Chang, Phys. Rev. **162**, 1497 (1967).

[24] The two "orthogonal" constants which contribute to $\pi\pi$ scattering are retained in the hard-pion Ward identity analysis of four-point functions given by I. S. Gerstein and H. J. Schnitzer, Phys. Rev., this issue, **175**, 1876 (1968). In the Ward-identity approach, it appears to be more natural to retain these two constants, and set the rest to zero.

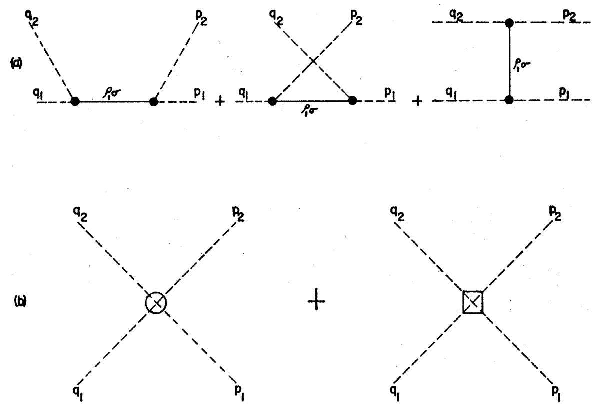

FIG. 1. (a) Pole diagrams which contribute to the $\pi\pi$ scattering. Black dots represent the vertices computed from current algebra. (b) Contribution to $\pi\pi$ amplitude from seagull terms as demanded by current algebra. The first diagram represents a set of terms involving $\pi$-, $\rho$-, and $A$-dependent coupling constants alone. The second diagram represents those terms which involve $\sigma$-dependent terms.

In Eqs. (3.4) we have made use of Eq. (2.15) to express $\lambda_{\sigma\pi\pi}$ in terms of $\lambda_{\sigma\pi A}$ and $\tilde{\lambda}_{\sigma\pi A}$ and only terms involving the pion field in $j_d^{(\mathrm{SG})}$ have been retained since only these contribute to Eq. (3.1) to the required second order.

The contribution of $j_d^{(\mathrm{SG})}$ to the matrix element of Eq. (3.1) can be immediately obtained by approximating the pion field by a free field. Since, however, $j_d^{(\mathrm{P})}$ is only linear in the coupling constants, one must expand the fields to one more order to get the required second-order contribution. Thus, from Eqs. (2.7)–(2.10) one finds that

$$[\eta^{\mu\lambda}(-\Box^2+m_\rho^2)+\partial^\mu\partial^\lambda]v_{\lambda a}(x) = -\epsilon_{abc}[g_{\pi\pi\rho}\varphi_b\varphi^\mu{}_c+\lambda_{\pi\pi\rho}\partial_\nu(\varphi^\mu{}_b\varphi^\nu{}_c)], \quad (3.5)$$

$$(-\Box^2+m_\sigma^2)\sigma(x)=\tfrac{1}{2}(g_{\sigma\pi\pi}-m_\pi^2\mu_{\sigma\pi\pi})\varphi_a\varphi_a + (\tfrac{1}{2}\lambda_{\sigma\pi\pi}-\mu_{\sigma\pi\pi})\varphi^\mu{}_a\varphi_{\mu a}, \quad (3.6)$$

where we have kept only those source terms on the right-hand side that give contributions to Eq. (3.1) to second order. Thus, upon inserting the solutions (to first order) of Eqs. (3.5) and (3.6) into $j_d^{(\mathrm{P})}$ one obtains the total $j_d^{(\mathrm{P})}$ valid to second order.[25] These structures give rise to the direct channel and crossed channel $\sigma$ and $\rho$ poles of Fig. 1 in Eq. (3.1) [as can be seen from the fact that the $\rho$ and $\sigma$ propagators arise upon solving Eqs. (3.5) and (3.6)].

The computation of the scattering matrix is now straightforward. It is convenient to define the function $M_{cd,ab}$ according to

$$\langle p_1c,p_2d|q_1a,q_2b\rangle = -i(2\pi)^4\delta^4(q_1+q_2-p_1-p_2) \times[16\omega_{q1}\omega_{q2}\omega_{p1}\omega_{p2}(2\pi)^{12}]^{-1/2}M_{cd,ab} \quad (3.7)$$

and introduce the invariants $s$, $t$, and $u$, where

$$s=-(q_1+q_2)^2=-(p_1+p_2)^2,\quad t=-(q_1-p_1)^2=-(q_2-p_2)^2,\quad u=-(q_1-p_2)^2=-(q_2-p_1)^2. \quad (3.8)$$

By virtue of isospin and crossing symmetry, $M$ can be written as

$$M_{cd,ab}(s,t,u)=\delta_{ab}\delta_{cd}A(s,t,u)+\delta_{ac}\delta_{bd}A(t,u,s) +\delta_{ad}\delta_{bc}A(u,s,t). \quad (3.9)$$

We may divide $A(s,t,u)$ into its pole and seagull parts:

$$A(s,t,u)=A^{\mathrm{P}}(s,t,u)+A^{\mathrm{SG}}(s,t,u). \quad (3.10a)$$

For the pole contribution one finds

$$A^{\mathrm{P}}(s,t,u)=\left\{[(g_{\pi\pi\rho})^2-u\lambda_{\pi\pi\rho}(g_{\pi\pi\rho}-\tfrac{1}{4}m_\rho^2\lambda_{\pi\pi\rho})]\times\frac{s-t}{u-m_\rho^2}+u\leftrightarrow t\right\}+[g_{\sigma\pi\pi}-m_\sigma^2\mu_{\sigma\pi\pi} +(\tfrac{1}{2}s-m_\pi^2)\lambda_{\sigma\pi\pi}][g_{\sigma\pi\pi}-s\mu_{\sigma\pi\pi} +(\tfrac{1}{2}s-m_\pi^2)\lambda_{\sigma\pi\pi}]\frac{1}{s-m_\sigma^2} \quad (3.10b)$$

and for the seagull piece the result

$$A^{\mathrm{SG}}(s,t,u)=-\mu_{\sigma\pi\pi}[g_{\sigma\pi\pi}+\lambda_{\sigma\pi\pi}(\tfrac{1}{2}s-m_\pi^2)]+F_\pi^{-1}[-4m_\pi^2(F_\pi g_\rho^{-1}g_{\pi\pi\rho}+\tfrac{1}{2}m_A^2g_A^{-1}\lambda_{\sigma\pi\pi}\lambda_{\sigma\pi A}^{-1}) +m_A^2g_A^{-1}g_{\sigma\pi\pi}\lambda_{\sigma\pi A}^{-1}]+sF_\pi^{-2}[g_A^2m_A^{-4}(\lambda_{\sigma\pi A})^2-\tfrac{1}{2}+\tfrac{1}{2}\tilde{\lambda}_{\sigma\pi A}\lambda_{\sigma\pi A}^{-1}+3F_\pi(F_\pi g_\rho^{-1}g_{\pi\pi\rho} +\tfrac{1}{2}m_A^2g_A^{-1}\lambda_{\sigma\pi\pi}\lambda_{\sigma\pi A}^{-1})]. \quad (3.10c)$$

We note that the numerators of the $\rho$ and $\sigma$ poles are not constants. This is due to the fact that Eqs. (2.10) allow energy and momentum-transfer dependence in the $\pi\pi\rho$ and $\pi\pi\sigma$ vertices.

## IV. PARTIAL-WAVE AMPLITUDE

The $s$-channel $I=0$, 1, 2 isospin amplitudes $M^I(s,t)$ can be obtained from $A(s,t,u)$ according to the usual relations

$$M^0(s,t)=3A(s,t,u)+A(t,u,s)+A(u,s,t), \quad (4.1a)$$

$$M^1(s,t)=A(t,u,s)-A(u,s,t), \quad (4.1b)$$

$$M^2(s,t)=A(t,u,s)+A(u,s,t). \quad (4.1c)$$

$M^0$ contains the direct $\sigma$ pole and crossed $\sigma$ and $\rho$ poles, $M^1$ contains the direct $\rho$ pole and crossed $\sigma$ and $\rho$ poles, while $M^2$ contains only the crossed poles. The seagull terms contribute to all three amplitudes. One may

[25] The second-order contributions obtained from expanding the pion field $\varphi_a(x)$ to first order do not contribute to the matrix element in Eq. (3.1).

expand $M^I$ in partial-wave amplitudes:

$$M^I(s,t)=-16\pi\sum_l(2l+1)P_l(z)A^I{}_l(s)\,, \quad (4.2a)$$

where

$$A^I{}_l(s)\equiv s^{1/2}k^{-1}[\exp i\delta^I{}_l(s)]\sin\delta^I{}_l(s)$$
$$=-(32\pi)^{-1}\int_{-1}^{1}dz\,P_l(z)M^I(s,t)\,. \quad (4.2b)$$

In Eqs. (4.2), $z$ is the cosine of the $s$-channel scattering angle in the center-of-mass system and $2k\equiv(s-4m_\pi{}^2)^{1/2}$ is the relative momentum in this system.

