Hard-Pion Current-Algebra Calculation of Meson Processes—N-Point Functions*

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Hard-pion techniques are presented for calculating T products of an arbitrary number of vector and axial-vector currents under the assumptions of single σ-, π-, ρ-, and A_1-meson saturation of intermediate sums, chiral $SU(2)\times SU(2)$-algebra commutation relations, conservation of vector current (CVC), and partial conservation of axial-vector current (PCAC). The single-meson dominance hypothesis is shown to imply that one calculates the T products, keeping only certain generalized tree and seagull diagrams. Alternatively, the assumption can be replaced by requiring that one calculate with an "effective" interaction Lagrangian $\mathcal{L}_I$ to lowest nonvanishing order. The conditions that the remaining hypotheses (current commutation relations, CVC, and PCAC) impose on $\mathcal{L}_I$ are expressed in terms of functional differential equations to determine the form of $\mathcal{L}_I$. These equations are shown to be consistent with each other and may in fact be integrated order by order. The $\mathcal{L}_I$ needed to calculate any four-point function is given explicitly.

I. INTRODUCTION

DURING the past two years, it has become apparent that the assumption that the conserved vector current (CVC) and the partially conserved axial-vector current (PCAC) obey local current commutation relations[1] represents a powerful tool for calculating low-energy pion processes. The numerous successes of these ideas in the "soft-pion" approximation,[2] where one assumes that the transition amplitude is slowly varying as one limits the pion four-momentum q^μ to zero, has tended to strengthen belief in their physical validity. The soft-pion assumption itself, however, limits the possibility of exploring the domain of applicability of the current algebras. For, even if one accepts the premises of the approximation, the continuation cannot be expected to be gentle beyond a few hundred MeV above threshold. Thus the soft-pion calculations are essentially approximate-threshold theorems and cannot be applied to higher-energy data. Further, since the mass shell is a point and not a continuum, the continuation of the pion momenta off its mass shell is not unique.[3] This ambiguity is related to the value of the so-called σ term or equal-time axial-vector current commutator $[\partial_\mu A^\mu{}_a, A^0{}_b]$.

One of the strengths of the solft-pion technique is that one does not need to know the detailed dynamical structure of the interaction to determine the scattering amplitude to the desired approximation. This is due to the fact that the limit $q^\mu \to 0$ suppresses most of the dynamical details. For example, consider the characteristic S-matrix element, $\langle\beta; q,b \text{ out}|\alpha; k,a \text{ in}\rangle$, which contains one pion of isotopic type a in the "in" state and one of type b in the "out" state. Contracting down both pions simultaneously, and using the fact that $\partial_\mu A^\mu{}_a$ is proportional to an interpolating field for the pion, one finds in the usual fashion[4] that the above amplitude is rigorously proportional to

$$\int d^4x d^4y\, e^{-iqy} e^{ikx}(q^2+m_\pi^2)(k^2+m_\pi^2)\{-iq_\nu\delta(x^0-y^0)$$
$$\times\langle\beta|[A^0{}_a(x),A^\nu{}_b(y)]|\alpha\rangle+q_\nu k_\mu\langle\beta|T(A^\mu{}_a(x)A^\nu{}_b(y))|\alpha\rangle$$
$$-\delta(x^0-y^0)\langle\beta|[A^0{}_b(y),\partial_\mu A^\mu{}_a(x)]|\alpha\rangle\}\,, \quad (1.1)$$

where m_π is the pion mass. In Eq. (1.1) the mass-shell limit $q^2,k^2\to -m_\pi^2$ is understood. The soft-pion approximation assumes, however, that one may evaluate Eq. (1.1) by limiting q^μ and k^μ to zero, keeping only the leading terms. Thus the first term can be evaluated by the current-algebra relation[5] (neglecting Schwinger terms)

$$\delta(x^0-y^0)[A^0{}_a(x),A^\nu{}_b(y)]=i\epsilon_{abc}V^\nu{}_c(x)\delta^4(x-y) \quad (1.2)$$

and is thus $O(q)$. The second term is of quadratic order in the momenta (aside from any poles that might develop in the matrix element at the unphysical point $q^\mu=0=k^\mu$) and is generally negligible compared to the first term.[6] The Adler consistency condition[7] may be used to show that the third term is comparable to the second in magnitude and hence also negligible. It is clear, however, that the dynamical details are hidden in the unevaluated T-product term since it is this term (rather than the commutator) that contributes on the mass shell. However, in the soft-pion limit it may be neglected.

The above discussion points up the fact that if one wishes to eliminate the soft-pion approximation and

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[1] M. Gell-Mann, Physics 1, 63 (1964).

[2] A summary of some of the achievements of the soft-pion current-algebra analyses can be found in the talk by R. F. Dashen, in *Proceedings of the Thirteenth International Conference on High-Energy Nuclear Physics* (University of California Press, Berkeley, 1967).

[3] L. S. Kisslinger, Phys. Rev. Letters 18, 861 (1967).

[4] See, for example, S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

[5] Our currents are normalized as in Eq. (1.2) to obey the usual chiral $SU(2)\times SU(2)$ algebra and PCAC is written as $\partial_\mu A^\mu{}_a = F_\pi m_\pi^2 \varphi_a$, where φ_a is the pion field. Thus the experimental value of F_π is 94 MeV. We use a metric with signature $+2$.

[6] An exception occurs for π-π scattering as discussed in Ref. 4.

[7] S. L. Adler, Phys. Rev. 137, B1022 (1965); 139, B1638 (1965).

deal only with "hard" or mass-shell pions, it is necessary to make some sort of further dynamical assumption. One possibility, first proposed by Weinberg in his treatment of T products of two current operators,[8] is that one saturate intermediate sums over states with single π-, ρ-, and A_1-meson states. In previous work,[9,10] vacuum-expectation values of T products of three current operators have been constructed, using the assumptions of CVC, PCAC, that the currents obey the $SU(2)\times SU(2)$ chiral algebra without q-number Schwinger terms (CCR), and the dynamical assumption of single-meson saturation of intermediate states. These T products allow one to calculate hard-meson vertex functions involving the π, ρ, and A_1 particles which appear (as discussed in II) to be in good agreement with experiment for a number of vertices having momentum transfer ranging from 0 to 1 GeV. It is the purpose of this paper to extend these ideas to treat T products of an arbitrary number of current operators to allow the calculation of meson scattering and production amplitudes. It will be seen that in fact the higher N-point functions can be calculated, assuming CVC, PCAC, current commutation relations (CCR), and single-meson saturation, and that the ability to do this depends upon an internal consistency between these physically disparate assumptions.

The usefulness of the T products of current operators resides in the fact that these operators may be used as interpolating fields for the π, ρ, and A_1 mesons. The vector current $V^\mu{}_a(x)$ can be viewed as being proportional to a ρ-meson field, while $A^\mu{}_a$ can be used to interpolate the π and A_1 fields. Thus the π-π scattering amplitude is proportional to

$$\int d^4xd^4yd^4zd^4\omega\; e^{-ik'x}e^{-iq'y}K(x)K(y)K(z)K(\omega) \times\langle 0|T(\partial_\mu A^\mu{}_a(x)\partial_\nu A^\nu{}_b(y)\partial_\alpha A^\alpha{}_c(z)\partial_\beta A^\beta{}_d(\omega))|0\rangle \times e^{ikz}e^{iq\omega}, \quad (1.3)$$

where $K(x)\equiv-\Box^2+m_\pi^2$. Similar rigorous expression holds for other scattering and decay amplitudes. In the following paper,[11] an application of the results given here will be made to π-π scattering. Theory and experiment appear to be in good agreement from threshold to 1 GeV.

Section II consists of a review of the results of paper I concerning the conditions imposed on the T products of three currents by the single-meson saturation assumption. These results are extended here to the N-point functions in Sec. III. It will be seen that this dynamical assumption can be rephrased in terms of writing an effective Lagrangian which is to be used only to lowest-order perturbation theory. The condition of single-particle saturation thus reduces simply to the statement that one only calculates tree and generalized sea-gull diagrams. Section IV is concerned with the constraints imposed on the effective Lagrangian by the current-commutation relations, CVC, and PCAC conditions. These requirements are expressed as a set of functional differential equations to be satisfied by the Lagrangian. Existence of solutions of these equations is demonstrated in Appendix A and the general structure of the resultant Lagrangian is discussed in Sec. V. A power-series solution of the equations, good for any N-point function, may be straightforwardly obtained. A partial closed-form integration of the equations is given in Appendix B. The explicit form of the effective Lagrangian appropriate for calculating any four-point scattering amplitude is given in Appendix C.

II. SINGLE-MESON SATURATION CONDITION—THREE-POINT FUNCTION

In previous work,[9] a description was given of the conditions imposed on T products of three currents by the assumption that intermediate sums are saturated by single π-, ρ-, and A_1- meson states. It was seen there that these conditions could be rephrased in terms of an effective Lagrangian. In this section we review this analysis for the convenience of the reader. (In Sec. III, the extension to the case of N-point functions of vector and axial-vector currents will be given.) Consider, for example, the T product

$$F^{\alpha\mu\beta}(x,y,z)\equiv\langle 0|T(A^\alpha{}_a(x)V^\mu{}_c(z)A^\beta{}_b(y))|0\rangle. \quad (2.1)$$

This quantity may be expanded in terms of its six time orderings. For the case $x^0>z^0>y^0$ one has

$$F^{\alpha\mu\beta}=\sum_{n,m}\langle 0|A^\alpha{}_a|n\rangle\langle n|V^\mu{}_c|m\rangle\langle m|A^\beta{}_b|0\rangle. \quad (2.2)$$

The only single-meson states that can enter in the sums of Eq. (2.2) are the π and A_1 states. For the π states, Eq. (2.2) reduces to

$$F^{\alpha\mu\beta}=\sum_{a_1,a_2}\int d^3q_1d^3q_2\langle 0|A^\alpha{}_a|\pi q_1a_1\rangle \times\langle\pi q_1a_1|V^\mu{}_c|\pi q_2a_2\rangle\langle\pi q_2a_2|A^\beta{}_b|0\rangle, \quad (2.3)$$

[8] S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

[9] R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters 19, 1085 (1967); Phys. Rev. 174, 1999 (1968) (hereafter referred to as I); 174, 2008 (1968) (hereafter referred to as II).

[10] Results equivalent to those of Ref. 9 (for the π-ρ-A_1 system) but using different techniques have also been obtained by H. Schnitzer and S. Weinberg [Phys. Rev. 164, 1828 (1967)] and by S. G. Brown and G. B. West [Phys. Rev. Letters 19, 812 (1967); and Phys. Rev. 168, 1605 (1968)]. Equivalent results for higher point functions have been obtained by I. S. Gerstein and H. J. Schnitzer, Phys. Rev. 170, 1638 (1968). Similar results, but using different physical assumptions, have been obtained by J. Schwinger [Phys. Letters 24B, 473 (1967); Phys. Rev. 167, 1432 (1968)], J. Wess and B. Zumino [Phys. Rev. 163, 1727 (1967)], and B. W. Lee and H. T. Nieh [Phys. Rev. 166, 1507 (1968)] using a "phenomenological" Lagrangian approach; and by T. Das, V. S. Mathur, and S. Okubo [Phys. Rev. Letters 19, 812 (1967)] and D. A. Greffen [Phys. Rev. Letters 19, 770 (1967)] using dispersion-relation techniques.

[11] R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, following paper, Phys. Rev. 175, 1820 (1968). Some of the results in this paper are reported in Phys. Rev. Letters 20, 475 (1968).

with a similar form for the A_1 state. Similarly, the time ordering $y^0 > x^0 > z^0$ gives rise to the expression

$$F^{\alpha\mu\beta} = \sum_{n,m} \langle 0 | A^\beta{}_b | n \rangle \langle n | A^\alpha{}_a | m \rangle \langle m | V^\mu{}_c | 0 \rangle . \quad (2.4)$$

In this case the state $|n\rangle$ may be a π or A_1 state and $|m\rangle$ must be a single-ρ state. A diagrammatic representation of these two time orderings is given in Figs. 1(a) and 1(b). In addition to single-particle states of the type considered above, Lorentz convariance (and hence crossing) require that one also include all two-particle intermediate states where one of the two mesons is a "spectator," i.e., its momentum is not summed over. Thus for the time ordering of Eq. (2.2), one must also include into the evaluation of $F^{\alpha\mu\beta}$ the term

$$F^{\alpha\mu\beta} = \sum_{a_1 a_2 a_3} \int d^3q_1 d^3q_2 d^3p_1 \langle 0 | A^\alpha{}_a | \pi q_1 a_1, \rho q_2 b_2 \rangle$$
$$\times \langle \pi q_1 a_1, \rho p_1 a_3 | V^\mu{}_c | \pi q_2 a_2 \rangle \langle \pi q_2 a_2 | A^\beta{}_b | 0 \rangle , \quad (2.5a)$$

where the vector-current matrix element is to be approximated by

$$\langle \pi q_1 a_1, \rho p_1 a_3 | V^\mu{}_c | \pi q_2 a_2 \rangle = \delta^3(\mathbf{q}_1 - \mathbf{q}_2) \delta_{a_1 a_2} \langle \rho p_1 a_3 | V^\mu{}_c | 0 \rangle . \quad (2.5b)$$

Thus, actually one sums over the momentum of only a *single* particle in each sum. The diagram representing Eqs. (2.5) is shown in Fig. 1(c) and is the crossed diagram to Fig. 1(b).

If one retains the single-meson states and all the two-particle states in which one meson is a spectator, one may obtain a covariant and crossing symmetric approx-

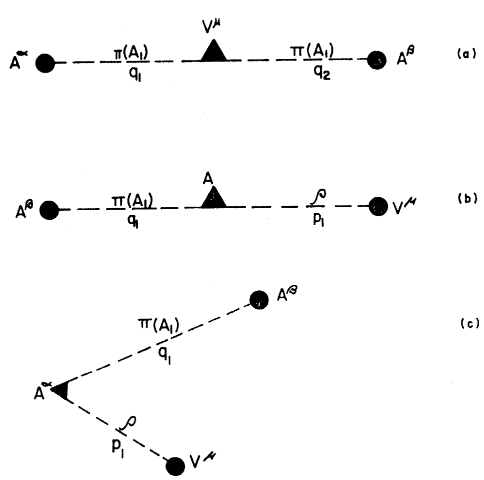

FIG. 1. (a) Diagram representing the time ordering $x^0 > z^0 > y^0$ in the three-point function $\langle 0 | T[A^\alpha{}_a(x) V^\mu{}_c(z) A^\beta{}_b(y)] | 0 \rangle$ for the case of π or A_1 intermediate states. The circles represent the vacuum to one-particle matrix elements of the current, while the solid triangles the one-particle matrix elements. (b) Diagram for time ordering $x^0 > y^0 > z^0$ with a π or A_1 for the first intermediate state and a ρ for the second. (c) Diagram with time ordering of (a) for two-body intermediate state where π (or A_1) is a "spectator." This diagram is the crossed diagram of (b).

imation. In the various time orderings making up the three-point T product one encounters only the vacuum–one-particle matrix elements,

$$\langle 0 | A^\mu{}_a(0) | \pi q, b \rangle = i N_\pi \delta_{ab} F_\pi q^\mu , \quad (2.6a)$$

$$\langle 0 | A^\mu{}_a(0) | A_1, q, b, \sigma \rangle = N_A \delta_{ab} g_A \, {}_A\epsilon^{\mu\sigma}(q) , \quad (2.6b)$$

$$\langle 0 | V^\mu{}_a(0) | \rho, q, b, \sigma \rangle = N_\rho \delta_{ab} g_\rho \, {}_\rho\epsilon^{\mu\sigma}(q) , \quad (2.6c)$$

and the one-particle matrix elements of $V^\mu{}_a$ and $A^\mu{}_a$, or their crossing related vacuum–two-meson matrix elements. [In Eqs. (2.6), N_π, etc., are the conventional Bose normalizing factors and $\epsilon^{\mu\sigma}$ is the vector-meson polarization vector of helicity[12] σ.] Thus Eqs. (2.6) are essentially the defining equations for F_π, g_A, and g_ρ. Single-particle saturation suggests that in the one-particle matrix elements, the vector current links to the particles only through the ρ meson while the axial-vector current links to the particles only through the π and A_1 mesons. One may write, without loss of generality,

$$\langle Bq_1 a | V^\mu{}_c(0) | Cq_2 b \rangle = [i \epsilon_{abc} \, {}_\rho\Delta^\mu{}_\lambda(k) \Gamma^\lambda{}_{B\rho C}(q_1, q_2)] N_B N_C , \quad (2.7a)$$

and

$$\langle Bq_1 a | A^\mu{}_c(0) | Cq_2 b \rangle = \epsilon_{abc} [{}_A\Delta^\mu{}_\lambda(k) \Gamma^\lambda{}_{BAC}(q_1, q_2) + {}_\pi\Delta(k) \Gamma^\mu{}_{B\pi C}(q_1, q_2)] N_B N_C , \quad (2.7b)$$

where ${}_\rho\Delta^\mu{}_\lambda$ and ${}_A\Delta^\mu{}_\lambda$ are the ρ and A_1 propagators (with physical ρ and A_1 masses)

$${}_\rho\Delta^\mu{}_\lambda(k) \equiv (k^2 + m_\rho^2)^{-1} (\delta^\mu{}_\lambda + k^\mu k_\lambda m_\rho^{-2}) , \quad (2.8a)$$

$${}_A\Delta^\mu{}_\lambda(k) \equiv (k^2 + m_A^2)^{-1} (\delta^\mu{}_\lambda + k^\mu k_\lambda m_A^{-2}) , \quad (2.8b)$$

and ${}_\pi\Delta \equiv (k^2 + m_\pi^2)^{-1}$ is the pion propagator. In Eqs. (2.7), we use B, C to be the particle labels (π, ρ, or A_1) allowed by G parity and $k^\mu \equiv q_1{}^\mu - q_2{}^\mu$. For low momentum transfer, the particle vertex functions appearing in Eqs. (2.7) presumably can be expanded in a power series. For example, one might write for the π-ρ-π vertex

$$\Gamma^\mu{}_{\pi\rho\pi}(q_1, q_2) = (q_1{}^\mu + q_2{}^\mu)(\alpha_1 + \alpha_2 k^2 + \cdots) \quad (2.9)$$

and similarly for the other vertex functions.

It is now straightforward to introduce, in a phenomenological fashion, field operators that reproduce the matrix elements of Eqs. (2.6) and (2.7). Thus, let $\tilde{\varphi}_a(x)$, $\tilde{v}^\mu{}_a(x)$, and $\tilde{a}^\mu{}_a(x)$ be π, ρ, and A_1 in-field operators that annihilate and create the corresponding physical particles. Then clearly the right-hand sides of Eqs. (2.6a) and (2.6b) will be reproduced if $A^\mu{}_a$ is replaced by $g_A \tilde{a}^\mu{}_a + F_\pi \partial^\mu \tilde{\varphi}_a$, and the right-hand side of Eq. (2.6c) will be obtained if $V^\mu{}_a$ is replaced by $g_\rho \tilde{v}^\mu{}_a$. The one-particle matrix elements of Eq. (2.7) can similarly be obtained if $V^\mu{}_c$ and $A^\mu{}_c$ are replaced by operators bilinear in the in-fields. Thus, for the π-ρ-π vertex case

[12] We normalize states so that $N_\pi{}^{(q)} = [2\omega_q(2\pi)^3]^{-1/2}$, where $\omega_q \equiv (\mathbf{q}^2 + m_\pi{}^2)^{1/2}$. The polarization vectors $\epsilon^{\mu\sigma}$ obey the conditions $q_\mu \epsilon^{\mu\sigma} = 0$ and $\epsilon^{\mu\sigma *} \epsilon_\mu{}^{\sigma'} = \delta^{\sigma\sigma'}$.

of Eq. (2.7a), $V^\mu{}_c(x)$ must be bilinear in $\tilde\varphi_a$, and to reproduce explicitly the series form of Eq. (2.9) one may use the operator

$$\int d^4y\;{}_\rho\Delta^\mu{}_\lambda(x-y)\,\epsilon_{abc}[\alpha_1-\alpha_2\Box^2+\cdots]\times\tilde\varphi_a(y)\partial_\mu\tilde\varphi_b(y). \quad (2.10)$$

Analogously, to obtain the $B=\pi$, $C=\rho$ matrix element in Eq. (2.7b) one would use for $A^\mu{}_c(x)$ the operator

$$\epsilon_{abc}\int d^4y\{{}_A\Delta^\mu{}_\lambda(x-y)[\beta_1\tilde\varphi_a(y)\tilde v^\lambda{}_b(y)+\cdots]+{}_\pi\Delta(x-y)[\gamma_1\tilde\varphi_a(y)\tilde v^\mu{}_b(y)+\cdots]\}. \quad (2.11)$$

In general, then, all the matrix elements of Eqs. (2.6) and (2.7) can phenomenologically be reproduced by the choices

$$V^\mu{}_c(x)=g_\rho\tilde v^\mu{}_c+\epsilon_{abc}\int d^4y_\rho\Delta^\mu{}_\lambda(x-y)\times[\alpha_1-\alpha_2\Box^2+\cdots]\tilde\varphi_a(y)\partial^\mu\tilde\varphi_b(y)+\cdots, \quad (2.12a)$$

and

$$A^\mu{}_c(x)=g_A\tilde a^\mu{}_c+F_\pi\partial^\mu\tilde\varphi_a+\epsilon_{abc}\int d^4y\times\{{}_A\Delta^\mu{}_\lambda(x-y)[\beta_1\tilde\varphi_a\tilde v^\lambda{}_b+\cdots]+{}_\pi\Delta(x-y)[\gamma_1\tilde\varphi_a\tilde v^\mu{}_b+\cdots]\}+\cdots, \quad (2.12b)$$

where the omitted terms represent the other bilinear structures needed to obtain all the matrix elements of Eqs. (2.7). Note that the form of the operators $V^\mu{}_c$ and $A^\mu{}_c$ automatically guarantees the crossing symmetry of the matrix elements.

It is more convenient to replace the in-field characterization of $V^\mu{}_c$ and $A^\mu{}_c$ by one involving Heisenberg fields. Thus let $v^\mu{}_c(x)$, $a^\mu{}_c(x)$, and $\varphi_a(x)$ be a set of ρ, A_1, and π Heisenberg field operators obeying the Heisenberg field equations

$${}_\rho K^\mu{}_\lambda(x)v^\lambda{}_c(x)=g_\rho^{-1}\epsilon_{abc}[\alpha_1-\alpha_2\Box^2+\cdots]\times\varphi_a(x)\partial^\mu\varphi_b(x)+\cdots, \quad (2.13a)$$

$${}_A K^\mu{}_\lambda(x)a^\lambda{}_c(x)=g_A^{-1}\epsilon_{abc}[\beta_1\varphi_a(x)v^\mu{}_b(x)+\cdots]+\cdots, \quad (2.13b)$$

$${}_\pi K(x)\varphi_c(x)=F_\pi^{-1}\epsilon_{abc}[\gamma_1\varphi_a(x)v^\mu{}_b(x)+\cdots]+\cdots, \quad (2.13c)$$

where ${}_\rho K^\mu{}_\lambda$ and ${}_A K^\mu{}_\lambda$ are the ρ and A_1 Proca operators $[{}_\rho K^\mu{}_\lambda\equiv(-\Box^2+m_\rho^2)\delta^\mu{}_\lambda+\partial^\mu\partial_\lambda]$ and ${}_\pi K\equiv-\Box^2+m_\pi^2$. Then Eqs. (2.12) are clearly equivalent to the assumption that

$$V^\mu{}_c(x)=g_\rho v^\mu{}_c(x), \quad (2.14a)$$

$$A^\mu{}_c(x)=g_A a^\mu{}_c(x)+F_\pi\partial^\alpha\varphi_c(x), \quad (2.14b)$$

provided we agree to solve the Heisenberg equations (2.13) in an in-field expansion only to *first order* in the "coupling constants" α_i, β_i, γ_i. Returning now to the T product of Eq. (2.1), one may verify that the requirement of single-meson saturation in the intermediate sums such as Eqs. (2.2), (2.4), (2.5), etc., implies that one is to evaluate the T product by adding up all contributions where two of the currents are replaced by the terms linear in the in-fields in Eqs. (2.12) and one current by a bilinear structure. Equivalently, one arrives at the final result: Single-meson saturation implies that one may evaluate Eq. (2.1) by replacing the currents by the Heisenberg fields according to Eq. (2.14) and then calculating the T product to first order in the coupling constants α_i, β_i, γ_i, etc., using the field equations[13] (2.13).

The currents of Eqs. (2.12) [or equivalently those obtained from the in-field expansion in Eqs. (2.13) and (2.14)] are not, in general, local-field operators whose commutators vanish for spacelike separations. This is due to the presence of the nonlocal propagator factors ${}_A\Delta^{\mu\nu}(x-y)$, ${}_\rho\Delta^{\mu\nu}(x-y)$, and ${}_\pi\Delta(x-y)$. On the other hand, we wish to impose the requirement that $V^\mu{}_c$ and $A^\mu{}_c$ obey a local-current algebra. This can clearly be achieved by requiring that the Heisenberg operators $v^\mu{}_a(x)$, $a^\mu{}_a(x)$, and $\varphi_a(x)$ be *local*-field operators. A straightforward (and perhaps the only) way of guaranteeing this locality condition is to require that the Heisenberg equations be derivable from a local-field Lagrangian. Since bilinear source terms appear on the right in Eqs. (2.13), the appropriate interaction Lagrangian must therefore be cubic. We therefore chose the following *effective Lagrangian* to simulate Eqs. (2.13):

$$\mathfrak{L}=\mathfrak{L}_{0\pi}+\mathfrak{L}_{0\rho}+\mathfrak{L}_{0A}+\mathfrak{L}_{(3)\pi\rho A}, \quad (2.15a)$$

where the free-particle Lagrangian is

$$\mathfrak{L}_{0\pi}+\mathfrak{L}_{0\rho}+\mathfrak{L}_{0A}=-\varphi^\mu{}_a\partial_\mu\varphi_a+\tfrac{1}{2}(\varphi^\mu{}_a\varphi_{\mu a}-m_\pi^2\varphi_a^2)-\tfrac{1}{2}G^{\mu\nu}{}_a(\partial_\mu v_{\nu a}-\partial_\nu v_{\mu a})+\tfrac{1}{4}G^{\mu\nu}{}_aG_{\mu\nu a}-\tfrac{1}{2}m_\rho^2v^\mu{}_av_{\mu a}$$
$$-\tfrac{1}{2}H^{\mu\nu}{}_a(\partial_\mu a_{\nu a}-\partial_\nu a_{\mu a})+\tfrac{1}{4}H^{\mu\nu}{}_aH_{\mu\nu a}-\tfrac{1}{2}m_A^2a^\mu{}_aa_{\mu a}, \quad (2.15b)$$

and the interaction Lagrangian is

$$\mathfrak{L}_{(3)\pi\rho A}=\tfrac{1}{2}\epsilon_{abc}[2g_{\pi\pi\rho}\varphi^\mu{}_b\varphi_cv_{\mu a}+\lambda_{\pi\pi\rho}\varphi_{\mu a}\varphi_{\nu b}G^{\nu\mu}{}_c+2g_{\pi\rho A}v_{\mu a}\varphi_ba^\mu{}_c+2\mu_{\pi\rho A}\varphi_aG^{\mu\nu}{}_bH_{\mu\nu c}+2\lambda_{\pi\rho A}v_{\mu a}\varphi_{\nu b}H^{\mu\nu}{}_c$$
$$+2\tilde\lambda_{\pi\rho A}a_{\mu a}\varphi_{\nu b}G^{\mu\nu}{}_c+g_{\rho\rho\rho}v_{\mu a}v_{\nu b}G^{\nu\mu}{}_c+2g_{\rho AA}v_{\mu a}a_{\nu b}H^{\nu\mu}{}_c$$
$$+\lambda_{\rho AA}a_{\mu a}a_{\nu b}G^{\nu\mu}{}_c+\mu_{\rho\rho\rho}G_{\mu\nu a}G^{\nu\lambda}{}_bG_\lambda{}^\mu{}_c+\mu_{\rho AA}G_{\mu\nu a}H^{\nu\lambda}{}_bH_\lambda{}^\mu{}_c]. \quad (2.15c)$$

[13] Note that the current-field association of Eq. (2.14) is merely a phenomenological consequence of the single-meson saturation approximation rather than a fundamental postulate. In this respect our work differs from that of T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).

In Eqs. (2.15) we have used (for later convenience) the "first-order" form of the Lagrangian, where (φ_μ,φ), $(G_{\mu\nu},v_\mu)$, and $(H_{\mu\nu},a_\mu)$ are to be varied independently to yield coupled first-order differential field equations.[14] The coupling constants $g_{\pi\pi\rho}$, $\lambda_{\pi\pi\rho}$, etc., are, at this point in the analysis, arbitrary and can, of course, be related to the constants α_i, β_i, γ_i, etc., appearing in Eqs. (2.12) and (2.13). $\mathcal{L}_{(3)\pi\rho A}$ is the most general cubic interaction Lagrangian involving the π, ρ, and A_1 fields without explicit derivatives. Since the number of derivatives in $\mathcal{L}_{(3)}$ controls the number of derivatives on the right-hand sides of Eq. (2.13), this restriction controls the amount of momentum transfer assumed in the vertex functions of Eq. (2.7).[15] It should be stressed that no *a priori* fundamental significance is attached to the Lagrangian of Eqs. (2.15). Rather, it is introduced as a convenient mathematical device to guarantee locality of the current operators of Eqs. (2.14) and (2.13). The single-meson saturation condition can now be rephrased by requiring that the T product of Eq. (2.1) is to be evaluated using the effective Lagrangian of Eq. (2.15) [and the defining relations (2.14)] to *first order* in the coupling constants.

III. SINGLE-MESON SATURATION CONDITION—*N*-POINT FUNCTIONS

In this section we extend the previous results and investigate the conditions imposed on T products of an arbitrary number of current operators by the single-meson saturation assumption. We begin by considering

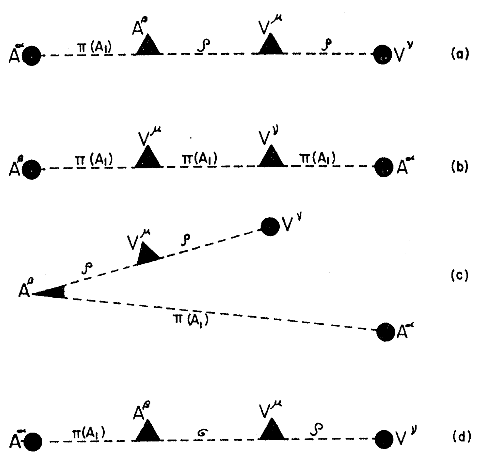

FIG. 2. (a) Four-point T product of Eq. (3.1) for the time ordering $x^0>y^0>z^0>\omega^0$ assuming single π-, A_1-, or ρ-meson intermediate states. (b) T product of Eq. (3.1) for the time ordering $y^0>z^0>\omega^0>x^0$. (c) Time ordering $y^0>z^0>\omega^0>x^0$ with two-body π (or A_1)–ρ states where the π (or A_1) is a spectator particle. (d) Diagram for time ordering $x^0>y^0>z^0>\omega^0$ with σ-meson intermediate state.

a characteristic four-point function

$$F^{\alpha\beta\mu\nu}(x,y,z,\omega)\equiv\langle T(A^\alpha{}_a(x)A^\beta{}_b(y)V^\mu{}_c(z)V^{\nu\lambda}(\omega))\rangle \quad (3.1)$$

and proceed in a fashion similar to the analysis of Sec. II. For the time ordering $x^0>y^0>z^0>\omega^0$, one has

$$F^{\alpha\beta\mu\nu}=\sum_{n,m,r}\langle 0|A^\alpha{}_a|n\rangle\langle n|A^\beta{}_b|m\rangle \times\langle m|V^\mu{}_c|r\rangle\langle r|V^\nu{}_d|0\rangle. \quad (3.2)$$

Assuming single-π, -ρ, or -A_1 intermediate states implies that the sum over n be restricted to single-π or single-A_1 states while the remaining two sums involve single-ρ states. This situation is represented diagrammatically in Fig. 2(a). Analogously, for the time ordering $y^0>z^0>\omega_0>x^0$ one has

$$F^{\alpha\beta\mu\nu}=\textstyle\sum_{n,m,r}\langle 0|A^\beta{}_b|n\rangle\langle n|V^\mu|m\rangle \times\langle m|V^\nu|r\rangle\langle r|A^\alpha{}_a|0\rangle. \quad (3.3)$$

The single-intermediate-meson condition requires all sums to involve either a single π or A_1 meson, as shown in Fig. 2(b) Lorentz convariance and crossing symmetry imply (as in the three-point functions) that one also include all two-meson intermediate states where one meson is a "spectator," i.e., its momentum is not summed over. Thus, for the time ordering of Eq. (3.3), one would include the case where $|n\rangle$ and $|m\rangle$ constitute a two-particle π-ρ state and $|r\rangle$ is a π state. One has for this contribution

$$\begin{aligned}F^{\alpha\beta\mu\nu}=&\textstyle\sum\langle 0|A^\beta{}_b|\pi,q_1a_1;\rho,p_1a_2\rangle\\&\times\langle\pi,q_1a_1;\rho,p_1b_1|V^\mu{}_c|\pi,q_2a_2;\rho,p_2b_2\rangle\\&\times\langle\pi,q_2a_2;\rho p_2b_2|V^\nu{}_d|\pi q_3a_3\rangle\langle\pi q_3a_3|A^\alpha{}_a|0\rangle,\end{aligned} \quad (3.4)$$

where the π mesons in the first two sums are spectators, i.e., one makes the approximation

$$\langle\pi,q_1a_1;\rho,p_1b_1|V^\mu{}_c|\pi,q_2a_2;\rho p_2b_2\rangle =\delta^3(\mathbf{q}_1-\mathbf{q}_2)\delta a_1a_2\langle\rho,p_1b_1|V^\mu{}_c|\rho,p_2b_2\rangle, \quad (3.5a)$$

$$\langle\pi,q_2a_2;\rho,p_2b_2|V^\nu{}_d|\pi,q_3a_3\rangle =\delta^3(\mathbf{q}_2-\mathbf{q}_3)\delta a_2a_3\langle\rho,p_2b_2|V^\nu{}_d|0\rangle. \quad (3.5b)$$

The diagrammatic representation is given in Fig. 2(c) and represents the crossed diagram to Fig. 2(a).