As a consequence of Bose statistics, the only nonzero $S$- and $P$-wave amplitudes are $A^0{}_0$, $A^2{}_0$, and $A^1{}_1$. These quantities can be obtained directly from Eqs. (3.10), (4.1), and (4.2) and we record their values here. The $S$-wave $I=0$ amplitude is given by

$$\begin{aligned}-16\pi A^0{}_0=&\tfrac{3}{4}m_\sigma{}^2F_\pi{}^2\{m_\sigma{}^2(\lambda_1-\lambda_2+2\epsilon_\sigma\lambda_2)^2(s-m_\sigma{}^2)^{-1}+[(\lambda_1-\lambda_2)^2-4\epsilon_\sigma(\lambda_1{}^{-1}+\lambda_2)\lambda_2-4]\\&+sm_\sigma{}^{-2}[(\lambda_1-\lambda_2)^2+2+2\lambda_1{}^{-1}\lambda_2-2(\lambda_1-\lambda_2)\lambda_1]\}+F_\pi{}^{-2}\{2(1-x^4y^2\tfrac{1}{4}\lambda_A)^2z^2\\&\times(2s+m_\rho{}^2-4m_\pi{}^2)\tfrac{1}{2}x_\rho L^{(0)}(x_\rho)+\tfrac{1}{2}m_\sigma{}^2(\lambda_1-\lambda_2+2\epsilon_\sigma\lambda_2)^2\tfrac{1}{2}x_\sigma L^{(0)}(x_\sigma)-2m_\sigma{}^2\\&+m_\pi{}^2[4(1-x^4y^2\tfrac{1}{4}\lambda_A)(1+z^2-x^4y^2z^2\tfrac{1}{4}\lambda_A)+(\lambda_1-\lambda_2)^2+2(1-\lambda_1{}^2)-2\epsilon_\sigma\lambda_2{}^2]\\&-\tfrac{1}{2}s[1+2(1-x^4y^2\tfrac{1}{4}\lambda_A)(3+z^2-x^4y^2z^2\tfrac{1}{4}\lambda_A)+\tfrac{1}{2}(\lambda_1-\lambda_2)^2+\lambda_1{}^{-1}\lambda_2-(\lambda_1-\lambda_2)\lambda_1]\}\\&+F_\pi{}^{-2}\{5m_\sigma{}^2+m_\pi{}^2[-5+x^4y^2\lambda_A+5(\lambda_1+\lambda_1{}^{-1})\lambda_2+2\lambda_1{}^2-2\lambda_1{}^{-1}\lambda_2-2\lambda_1\lambda_2]\\&+s[-\tfrac{3}{4}x^4y^2\lambda_A+\lambda_1{}^2-\lambda_1{}^{-1}\lambda_2-\lambda_1\lambda_2]\}\,. \quad (4.3)\end{aligned}$$

For the $I=2$, $S$-wave amplitude we find that

$$\begin{aligned}-16\pi A^2{}_0=&F_\pi{}^{-2}\{-(1-x^4y^2\tfrac{1}{4}\lambda_A)^2z^2(2s+m_\rho{}^2-4m_\pi{}^2)\tfrac{1}{2}x_\rho L^{(0)}(x_\rho)+\tfrac{1}{2}m_\sigma{}^2(\lambda_1-\lambda_2+2\epsilon_\sigma\lambda_2)^2\\&\times\tfrac{1}{2}x_\sigma L^{(0)}(x_\sigma)-2m_\sigma{}^2+m_\pi{}^2[-2(1-x^4y^2\tfrac{1}{4}\lambda_A)(1+z^2-x^4y^2z^2\tfrac{1}{4}\lambda_A)+(\lambda_1-\lambda_2)^2\\&+2(1-\lambda_1{}^2)-2\epsilon_\sigma\lambda_2{}^2]-\tfrac{1}{2}s[1-(1-x^4y^2\tfrac{1}{4}\lambda_A)(3+z^2-x^4y^2z^2\tfrac{1}{4}\lambda_A)+\tfrac{1}{2}(\lambda_1-\lambda_2)^2+\lambda_1{}^{-1}\lambda_2\\&-(\lambda_1-\lambda_2)\lambda_1]\}+F_\pi{}^{-2}\{2m_\sigma{}^2+m_\pi{}^2(-2-\tfrac{1}{2}x^4y^2\lambda_A+2\lambda_1{}^2)-\tfrac{1}{2}s(-\tfrac{3}{4}x^4y^2\lambda_A+\lambda_1{}^2-\lambda_1{}^{-1}\lambda_2-\lambda_1\lambda_2)\}\,, \quad (4.4)\end{aligned}$$

where $\epsilon_\sigma\equiv m_\pi{}^2/m_\sigma{}^2$, $x_\sigma\equiv 4k^2/m_\sigma{}^2$, $x_\rho\equiv 4k^2/m_\rho{}^2$, and $\frac{1}{2}\xi L^{(0)}=1-\xi^{-1}\ln(1+\xi)$. The parameters $x$, $y$, $z$ are defined in Eq. (2.14). In Eq. (4.3), the first curly bracket arises from the direct $\sigma$ pole, the second is from the crossed channel $\sigma$ and $\rho$ poles, and the third from the seagull diagrams. In Eq. (4.4) the first curly bracket comes from the crossed channel $\sigma$ and $\rho$ poles, while the second gives the contribution of the seagull terms. No direct pole occurs in this channel as no $I=2$, $J=0$ meson appears to exist in the low-energy domain we are considering.

The $P$-wave amplitude is given by

$$\begin{aligned}-12\pi k^{-2}A^1{}_1=&F_\pi{}^{-2}(1-x^4y^2\tfrac{1}{4}\lambda_A)\{(1-x^4y^2\tfrac{1}{4}\lambda_A)z^2m_\rho{}^2(s-m_\rho{}^2)^{-1}-(1-z^2+x^4y^2z^2\tfrac{1}{4}\lambda_A)\}+F_\pi{}^{-2}\{(1-x^4y^2\tfrac{1}{4}\lambda_A)^2z^2\\&\times[m_\rho{}^{-2}(s+\tfrac{1}{2}m_\rho{}^2-2m_\pi{}^2)x_\rho L^{(1)}(x_\rho)-sm_\rho{}^{-2}+2\epsilon_\rho]-\tfrac{1}{2}(1-x^4y^2\tfrac{1}{4}\lambda_A)-\epsilon_\sigma\lambda_1\lambda_2-\tfrac{1}{2}(1+\lambda_1{}^2-\lambda_1\lambda_2-\lambda_2\lambda_1{}^{-1})\\&+(\lambda_1-\lambda_2+2\epsilon_\sigma\lambda_2)^2x_\sigma L^{(1)}(x_\sigma)\}-\tfrac{1}{2}F_\pi{}^{-2}\{-\tfrac{3}{4}x^4y^2\lambda_A+\lambda_1(\lambda_1-\lambda_2)-\lambda_2\lambda_1{}^{-1}\}\,, \quad (4.5a)\end{aligned}$$

where

$$\xi L^{(1)}(\xi)\equiv\xi+12\xi^{-1}-(6\xi^{-1}+12\xi^{-2})\ln(1+\xi)\,. \quad (4.5b)$$

The first curly bracket in Eq. (4.5a) comes from the direct channel $\rho$ pole, the second from the crossed channel $\sigma$ and $\rho$ poles, and the third from the seagull diagrams. The functions $L^{(1)}(\xi)$ has been defined so that $L^{(1)}(0)=1$.

## V. COMPARISON WITH EXPERIMENT

We consider in this section the comparison of the above results with experiment in the energy range from threshold to 1 GeV.

### A. S-Wave Amplitudes

(*1*) *Threshold parameters.* The scattering length $a^I{}_l$ and effective range $r^I{}_l$ are conventionally defined by the threshold expansion

$$2m_\pi k^{2l}\,\mathrm{Re}(A^I{}_l)^{-1}=(a^I{}_l)^{-1}+\tfrac{1}{2}r^I{}_lk^2+\cdots. \quad (5.1)$$

Neglecting small terms of order $(\epsilon_\sigma)^2$, the $S$-wave scattering lengths obtained from Eqs. (4.3) and (4.4) are

$$32\pi m_\pi a^0{}_0=7m_\pi{}^2F_\pi{}^{-2}\{1+(1/7)\epsilon_\sigma[12\lambda_1(\lambda_1-\lambda_2)+5\lambda_2{}^2]\}\,, \quad (5.2)$$

$$32\pi m_\pi a^2{}_0=-2m_\pi{}^2F_\pi{}^{-2}(1-\epsilon_\sigma\lambda_2{}^2)\,, \quad (5.3)$$

while the effective ranges are given by

$$2\pi m_\pi(a^0{}_0)^2r^0{}_0=-F_\pi{}^{-2}[1+\epsilon_\sigma(\lambda_1-\lambda_2)(3\lambda_1-2\lambda_2)-2\epsilon_\rho(1-x^4y^2\tfrac{1}{4}\lambda_A)^2z^2]\,, \quad (5.4)$$

$$4\pi m_\pi(a^2{}_0)^2r^2{}_0=F_\pi{}^{-2}[1+\epsilon_\sigma(\lambda_1-\lambda_2)\lambda_2-2\epsilon_\rho(1-x^4y^2\tfrac{1}{4}\lambda_A)^2z^2]\,, \quad (5.5)$$

where $\epsilon_\rho\equiv m_\pi{}^2/m_\rho{}^2$. The leading terms in Eqs. (5.2)–(5.5) (obtained by neglecting $\epsilon_\sigma$ and $\epsilon_\rho$ corrections) are precisely the soft-pion threshold results obtained by Weinberg.[5] As discussed in Sec. II, analysis of meson vertex functions[12,19] yields $x\simeq y\simeq z\simeq 1$ and $\lambda_A\simeq\frac{1}{2}$. Since $\epsilon_\sigma\equiv m_\pi{}^2/m_\sigma{}^2$ and $\epsilon_\rho\simeq 1/30$, the hard-pion corrections to the threshold parameters are only a few percent unless $\lambda_1$ and $\lambda_2$ are anomalously large. We will see below that in fact the combination

$$\lambda\equiv\lambda_1-\lambda_2+2\epsilon_\sigma\lambda_2\,, \quad (5.6)$$

which governs the $\sigma$ width, has the experimental value $\lambda\simeq 1$. No direct separate measurement of $\lambda_1$ and $\lambda_2$ exists. However, the soft-pion analyses of threshold single-pion production by Chang and by Olsson and Turner[23] are consistent with the Weinberg scattering lengths and imply no major deviation from these values. It would be of interest to have a hard-pion calculation of this process performed since, if the data improve, it would represent a method of determining the $\pi\pi$ parameters. In the following considerations we will assume that $\lambda_1$ and $\lambda_2$ are separately $\approx 1$.

An interesting property of the amplitudes (4.3) and (4.4) is the large amount of cancellation that occurs at threshold. Thus both $A^0{}_0$ and $A^2{}_0$ contain terms of $O(m_\sigma^2/F_\pi^2)$ which are $\approx 5$–10 times as large as the leading $m_\pi^2/F_\pi^2$ terms appearing in Eqs. (5.2) and (5.3). It is the cancellation of these structures at threshold that gives rise to the smallness of the $S$-wave scattering lengths. Further, all the model-dependent parameters $\lambda_1$, $\lambda_2$, $\lambda_A$, $x$, $y$, and $z$ cancel from the remaining leading $m_\pi^2/F_\pi^2$ term in $a^I{}_l$ and similarly from the leading $F_\pi^{-2}$ term of $m_\pi(a^I{}_l)^2 r^I{}_l$. These cancellations reflect the strong interdependence between the pole and seagull parts of the amplitude required by the current-algebra constraints.[26]

(2) *Low-energy region.* The expansion (5.1) is valid only in a very small region around threshold. This is, in fact, a low-energy spurious zero in $(A^I{}_0)^{-1}$ in the effective-range approximation. On the other hand, the expansion of $A^I{}_0$ itself,

$$A^I{}_0=2m_\pi[a^I{}_0-\tfrac{1}{2}(a^I{}_0)^2r^I{}_0k^2+\cdots], \qquad (5.7)$$

converges reasonably rapidly in the low-energy region. The "shape parameter" coefficients of the $k^4$ and higher terms in the series depend on $\lambda_1$ and $\lambda_2$ only in the combination $\lambda$ of Eq. (5.6) [as the $\lambda_1$ and $\lambda_2$ contributions to the higher terms arise only from the expansion of $(s-m_\sigma^2)^{-1}$ and $L^0(x_\sigma)$ in Eqs. (4.3) and (4.4)]. For $\lambda^2\lesssim 3$, the first two terms in Eq. (5.7) represent a good approximation to the rigorous amplitudes for $\sqrt{s}\lesssim 500$ MeV. Since, as discussed above, the scattering lengths and effective ranges depend only weakly on $\lambda_1$ and $\lambda_2$, the results are essentially model-independent in this region. As pointed out by Weinberg,[5] a strong low-energy $\pi\pi$ interaction appears to be inconsistent with the success of the various soft-pion current-algebra calculations where $\pi\pi$ interactions are neglected. The hard-pion calculation yields $\delta^0{}_0\lesssim 20°$ and $|\delta^2{}_0|\lesssim 10°$ for $\sqrt{s}\lesssim 450$ MeV in agreement with the above requirement.

At the $K$-meson mass, Eqs. (4.3) and (4.4) give

$$\delta\equiv\delta^0{}_0-\delta^2{}_0\simeq 35°. \qquad (5.8)$$

This result varies by no more than about 10% as $\lambda^2$ varies from 1 to 3 and $m_\sigma$ from 700 MeV to 1 GeV. It is thus essentially model-independent. Thus,[27] for $\lambda^2=1$, $\lambda_2=0$, $m_\sigma=930$ MeV we find $\delta=35°$; for $\lambda^2=0.75$, $\lambda_2=0$, $m_\sigma=730$ MeV, $\delta=36°$; and for $\lambda^2=0$, $\lambda_2=0$, $\delta=31°$. Several estimates of $\delta$ obtained from $K^0$-$\bar{K}^0$ decay parameters exist in the literature. The Yen type-I solution[28] gives $\delta=(30_{-21}^{+25})°$ while the analysis of Glashow[29] yields $\delta=(35\pm 25)°$ consistent with our value. While the determinations of $\delta$ still have rather large errors, it does now appear that $\delta$ is positive.

(3) *Resonant region.* As one goes to higher energies and approaches the $\sigma$ resonance, one must make use of the full expressions of Eqs. (4.3) and (4.4) for the amplitudes. Here, the $\sigma$ pole dominates the $I=0$ channel and the threshold terms of Eq. (5.7) make only a small contribution. Since, further, $\lambda_1$ and $\lambda_2$ enter only weakly in the scattering length and effective range, the amplitude is effectively controlled by the single parameter $\lambda^2$ (and the $\sigma$ mass value). For simplicity, we will set $\lambda_2$ to zero in the following discussions.

The extraction of the phase shifts from the current-algebra results is not straightforward in this energy region, however, since unitarity has not been imposed. How to impose unitarity within the framework of current algebra remains an important problem for future consideration. Here, we will take the simplest procedure of imposing two-body unitarity on each partial-wave amplitude separately, i.e., we will assume that Eqs. (4.3) and (4.4) yield correct results for $\mathrm{Re}(A^I{}_0)^{-1}$ but must be appropriately modified for $\mathrm{Im}(A^I{}_0)^{-1}$ to restore unitarity. Effectively, this implies that one replace $ks^{-1/2}A^I{}_0$ by $\tan\delta^I{}_0$ in Eqs. (4.3) and (4.4).[30]

We consider first the analysis of the experimental data of Walker *et al.*[3] These results imply $m_\sigma\cong 930$ MeV. The theoretical curves for several values of $\lambda^2$ are plotted in Fig. 2(a). Good fits with the data for both $I=0$ and $I=2$ phase shifts can be obtained for $\lambda^2\cong 1$ to 3, except in the low-energy $I=0$ channel. However, very likely, the experimental analysis overestimates the phase shift in this region since the peripheral pion-exchange diagram no longer dominates the pion-production process at low energies. In fact, as pointed out above, the soft-pion analyses of single-pion

[26] The existence of the some of these threshold cancellations has been noted previously by S. Weinberg, Phys. Rev. **166**, 1568 (1968).