The intermediate-state analysis of $F^{\alpha\beta\mu\nu}$ given above is the direct generalization of the three-point analysis of Sec. II. In addition, there exist two new classes of states to be included here which are not found in the vertex functions. First, there could exist intermediate states in the sums of Eqs. (3.2), (3.3), etc., with angular momentum zero and isospin zero or 2. Analysis of the single and double π-meson production data in pion-nucleon scattering now seems to indicate the existence of a resonance with the quantum numbers $J=0=I$ (and positive G parity) somewhere between 700 MeV and 1 GeV.[16] It is natural then to include this particle

[14] Thus, (φ_{0a},φ_a), (G_{0ia},v_{ia}), and (H_{0ia},a_{ia}) are the canonically conjugate pairs of variables for the π, ρ, and A_1 fields.

[15] An equivalent assumption on the amount of momentum transfer appearing in the vertex functions is also explicitly made in the methods of Ref. 10.

[16] W. D. Walker, J. Carroll, A. Garfinkel, and B. Y. Oh, Phys. Rev. Letters **18**, 630 (1967); P. E. Schlein, *ibid.* **119**, 1052 (1967); E. Malamud and P. E. Schlein, *ibid.* **19**, 1056 (1967).

(which we will refer to as the σ meson) in our intermediate sums on the same basis that one includes the π, ρ, and A_1 mesons.[17] (We will see in Sec. IV that the inclusion of the σ meson is also strongly suggested by the current-algebra conditions.) Accepting the existence of the $I=0$ σ meson implies that one must include it in intermediate sums in all places not forbidden by the usual selection rules. An example is shown in Fig. 2(d). This implies the existence of new vertex functions involving the σ particle and thus the particle labels B, C in Eqs. (2.7) must now be allowed to range over σ, π, ρ, A_1.

The second class of new states appearing in the intermediate sums of four-point functions involves the spectator particles. In three-point functions, there can be at most two-particle intermediate states with one spectator, and an example of this for a four-point function was given in Fig. 2(c). However, the presence of four current operators in $F^{\alpha\beta\mu\nu}$ allows one to consider three-body intermediate states where *two* of the mesons are spectators, so that still at most one meson's momentum is being summed over in a given intermediate summation (and the single-particle saturation condition is not violated in the remaining intermediate sums). Thus, for the time ordering of Eq. (3.2), one can have the intermediate states given by

$$F^{\alpha\beta\mu\nu}=\sum\langle 0|A^{\alpha}{}_a|\pi,q_1a_1\rangle\langle\pi,q_1a_1|A^{\beta}{}_b|\pi,q_2a_2;A_1,q_3a_3\rangle \times\langle\pi,q_2a_2;A_1,q_3a_3|V^{\mu}{}_c|\pi,q_4a_4;A_1,q_5a_5;\rho,p_1b_1\rangle \times\langle\pi,q_4a_4;A_1,q_5a_5;\rho,p_1b_1|V^{\nu}{}_d|0\rangle\,,\quad(3.6\text{a})$$

where we make the spectator approximations

$$\langle\pi,q_1a_1|A^{\beta}{}_b|\pi,q_2a_2A_1,\,q_3a_3\rangle =\delta^3(\mathbf{q}_1-\mathbf{q}_2)\delta a_1a_2\langle 0|A^{\beta}{}_b|A_1,q_3a_3\rangle\quad(3.6\text{b})$$

and

$$\langle\pi,q_2a_2;A_1,q_3a_3|V^{\mu}{}_c|\pi,q_4a_4;\,q_5a_5;\rho,p_1b_1\rangle =\delta^3(\mathbf{q}_2-\mathbf{q}_4)\delta a_2a_4\delta^3(\mathbf{q}_3-\mathbf{q}_5)\delta a_3\delta a_5 \times\langle 0|V^{\mu}{}_c|\rho,p_1b_1\rangle.\quad(3.6\text{c})$$

The diagrammatic representation of Eqs. (3.6) is given in Fig. 3(a) and involves a direct four-point "vertex" diagram. Such diagrams can also appear crossed as in Fig. 3(b).

The above discussion characterizes all the possible types of intermediate states for a four-point function within the framework of a covariant, crossing-symmetric single-particle saturation assumption. We next rephrase this in terms of equivalent field operators. The terms contributing to $F^{\alpha\beta\mu\nu}$ of the type appearing in Figs. 2(a)–2(c) involve only the matrix elements of Eqs. (2.6) and (2.7) which already have been discussed with respect to the three-point functions. Thus, all terms of this type can be obtained by using the currents of

[17] Note that isospin invariance forbids the presence of an $I=0$ or $I=2$ S-wave meson in the intermediate sums for the T product of three current operators. Thus the particle makes an explicit appearance only in the four- or higher-point functions.

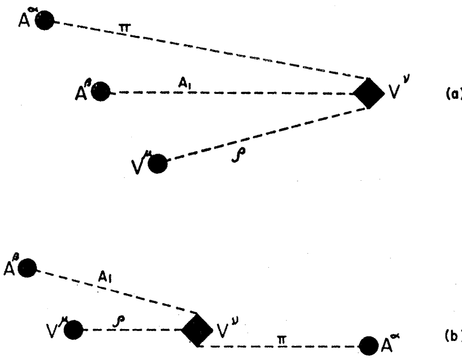

FIG. 3. (a) Diagram for time ordering of Eq. (3.1) with $x^0>y^0>z^0>\omega^0$ when two particles (π and A_1 mesons) are simultaneously spectators. (b) Diagram (a) with π meson crossed.

Eq. (2.12) [or equivalently the Lagrangian of Eq. (2.15)]. Since two vertex functions appear in each possible diagram, one must now calculate the four-point function to *second order* in the coupling constants (since each vertex is linear in the coupling constants). In order to include in the terms with intermediate σ mesons [e.g., Fig. 2(d)] one may introduce an $I=0$ scalar Heisenberg-field operator $\sigma(x)$. Since $\langle 0|V^{\mu}{}_c|\sigma,q\rangle$ and $\langle 0|A^{\mu}{}_c|\sigma,q\rangle$ vanish by isospin invariance, one needs field operators only to simulate the vertex analogous of Eqs. (2.7), i.e.,

$$\langle B,q_1a|V^{\mu}{}_c(0)|\sigma,q_2\rangle=\delta_{ac}\,{}_{\rho}\Delta^{\mu}{}_{\lambda}(k) \times\Gamma^{\lambda}{}_{B\rho\sigma}(q_1,q_2)N_BN_{\sigma}\,,\quad(3.7\text{a})$$

and

$$\langle B,q_1a|A^{\mu}{}_c(0)|\sigma,q_2\rangle=\delta_{ac}[{}_A\Delta^{\mu}{}_{\lambda}(k)\Gamma^{\lambda}{}_{BA\sigma}(q_1,q_2) +{}_{\pi}\Delta(k)\Gamma^{\mu}{}_{B\pi\sigma}(q_1,q_2)]N_BN_{\sigma}.\quad(3.7\text{b})$$

Thus, to simulate Eq. (3.7a) for the case $B=\rho$, one must add to the expansion of Eq. (2.12a) the term

$$\int d^4y\;{}_{\rho}\Delta^{\mu}{}_{\lambda}(x-y)[\delta_1\bar{v}^{\lambda}{}_c(y)\bar{\sigma}(y)+\cdots],\quad(3.8)$$

where $\bar{\sigma}(y)$ is the σ-meson in-field. Similarly, bilinear field structures (containing a single factor of $\bar{\sigma}$) are needed to reproduce the other matrix elements of Eqs. (3.7). Equivalently, one must add the term

$$g_{\rho}^{-1}[\delta_1v^{\mu}{}_c(x)\sigma(y)+\cdots]\quad(3.9)$$

to the right-hand side of the Heisenberg field equation (2.13a) with similar σ field structures included in the other equations (2.13). The locality requirement on the σ contributions to the current can then be guaranteed by requiring that structures such as Eq. (3.9) arise from varying a local cubic Lagrangian. To include the σ meson we therefore add to the Lagrangian of Eq. (2.15a) the additional general structure $\mathcal{L}_{0\sigma}+\mathcal{L}_{(3)\sigma}$, where

$$\mathcal{L}_{0\sigma}=-\sigma^{\mu}\partial_{\mu}\sigma+\tfrac{1}{2}(\sigma^{\mu}\sigma_{\mu}-m_{\sigma}^2\sigma^2)\,,\quad(3.10\text{a})$$

and

$$\mathfrak{L}_{(3)\sigma}=\tfrac{1}{2}g_{\sigma\pi\pi}\varphi_a\varphi_a\sigma+\tfrac{1}{2}\lambda_{\sigma\pi\pi}\varphi^{\mu}{}_a\varphi_{\mu a}\sigma+\tfrac{1}{2}g_{\sigma\rho\rho}v^{\mu}{}_a v_{\mu a}\sigma$$
$$+\tfrac{1}{4}\lambda_{\sigma\rho\rho}G^{\mu\nu}{}_a G_{\mu\nu a}\sigma+\tfrac{1}{2}g_{\sigma AA}a^{\mu}{}_a a_{\mu a}\sigma+\tfrac{1}{4}\lambda_{\sigma AA}H^{\mu\nu}{}_a H_{\mu\nu a}\sigma$$
$$+\lambda_{\sigma\pi A}\varphi_a a^{\mu}{}_a\sigma_{\mu}+\tilde{\lambda}_{\sigma\pi A}a^{\mu}{}_a\varphi_{\mu a}\sigma+\mu_{\sigma\rho\rho}v_{\mu a}G^{\mu\nu}{}_a\sigma_{\nu}$$
$$+\mu_{\sigma AA}a_{\mu a}H^{\mu\nu}{}_a\sigma_{\nu}+\mu_{\sigma\pi A}\varphi_{a\mu}H^{\mu\nu}{}_a\sigma_{\nu}+\mu_{\sigma\pi\pi}\varphi_a\varphi^{\nu}{}_a\sigma_{\nu}$$
$$+g_{\sigma\sigma\sigma}\sigma\sigma\sigma+\lambda_{\sigma\sigma\sigma}\sigma\sigma_{\mu}\sigma^{\mu}. \quad (3.10b)$$

Equation (3.10b) is the most general cubic σ-dependent Lagrangian involving the σ, π, ρ, and A_1 fields and containing no derivatives in first-order formalism. Thus, all the diagrams contributing to the four-point functions $F^{\alpha\beta\mu\nu}$ of the type shown in Fig. 2 involving vertex functions of σ, π, ρ, and A_1 mesons are to be calculated by replacing the currents by the Heisenberg fields according to Eq. (3.14) and then calculating the T product to *second* order in the coupling constants using the Lagrangian $\mathfrak{L}_0+\mathfrak{L}_{(3)\pi\rho A}+\mathfrak{L}_{(3)\sigma}$.

The remaining contributions to the four-point function within the framework of single-particle saturation are the diagrams of Fig. 3 which involve the three-particle matrix elements of the currents. Single-particle saturation again implies that the currents link to the particles only via the ρ, π, and A_1 mesons and so one can write

$$\langle 0|V^{\mu}{}_c(0)|Bq_1,Cq_2,Dq_3\rangle$$
$$=[{}_{\rho}\Delta^{\mu}{}_{\lambda}(k)Z^{\lambda}{}_{\rho BCD}(q_1,q_2,q_3)]N_BN_CN_D\,, \quad (3.11a)$$

$$\langle 0|A^{\mu}{}_c(0)|Bq_1,Cq_2,Dq_3\rangle$$
$$=[{}_{A}\Delta^{\mu}{}_{\lambda}(k)Z^{\lambda}{}_{ABCD}(q_1,q_2,q_3)$$
$$+{}_{\pi}\Delta(k)Z^{\mu}{}_{\pi BCD}(q_1,q_2,q_3)]N_BN_CN_D\,, \quad (3.11b)$$

where $k^{\mu}\equiv(q_1+q_2+q_3)^{\mu}$ and B, C, D represent σ, π, ρ, or A_1 particles. Similar expressions hold for $\langle B|V^{\mu}{}_c|CD\rangle$, etc., and are related to the forms of Eqs. (3.11) by crossing. To simulate such matrix elements, one must add additional structures to the right-hand side of Eqs. (2.12) that are cubic in the in-fields. For example, Eq. (3.11a) for the case $B=C=\pi$, $D=\rho$ requires structures in V^{μ} of the form

$$\int {}_{\rho}\Delta^{\mu}{}_{\lambda}(x-y)Z^{\lambda\nu}(y-x_1,\,y-x_2,\,y-x_3)$$
$$\times\tilde{\varphi}_a(x_1)\tilde{\varphi}_a(x_2)\tilde{v}_{\nu c}(x_3)\,, \quad (3.12)$$

where $Z^{\lambda\nu}$ is proportional to the Fourier transform of $Z^{\lambda}{}_{\rho\pi\pi\rho}$. To investigate the form of this contribution, one may contract down the three particles appearing in the matrix elements of Eqs. (3.11). Using the fact that $V^{\mu}{}_a$ is an interpolating field for the ρ meson and $A^{\mu}{}_a$ for the A_1 and π mesons, one arrives at a structure which is proportional to a four-point function. Thus, for the case of Eq. (3.12), the matrix element of Eq. (3.11a) depends upon

$$\langle 0|T(V^{\mu}{}_c\partial_{\alpha}A^{\alpha}{}_a\partial^{\beta}A^{\beta}{}_bV^{\nu}{}_d)|0\rangle. \quad (3.13)$$

As discussed above, the diagrammatic contribution to this term will include the single-pole diagrams of Fig. 2 obtained from second-order perturbation analysis using $\mathfrak{L}_{(3)}$. This contribution corresponds to a *nonlocal* piece of $Z^{\lambda\nu}$ due to the presence of the propagator factors for the intermediate particle poles. In addition, for generality, we will assume that Eq. (3.12) possesses a contribution *local* in the field variables. This can be achieved by including into the effective Lagrangian a contribution $\mathfrak{L}_{(4)}$ quartic in the Heisenberg fields. Thus to simulate an additional local contribution in Eq. (3.12) requires an $\mathfrak{L}_{(4)}$ of the form

$$\mathfrak{L}_{(4)}=a_1\varphi^{\mu}{}_a\varphi_{\mu a}v^{\nu}{}_bv_{\nu b}+a_2\varphi_a\varphi_{\mu a}G^{\mu\nu}{}_bv_{\nu b}+\cdots. \quad (3.14)$$

In calculating the contribution of $\mathfrak{L}_{(4)}$ to the in-field expansion of Eq. (3.12) it is understood that one carries the analysis to only first order in the coupling constants a_1, a_2, $\cdots$ (since already this gives rise to structures cubic in the in-fields). $\mathfrak{L}_{(4)}$ clearly produces seagull-type diagrams to the four-point functions.[18] We will see in Sec. IV that such terms are essential in satisfying the current-algebra conditions.

The above discussion can now be summarized as follows. The conditions of single-particle saturation and locality of the current operators imply that one may calculate four-point functions such as (3.1) by replacing the currents by the fields according to Eqs. (2.14) and then calculating the resulting T products *to first nonvanishing* order[19] using the Lagrangian

$$\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}_{(3)}+\mathfrak{L}_{(4)}\,, \quad (3.15a)$$

where

$$\mathfrak{L}_0=\mathfrak{L}_{0\sigma}+\mathfrak{L}_{0\pi}+\mathfrak{L}_{0\rho}+\mathfrak{L}_{0A}\,, \quad (3.15b)$$

$\mathfrak{L}_{(3)}=\mathfrak{L}_{(3)\pi\rho A}+\mathfrak{L}_{(3)\sigma}$ [of Eqs. (2.15c) and (3.10b)], and $\mathfrak{L}_{(4)}$ is the quartic Lagrangian of type (3.14). Alternately, if we adopt the convention that the coupling constants α_1, α_2, $\cdots$ of $\mathfrak{L}_{(4)}$ are of second order compared to those of $\mathfrak{L}_{(3)}$, the prescription is that one calculate the four-point function using the effective Lagrangian of Eq. (3.15) to *second order* in the coupling constants.

The generalization of the above analysis to higher-point functions is now straightforward. Thus, for the five-point functions, single-particle saturation allows three general types of diagrams to contribute; (a)

[18] Actually, because we are using a first-order formalism, the use of $L_{(3)}$ to second-order perturbation theory automatically introduces some seagull terms (as well as the nonlocal contributions to $Z^{\lambda\nu}$ mentioned in text). One can see this by examining the field equations (4.5)–(4.8). Thus, in the ρ Proca equation (4.6b) $\delta\mathfrak{L}_{(3)}/\delta v_{\mu a}$ contains a term proportional to the pion isospin current $\epsilon_{abc}\varphi_b\varphi^{\mu}{}_c$. From Eq. (4.5a), one sees that the cubic local contribution $\epsilon_{abc}\varphi_b(\delta\mathfrak{L}_{(3)}/\delta\varphi_{\mu c})$ will then arise in the source of the ρ equation and produce seagull diagrams. Since at this stage $\mathfrak{L}_{(4)}$ has completely general coupling constants, there is no harm in including these additional seagull terms (depending on the constants of $\mathfrak{L}_{(3)}$). It is also highly convenient to do so for the current-algebra analysis of the next section.

[19] We have neglected disconnected diagrams in the above analysis as they do not contribute to any of the physical scattering amplitudes obtainable from the N-point functions.

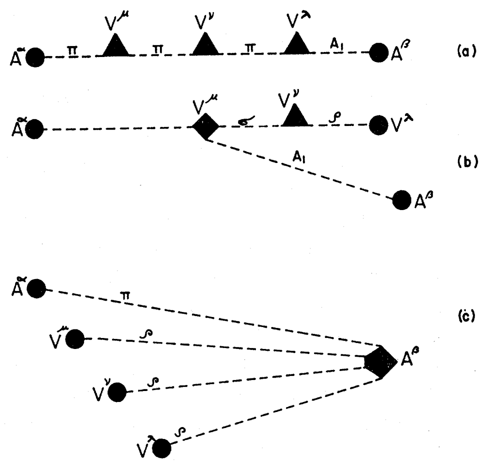

FIG. 4. Possible diagrams for the five-point function $\langle T(A^{\alpha}{}_a V^{\mu}{}_b V^{\nu}{}_c V^{\lambda}{}_d A^{\beta}{}_e)\rangle$.
(a) Tree diagrams involving only three-point vertices. (b) Tree diagrams with one four-point vertex. (c) Five-point vertex (with as many as three spectator particles in the intermediate states).

"tree" diagrams which involve a succession of three-point vertices [Fig. 4(a)]; (b) tree diagrams involving one four-point vertex [Fig. 4(b)]; and (c) five-point diagrams [Fig. 4(c)]. Diagrams of types (a) and (b) can be characterized using the $\mathcal{L}_{(3)}$ and $\mathcal{L}_{(4)}$ parts of the effective Lagrangian previously constructed. For type (c) one must, for generality, introduce an additional term $\mathcal{L}_{(5)}$ into the Lagrangian which is quartic in the fields. This contributes a local five-point vertex or "flower" diagram. The total effective Lagrangian is then to be used to first nonvanishing order (discarding disconnected diagrams) to calculate the T product. More generally, for an N-point function, one has appearing generalized tree diagrams containing 3-, 4-, $\cdots$, N-point vertices characterized by interaction Lagrangians $\mathcal{L}_{(3)}$, $\mathcal{L}_{(4)}$, $\cdots$, $\mathcal{L}_{(N)}$ and again one is to calculate the T product to first nonvanishing order. Equivalently, if we adopt the convention that the coupling constants of $\mathcal{L}_{(r)}$ are of order $r-2$ relative to $\mathcal{L}_{(3)}$, then the T product is to be calculated using the total effective Lagrangian to order $N-2$.

IV. CURRENT-ALGEBRA CONDITIONS

In the previous section, the general form of an effective Lagrangian needed to reproduce the N-point functions under the assumption of single-meson saturation has been given. The total interaction Lagrangian $\mathcal{L}_I$ is a polynomial of degree N in the field variables,

$$\mathcal{L}_I=\mathcal{L}_{(3)}+\mathcal{L}_{(4)}+\cdots+\mathcal{L}_{(N)}, \tag{4.1}$$

and is to be used to first nonvanishing order in perturbation theory. The general form of $\mathcal{L}_{(3)}\equiv\mathcal{L}_{(3)\pi\rho A}+\mathcal{L}_{(3)\sigma}$ has been exhibited in Eqs. (2.15c) and (3.10b). In this section we determine the constraints that are imposed on $\mathcal{L}_I$ by the additional conditions of the chiral $SU(2)\times SU(2)$ current algebra, CVC, and PCAC. We note that in the approach being adopted here, the Lagrangian has been introduced merely as a covenient mathematical tool for achieving local current operators. It is to be emphasized that no assumption of chiral symmetry or partial chiral symmetry has been imposed on $\mathcal{L}_I$. The physical conditions of CCR, CVC, and PCAC determine the amount of chiral $SU(2)\times SU(2)$ symmetry remaining in the resulting T products.

The current commutation relations read

$$\delta(x^0-y^0)[V^0{}_a(x),V^{\mu}{}_b(y)] =i\epsilon_{abc}V^{\mu}{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}, \tag{4.2a}$$

$$\delta(x^0-y^0)[V^0{}_a(x),A^{\mu}{}_b(y)] =i\epsilon_{abc}A^{\mu}{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}, \tag{4.2b}$$

$$\delta(x^0-y^0)[A^0{}_a(x),V^{\mu}{}_b(y)] =i\epsilon_{abc}A^{\mu}{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}, \tag{4.2c}$$

$$\delta(x^0-y^0)[A^0{}_a(x),A^{\mu}{}_b(y)] =i\epsilon_{abc}V^{\mu}{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}, \tag{4.2d}$$

where "c-No. S.T." stands for c-number Schwinger terms. The conservation conditions are

$$\partial_{\mu}V^{\mu}{}_a(x)=0, \tag{4.3}$$

and

$$\partial_{\mu}A^{\mu}{}_a(x)=F_{\pi}m_{\pi}^2\varphi_a(x). \tag{4.4}$$

We begin by first reviewing the results of I and discuss the constraints Eqs. (4.2)–(4.4) impose on the three-point functions. For these T products, only $\mathcal{L}_{(3)}$ contributes to $\mathcal{L}_I$, and single-particle saturation requires that one carry out the analysis only to first order in the coupling constants. The perturbation expansion of $V^{\mu}{}_a$ and $A^{\mu}{}_a$ in terms of in-fields has been exhibited to first order in Eqs. (2.12) (terms linear in the in-fields are zeroth order, and those quadratic are first order). In calculating the left-side commutators of Eqs. (4.2), then, it is necessary only to include terms as high as the cross terms between linear and quadratic pieces.[20] While a direct calculation using the in-field expansions of Eqs. (2.12) is feasible, a much more convenient method involves using the field equations and expressing the current in a series increasingly nonlinear in the *Heisenberg canonical variables*.[21] Thus, the field equations for the general Lagrangian $\mathcal{L}_0+\mathcal{L}_I$, where $\mathcal{L}_0$ is given by Eqs. (2.15b) and (3.10a), read, for the pion fields,

$$\varphi_{\mu a}=\partial_{\mu}\varphi_a-\delta\mathcal{L}_I/\delta\varphi^{\mu}{}_a, \tag{4.5a}$$

$$-\partial_{\mu}\varphi^{\mu}{}_a+m_{\pi}^2\varphi_a=\delta\mathcal{L}_I/\delta\varphi_a, \tag{4.5b}$$

[20] Actually, the commutators between two linear pieces are just c-number Schwinger terms.

[21] That one can use the field equations to directly calculate current commutators for currents obeying Eqs. (2.14) was previously pointed out in Ref. 13.

for the ρ fields,

$$G_{\mu\nu a}=\partial_\mu v_{\nu a}-\partial_\nu v_{\mu a}-2\delta\mathcal{L}_I/\delta G^{\mu\nu}{}_a\,, \tag{4.6a}$$

$$\partial_\nu G^{\mu\nu}{}_a+m_\rho{}^2 v^\mu{}_a=\delta\mathcal{L}_I/\delta v_{\mu a}\,, \tag{4.6b}$$

for the A_1 fields,

$$H_{\mu\nu a}=\partial_\mu a_{\nu a}-\partial_\nu a_{\mu a}-2\delta\mathcal{L}_I/\delta H^{\mu\nu}{}_a\,, \tag{4.7a}$$

$$\partial_\nu H^{\mu\nu}{}_a+m_A{}^2 a^\mu{}_a=\delta\mathcal{L}_I/\delta a\mu_a\,, \tag{4.7b}$$

and for the σ fields,

$$\sigma_\mu=\partial_\mu\sigma-\delta\mathcal{L}_I/\delta\sigma^\mu\,, \tag{4.8a}$$

$$-\partial_\mu\sigma^\mu+m_\sigma{}^2\sigma=\delta\mathcal{L}_I/\delta\sigma\,. \tag{4.8b}$$

Since (φ_{0a},φ_a), (G_{0ia},v_{ia}), (H_{0ia},a_{ia}), and (σ_0,σ) are canonically conjugate pairs of variables, one has from Eqs. (2.14) that $V^i{}_a$ and $A^i{}_a$ are linear in the canonical variables. φ_{ia}, G_{ija}, H_{ija}, and σ_i are constraint variables which can be eliminated in terms of the other variables by Eqs. (4.5a), (4.6a), (4.7a), and (4.8a). The remaining quantities $v^0{}_a$ and $a^0{}_a$ are also constraint variables which may be expressed in terms of the canonical variables using the $\mu=0$ components of Eqs. (4.6b) and (4.7b). Thus, one can write the time components of currents of Eqs. (2.14) as

$$V^0{}_a(x)=g_\rho m_\rho{}^{-2}\partial_i G_{0ia}+g_\rho m_\rho{}^{-2}\delta\mathcal{L}_I/\delta v_{0a}\,, \tag{4.9a}$$

$$A^0{}_a(x)=(g_A m_A{}^{-2}\partial_i H_{0ia}-F_\pi\varphi_{0a})+g_A m_A{}^{-2}\delta\mathcal{L}_I/\delta a_{0a}+F_\pi\delta\mathcal{L}_I/\delta\varphi_{0a}\,. \tag{4.9b}$$

On the right-hand side, the first structures are clearly linear in the canonical variables while the variational derivative terms involve nonlinear structures of both canonical and constraint variables. The latter can be eliminated in terms of the canonical variables using the constraint equations as described above. Since, in fact, constraint variables appear also in the nonlinear functional derivatives on the right-hand side of the constraint equations, the constraint equations are actually coupled algebraic equations which may be solved by iteration. Inserting their solutions into Eqs. (4.9) then expresses $V^0{}_a$ and $A^0{}_a$ in a power series of canonical variables, each term being increasingly higher order in the coupling constants. One has in this fashion expressed $V^\mu{}_a$ and $A^\mu{}_a$ completely in terms of canonical variables and so the commutators of Eqs. (4.2) can be directly evaluated.

The conservation laws can also be expressed in terms of Heisenberg operators. Thus Eq. (4.3) becomes, upon using Eq. (4.6b),

$$g_\rho m_\rho{}^{-2}\partial_\mu(\delta\mathcal{L}_I/\delta v_{\mu a})=0\,, \tag{4.10}$$

while Eqs. (4.5) and (4.7b) reduce the PCAC condition to

$$g_A m_A{}^{-2}\partial_\mu(\delta\mathcal{L}_I/\delta a_{\mu a})+F_\pi\partial_\mu(\delta\mathcal{L}_I/\delta\varphi_{\mu a})=F_\pi(\delta\mathcal{L}_I/\delta\varphi_a)\,. \tag{4.11}$$

Returning to the three-point functions where $\mathcal{L}_I=\mathcal{L}_{(3)}$, one must do the analysis only to first order in the coupling constants. To this approximation $V^0{}_a$ and $A^0{}_a$ are quadratic functions of the canonical variables, the bilinear pieces being first order in the coupling constants of $\mathcal{L}_{(3)}$. As in the in-field expansion, one need include only terms up to the cross terms between linear and quadratic pieces[20] in the commutators of Eqs. (4.2). Since these commutators will be linear in the coupling constants, satisfaction of Eqs. (4.2) will thus produce constraints on these quantities. Replacing $\mathcal{L}_I$ by $\mathcal{L}_{(3)}$ in the conservation conditions (4.10) and (4.11) gives additional conditions on the coupling constants. By direct calculation[9] one finds that all the constants except $\mu_{\rho\rho\rho}$ and $\mu_{\rho AA}$ in the $\mathcal{L}_{(3)\pi\rho A}$ of Eq. (2.15c) can be expressed in terms of one independent constant, the anomalous magnetic moment of the A_1 meson,

$$\lambda_A\equiv g_\rho m_\rho{}^{-2}\lambda_{\rho AA}\,, \tag{4.12}$$

and three parameters

$$x\equiv\sqrt{2}m_\rho/m_A\,,\quad y\equiv g_A/g_\rho\,,\quad z\equiv g_\rho/\sqrt{2}m_\rho F_\pi\,. \tag{4.13}$$

We find

$$\begin{aligned} g_{\rho\rho\rho}&=g_{\rho AA}=g_{\pi\pi\rho}=m_\rho{}^2 g_\rho{}^{-1}\,,\\ g_{\pi\rho A}&=-m_A{}^2\lambda_{\pi\rho A}=m_\rho{}^2(F_\pi x^2yz^2)^{-1}\,,\\ g_\rho\lambda_{\pi\pi\rho}&=x^4y^2z^2\tfrac{1}{2}\lambda_A+2(1-z^2)\,,\\ F_\pi\bar{\lambda}_{\pi\rho A}&=-y(1-x^2\tfrac{1}{2}\lambda_A)\,,\\ 2F_\pi\mu_{\pi\rho A}&=y-y^{-1}\,. \end{aligned} \tag{4.14}$$

The absence of q-number Schwinger terms imposes one relation between x, y, and z, the first Weinberg sum rule[8]

$$x^2y^2z^2-2z^2+1=0\,. \tag{4.15}$$

Experimentally, one has[9] $x\cong y\cong z\cong 1$ and $\lambda_A\simeq 0.4\pm 0.2$. Since the σ field does not enter directly into the currents, the current commutators and conservation conditions determine fewer of the coupling constants in $\mathcal{L}_{(3)\sigma}$ of Eq. (3.10b). In I it is shown that six relations emerge between ten of the 14 coupling constants of $\mathcal{L}_{(3)\sigma}$;

$$\begin{aligned} F_\pi g_{\sigma\pi\pi}&=m_\sigma{}^2(\lambda_3-\epsilon_\sigma\lambda_1)\,,\\ F_\pi\lambda_{\sigma\pi\pi}&=-(\lambda_1+\lambda_2)\,,\\ F_\pi g_{\sigma AA}&=(x^2yz)^{-2}2m_\rho{}^2(\lambda_1-\lambda_2)\,,\\ \sqrt{2}m_\rho\mu_{\sigma\pi A}&=-x^2yz\mu_{\sigma AA}\,,\\ g_{\sigma\rho\rho}&=0=\mu_{\sigma\rho\rho}\,, \end{aligned} \tag{4.16}$$

where $\epsilon_\sigma\equiv(m_\pi/m_\sigma)^2$, and

$$\begin{aligned} \lambda_1&\equiv(g_A m_A{}^{-2})\lambda_{\sigma\pi A}\,,\\ \lambda_2&\equiv(g_A m_A{}^{-2})\bar{\lambda}_{\sigma\pi A}\,,\\ \lambda_3&\equiv\lambda_1+F_\pi\mu_{\sigma\pi\pi}\,. \end{aligned} \tag{4.17}$$

The remaining four constants ($\lambda_{\sigma\rho\rho}$, $\lambda_{\sigma AA}$, $g_{\sigma\sigma\sigma}$, and $\lambda_{\sigma\sigma\sigma}$) are totally unconstrained.[22]

Returning now to Eq. (4.9a), $V^0{}_a$ becomes (good to first order in the coupling constants)

$$V^0{}_a = g_\rho m_\rho{}^{-2}\partial_k G_{0ka} + g_\rho m_\rho{}^{-2}\epsilon_{abc}[g_{\pi\pi\rho}\varphi_b\varphi_{0c} + g_{\rho\rho\rho}v_{kb}G_{0kc} + g_{\rho AA}a_{kb}H_{0kc} - \lambda_{\pi\rho A}\partial_k(\varphi_b H_{0kc})], \quad (4.18)$$

and an analogous result from Eq. (4.9b) holds for $A^0{}_a$. Inserting in Eqs. (4.14) reduces $V^0{}_a$ to the relatively simple form

$$V^0{}_a = \epsilon_{abc}[v_{kb}G_{0kc} + a_{kb}H_{0kc} + \varphi_b\varphi_{0c}] + \partial_k[g_\rho m_\rho{}^{-2}G_{0ka} + F_\pi g_A{}^{-1}\epsilon_{abc}\varphi_b H_{0kc}]. \quad (4.19a)$$

Similarly, using Eqs. (4.14)–(4.17) to eliminate the coupling constants in the quadratic pieces of $A^0{}_a$ of Eq. (4.9b) yields

$$A^0{}_a = g_A m_A{}^{-2}\partial_k H_{0ka} - F_\pi\varphi_{0a} + \epsilon_{abc}[y^{-1}v_{kb}H_{0kc} + y a_{kb}G_{0kc} + F_\pi g_\rho{}^{-1}\partial_k\varphi_b G_{0kc}] + F_\pi g_A{}^{-1}\lambda_1\partial_k H_{0ka}\sigma + \lambda_1\varphi_{0a}\sigma - \lambda_3\varphi_a\sigma_0. \quad (4.19b)$$

We note that $V^0{}_a$ differs from the usual isotopic current density by a divergence, so that $\int d^3x\, V^0{}_a(x)$ is correctly the total isotopic spin. The divergence is necessary to exclude q-No. Schwinger terms. Equations (4.14), (4.15), and (4.17) represent the full content of the current-algebra constraints (4.2)–(4.4) on the Lagrangian $\mathcal{L}_0 + \mathcal{L}_{(3)}$ used to calculate the three-point functions.