[27] Since the phase shifts are still fairly small for $s^{1/2}\lesssim 500$ MeV it does not matter much whether one approximates $ks^{-1/2}A^I{}_0$ by $\delta^I{}_0$, $\sin\delta^I{}_0$, or $\tan\delta^I{}_0$. The results are quoted for the "unitarized" choice of $\tan\delta^I{}_0$.

[28] E. Yen, Phys. Rev. Letters **18**, 513 (1967).

[29] S. Glashow, Phys. Rev. Letters **18**, 524 (1967).

[30] While one cannot of course make any clear-cut justification for this step, one may argue as follows. The current-algebra results of Eqs. (4.3) and (4.4) are rigorously crossing-symmetric. In the low-energy region, where $\delta^I{}_0$ is small the replacement of $ks^{-1/2}A^I{}_0$ by $\tan\delta^I{}_0$ produces only a very small change in the amplitude and thus gives rise to an amplitude that is both crossing-symmetric and unitary to a good approximation. At higher energies, this procedure of course maintains unitarity but in general destroys crossing. Close to the $\sigma$ resonance, however, the pole may be expected to dominate and the loss of crossing to produce small effects. Thus, one presumably may be making an error only in the in-between region.

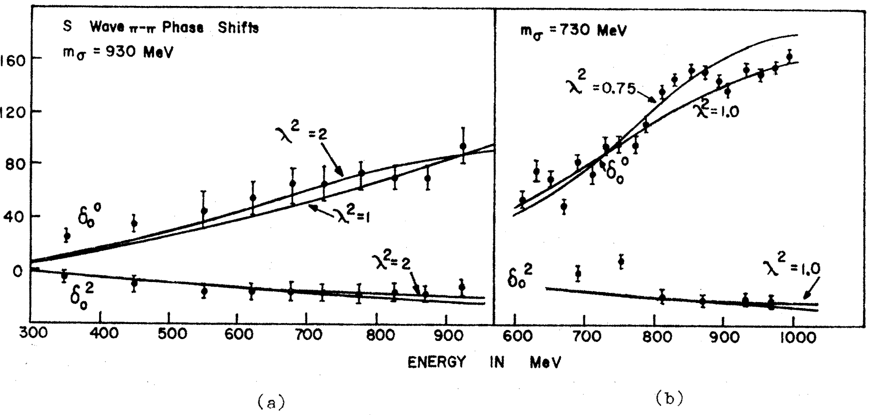


FIG. 2. $S$-wave phase shifts (upper curve for $I=0$, lower curve for $I=2$). (a) $m_\sigma=930$ MeV; the experimental points are from Walker *et al.* (Ref. 3). (b) $m_\sigma=730$ MeV; the experimental points are the Malamud and Schlein "up-up" solution (Ref. 4).

production at threshold,[23] which includes *both* the peripheral and seagull diagrams, are consistent with the data there.

We consider next the alternate results of Malamud and Schlein.[4] Here, the analysis of the data implies $m_\sigma \simeq 730$ MeV. Theoretical curves for various values of $\lambda^2$ are plotted in Fig. 2(b), the experimental points being the "up-up" solution. Reasonable fits are obtained for $\lambda^2\simeq 1$. The theoretical curves appear to follow the general trend of the experimental points,[31] but do not reproduce the "oscillations" of the data, particularly the somewhat remarkable break at $\simeq 750$ MeV in the $I=2$ curve.

The parameter $\lambda^2$ governs the width $\Gamma_\sigma$ of the $\sigma\to 2\pi$ decay. One can calculate $\Gamma_\sigma$ directly from Eq. (2.10b). We find

$$\Gamma_\sigma=3\lambda^2(m_\sigma)^3(128\pi F_\pi^2)^{-1}. \tag{5.9}$$

Thus for the $\sigma$ meson of Malamud and Schlein

$$\Gamma_\sigma=245\text{ MeV},\quad m_\sigma=730\text{ MeV},\quad \lambda^2=0.75; \tag{5.10}$$

and for the $\sigma$ meson of Walker *et al.*

$$\Gamma_\sigma=650\text{ MeV},\quad m_\sigma=930\text{ MeV},\quad \lambda^2=0.9. \tag{5.11}$$

In general, one expects an $S$-wave resonance to have a relatively large width due to the absence of a centrifugal barrier. On the other hand, the treatment of the $\sigma$ meson as a resonance pole would become unreasonable unless $\Gamma_\sigma^2/m_\sigma^2 \lesssim 1$. The fact that values of $\lambda^2$ exist that fit the data without violating this criterion too seriously represents a theoretical consistency of the analysis. In general, it appears that the experimental situation in this energy domain is still quite ambiguous and that the current-algebra analysis does not definitely prefer either of the phase-shift analyses.

As mentioned above, the $\sigma$-meson pole plays an important role in the high-energy region. This can be seen most clearly by deleting the $\sigma$ meson from the amplitudes of Eqs. (4.3) and (4.4). The simplest way of doing this is to remove the pole to infinity,[32] i.e., let $m_\sigma\to\infty$ in these formulas. We find then

$$\begin{aligned}-16\pi A^0{}_0 \to{}& (2/F_\pi^2)(1-\tfrac14\lambda_A)^2(2s+m_\rho^2-4m_\pi^2)\\ &\times\{1-m_\rho^2(s-4m_\pi^2)^{-1}\ln[1+m_\rho^{-2}\\ &\times(s-4m_\pi^2)]\}+(m_\pi^2/F_\pi^2)[8-2\lambda_A\\ &\times(1-\tfrac18\lambda_A)]-\tfrac12(s/F_\pi^2)[9-\lambda_A\\ &\times(1-\tfrac18\lambda_A)],\end{aligned} \tag{5.12}$$

$$\begin{aligned}-16\pi A^2{}_0 \to{}& -(1/F_\pi^2)(2s+m_\rho^2-4m_\pi^2)(1-\tfrac14\lambda_A)^2\\ &\times\{1-m_\rho^2(s-4m_\pi^2)^{-1}\ln[1+m_\rho^{-2}\\ &\times(s-4m_\pi^2)]\}-(m_\pi^2/F_\pi^2)[4-\lambda_A\\ &\times(1-\tfrac18\lambda_A)]-(s/F_\pi^2)[-\tfrac32+\tfrac14\lambda_A\\ &\times(1-\tfrac18\lambda_A)].\end{aligned} \tag{5.13}$$

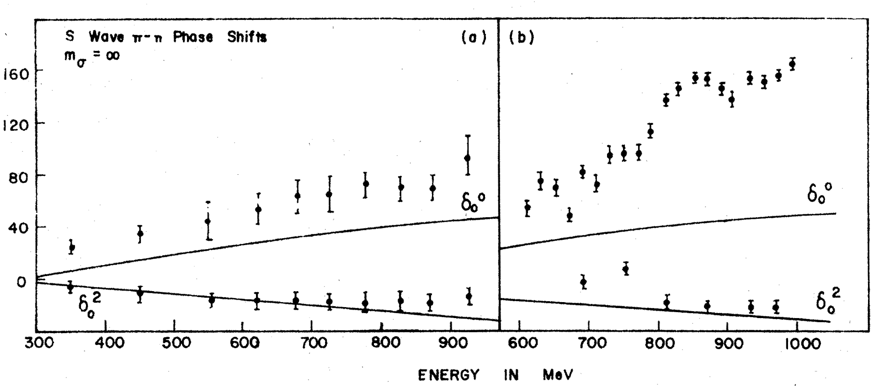


FIG. 3. $S$-wave phase shifts in the limit $m_\sigma\to\infty$. These amplitudes are independent of the value of $\lambda_1$ and $\lambda_2$. (a) The experimental points are from Walker *et al.* (Ref. 3). (b) The experimental points are the Malamud and Schlein "up-up" solution (Ref. 4).