We next extend the above results to the higher-point functions. The discussion of Eqs. (4.5)–(4.11) is of course valid for the general case with $\mathcal{L}_I$ now of the form Eq. (4.1). As can be seen, for example, in Fig. 2, the existence of the $\mathcal{L}_{(3)}$ piece in the $\mathcal{L}_I$ for the four-point function is due to the presence of one-particle matrix elements of the currents, e.g., $\langle\pi|A^\beta{}_b|\rho\rangle$, etc., factors. If one contracts the π and ρ mesons and uses $(F_\pi m_\pi{}^2)^{-1} \times \partial_\alpha A^\alpha{}_a$ and $(g_\rho)^{-1}V^\mu{}_c$ as the respective interpolating fields, such functions reduce to three-point functions. Thus if single-particle saturation, CCR, CVC, and PCAC are to be satisfied for the vertex insertions in the four-point functions, the $\mathcal{L}_{(3)}$ to be used in this analysis must be precisely the one determined above in the three-point analysis. While the same argument would not apply for the vertex $\langle\pi|A^\beta{}_b|\sigma\rangle$ since no single current can be used as an interpolating field for the σ meson, one would expect that the σ piece $\mathcal{L}_{(3)\sigma}$ to be used in the three- and four-point functions are identical if the latter is to satisfy the current-algebra conditions. One may in fact show that this is so, though we shall not give a formal demonstration here. We will, therefore, choose the $\mathcal{L}_{(3)}$ appearing in the effective Lagrangians for the three- and four-point functions to be the same quantity. Similarly, one is led to the result that the $\mathcal{L}_{(3)}$ and $\mathcal{L}_{(4)}$ of the five-point function Lagrangian are identical to the ones determined by the lower-point cases, etc. In general, then, we will try to satisfy the current-algebra conditions for the arbitrary T product by finding a *single* $\mathcal{L}_I$ with an infinite series of terms

$$\mathcal{L}_I = \mathcal{L}_{(3)} + \mathcal{L}_{(4)} + \cdots, \quad (4.20)$$

which is of course to be used to lowest nonvanishing order (i.e., to order $N-2$) in a given N-point function. (Only the first $N-2$ terms of $\mathcal{L}_I$ will contribute to the calculation of the T product of N currents.) Further, if in the theory described by the $\mathcal{L}_I$ of Eq. (4.20), the current-algebra conditions of Eqs. (4.2)–(4.4) are satisfied rigorously, they will then clearly be satisfied up to order $N-2$ as required in the determination of a given N-point function. (We note that one need not consider the "implicit" q-number Schwinger terms in the current commutation relations arising due to the dynamics, since closed loops, etc., do not contribute to the order needed in calculating a given N-point function.)

In the series of Eq. (4.20) only $\mathcal{L}_{(3)}$ has been determined up to now. This has led to the determination in Eq. (4.19) of $V^0{}_a$ and $A^0{}_a$ valid to first order in the coupling constants. In the analysis leading to Eqs. (4.14)–(4.17), the commutators of Eqs. (4.2) were calculated only to linear order in the coupling constants (i.e., only up to the cross terms between a linear term and a zeroth-order term). It is interesting first to ask whether Eqs. (4.19) treated as the *rigorous* results for $V^0{}_a$ and $A^0{}_a$ can satisfy Eqs. (4.2) (when combined with the relations $V^k{}_a = g_\rho v^k{}_a$ and $A^k{}_a = g_A a^k{}_a + F_\pi\partial^k\varphi_a$ which rigorously express the spatial components of the currents in terms of the canonical variables). By direct calculation, one can easily verify that in fact Eqs. (4.19) do indeed rigorously satisfy Eqs. (4.2) provided λ_3 is chosen to obey[23]

$$\lambda_3 = 1/\lambda_1. \quad (4.21)$$

Note that Eq. (4.21) implies that the current commutation relations forbid λ_1 to vanish. Thus, at least some of the σ couplings cannot be zero.[24]

The fact that the currents of Eqs. (4.19) (which were obtained from $\mathcal{L}_{(3)}$ only) already satisfy the current commutation relations suggests the possibility that in fact these expressions are the correct currents of the full Lagrangian (4.20). While this is not the only way of constructing an effective Lagrangian, it appears to

[22] We have omitted from our analysis the $I=2$, S-wave σ meson since experimentally it does not appear to be present in the energy range under analysis. The current-algebra relations for three-point functions do not, however, forbid its presence and the results analogous to Eq. (4.16) for such a particle are given in I.

[23] The only commutator that is not automatically fulfilled by Eqs. (4.19) is $\delta(x^0-y^0)[A^0{}_a(x), A^0{}_b(y)] = i\epsilon_{abc}V^0{}_c(x)\delta^4(x-y)$.

[24] We remark that if an $I=2$ σ meson is included, Eqs. (4.19) cannot satisfy Eqs. (4.2). Within the framework of this paper, Eqs. (4.19) appear to be essential to obtain a total $\mathcal{L}_I$ satisfying the current-algebra conditions. Thus the current algebra suggests the presence of an $I=0$ σ meson and the absence of an $I=2$ meson, as also appears to be the case experimentally (Ref. 16).

be the simplest possibility.[25] We therefore investigate next what conditions must be imposed on the total $\mathfrak{L}_I$ to guarantee that Eqs. (4.19) are the rigorous currents. For if a consistent set of conditions can be achieved, one would have then obtained an effective Lagrangian satisfying Eqs. (4.2). From Eqs. (2.14a) and (4.6b) one finds

$$V^\mu{}_a = g_\rho m_\rho^{-2}\partial_\nu G^{\nu\mu}{}_a + g_\rho m_\rho^{-2}(\delta\mathfrak{L}_{(3)}/\delta v_{\mu a} + \delta\tilde{L}/\delta v_{\mu a}), \quad (4.22)$$

where we have written

$$\mathfrak{L}_I \equiv \mathfrak{L}_{(3)} + \tilde{L}. \quad (4.23)$$

Inserting in the value of $\mathfrak{L}_{(3)}$ from Eqs. (2.15c) and (3.10) gives

$$\begin{aligned} V^\mu{}_a = {} & g_\rho m_\rho^{-2}\partial_\nu G^{\nu\mu}{}_a + g_\rho m_\rho^{-2}\epsilon_{abc}[g_{\pi\pi\rho}\varphi^\mu{}_b\varphi_c + g_{\rho\rho\rho}v_{\nu b}G^{\nu\mu} \\ & + g_{\rho AA}a_{\nu b}H^{\nu\mu}{}_c + \lambda_{\pi\rho A}\varphi_{\nu b}H^{\mu\nu}{}_c + g_{\pi\rho A}\varphi_b a^\mu{}_c] + g_\rho m_\rho^{-2} \\ & \times[g_{\sigma\rho\rho}v^\mu{}_a\sigma + \mu_{\sigma\rho\rho}G^{\mu\nu}{}_a\sigma_\nu] + g_\rho m_\rho^{-2}\delta\tilde{L}/\delta v_{\mu a}. \quad (4.24) \end{aligned}$$

Now by Eqs. (4.5a) and (4.7b) one has

$$\begin{aligned} \varphi_{\nu b}H^{\mu\nu}{}_c = {} & \partial_\nu(\varphi_b H^{\mu\nu}{}_c) - (\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)H^{\mu\nu}{}_c \\ & + m_A{}^2\varphi_b a^\mu{}_c - \varphi_b(\delta\mathfrak{L}_I/\delta a_{\mu c}). \quad (4.25) \end{aligned}$$

Inserting Eq. (4.25) and the $\mathfrak{L}_{(3)}$ coupling-constant conditions (4.14)–(4.17) into Eq. (4.24) yields

$$\begin{aligned} V^\mu{}_a = {} & \epsilon_{abc}[\varphi^\mu{}_b\varphi_c + v_{\nu b}G^{\nu\mu}{}_c + a_{\nu b}H^{\nu\mu}{}_c] + \partial_\nu[g_\rho m_\rho^{-2}G^{\nu\mu}{}_a \\ & - F_\pi g_A{}^{-1}\epsilon_{abc}\varphi_b H^{\mu\nu}{}_c] + [g_\rho m_\rho^{-2}\delta\tilde{L}/\delta v_{\mu a} + F_\pi g_A{}^{-1} \\ & \times \epsilon_{abc}\{(\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)H^{\mu\nu}{}_c + \varphi_b(\delta\mathfrak{L}_I/\delta a_{\mu c})\}]. \quad (4.26) \end{aligned}$$

The first two brackets of Eq. (4.26) are precisely Eq. (4.19a) for $V^0{}_a$. Thus the condition that makes Eq. (4.19a) a rigorous result is

$$\begin{aligned} \delta\tilde{L}/\delta v_{\mu a} = {} & -F_\pi m_\rho{}^2(g_A g_\rho)^{-1}\epsilon_{abc}[(\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)H^{\mu\nu}{}_c \\ & + \varphi_b(\delta\mathfrak{L}_I/\delta a_{\mu c})]. \quad (4.27a) \end{aligned}$$

A similar analysis can be carried out for $A^\mu{}_a$. Thus using Eq. (2.14b), the field equations (4.5a) and (4.7b), and the coupling-constant conditions (4.14)–(4.17) and (4.21), the requirement that the expression for $A^0{}_a$ of Eq. (4.19b) be exact is

$$\begin{aligned} g_A m_A{}^{-2}(\delta\tilde{L}/\delta a_{\mu a}) + F_\pi(\delta\tilde{L}/\delta\varphi_{\mu a}) = {} & -F_\pi g_\rho{}^{-1}\epsilon_{abc}(\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)G^{\mu\nu}{}_c \\ & - F_\pi g_A{}^{-1}\lambda_1(\delta\mathfrak{L}_I/\delta a_{\mu a})\sigma. \quad (4.27b) \end{aligned}$$

Any $\tilde{L}$ chosen to satisfy Eqs. (4.27) will automatically satisfy the current commutation relations.

We turn next to consider the constraints on $\tilde{L}$ due to the conservation conditions (4.3) and (4.4). From Eqs. (4.26) and (4.27a) one sees that $V^\mu{}_a$ consists of the isotopic current plus an additional four-divergence. The latter is conserved identically because of the antisymmetry of $G^{\mu\nu}{}_a$ and $H^{\mu\nu}{}_a$. Since the isotopic current will be conserved provided merely that $\mathfrak{L}_I$ is an isotopic scalar, we see that Eq. (4.27a) also guarantees Eq. (4.3). The PCAC condition (4.4) is not so trivially satisfied, however. Inserting Eq. (4.23) into Eq. (4.11) gives

$$\begin{aligned} \delta\tilde{L}/\delta\varphi_a = {} & \partial_\mu[g_A(m_A{}^2F_\pi)^{-1}(\delta\mathfrak{L}_{(3)}/\delta a_{\mu a}) + (\delta\mathfrak{L}_{(3)}/\delta\varphi_{\mu a})] \\ & + \partial_\mu[g_A(m_A{}^2F_\pi)^{-1}(\delta\tilde{L}/\delta a_{\mu a}) + (\delta\tilde{L}/\delta\varphi_{\mu a})]. \quad (4.28) \end{aligned}$$

This may be simplified further by using the current commutator condition (4.27b):

$$\begin{aligned} \delta\tilde{L}/\delta\varphi_a = {} & \partial_\mu[g_A(m_A{}^2F_\pi)^{-1}(\delta\mathfrak{L}_{(3)}/\delta a_{\mu a}) + (\delta\mathfrak{L}_{(3)}/\delta\varphi_{\mu a})] \\ & - \partial_\mu[F_\pi g_\rho{}^{-1}\epsilon_{abc}(\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)G^{\mu\nu}{}_c \\ & + F_\pi g_A{}^{-1}\lambda_1(\delta\mathfrak{L}_I/\delta a_{\mu a})\sigma]. \quad (4.29) \end{aligned}$$

The first term may be explicitly evaluated using the known form of $\mathfrak{L}_{(3)}$ and the field equations (4.5)–(4.8). For example, from Eq. (2.15c) one sees that there is a contribution to $\delta\mathfrak{L}_{(3)}/\delta a_{\mu a}$ proportional to $\epsilon_{abc}\varphi_{\nu b}G^{\mu\nu}{}_c$ and by the field equations one has

$$\begin{aligned} \partial_\mu(\varphi_{\nu b}G^{\mu\nu}{}_c) = {} & \partial_\mu\varphi_{\nu b}G^{\mu\nu}{}_c + \varphi_{\nu b}\partial_\mu G^{\mu\nu}{}_c \\ = {} & -\partial_\mu(\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)G^{\mu\nu}{}_c + m_\rho{}^2\varphi_{\nu b}v^\nu{}_c \\ & - \varphi_{\nu b}(\delta\mathfrak{L}_I/\delta v_{\nu c}). \quad (4.30) \end{aligned}$$

The terms in the second bracket of Eq. (4.29) may similarly be rearranged. For example, from Eq. (4.6b) one has

$$\begin{aligned} \partial_\mu[(\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)G^{\mu\nu}{}_c] = {} & \partial_\mu(\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)G^{\mu\nu}{}_c - m_\rho{}^2(\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)v^\nu{}_c \\ & - (\delta\mathfrak{L}_I/\delta\varphi^\nu{}_b)(\delta\mathfrak{L}_I/\delta v_{\nu c}). \quad (4.31) \end{aligned}$$

The other terms may be similarly treated, and after a straightforward calculation, Eq. (4.29) reduces to the rather lengthy result:

$$\begin{aligned} \delta\tilde{L}/\delta\varphi_a = {} & \epsilon_{abc}[g_\rho(F_\pi g_A)^{-1}H^{\mu\nu}{}_b(\delta\mathfrak{L}_I/\delta G^{\mu\nu}{}_c) + g_\rho(F_\pi g_A)^{-1}v^\mu{}_b(\delta\mathfrak{L}_I/\delta a^\mu{}_a) + g_A(F_\pi g_\rho)^{-1}G^{\mu\nu}{}_b(\delta\mathfrak{L}_I/\delta H^{\mu\nu}{}_c) \\ & + g_A(F_\pi g_\rho)^{-1}a^\mu{}_b(\delta\mathfrak{L}_I/v^\mu{}_c) + m_\rho{}^2g_\rho{}^{-1}v^\mu{}_b(\delta\mathfrak{L}_I/\delta\varphi^\mu{}_c) + g_\rho{}^{-1}\varphi^\mu{}_b(\delta\mathfrak{L}_I/\delta v^\mu{}_c) + g_\rho{}^{-1}(\delta\mathfrak{L}_I/\delta\varphi_{\mu b})(\delta\mathfrak{L}_I/\delta v^\mu{}_c)] \\ & + [(F_\pi\lambda_1)^{-1}\sigma^\mu(\delta\mathfrak{L}_I/\delta\varphi^\mu{}_a) - (F_\pi\lambda_1)^{-1}\varphi_a(\delta\mathfrak{L}_I/\delta\sigma) + \lambda_1F_\pi{}^{-1}\sigma(\delta\mathfrak{L}_I/\delta\varphi_a) - \lambda_1F_\pi{}^{-1}\varphi^\mu{}_a(\delta\mathfrak{L}_I/\delta\sigma^\mu) \\ & - \lambda_1g_A{}^{-1}\sigma_\mu(\delta\mathfrak{L}_I/\delta a_{\mu a}) + m_A{}^2\lambda_1g_A{}^{-1}a^\mu{}_a(\delta\mathfrak{L}_I/\delta\sigma^\mu) - \lambda_1g_A{}^{-1}(\delta\mathfrak{L}_I/\delta a_{\mu a})(\delta\mathfrak{L}_I/\delta\sigma^\mu)]. \quad (4.32) \end{aligned}$$

Equation (4.32) represents the constraint imposed on $\tilde{L}$ by the PCAC condition. The complexity of Eq. (4.32) indicates the complexity of this requirement.