[31] Fits of about the same quality can also be obtained to the less preferred "down-up" solution of Malamud and Schlein (Ref. 4).
[32] The $\pi\pi$ amplitudes given in Ref. 24 have deleted the $\sigma$ meson in this fashion.

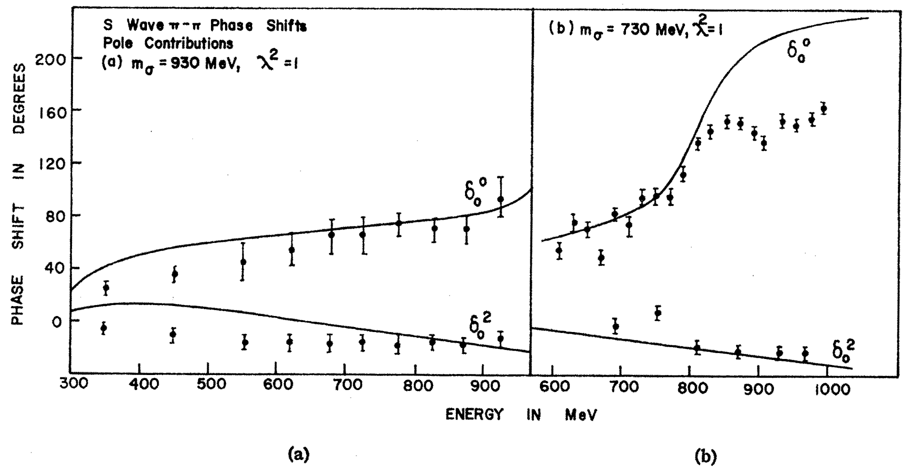


FIG. 4. $S$-wave phase shifts obtained from direct-channel $\sigma$ and crossed-channel $\sigma$ and $\rho$ poles only. (a) $m_\sigma=930$ MeV, $\lambda^2=1$. The experimental points are from Walker *et al.* (Ref. 3). (b) $m_\sigma=730$ MeV, $\lambda^2=1$. The experimental points are the Malamud and Schlein "up-up" solution (Ref. 4).

Note that the $\sigma$ coupling constants $\lambda_1$ and $\lambda_2$ have disappeared from the amplitudes in this limit. Equations (5.12) and (5.13) are plotted in Fig. 3 and are seen to be in sharp disagreement with both sets of data. Essentially, deleting the $\sigma$ meson in the hard-pion amplitude yields a result that is not much improved over the original soft-pion amplitude.[5] On the other hand, it would not be correct to say that the current-algebra conditions produce no significant effects beyond the threshold region. The hard-pion amplitudes (4.3) and (4.4) have two additional sets of terms characteristic of the current algebra and not found in a simple-pole picture of the scattering: the seagull terms and the energy and momentum-transfer dependence in the numerator functions of the $\rho$ and $\sigma$ poles. These additional terms tend to largely cancel the crossed $\rho$ and $\sigma$ pole contributions in the $I=0$ channel, leaving the direct $\sigma$ pole as the dominant structure in the high-energy region. This can be seen by considering an amplitude consisting purely of direct and crossed "dispersion theoretical" $\rho$ and $\sigma$ poles (i.e., poles with constant residues):

$$-16\pi A^0{}_0'=3\beta_0(s-m_\sigma^2)^{-1}+\{2\beta_0(4m_\pi^2-s)^{-1}\times\ln[1+(s-4m_\pi^2)m_\sigma^{-2}]+4n_0+4n_0(2s+m_\rho^2-4m_\pi^2)(4m_\pi^2-s)^{-1}\times\ln[1+(s-4m_\pi^2)m_\rho^{-2}]\},\quad(5.14)$$

$$-16\pi A^2{}_0'=2\beta_0(4m_\pi^2-s)\ln[1+(s-4m_\pi^2)m_\sigma^{-2}]-2n_0-2n_0(2s+m_\rho^2-4m_\pi^2)(4m_\pi^2-s)\times\ln[1+(s-4m_\pi^2)m_\rho^{-2}],\quad(5.15)$$

where

$$n_0\equiv m_\rho^2(2F_\pi^2)^{-1}(1-\tfrac{1}{4}\lambda_A)^2,\quad \beta_0\equiv m_\sigma^4(4F_\pi)^{-1}\lambda^2.\quad(5.16)$$

These amplitudes no longer obey the current-algebra conditions as the seagull parts and energy dependence in the numerators were needed for their satisfaction. From Fig. 4 one can see that the agreement with the data is considerably worsened, the theoretical curve overshooting the data for the $m_\sigma=930$ MeV case below the resonance due to the failure now of the above-mentioned cancellation. (As mentioned above, the analysis most likely already overestimates the phase shifts at the low-energy end.) Similarly, the curves overshoot the data above the resonance for the $m_\sigma=730$ MeV analysis.

### B. *P*-Wave Amplitudes

The $P$-wave amplitude of Eqs. (4.5) may be expanded near threshold to obtain the scattering length and effective range as defined by Eq. (5.1). We find for these parameters

$$24\pi m_\pi a^1{}_1=F_\pi^{-2}[1+\epsilon_\sigma\lambda_2(\lambda_1-\lambda_2)+6\epsilon_\rho(1-x^4y^2\tfrac{1}{4}\lambda_A)^2z^2],\quad(5.17)$$

$$2\pi m_\pi m_\rho^2(a^1{}_1)^2r^1{}_1=-F_\pi^{-2}[(1-x^4y^2\tfrac{1}{4}\lambda_A)^2z^2\times(1+4\epsilon_\rho)-(m_\rho^2/6m_\sigma^2)\lambda^2].\quad(5.18)$$

The leading term in the scattering length is again precisely the soft-pion result,[5] though here the $\epsilon_\rho$ corrections are a little larger than usual. The leading terms of the effective range depend on $x$, $y$, $z$, $\lambda_A$, and $m_\sigma$, showing that this is intrinsically a hard-pion result. While the total amplitude (when unitarized) has a resonance at the $\rho$ mass (from the direct-channel $\rho$ pole), it is interesting to note that the effective-range approximation (5.1) does not produce a resonance at the $\rho$ mass.[33] Thus for the case $m_\sigma=730$ MeV, $\lambda^2=0.75$, $\lambda_A=0.5$, $x=y=z=1$, Eq. (5.1) implies a resonance at $s=1.21m_\rho^2$. The fact that one is at all close to the $\rho$ is due to the numerical "accident" that $\lambda_A$ is so small.

The amplitude Eq. (4.5) can be unitarized in the fashion used for the $S$ waves. Away from the immediate region of threshold, this unitarized amplitude is well approximated by a simple direct-channel $\rho$ pole. This can be seen in Fig. 5 where the total amplitude is plotted in curve A while curve B represents that amplitude

[33] That one should choose $r^1{}_1$ so that the effective-range approximation resonates at the $\rho$ mass has been suggested by L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 346 (1968).

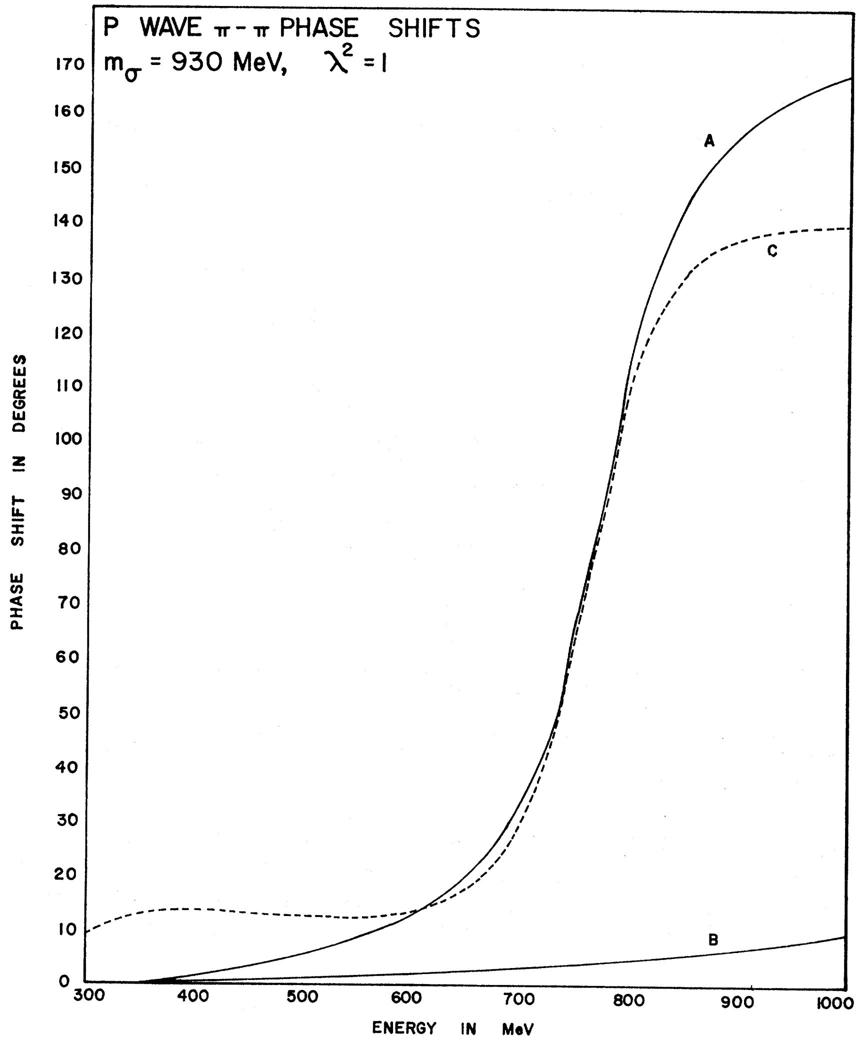


FIG. 5. $P$-wave phase shift. Curve A is the total theoretical amplitude. Curve B is the total amplitude with direct-channel $\rho$ pole deleted. Curve C is the amplitude obtained from direct-channel $\rho$ and crossed-channel $\rho$ and $\sigma$ poles only.

with the $\rho$ pole deleted. In the $I=0$ $S$-wave case, the seagull terms and crossed $\rho$ and $\sigma$ pole terms effectively canceled at high energies making the direct-channel $\sigma$ pole dominant in that region. For the $P$-wave case, both the seagull and crossed-pole pieces are fairly small in the entire region below the $\rho$ resonance. Above the $\rho$ they tend to become larger but again substantially cancel, leaving the direct $\rho$ pole as the dominant term in the entire energy range. Thus if we delete the seagull pieces (and the energy dependence of the numerators in the pole terms) we obtain the amplitude

$$\begin{aligned}-16\pi A^1{}_1{}' = {} & \tfrac{2}{3} n_0 (s-4m_\pi^2)(s-m_\rho^2) \\ & +\{2\beta_0(4m_\pi^2-s)^{-1}[(1+2m_\sigma^2(s-4m_\pi^2)^{-1}) \\ & \times\ln(1+(s-4m_\pi^2)m_\sigma^{-2})-2] \\ & +2n_0(2s+m_\rho^2-4m_\pi^2)(4m_\pi^2-s)^{-1} \\ & \times[(1+2m_\rho^2(s-4m_\pi^2)^{-1}) \\ & \times\ln(1+(s-4m_\pi^2)m_\rho^{-2})-2]\}, \quad (5.19)\end{aligned}$$

where $\beta_0$ and $n_0$ are defined in Eq. (5.16). The phase shift obtained from unitarizing Eq. (5.19) is plotted on graph C of Fig. 5. A significant deviation from the total amplitude above the $\rho$ results. The cancellation between seagull and crossed-pole contributions represents a remarkable feature of the hard-pion current-algebra amplitudes. Experimentally, the $P$-wave amplitude above 400 MeV is found to be well represented by a simple Breit-Wigner form.[34] Thus the hard-pion theory and experiment are in good agreement here.

*Note added in proof*. Recently, Gutay *et al.*[35] have given a general analysis of $\pi$-$\pi$ phase-shift data. They point out that there exists an ambiguity in extracting phase shifts from the data resulting in three roughly equally probable $I=0=J$ sets of solutions related according to

$$_{(III)}\delta^0{}_0=\tfrac{1}{2}\pi-[_{(I)}\delta^0{}_0-\delta^1{}_1], \quad _{(II)}\delta^0{}_0={}_{(I)}\delta^0{}_0-\pi.$$

[34] We note that this form need not be *a priori* assumed but can in fact be deduced as a result of the phase-shift analysis; P. E. Schlein (private communication).

[35] L. J. Gutay, D. D. Carmony, P. L. Czonka, F. J. Loeffler, and F. T. Meiere, Phys. Rev. (to be published).

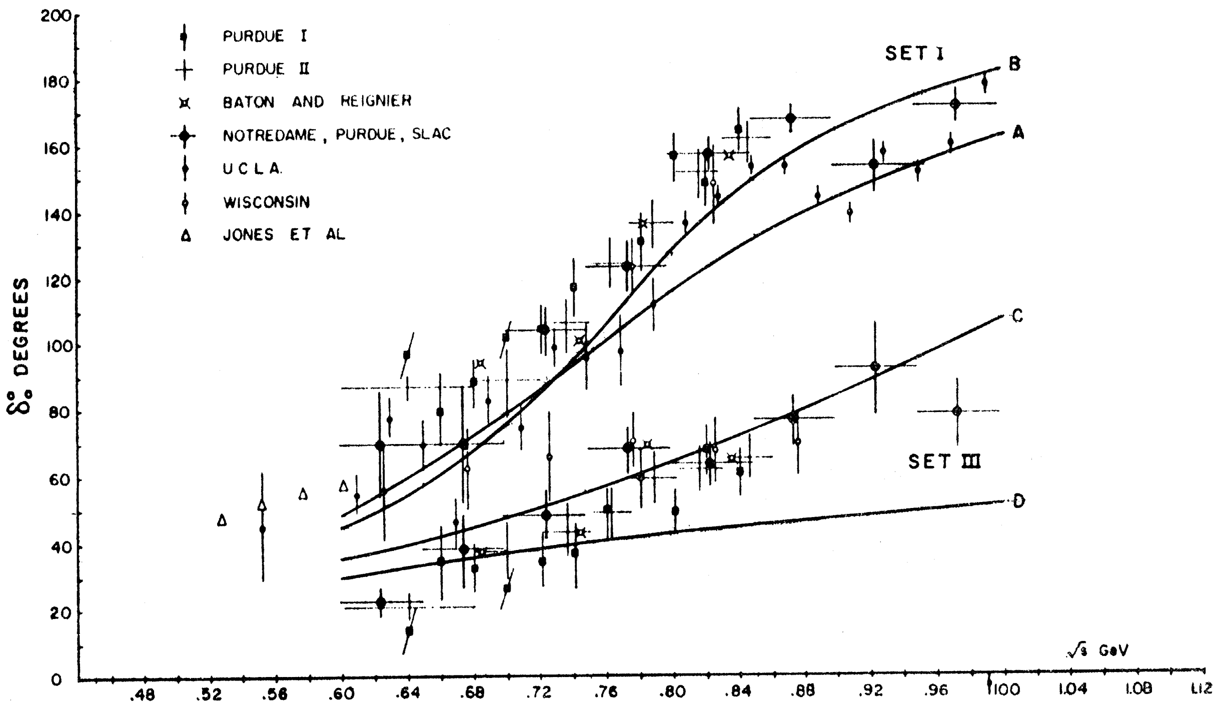


FIG. 6. $S$-wave phase shifts $\delta^0_0$. Data points taken from the analysis of Gutay *et al.* (Ref. 35). Curve A represents theoretical predictions with $m_\sigma=730$ MeV, $\lambda^2=1.0$, curve B with $m_\sigma=730$ MeV, $\lambda^2=0.75$, curve C with $m_\sigma=930$ MeV, $\lambda^2=0.75$, and curve D with $m_\sigma=\infty$.

Since ${}_{(II)}\delta^0_0<0$, it is inconsistent with the current-algebra solution. The theoretical curves are plotted against ${}_{(I)}\delta^0_0$ and ${}_{(III)}\delta^0_0$ in Fig. 6. A further analysis by Maratech *et al.*[36] in the region 500–900 MeV has resolved the ambiguity between ${}_I\delta^0_0$ and ${}_{III}\delta^0_0$ in favor of the Set-I solution. At the $K$-meson mass, these authors also find $\delta^0_0-\delta^2_0=30°\pm10°$ in good agreement with our Eq. (5.8).