V. PROPERTIES OF L_I

In the previous section, a set of conditions were obtained [Eqs. (4.27) and (4.32)] which if satisfied would guarantee that the effective Lagrangian obeyed

[25] The currents of Eqs. (4.19) are particularly simple by being only quadratic in the canonical variables. This choice corresponds to using the σ meson itself to characterize chiral breakdown, as is discussed in the conclusions. It is possible to add cubic (and higher) structures to $A^0{}_a$ to obtain more complicated chiral-breakdown assumptions.

the current commutation relations, CVC, and PCAC. In this section we investigate some of the properties the resultant interaction Lagrangian possesses.

The significance of Eq. (4.27b), the current commutation condition on $A^0{}_a$, can be made more apparent by introducing a new set of variables to replace $a^\mu{}_a$ and $\varphi^\mu{}_a$:

$$\beta_{\mu a}=\tfrac{1}{2}(m_A{}^2g_A{}^{-1}a_{\mu a}-F_\pi{}^{-1}\varphi_{\mu a}), \qquad (5.1a)$$

$$\alpha_{\mu a}=\tfrac{1}{2}(m_A{}^2g_A{}^{-1}R_1a_{\mu a}+F_\pi{}^{-1}R_2\varphi_{\mu a}), \qquad (5.1b)$$

where R_1 and R_2 are arbitrary constants normalized by

$$R_1+R_2=1. \qquad (5.1c)$$

In terms of $\alpha_{\mu a}$ and $\beta_{\mu a}$, Eq. (4.27b) reduces to

$$\begin{aligned}\delta\tilde{L}/\delta\alpha_{\mu a}=&-g_\rho{}^{-1}\epsilon_{abc}[R_2(\delta\mathfrak{L}_I/\delta\alpha^\nu{}_b)-(\delta\mathfrak{L}_I/\delta\beta^\nu{}_b)]G^{\mu\nu}{}_c\\&-F_\pi\lambda_1m_A{}^2g_A{}^{-2}[R_1(\delta\mathfrak{L}_I/\delta\alpha_{\mu a})+(\delta\mathfrak{L}_I/\delta\beta_{\mu a})]. \qquad (5.2)\end{aligned}$$

The transformation (5.1) is the most general linear one that reduces the left-hand side of Eq. (4.27b) to a single term and any value of R_1 and R_2 obeying Eq. (5.1c) may be chosen.

Equations (4.27a), (4.32), and (5.2) may now be viewed as differential equations to determine the $v^\mu{}_a$, $\alpha^\mu{}_a$, and φ_a dependence of the function $\tilde{L}$ defined in Eq. (4.23). One may straightforwardly attempt to integrate these equations order by order. For example, to determine $\mathfrak{L}_{(4)}$, the contribution to $\mathfrak{L}_I$ quartic in the fields, one need only insert the known value of $\mathfrak{L}_{(3)}$ [Eqs. (2.15c) and (3.10b)] into the right-hand side of these equations and integrate once. One may then iterate to obtain $\mathfrak{L}_{(5)}$, etc., in a similar fashion. (For a given N-point function, one needs to know $\mathfrak{L}_I$ only up to $\mathfrak{L}_{(N)}$.) Since one is dealing with a set of coupled first-order differential equations, however, it is not *a priori* obvious that they are consistent and that the above integration procedure can actually be carried out. That is, Eqs. (4.27a), (4.32), and (5.2) must obey a set of integrability conditions on the second derivatives of $\tilde{L}$. Thus, if one abbreviates these equations as

$$\delta\tilde{L}/\delta v_{\mu a}=F_{\mu a}, \quad \delta\tilde{L}/\delta\alpha_{\mu a}=G_{\mu a}, \quad \delta\tilde{L}/\delta\varphi_a=H_a, \quad (5.3)$$

where $F_{\mu a}$, $G_{\mu a}$, and H_a stand for the right-hand sides of Eqs. (4.27a), (5.2), and (4.32), one must have

$$\begin{aligned}&\delta F_{\mu a}/\delta v^\nu{}_b=\delta F_{\nu b}/\delta v^\mu{}_a, \quad \delta G_{\mu a}/\delta v^\nu{}_b=\delta F_{\nu b}/\delta\alpha^\mu{}_a,\\&\delta F_{\mu a}/\delta\varphi_b=\delta H_b/\delta v^\mu{}_a, \quad \text{etc.} \qquad (5.4)\end{aligned}$$

Since the current commutation conditions and the PCAC requirement have no apparent interconnection, there is no a priori *guarantee that Eqs. (5.4) will be satisfied* and they must be checked explicitly. The verification of the integrability conditions is carried out in Appendix A.[26] The fact that they are satisfied appears to be nontrivial and depends explicitly on the values of the coupling constants determined in $\mathfrak{L}_{(3)}$, i.e., Eqs. (4.14)–(4.17) and (4.21). This consistency between the current commutation relations and PCAC supports the possibility of the existence of a more fundamental theory underlying both hypotheses.

With the integrability conditions satisfied, Eqs. (4.27a), (4.32), and (5.2) can be solved to determine the $v^\mu{}_a$, $\alpha^\mu{}_a$, and φ_a dependence of $\tilde{L}$, at least in a perturbation series.[27] Thus $\mathfrak{L}_I=\sum\mathfrak{L}_{(n)}$, where $\mathfrak{L}_{(n)}$ takes the form (for $n\geq4$)

$$\begin{aligned}\mathfrak{L}_{(n)}=F_{(n)}(v^\mu{}_a,\varphi_a,\alpha^\mu{}_a;\beta^\mu{}_a,G^{\mu\nu}{}_a,H^{\mu\nu}{}_a,\sigma,\sigma^\mu)\\+I_{(n)}(\beta^\mu{}_a,G^{\mu\nu}{}_a,\sigma,\sigma^\mu,H^{\mu\nu}{}_a). \qquad (5.5)\end{aligned}$$

Here $F_{(n)}$ is a known function depending only on the constants appearing in $\mathfrak{L}_{(3)}, \cdots, \mathfrak{L}_{(n-1)}$, whereas $I_{(n)}$ is a function of integration and hence arbitrary. Thus a great deal of the Lagrangian for the higher-point functions is *not* determined by the current-algebra conditions and additional physical assumptions must be made to calculate further. The most obvious postulate to make is to assume a "minimal" choice for $\mathfrak{L}_I$ and set the functions $I_{(n)}$ equal to zero. However, there are certain pieces of $I_{(n)}$ for which this cannot be done unambiguously. For as we have seen above, while the current algebra determines the $\alpha^\mu{}_a$ dependence of $\mathfrak{L}_I$, it does not determine $\alpha_{\mu a}$ itself uniquely in terms of of $a^\mu{}_a$ and $\varphi^\mu{}_a$. Thus, one gets a solution of the type (5.5) for every choice of the constant R_1 of Eq. (5.1b) and different values of R_1 just correspond to different choices of the function of integration $I_{(n)}$. Setting $I_{(n)}$ equal to zero for one value R_1 gives a *different* Lagrangian than the choice $I_n=0$ for another value of R_1. More precisely, one may decompose $I_{(n)}$ into two parts,

$$I_{(n)}=A_{(n)}+B_{(n)}. \qquad (5.6)$$

Here $A_{(n)}$ has the identical functional form (but with arbitrary coupling constants) as pieces in $F_{(n)}$ containing $\alpha^\mu{}_a$ *but with* $\alpha^\mu{}_a$ *replaced by* $\beta^\mu{}_a$. The function $B_{(n)}$ is the remaining arbitrary part of $I_{(n)}$. As one varies the values of R_1 in $F_{(n)}$, the same Lagrangian can be maintained by varying the coupling constants in $A_{(n)}$, keeping $B_{(n)}$ unchanged. Thus it is meaningful to say that $B_{(n)}=0$ represents a "minimal" coupling choice but any *a priori* choice of the coupling constants in $A_{(n)}$ (such as zero) would be arbitrary.[28] As an example

[26] The analysis in text is actually done for the c-number theory. For the q-number theory, the products of operators appearing on the right-hand sides of Eqs. (4.27a), (4.32), and (5.2) must be replaced by one-half the anticommutator (to make it Hermitian) and care must be taken to correctly keep the order of the operators. For $\mathfrak{L}_{(n)}$ with $n\gtrsim5$, however, multiple commutators arise in the q-number analysis of the integrability conditions. However, the discussion of Appendix A shows that these cause no difficulty.

[27] Closed form solutions for the linear equations (4.27a) and (5.2) are given in Appendix B. We have not been able to obtain a closed-form integration of the nonlinear PCAC condition (4.32).

[28] A similar phenomenon occurs in $\mathfrak{L}_{(3)}$. Thus the constants $\mu_{\rho\rho\rho}$ and $\mu_{\rho AA}$ of Eq. (2.15c) do not enter at all in the coupling constant constraints of Eqs. (4.14) and so the minimal choice $\mu_{\rho\rho\rho}=0=\mu_{\rho AA}$ can unambiguously be made. The last two terms of Eq. (2.15c) are B-type pieces. On the other hand, while λ_A is not determined by the current-algebra conditions, it is related to many of the other coupling constants by the current algebra, and so the choice $\lambda_A=0$ would be arbitrary. Experimentally, in fact, one finds (Ref. 9) $\lambda_A\simeq0.4\pm0.2$. No experimental determination of $\mu_{\rho\rho\rho}$ or $\mu_{\rho AA}$ currently exists.

of this phenomenon we consider the case of $\mathcal{L}_{(4)}$ which is given in Appendix C. Here, $F_{(4)}$ contains structures of the form $\alpha_{\mu a}G^{\nu\mu}{}_b\alpha^\lambda{}_c G_{\lambda\mu d}$ and $\alpha_{\nu a}G^{\nu\mu}{}_b\beta^\lambda{}_c G_{\lambda\mu d}$, with coupling constants fixed by the current-algebra conditions. One must, therefore, include into $\mathcal{L}_{(4)}$ an $A_{(4)}$ piece of the form $c_1\beta_{\nu a}G^{\nu\mu}{}_b\beta^\lambda{}_c G_{\lambda\mu d}$, where c_1 is undetermined (and is the arbitrariness of this part of the function of integration). If one then fixes the value of R_1, and eliminates $\alpha^\mu{}_a$ and $\beta^\mu{}_a$ back in terms of $a^\mu{}_a$ and $\varphi^\mu{}_a$, these structures contribute terms to $\mathcal{L}_{(4)}$ of the form

$$a_1\varphi_{\nu a}G^{\nu\mu}{}_b\varphi^\lambda{}_c G_{\lambda\mu d}+a_2 a_{\nu a}G^{\nu\mu}{}_b\varphi^\lambda{}_c G_{\lambda\mu d} +a_3 a_{\nu a}G^{\nu\mu}{}_b a^\lambda{}_c G_{\lambda\mu d}\,, \quad (5.7)$$

where a_1, a_2, and a_3 depend on one undetermined constant c_1. Alternatively, there exist only two relations between the three constants a_1, a_2, a_3. As shown in Appendix C, $\mathcal{L}_{(4)}$ contains five such additional undetermined constants, even for the minimal solution where $B_{(4)}$ has been set to zero. Four of them involve σ-meson channels and so are currently not of physical interest. However, the fifth additional constant, exhibited in Eq. (5.7), will contribute to π-ρ scattering and hence affects the value of the π^+-π^0 electromagnetic mass difference.[29] Similar additional undetermined constants appear in the higher parts of the Lagrangian, $\mathcal{L}_{(5)}$, $\mathcal{L}_{(6)}$, etc.

VI. CONCLUSIONS

In the preceding sections, a hard-pion current-algebra method has been described for calculating T products of an arbitrary number of vector and axial-vector current operators. The analysis was based on the assumptions of single σ-, π-, ρ-, and A_1-meson saturation of intermediate sums, the chiral $SU(2)\times SU(2)$ current commutation relations, CVC, and PCAC. The single-meson saturation hypothesis implied that the T products could be calculated from a set of generalized tree diagrams and generalized seagull diagrams (i.e., "flower" diagrams). This result could equivalently be rephrased in terms of an effective Lagrangian. Thus, to calculate a given N-point function one uses an effective interaction Lagrangian $\mathcal{L}_I$ (which is a polynomial of order N in the meson fields) to lowest nonvanishing order. *A priori*, no symmetry conditions were imposed on $\mathcal{L}_I$. These arose from the constraints of the current commutation relations, CVC, and PCAC.

The current-algebra constraints on $\mathcal{L}_I$ have been represented by a set of differential equations to determine the form of that function. In arriving at this result we have made one additional postulate of "simplicity": that the time components $V^0{}_a$ and $A^0{}_a$ be at most quadratic in the meson-field canonical variables. The physical significance of this assumption becomes clearer from the result of Weinberg[30] that the "σ commutator"

$$\delta(x^0-y^0)[\varphi_a(x),A^0{}_b(y)]=i\delta^4(x-y)\sigma_{ab}(x)\,, \quad (6.1)$$

where $\varphi_a\equiv c^{-1}\partial_\mu A^\mu{}_a$, governs in part the breakdown of chiral invariance. This commutator is not determined by the current-algebra conditions, and some assumption must be made about it to get a well-defined theory. In view of the previous assumption of single-meson saturation (which is a generalized pole-dominance hypothesis), it is natural to assume that $\sigma_{ab}(x)$ is dominated by a resonance pole (just as $\partial_\mu A^\mu{}_a$ is dominated by the pion pole). The only resonance with the proper quantum numbers that appears to be available in the low-energy domain is the $I=0=J$ σ meson itself. This would imply

$$\sigma_{ab}\sim\delta_{ab}\sigma(x)\,. \quad (6.2)$$

The quadratic $A^0{}_a(x)$ of Eq. (4.19b) precisely produces Eq. (6.2). While it is indeed possible to add cubic (and presumably higher) structures to $A^0{}_a(x)$ which will then give more general (nonpole) contributions to σ_{ab},[31] Eq. (6.2) has the physical appeal of relating chiral breakdown to a physical particle, the σ meson (just as isospin breakdown is related to the existence of the photon). The fact that the integrability conditions are satisfied, implies that Eq. (6.2) is consistent with PCAC and the current commutation relations. On the other hand, one cannot consistently add an $I=2$, $J=0$ pole to σ_{ab} (and such a meson does not appear to exist experimentally). Further, Eq. (6.2) leads, in the soft-pion approximation, to the Weinberg[4] S-wave π-π scattering lengths, which now appear to be in agreement with current data.[32] Finally, we note that an assumption such as Eq. (6.2) is directly extendable to chiral $SU(3)$ currents.

As discussed in Sec. V, the differential equations for $\mathcal{L}_I$ do not determine it uniquely, and a great deal of ambiguity exists in the higher-point functions. Even if one assumes a "minimal" coupling choice, there is still an increasing number of undetermined coupling constants as one goes higher and higher in the series of $\mathcal{L}_I$. Additional physical assumptions, outside the framework of current algebra, are needed to obtain unique theoretical predictions. It is tempting to speculate that conditions on the high-energy behavior of the theory might furnish the necessary hypotheses. Thus, one might argue that vertices with particularly large numbers of momentum factors in the numerator

[29] This effect appears to have been omitted in the calculation of the π^+-π^0 mass difference of I. S. Gerstein, B. W. Lee, H. T. Nieh, and H. J. Schnitzer, Phys. Rev. Letters **19**, 1064 (1967). A detailed hard-pion calculation of this mass difference will be considered elsewhere.

[30] S. Weinberg, Phys. Rev. **166**, 1568 (1968).

[31] The inclusion of cubic structures to $A^0{}_a$ allows one to relax Eq. (4.21) and leave λ_1 and λ_3 arbitrary. Conversely, if one imposes Eq. (6.2) on the $A^0{}_a$ with cubic terms, one automatically gets back Eqs. (4.19b) and (4.21).

[32] M. G. Olsson and L. Turner, Phys. Rev. Letters **20**, 1127 (1968). The π-π scattering lengths enter importantly at threshold for the process $\pi+N\to 2\pi+N$. Current data appear to favor the Weinberg values.

should be set to zero. For example, in the part of $\mathcal{L}_{(3)}$ of Eq. (2.15c), the term proportional to $\mu_{\pi\rho A}\varphi_a G^{\mu\nu}{}_b H_{\mu\nu c}$ possesses more momentum factors (due to the gradients in $G^{\mu\nu}{}_b$ and $H_{\mu\nu c}$) than the $g_{\pi\rho A}$ term. The assumption that this term must vanish, i.e., that $\mu_{\pi\rho A}=0$, then yields, by the current-algebra conditions (4.14), the result

$$g_A = g_\rho\,, \tag{6.3}$$

which is just the second Weinberg sum rule.[8] On similar grounds, one might assume $\mu_{\rho\rho\rho}=0=\mu_{\rho AA}$, which is essentially the "minimal" solution discussed in Sec. V.[28] A more ambitious idea arises if one wishes to take the effective Lagrangian seriously. Thus, one may try to determine the additional coupling constants in such a way that the theory is renormalizable (or even finite). This is not *a priori* inconceivable since the type of theory that has arisen here is different from the conventional field theories in that $\mathcal{L}_I$ possesses an infinite series of terms. Thus, the theory may furnish its own infinite sequence of counter terms. Of course, suggestions of this type imply using $\mathcal{L}_I$ beyond its original domain of validity (which was to only first nonvanishing order of perturbation theory).

As was pointed out in Sec. I, one of the uses of the T products of currents comes from the fact that the currents themselves represent interpolating fields for the mesons. Thus the T products are essentially scattering amplitudes. Having obtained the effective Lagrangian, however, one may discard the currents completely and calculate scattering and production processes directly from the Lagrangian, using the phenomenological Heisenberg meson fields appearing there. In a certain sense then, the currents have disappeared from the final form of the theory.

It is of interest to compare the results given here with other discussions. The work closest in spirit to this paper is the Ward's identity analysis of Gerstein and Schnitzer[10] which is also based on the same current-algebra conditions of Eqs. (4.2)–(4.4). However, there exist several differences in assumptions independent of the current algebra. Thus, these authors are able to obtain general amplitudes without imposing single-meson saturation, but omit the σ meson when they later make this assumption in their π-π and π-ρ scattering amplitudes.[33] Consequently, a nonpole assumption instead of Eq. (6.2) is made for the σ commutator. Finally, two of the "nonminimal" couplings of $\mathcal{L}_{(4)}$,

$$\xi_1(\beta^\mu{}_a\beta_{\mu a})^2+\xi_2\beta^\mu{}_a\beta^\nu{}_b\beta_{\mu b}\beta_{\nu a}\,, \tag{6.4}$$

are retained while a definite value is assigned to the constant a_3 of Eqs. (C4) and (C5) (which is arbitrary in our analysis for reasons discussed in Sec. V).

The "phenomenological" Lagrangian approach[10] gives similar results to those obtained here, though it is based on a different set of principles. These authors assume *a priori* a chiral-invariant Lagrangian and a postulate for breakdown of this symmetry equivalent to a nonpole choice for the σ commutator.[34] In contrast, the analysis given here assumes only that chiral invariance required by the current-algebra conditions. As a consequence, the $\mathcal{L}_{(3)}$ and $\mathcal{L}_{(4)}$ so obtained have a number of additional chiral-asymmetric and chiral-symmetric couplings not included in the phenomenological Lagrangians. Whether these couplings are required in nature remains to be seen.[35]

[33] This corresponds to taking the limit $m_\sigma\to\infty$ in the amplitudes obtained from the $\mathcal{L}_{(4)}$ of this paper.

APPENDIX A: INTEGRABILITY OF CURRENT-ALGEBRA CONDITIONS

In Sec. IV it was shown that one could reexpress the current commutation relations and conservation conditions (4.2)–(4.4) as first-order functional differential equations to be satisfied by the interaction Lagrangian $\mathcal{L}_I$. These equations, (4.23), (4.27a), (4.32), and (5.2), determine the $v^\mu{}_a$, φ_a, and $\alpha^\mu{}_a$ dependence of $\mathcal{L}_I$ [where $\alpha^\mu{}_a$ is defined in Eq. (5.1b)]. In this Appendix, we verify that in fact the coupled differential equations correctly obey the integrability conditions (5.4), at least for the classical theory.

As discussed in Sec. V, Eqs. (4.27a), (4.32), and (5.2) determine $\mathcal{L}_I$ in a power series in the field variables,

$$\mathcal{L}_I=\sum_{n=3}\mathcal{L}_{(n)}\,, \tag{A1}$$

where $\mathcal{L}_{(n)}$ is of nth order in the fields. In the following it is convenient to introduce the symbol D_n to represent the functional derivative of $\mathcal{L}_{(n)}$. Thus

$$D_nG_{\mu\nu a}\equiv\delta\mathcal{L}_{(n)}/\delta G^{\mu\nu}{}_a\,,\quad\text{etc.} \tag{A2}$$

Similarly, second derivatives of $\mathcal{L}_{(n)}$ will be denoted by D_n followed by two field variables, e.g.,

$$D_n(\varphi_{\mu a}v_{\nu b})\equiv\frac{\delta}{\delta\varphi^\mu{}_a}\left(\frac{\delta\mathcal{L}_{(n)}}{\delta v^\nu{}_b}\right). \tag{A3}$$

We will also use the symbol A_q or A_p to represent any isotopic and Lorentz component of the three fields $v_{\mu a}$, $\alpha_{\mu a}$, and φ_a while A_s or A_t will denote a component of any field (i.e., of $v_{\mu a}$, $\alpha_{\mu a}$, φ_a, $\beta_{\mu a}$, $G_{\mu\nu a}$, $H_{\mu\nu a}$, σ, and σ_μ).

Consider now, for generality, the quantum case where the product of factors in the differential equations (4.27a), (4.32), and (5.2) are to be symmetrized (to ensure Hermiticity). Inserting in Eq. (A1), one obtains

[34] The use by these authors of a nonlinear relation between currents and fields is not a fundamental distinction as one is always free to make a change of field variables to restore the linear current-field identity. The σ meson is not included in any of the papers of Ref. 10 (and hence a σ-pole assumption for σ_{ab} is not assumed) but could be included in a chiral symmetric way. We also note that while *a priori* chiral invariance would require only the *integrated* current algebra to hold, the phenomenological Lagrangians actually satisfy the local current commutation relations with only c-number Schwinger terms [B. Zumino (private communication)].

[35] We note that some of these additional couplings can be used to make the meson electromagnetic mass shifts finite.

a set of differential equations to determine the A_q dependence of $\mathcal{L}_{(n)}$. These have the general form

$$D_n A_q = F_{nq}\,, \quad n \geq 4 \tag{A4a}$$

where

$$F_{nq} = \sum_{s,t} C^q{}_{st} \tfrac{1}{2}\{A_s, D_{n-1}A_t\} + \sum_{s,t} K^q{}_{st} \sum_{r=2}^{n-3} \tfrac{1}{2}\{D_{n-r}A_s, D_{r+1}A_t\} \tag{A4b}$$

and $C^q{}_{st}$, $K^q{}_{st} \equiv K^q{}_{ts}$ are a set of numerical coefficients. [The second sum in Eq. (A4b) is defined to be zero for the case $n=4$.] We wish to show that Eqs. (A4) satisfy the integrability conditions

$$\delta F_{nq}/\delta A_p = \delta F_{np}/\delta A_q\,, \tag{A5a}$$

i.e., that these equations are consistent with the requirement

$$D_n(A_q A_p) = D_n(A_p A_q)\,. \tag{A5b}$$

This would then imply that one could integrate Eq. (A4a) to obtain a single function $\mathcal{L}_{(n)}$ satisfying these equations. Consequently, Eqs. (A5) would also imply

$$D_n(A_s A_t) = D_n(A_t A_s)\,. \tag{A6}$$

Now, $\mathcal{L}_{(3)} = \mathcal{L}_{(3)\pi\rho A} + \mathcal{L}_{(3)\sigma}$ has been explicitly exhibited in Eqs. (2.15c) and (3.10b) and so, of course, Eq. (A6) holds for $n=3$. Using this $\mathcal{L}_{(3)}$, one may integrate Eqs. (A4) for the case $n=4$ to find a single function $\mathcal{L}_{(4)}$ satisfying the equations. This solution is explicitly exhibited in Appendix C. Thus, by construction Eqs. (5) and (6) are satisfied for the case $n=4$. To verify them for $n \geq 5$ we will proceed by induction and assume that Eqs. (A5) and (A6) hold for all m between $3 \leq m \leq n-1$ and from this establish that Eqs. (A5) holds for $m=n$.

Differentiating Eqs. (4) yields

$$D_n(A_p A_q) = \sum_{s,t} C^q{}_{st} \tfrac{1}{2}\{A_s, D_{n-1}(A_p A_t)\} + F_{n,pq}\,, \tag{A7a}$$

where

$$F_{n,pq} \equiv \sum_t C^q{}_{pt} D_{n-1} A_t + \sum_{s,t} K^q{}_{st} \sum_{r=2} \{D_{n-r}(A_p A_s), D_{r+1}A_t\}\,. \tag{A7b}$$

By hypothesis, $D_{n-1}(A_p A_t) = D_{n-1}(A_t A_p)$ and since $n-1 \geq 4$ one may differentiate Eqs. (4) with respect to A_t to eliminate this quantity. One finds

$$D_n(A_p A_q) = \tfrac{1}{4} \sum_{s,t,u,v} C^q{}_{st} C^p{}_{uv} \{A_s, \{A_u, D_{n-2}(A_t A_v)\}\} + \tfrac{1}{2} \sum_{s,t} C^q{}_{st} \{A_s, F_{n-1,tp}\} + F_{n,pq}\,. \tag{A8}$$

For the linear conditions, Eqs. (4.27a) and (5.2), one has that $F=0$ and so Eq. (A5a) reduces to

$$\sum_{stuv} C^q{}_{st} C^p{}_{uv} [[A_s, A_u], D_{n-2}(A_t A_v)] = 0\,. \tag{A9}$$

This condition is thus automatically satisfied for the c-number theory. The analysis involving the nonlinear PCAC condition is more complicated since F is nonzero. Inserting Eq. (A7b) into Eq. (A8) yields

$$\begin{aligned} D_n(A_p A_q) - D_n(A_q A_p) = {} & \tfrac{1}{4} \sum_{stuv} C^q{}_{st} C^p{}_{uv} [[A_s, A_u], D_{n-1}(A_t A_v)] + \tfrac{1}{2} \sum_{stu} (C^q{}_{st} C^p{}_{tu} - C^p{}_{st} C^q{}_{tu}) \{A_s, D_{n-2}A_u\} \\ & + \tfrac{1}{2} \sum_{stuv} (C^q{}_{st} K^p{}_{uv} - C^p{}_{st} K^q{}_{uv}) \{A_s, \sum_{r=2}^{n-4} \{D_{n-r-1} A_t A_u, D_{r+1} A_v\}\} + \sum_t (C^q{}_{pt} - C^p{}_{qt}) D_{n-1} A_t \\ & + \sum_{st} K^q{}_{st} \sum_{r=2}^{n-3} \{D_{n-r}(A_p A_s), D_{r+1} A_t\} - \sum_{st} K^p{}_{st} \sum_{r=2}^{n-3} \{D_{n-r}(A_q A_s), D_{r+1} A_t\}\,. \end{aligned} \tag{A10}$$

Using again the hypotheses that $D_{n-r}(A_q A_s) = D_{n-r}(A_s A_q)$ implies

$$\begin{aligned} \sum_{st} K^p{}_{st} \sum_{r=2}^{n-3} \{D_{n-r}(A_q A_s), D_{r+1} A_t\} = {} & \tfrac{1}{2} \sum_{s,t} K^p{}_{st} \sum_{r=2}^{n-4} \{\sum_{uv} C^q{}_{uv} \{A_u, D_{n-r-1}(A_s A_v)\}, D_{r+1} A_t\} \\ & + \sum_{s,t} K^p{}_{st} \sum_{r=2}^{n-4} \{F_{n-r,sq}, D_{r+1} A_t\} + \sum_{s,t} K^p{}_{st} \{D_3(A_q A_s), D_{n-2} A_t\}\,. \end{aligned} \tag{A11}$$

Thus, the third and last two terms of Eq. (A10) may be combined. Inserting in the value of F from Eq. (A7b) then reduces Eq. (A10) to

$$\begin{aligned} D_n(A_p A_q) - D_n(A_q A_p) = {} & \tfrac{1}{2} \sum_{s,t,u} (C^q{}_{st} C^p{}_{tu} - C^p{}_{st} C^q{}_{tu}) \{A_s, D_{n-2} A_u\} + \sum_t (C^q{}_{pt} - C^p{}_{qt})(D_{n-1} A_t) + \sum_{st} \{K^q{}_{st} D_3(A_p A_s) \\ & - K^p{}_{st} D_3(A_q A_s), D_{n-2} A_t\} + \sum_{stu} (K^q{}_{st} C^p{}_{su} - K^p{}_{st} K^q{}_{su}) \sum_{r=2}^{n-4} \{D_{n-r-1} A_u, D_{r+1} A_t\} \\ & + \sum_{stuv} [[\tfrac{1}{2} C^q{}_{st} A_s + K^q{}_{st} D A_s, \tfrac{1}{2} C^p{}_{uv} A_u + K^p{}_{uv} D A_u], D(A_t A_v)]_{n-2}\,, \end{aligned} \tag{A12}$$

where the subscript $n-2$ on the last term implies that one take only those parts of the double commutator containing $n-2$ field variables. Equation (A12) clearly reduces to the special case (A9) when $F=0$.

The verification that the right-hand side of Eq. (A12) (aside from the double commutator) vanish is now straightforward. It is necessary to use only the relations (4.14)–(4.17) and (4.21) between the coupling constants of $\mathfrak{L}_{(3)}$. Thus, for the case $A_q=\varphi_a$ and $A_p=\varphi_b$, the right-hand side of Eq. (A12) reduces to (neglecting the double commutator)

$$(F_\pi)^{-2}[H^{\mu\nu}{}_a(D_{n-2}H_{\mu\nu b})+G^{\mu\nu}{}_a(D_{n-2}G_{\mu\nu b})+v^\mu{}_a(D_{n-2}v_{\mu b}) +a^\mu{}_a(D_{n-2}a_{\mu b})+\varphi_a(D_{n-2}\varphi_b) +\varphi_a(D_{n-2}\varphi_{\mu b})]-[a\leftrightarrow b]\,. \quad \text{(A13)}$$

Since Eq. (A13) is manifestly antisymmetric in a and b one may multiply it by ϵ_{abc}. The result is then proportional to the first-order change in $\mathfrak{L}_{(n-2)}$ under an isotopic rotation. Thus Eq. (A13) vanishes by the isotopic invariance of $\mathfrak{L}_{(n-2)}$, i.e., by the CVC requirement for the lower-order Lagrangian.

The double commutator term in Eq. (A12) results from the fact that although the c-number equations are satisfied, an order of operators in a term in $\mathfrak{L}$ which satisfies one requirement (e.g., the requirement that the canonical expansion for $V^\mu{}_a$ contain terms at most quadratic in field operators) might not satisfy another requirement (e.g., the same requirement for $V^\mu{}_b$, $b\neq a$).

However, the order of operators in a term in the effective $\mathfrak{L}$ does not affect a calculation made according to the prescription resulting from the assumptions of this paper. Consider a calculation made with two Lagrangians, $\mathfrak{L}^{(1)}$ and $\mathfrak{L}^{(2)}$, where $\mathfrak{L}^{(2)}$ is identical to $\mathfrak{L}^{(1)}$ except for the ordering of certain terms. A calculation of an N-point process to order $N-2$ is a calculation of terms like

$$\langle N-a|\mathfrak{L}_{(N)}|a\rangle\,, \quad \text{(A14)}$$

and

$$\langle N-a|\mathfrak{L}_{(q)}|b\rangle\langle b|\mathfrak{L}_{(N-q+2)}|a-b+1\rangle\,, \quad \text{etc.,}$$

where the terms in the $\mathfrak{L}$'s above can be taken to be free since the terms have been expanded to the proper order in the coupling constant. The *difference* between terms (A14) calculated for $\mathfrak{L}^{(1)}$ and those calculated for $\mathfrak{L}^{(2)}$ will have c numbers replacing two of the field operators (arising from a canonical commutator), e.g.,

$$\langle N-a|\mathfrak{L}_{(N-2)}|a\rangle\times c\text{-No.}\,,$$

or

$$\langle N-a|\mathfrak{L}_{(q-2)}|b\rangle\langle b|\mathfrak{L}_{(N-q+2)}|a-b+1)\rangle\times c\text{-No.}\,,$$

or

$$\langle N-a|\mathfrak{L}_{(q)}|b\rangle\langle b|\mathfrak{L}_{(N-q)}|a-b+1\rangle\times c\text{-No.}$$

All such terms vanish since the field operators in $\mathfrak{L}$ have already been replaced by free-field operators. Thus whether or not a term in $\mathfrak{L}$ has a given ordering cannot affect a calculation.

APPENDIX B: CLOSED-FORM SOLUTIONS

In Sec. V, the general structure of the current-algebra differential equations (4.27a), (4.27b), and (4.32) was discussed. The analysis given there showed the nature of the power-series solutions. Actually, the two current commutation equations (4.27a) and (4.27b) can be solved in closed form (at least for the classical theory). We present here the results of this integration.