## APPENDIX

We give below the explicit form of the "minimal" $\mathcal{L}_{(4)}$ derived in III. All terms contributing to a given scattering process are displayed together below.

(1) $\pi+\pi\rightarrow\pi+\pi$:

$$\mathcal{L}_{(4)}=-\tfrac{1}{8}(F_\pi\lambda_1)^{-1}g_{\sigma\pi\pi}(\varphi_a{}^2)^2-\tfrac{1}{2}[g_\rho{}^{-1}g_{\pi\pi\rho}+\tfrac{1}{2}(F_\pi\lambda_1)^{-1}\lambda_{\sigma\pi\pi}](\varphi_a)^2(\varphi^\mu{}_b)^2+\tfrac{1}{2}[g_\rho{}^{-1}g_{\pi\pi\rho}-F_\pi{}^{-1}\lambda_1\mu_{\sigma\pi\pi}](\varphi_a{}^\mu\varphi_a)^2. \quad \text{(A1)}$$

(2) $A_1\rightarrow 3\pi$ and $\pi+\pi\rightarrow\pi+A_1$:

$$\mathcal{L}_{(4)}=F_\pi{}^{-2}[g_A{}^2(m_A{}^2g_\rho)^{-1}\lambda_{\pi\rho A}-g_\rho m_A{}^{-2}\tilde{\lambda}_{\pi\rho A}-g_Am_A{}^{-2}\lambda_1\mu_{\sigma AA}]\varphi_a\varphi_{\mu a}H^{\mu\nu}{}_b\varphi_{\nu b}-F_\pi{}^{-1}m_A{}^2g_A{}^{-1}[F_\pi{}^2g_\rho{}^{-1}g_{\pi\pi\rho}$$
$$+\tfrac{1}{2}F_\pi(\lambda_1)^{-1}\lambda_{\sigma\pi\pi}](a^\mu{}_a\varphi_a)(\varphi_{\mu b}\varphi_b)+m_A{}^2g_A{}^{-1}[F_\pi g_\rho{}^{-1}g_{\pi\pi\rho}-\lambda_1\mu_{\sigma\pi\pi}](\varphi_a)^2(a^\mu{}_b\varphi_{\mu b}). \quad \text{(A2)}$$

(3) $\pi+\rho\rightarrow\pi+\rho$ and $\rho\rightarrow\pi+\pi+\gamma(I=1)$:

$$\mathcal{L}_{(4)}=\epsilon_{abc}\epsilon_{cde}[\tfrac{1}{2}\lambda_{\pi\rho A}g_{\pi\rho A}v_{\mu a}\varphi_b v^\mu{}_c\varphi_d-\lambda_{\pi\rho A}\tilde{\lambda}_{\pi\rho A}v_{\nu a}\varphi_b\varphi_{\mu c}G^{\mu\nu}{}_d+a_1\varphi_{\nu a}G^{\nu\mu}{}_b\varphi^\lambda{}_cG_{\lambda\mu d}]-\tfrac{1}{2}g_A(F_\pi g_\rho)^{-1}\mu_{\pi\rho A}(\varphi_aG^{\mu\nu}{}_a)^2$$
$$+\tfrac{1}{2}[g_A(F_\pi g_\rho)^{-1}\mu_{\pi\rho A}-\tfrac{1}{4}(F_\pi\lambda_1)^{-1}\lambda_{\sigma\rho\rho}](\varphi_a)^2G^{\mu\nu}{}_bG_{\mu\nu b}. \quad \text{(A3)}$$

(4) $\pi+A_1\rightarrow\pi+A_1$:

$$\mathcal{L}_{(4)}=-m_A{}^2(F_\pi g_A)^{-1}[g_A{}^2(m_A{}^2g_\rho)^{-1}\lambda_{\pi\rho A}-g_\rho m_A{}^{-2}\tilde{\lambda}_{\pi\rho A}-g_Am_A{}^{-2}\lambda_1\mu_{\sigma AA}](a_{\mu a}\varphi_{\nu b}+\varphi_{\mu a}a_{\nu b})\varphi_aH^{\mu\nu}{}_b$$
$$+\tfrac{1}{2}m_A{}^2g_A{}^{-1}[F_\pi{}^2g_\rho{}^{-1}g_{\pi\pi\rho}-F_\pi\lambda_1\mu_{\sigma\pi\pi}](a^\mu{}_a\varphi_a)^2-\tfrac{1}{2}m_A{}^2g_A{}^{-1}[F_\pi{}^2g_\rho{}^{-1}g_{\pi\pi\rho}+\tfrac{1}{2}F_\pi(\lambda_1)^{-1}\lambda_{\sigma\pi\pi}](\varphi_a)^2(a^\mu{}_b)^2$$
$$-\tfrac{1}{2}[\tfrac{1}{4}(F_\pi\lambda_1)^{-1}\lambda_{\sigma AA}+g_\rho(F_\pi g_A)^{-1}\mu_{\pi\rho A}](\varphi_a)^2H^{\mu\nu}{}_bH_{\mu\nu b}+\tfrac{1}{2}g_\rho(F_\pi g_A)^{-1}\mu_{\pi\rho A}(\varphi_aH^{\mu\nu}{}_a)^2. \quad \text{(A4)}$$

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[36] S. Marateck, V. Hagopian, W. Selove, L. Jacobs, F. Oppenheimer, W. Schultz, L. J. Gutay, D. H. Miller, J. Prentice, E. West, and W. D. Walker (report of work prior to publication).

(5) $\pi+\rho\to\rho+A_1$ and $\rho+\rho\to\pi+A_1$:

$$\mathfrak{L}_{(4)}=\epsilon_{abe}\epsilon_{cde}[\lambda_{\pi\rho A}g_{\rho\rho\rho}v_{\nu a}\varphi_b v_{\mu c}H^{\mu\nu}{}_d+\lambda_{\pi\rho A}\lambda_{\rho AA}v_{\nu a}\varphi_b a_{\mu c}G^{\mu\nu}{}_d-\lambda_{\pi\rho A}\lambda_{\pi\pi\rho}v_{\nu a}H^{\nu\mu}{}_b\varphi^\lambda{}_c G_{\lambda\mu d}+a_2 a_{\nu a}G^{\nu\mu}{}_b\varphi^\lambda{}_c G_{\lambda\mu d}$$
$$-\tfrac{3}{2}g_\rho(F_\pi g_A)^{-1}\mu_{\rho\rho\rho}\varphi_a H^{\mu\nu}{}_b G_\mu{}^\rho{}_c G_{\rho\nu d}-g_A(F_\pi g_\rho)^{-1}\mu_{\rho AA}\varphi_a G^{\mu\nu}{}_b G_\mu{}^\rho{}_c H_{\rho\nu d}]. \quad \text{(A5)}$$

(6) $\pi+A_1\to A_1+A_1$:

$$\mathfrak{L}_{(4)}=\epsilon_{abe}\epsilon_{cde}[-\tfrac{1}{2}g_\rho(F_\pi g_A)^{-1}\mu_{\rho AA}]\varphi_a H^{\mu\nu}{}_b H_\mu{}^\rho{}_c H_{\rho\nu d}+m_A{}^2 g_A{}^{-1}[g_A{}^2(m_A{}^2 g_\rho)^{-1}\lambda_{\pi\rho A}-g_\rho m_A{}^{-2}\tilde{\lambda}_{\pi\rho A}-g_A m_A{}^{-2}\lambda_1\mu_{\sigma AA}]$$
$$\varphi_a a_{\mu a}H^{\mu\nu}{}_b a_{\nu b}. \quad \text{(A6)}$$

(7) $\rho+A_1\to\rho+A_1$:

$$\mathfrak{L}_{(4)}=\epsilon_{abe}\epsilon_{cde}[\tfrac{1}{2}(\lambda_{\pi\rho A})^2 v_{\nu a}H^{\nu\mu}{}_b v^\lambda{}_c H_{\lambda\mu d}+\lambda_{\pi\rho A}\tilde{\lambda}_{\pi\rho A}v_{\nu a}H^{\nu\mu}{}_b a^\lambda{}_c G_{\lambda\mu d}+a_3 a_{\nu a}G^{\nu\mu}{}_b a^\lambda{}_c G_{\lambda\mu d}]. \quad \text{(A7)}$$

(8) $\pi+\sigma\to\pi+\sigma$:

$$\mathfrak{L}_{(4)}=d_2\varphi^\mu{}_a\varphi_{\mu a}\sigma^2+\tfrac{1}{2}[F_\pi{}^{-1}\lambda_1 g_{\sigma\pi\pi}-3(F_\pi\lambda_1)^{-1}g_{\sigma\sigma\sigma}](\varphi_a)^2\sigma^2+\tfrac{1}{2}[(F_\pi\lambda_1)^{-1}\mu_{\sigma\pi\pi}-g_A{}^{-1}\lambda_1\lambda_{\sigma\pi A}$$
$$-(F_\pi\lambda_1)^{-1}\lambda_{\sigma\sigma\sigma}](\varphi_a)^2\sigma^\mu\sigma_\mu+[(F_\pi\lambda_1)^{-1}\lambda_{\sigma\pi\pi}+F_\pi{}^{-1}\lambda_1\mu_{\sigma\pi\pi}-g_A{}^{-1}\lambda_1\tilde{\lambda}_{\sigma\pi A}-2F_\pi{}^{-1}\lambda_1\lambda_{\sigma\sigma\sigma}]\varphi^\mu{}_a\varphi_a\sigma_\mu\sigma. \quad \text{(A8)}$$

(9) $A_1+\sigma\to A_1+\sigma$:

$$\mathfrak{L}_{(4)}=d_1 a^\mu{}_a a_{\mu a}\sigma^2+e_1 a_{\mu a}H^{\mu\nu}{}_a\sigma_\nu\sigma. \quad \text{(A9)}$$

(10) $\pi+\sigma\to A_1+\sigma$ and $\pi+A_1\to\sigma+\sigma$:

$$\mathfrak{L}_{(4)}=d_3 a^\mu{}_a\varphi_{\mu a}\sigma^2+e_2\varphi_{\mu a}H^{\mu\nu}{}_a\sigma_\nu\sigma+[(F_\pi\lambda_1)^{-1}\tilde{\lambda}_{\sigma\pi A}+F_\pi{}^{-1}\lambda_1\lambda_{\sigma\pi A}-g_A{}^{-1}\lambda_1 g_{\sigma AA}+2g_A{}^{-1}m_A{}^2\lambda_1\lambda_{\sigma\sigma\sigma}]a^\mu{}_a\varphi_a\sigma_\mu\sigma. \quad \text{(A10)}$$

(11) $\pi+A_1\to\rho+\sigma$:

$$\mathfrak{L}_{(4)}=\epsilon_{abc}[\lambda_{\pi\rho A}g_{\sigma AA}v_{\mu a}\varphi_b a^\mu{}_c\sigma+\lambda_{\pi\rho A}(\mu_{\sigma AA}+\mu_{\sigma\pi\pi})v_{\mu a}\varphi_b H^{\mu\lambda}{}_c\sigma_\lambda-\lambda_{\pi\rho A}\lambda_{\sigma\pi\pi}v_{\nu a}H^{\nu\mu}{}_b\varphi_{\mu c}\sigma+b_2 a_{\nu a}G^{\nu\mu}{}_b\varphi_{\mu c}\sigma]$$
$$+c_1\varphi_{\nu a}G^{\nu\mu}{}_b H_{\mu\lambda c}\sigma^\lambda+[-g_A(F_\pi g_\rho)^{-1}\mu_{\sigma AA}-(F_\pi\lambda_1)^{-1}\tilde{\lambda}_{\pi\rho A}-g_A{}^{-1}\lambda_1\lambda_{\rho AA}]\epsilon_{abc}a_{\nu a}G^{\nu\mu}{}_b\varphi_c\sigma_\mu$$
$$+[F_\pi{}^{-1}\lambda_1\mu_{\pi\rho A}+\tfrac{1}{2}g_A(F_\pi g_\rho)^{-1}\lambda_{\sigma AA}-\tfrac{1}{2}g_\rho(F_\pi g_A)^{-1}\lambda_{\sigma\rho\rho}]\epsilon_{abc}\varphi_a G_{\mu\nu b}H^{\mu\nu}{}_c\sigma. \quad \text{(A11)}$$

(12) $\pi+\pi\to\rho+\sigma$; $\pi+\sigma\to\pi+\rho$:

$$\mathfrak{L}_{(4)}=\epsilon_{abc}[\lambda_{\pi\rho A}\tilde{\lambda}_{\sigma\pi A}v_{\mu a}\varphi_b\varphi^\mu{}_c\sigma+b_1\varphi_{\nu a}G^{\nu\mu}{}_b\varphi_{\mu c}\sigma]+[-g_A(F_\pi g_\rho)^{-1}\mu_{\sigma\pi A}+(F_\pi\lambda_1)^{-1}\lambda_{\pi\pi\rho}+g_A{}^{-1}\lambda_1\tilde{\lambda}_{\pi\rho A}]$$
$$\times\epsilon_{abc}\varphi_{\nu a}G^{\nu\mu}{}_b\varphi_c\sigma_\mu. \quad \text{(A12)}$$

(13) $A_1+A_1\to\sigma+\rho$; $A_1+\rho\to A_1+\sigma$:

$$\mathfrak{L}_{(4)}=\epsilon_{abc}[-\lambda_{\pi\rho A}\tilde{\lambda}_{\sigma\pi A}v_{\nu a}H^{\nu\mu}{}_b a_{\mu c}\sigma-\lambda_{\pi\rho A}\mu_{\sigma\pi A}v_{\nu a}H^{\nu\mu}{}_b H_{\mu\lambda c}\sigma^\lambda+b_3 a_{\nu a}G^{\nu\mu}{}_b a_{\mu c}\sigma]+c_2 a_{\nu a}G^{\nu\mu}{}_b H_{\mu\lambda c}\sigma^\lambda. \quad \text{(A13)}$$

There is no contribution from the minimal $\mathfrak{L}_{(4)}$ for the following processes: $\rho+\rho\to\rho+\rho$; $A_1+A_1\to A_1+A_1$; $\rho+\rho\to\rho+\sigma$; $\rho+\sigma\to\rho+\sigma$; $\sigma+\sigma\to\sigma+\sigma$.

The constants $a_i$, $b_i$, $c_i$, $d_i$, $e_i$ appearing above are restricted by the following equations:

$$2F_\pi a_1+g_A m_A{}^{-2}a_2-F_\pi g_\rho{}^{-1}\lambda_{\pi\pi\rho}=0, \quad \text{(A14a)}$$

$$F_\pi a_2+2g_A m_A{}^{-2}a_3+F_\pi g_\rho{}^{-1}\tilde{\lambda}_{\pi\rho A}=0, \quad \text{(A14b)}$$

$$2F_\pi b_1+g_A m_A{}^{-2}b_2-F_\pi g_\rho{}^{-1}\lambda_{\sigma\pi\pi}-F_\pi m_A{}^{-2}\lambda_{\sigma\pi A}\tilde{\lambda}_{\pi\rho A}=0, \quad \text{(A14c)}$$

$$F_\pi b_2+2g_A m_A{}^{-2}b_3-F_\pi g_\rho{}^{-1}\tilde{\lambda}_{\sigma\pi A}+F_\pi m_A{}^{-2}\lambda_{\sigma\pi A}\lambda_{\rho AA}=0, \quad \text{(A14d)}$$

$$F_\pi c_1+g_A m_A{}^{-2}c_2-F_\pi g_\rho{}^{-1}\mu_{\sigma\pi A}=0, \quad \text{(A14e)}$$

$$2g_A m_A{}^{-2}d_1+F_\pi d_3+F_\pi m_A{}^{-2}\lambda_{\sigma\pi A}g_{\sigma AA}=0, \quad \text{(A14f)}$$

$$2F_\pi d_2+g_A m_A{}^{-2}d_3+F_\pi m_A{}^{-2}\lambda_{\sigma\pi A}\tilde{\lambda}_{\sigma\pi A}=0, \quad \text{(A14g)}$$

$$F_\pi e_1+g_A m_A{}^{-2}e_2+F_\pi m_A{}^{-2}\lambda_{\sigma\pi A}{}^2=0. \quad \text{(A14h)}$$