It is convenient to introduce the following notation for commonly occurring quantities:

$$\sigma'\equiv\lambda_1(F_\pi)^{-1}\sigma\,,\qquad \varphi_a'\equiv F_\pi^{-1}\varphi_a\,,$$
$$G'^{\mu\nu}{}_a\equiv g_\rho^{-1}G^{\mu\nu}{}_a\,,\qquad H'^{\mu\nu}{}_a=g_A^{-1}H^{\mu\nu}{}_a\,,\quad \text{(B1)}$$
$$\omega^2\equiv g_A{}^2/(F_\pi m_A)^2=x^2y^2z^2\,.$$

We will also make use of the isotopic-tensor quantities

$$\tilde{\varphi}_{ac}\equiv\epsilon_{abc}\varphi_b'\,,\quad \tilde{G}^{\mu\nu}{}_{ac}\equiv\epsilon_{a\lambda c}G'^{\mu\nu}{}_b\,,\quad \tilde{H}^{\mu\nu}{}_{ac}\equiv\epsilon_{abc}H'^{\mu\nu}{}_b\,.\quad \text{(B2)}$$

In the discussion of Sec. V it was pointed out that the series solution of the functional differential equations is facilitated by replacing $\varphi_{\mu a}$ and $a_{\mu a}$ by a new set of variables $\beta_{\mu a}$ and $\alpha_{\mu a}$ defined in Eqs. (5.1). Equations (4.27a) and (5.2) then determine the $v_{\mu a}$ and $\alpha_{\mu a}$ dependence of $\mathfrak{L}_I$, leaving the $\beta_{\mu a}$ dependence incompletely specified. A nonlinear generalization of Eqs. (5.1) is arrived at by introducing the variables $\eta_{\mu a}$ and $\zeta_{\mu a}$ defined by

$$\eta_{\mu a}=\tfrac{1}{2}\{m_A{}^2g_A{}^{-1}(1+\sigma'\omega^{-2})^{-1}[a_{\mu a}+\tfrac{1}{2}(yz^2)^{-1}\tilde{\varphi}_{ac}v_{\mu c}] +F_\pi{}^{-1}(1-\tilde{G})^{-1}{}_{\mu\nu ac}[\varphi^\nu{}_c+F_\pi m_\rho{}^2g_\rho{}^{-1}\tilde{H}^{\nu\lambda}{}_{cd}v_{\lambda d}]\}\,,\quad \text{(B3a)}$$

and

$$\zeta_{\mu a}=\tfrac{1}{2}\{g_Am_A{}^{-2}(1+\sigma'\omega^{-2})(\varphi_{\mu a}+F_\pi m_\rho{}^2g_\rho{}^{-1}\tilde{H}_{\mu\nu ad}v^\nu{}_d) -F_\pi(1-\tilde{G})_{\mu\nu ac}[a^\nu{}_c+\tfrac{1}{2}(yz^2)^{-1}\tilde{\varphi}_{cd}v^\nu{}_d]\}\,.\quad \text{(B3b)}$$

Here $(1-\tilde{G})^{-1}$ means the matrix inverse, i.e.,

$$(1-\tilde{G})^{-1\mu\nu}{}_{ac}(1-\tilde{G})_{\nu\lambda cd}=\delta^\mu{}_\lambda\delta_{ad}\,.\quad \text{(B4)}$$

Expanding $\eta_{\mu a}$ and $\zeta_{\mu a}$ in a power series in the fields gives

$$\eta_{\mu a}=\tfrac{1}{2}(m_A{}^2g_A{}^{-1}a_{\mu a}+F_\pi{}^{-1}\varphi_{\mu a})+\cdots\,,\quad \text{(B5a)}$$

and

$$\zeta_{\mu a}=-F_\pi g_Am_A{}^{-2}\tfrac{1}{2}(m_A{}^2g_A{}^{-1}a_{\mu a}-F_\pi{}^{-1}\varphi_{\mu a})+\cdots.\quad \text{(B5b)}$$

Thus, $\zeta_{\mu a}$ is the generalization of $\beta_{\mu a}$ while $\eta_{\mu a}$ is the generalization of $\alpha_{\mu a}$. As in the discussion of Sec. V, the choice of $\eta_{\mu a}$ is not unique and we have chosen here (for convenience) the analog of $R_1=R_2=\tfrac{1}{2}$ in Eq. (5.1b).

Equations (4.27a) and (4.27b) will now determine the $v_{\mu a}$ and ηv_a dependence of $\mathfrak{L}_I$ (and leave the $\zeta_{\mu a}$ dependence incompletely specified). In dealing with these equations, it is convenient to rearrange them slightly by adding $\mathfrak{L}_{(3)}$ and a piece of the free Lagrangian $\mathfrak{L}_{(0)}$ to $\tilde{L}$. Thus, for the (0) of Eqs. (2.15b) and (3.10a) we write

$$\mathfrak{L}_{(0)}=\mathfrak{L}_{(0)1}+\mathfrak{L}_{(0)2}\,, \tag{B6}$$

where $\mathfrak{L}_{(0)1}$ contains those pieces of the free Lagrangian having no derivatives of field variables and $\mathfrak{L}_{(0)2}$ those terms depending on derivatives. Then, if we define

$$\mathfrak{L}'\equiv\mathfrak{L}_{01}+g_AF_\pi^{-1}\varphi^\mu{}_a a_{\mu a}+\mathfrak{L}_I\,, \tag{B7}$$

Eqs. (4.27) reduce to

$$\delta\mathfrak{L}'/\delta v^\mu{}_a=-m_\rho^2(1-\tilde{G})_{\mu\nu ab}v^\nu{}_b\,, \tag{B8}$$

and

$$\delta\mathfrak{L}'/\delta\eta^\mu{}_a=g_\rho^2m_\rho^{-2}(1-\tilde{G})_{\mu\nu ab}\eta^\nu{}_b+2g_\rho(x^2yF_\pi)^{-1}\times(1+\sigma'\omega^{-2})^{-1}\zeta_{\mu a}+\lambda_1^{-1}\varphi_a\sigma_\mu\,. \tag{B9}$$

These equations may be directly integrated to yield the total Lagrangian

$$\begin{aligned}\mathfrak{L}=\mathfrak{L}_{(0)2}&-g_AF_\pi^{-1}\varphi^\mu{}_a a_{\mu a}-\tfrac{1}{2}m_\rho^2(1-\tilde{G})^{\mu\nu}{}_{ab}v_{\mu a}v_{\nu b}\\&+\tfrac{1}{2}g_\rho^2m_\rho^{-2}(1-\tilde{G})^{\mu\nu}{}_{ab}\eta_{\mu a}\eta_{\nu b}+2g_\rho(x^2yF_\pi)^{-1}\\&\times(1+\sigma'\omega^{-2})^{-1}\eta^\mu{}_a\xi_{\mu a}\lambda_1^{-1}\varphi_a\eta_{\mu a}\sigma^\mu\\&+f(\varphi_a,G^{\mu\nu}{}_a,H_{\mu\nu a},\zeta_{\mu a},\sigma,\sigma_\mu)\,,\end{aligned} \tag{B10}$$

where f is an arbitrary function of integration.

The quadratic part of f may be chosen so that $\mathfrak{L}$ has the correct total free Lagrangian contribution $\mathfrak{L}_{(0)}$. Similarly, the cubic pieces are to be chosen so that $\mathfrak{L}$ has the total $\mathfrak{L}_{(3)}$ of Eqs. (2.15c) and (3.10a). The φ_a dependence of the remainder of f is determined by the PCAC equation (4.32). In dealing with this equation it is slightly more convenient to introduce the quantity

$$\mathfrak{L}''\equiv\mathfrak{L}_{(0)1}+\mathfrak{L}_I\,. \tag{B11}$$

Equation (4.32) then becomes

$$\begin{aligned}(1-\sigma')D\varphi_a'-\epsilon_{abc}[&H'^{\mu\nu}{}_bDG'^{\mu\nu}{}_c+G'^{\mu\nu}{}_bDH'^{\mu\nu}{}_c+g_\rho g_A^{-1}v^\mu{}_bDa^\mu{}_c+g_Ag_\rho^{-1}a^\mu{}_bDv^\mu{}_c+F_\pi m_\rho^2g_\rho^{-1}v^\mu{}_bD\varphi_{\mu c}\\&+F_\pi g_\rho^{-1}\varphi^\mu{}_bDv^\mu{}_c+F_\pi g_\rho^{-1}(D\varphi^\mu{}_b)(Dv_{\mu c})]-\lambda_1^{-1}\sigma^\mu D\varphi_{\mu a}+\varphi_a'D\sigma'+\lambda_1\varphi^\mu{}_aD\sigma_\mu\\&+F_\pi\lambda_1g_A^{-1}\sigma^\mu Da_{\mu a}-F_\pi m_A^2\lambda_1g_A^{-1}a^\mu{}_aD\sigma_\mu+\lambda_1F_\pi g_A^{-1}(Da^\mu{}_a)(D\sigma_\mu)\\&=\epsilon_{abc}3m_\rho^2F_\pi g_\rho^{-1}v^\mu{}_b\varphi_{\mu c}+3m_A^2\lambda_1F_\pi g_A^{-1}a^\mu{}_a\sigma_\mu+(\lambda_1^{-1}-\lambda_1)\varphi^\mu{}_a\sigma_\mu-F_\pi^2m_\pi^2\varphi_a'\,,\end{aligned} \tag{B12}$$

where $D\varphi_a\equiv\delta\mathfrak{L}''/\delta\varphi_a$, etc. The nonlinearity and general complexity of this equation makes it rather intractible. We note, however, that the following structures are solutions of the *homogeneous* equation:

$$h(\sigma'-\tfrac{1}{2}\sigma'^2-\tfrac{1}{2}\varphi'^2)\,, \tag{B13}$$

where $h(x)$ is an arbitrary function,

$$\begin{aligned}H'^{\mu\nu}{}_aH'_{\mu\nu a}+[&2\epsilon_{abc}H'^{\mu\nu}{}_a\varphi_bG'_{\mu\nu c}(1-\sigma')+\epsilon_{abc}\epsilon_{cde}\\&\times(H'^{\mu\nu}{}_a\varphi_b'\varphi_d'H'_{\mu\nu e}-G'^{\mu\nu}{}_a\varphi_b'\varphi_d'G'_{\mu\nu e})]\\&\times[(1-\sigma')^2+\varphi'^2]^{-1},\end{aligned} \tag{B14}$$

and Eq. (B14) with $G_{\mu\nu a}$ and $H_{\mu\nu a}$ interchanged. These terms may be added to Eq. (B10) and represent a partial determination of the function of integration f.

APPENDIX C: FORM OF $\mathfrak{L}_{(4)}$

In this Appendix we record the form of the "minimal" solution of the current-algebra conditions (4.27a), (5.2), and (4.32) (as defined in Sec. V) for the quartic part of the interaction Legrangian $\mathfrak{L}_{(4)}$. This quantity is obtained by inserting in the known value of $\mathfrak{L}_{(3)}$ on the right-hand side of Eqs. (4.27a), (5.2), and (4.32) and integrating once. A convenient procedure is to first integrate the current commutation equations (4.27a) and (5.2). As discussed in Sec. V, these equations determine the $v_{\mu a}$ and $\alpha_{\mu a}$ dependence of $\mathfrak{L}_{(4)}$ [where $\alpha_{\mu a}$ is defined in Eqs. (5.1)]. One obtains the result in terms of an arbitrary function of integration

$$f(\varphi_a,\beta_{\mu a},G_{\mu\nu a},H_{\mu\nu a},\sigma,\sigma_\mu)\,. \tag{C1}$$

For $\mathfrak{L}_{(4)}$, this function is of course quartic in the fields and the arbitrariness resides in the value of the coupling constants. The φ_a dependence of f is then determined by inserting this solution into the PCAC equation (4.32).

We divide the total $\mathfrak{L}_{(4)}$ into three pieces,

$$\mathfrak{L}_{(4)}=\mathfrak{L}_{(4)V}+\mathfrak{L}_{(4)A}+\mathfrak{L}_{(4)\mathrm{PCAC}}\,. \tag{C2}$$

Here $\mathfrak{L}_{(4)V}$ is the contribution required by the current commutation relations condition (4.27a) on the vector current $V^\mu{}_a$, $\mathfrak{L}_{(4)A}$ is the additional contribution required by the current commutation relations condition (4.27b) on the axial current $A^\mu{}_a$, and $\mathfrak{L}_{(4)\mathrm{PCAC}}$ is the additional φ_a dependence needed to satisfy the PCAC equation (4.32). We find for $\mathfrak{L}_{(4)V}$ the expression

$$\begin{aligned}\mathfrak{L}_{(4)V}=\epsilon_{abe}\epsilon_{cde}\lambda_{\pi\rho A}(&g_{\rho\rho\rho}v_{\nu a}\varphi_bv_{\mu c}H^{\mu\nu}{}_d+\lambda_{\rho AA}v_{\nu a}\varphi_ba_{\mu c}G^{\mu\nu}{}_d+\tfrac{1}{2}g_{\pi\rho A}v_{\nu a}\varphi_bv^\nu{}_c\varphi_d-\tilde{\lambda}_{\pi\rho A}v_{\nu a}\varphi_b\varphi_{\mu c}G^{\mu\nu}{}_d-\lambda_{\pi\pi\rho}v_{\nu a}H^{\nu\mu}{}_b\varphi^\lambda{}_cG_{\lambda\mu d}\\&+\tfrac{1}{2}\lambda_{\pi\rho A}v_{\nu a}H^{\nu\mu}{}_bv^\lambda{}_cH_{\lambda\mu d}+\tilde{\lambda}_{\pi\rho A}v_{\nu a}H^{\nu\mu}{}_ba^\lambda{}_cG_{\lambda\mu d})+\epsilon_{abc}\lambda_{\pi\rho A}[g_{\sigma AA}v_{\mu a}\varphi_ba^\mu{}_c\sigma+\tilde{\lambda}_{\sigma\pi A}v_{\mu a}\varphi_b\varphi^\mu{}_c\sigma\\&+(\mu_{\sigma AA}+\mu_{\sigma\pi\pi})v_{\mu a}\varphi_bH^{\mu\lambda}{}_c\sigma_\lambda-\lambda_{\sigma\pi\pi}v_{\nu a}H^{\nu\mu}{}_b\varphi_{\mu c}\sigma-\tilde{\lambda}_{\sigma\pi A}v_{\nu a}H^{\nu\mu}{}_ba_{\mu c}\sigma-\mu_{\sigma\pi A}v_{\nu a}H^{\nu\mu}{}_bH_{\mu\lambda c}\sigma^\lambda]\,.\end{aligned} \tag{C3}$$

The value of the $\mathfrak{L}_{(3)}$ coupling constants appearing in Eq. (C3) can be found in Eqs. (4.14)–(4.17) and (4.21).

The additional pieces of $\mathfrak{L}_{(4)}$ needed to satisfy Eq. (4.27b) is

$$\begin{aligned}\mathfrak{L}_{(4)A}=\epsilon_{abc}\epsilon_{cde}(&a_1\varphi_{\nu a}G^{\nu\mu}{}_b\varphi^\lambda{}_cG_{\lambda\mu d}+a_2a_{\nu a}G^{\nu\mu}{}_b\varphi^\lambda{}_cG_{\lambda\mu d}+a_3a_{\nu a}G^{\nu\mu}{}_ba^\lambda{}_cG_{\lambda\mu d})+\epsilon_{abc}(b_1\varphi_{\nu a}G^{\nu\mu}{}_b\varphi_{\mu c}\sigma+b_2a_{\nu a}G^{\nu\mu}{}_b\varphi_{\mu c}\sigma\\&+b_3a_{\nu a}G^{\nu\mu}{}_ba_{\mu c}\sigma)+[\epsilon_{abc}(c_1\varphi_{\nu a}G^{\nu\mu}{}_bH_{\mu\lambda c}\sigma^\lambda+c_2a_{\nu a}G^{\nu\mu}{}_bH_{\mu\lambda c}\sigma^\lambda)+(d_1a^\mu{}_aa_{\mu a}\sigma^2+d_2\varphi^\mu{}_a\varphi_{\mu a}\sigma^2+d_3a^\mu{}_a\varphi_{\mu a}\sigma^2)\\&+(e_1a_{\mu a}H^{\mu\nu}{}_a\sigma_\nu\sigma+e_2\varphi_{\mu a}H^{\mu\nu}{}_a\sigma_\nu\sigma)]+\epsilon_{abc}(k_1\varphi_{\nu a}G^{\nu\mu}{}_b\varphi_c\sigma_\mu+k_2a_{\nu a}G^{\mu\nu}{}_b\varphi_c\sigma_\mu)+(l_1a^\mu{}_a\varphi_a\sigma_\mu\sigma+l_2\varphi^\mu{}_a\varphi_a\sigma_\mu\sigma)\,.\end{aligned} \tag{C4}$$

The 17 new constants appearing in Eq. (C4) are constrained by ten equations. These are

$$g_A m_A^{-2} a_2 + 2F_\pi a_1 = F_\pi g_\rho^{-1} \lambda_{\pi\pi\rho}\,, \qquad 2g_A m_A^{-2} a_3 + F_\pi a_2 = -F_\pi g_\rho^{-1} \tilde{\lambda}_{\pi\rho A}\,, \tag{C5a}$$

$$g_A m_A^{-2} b_2 + 2F_\pi b_1 = F_\pi g_\rho^{-1} \lambda_{\sigma\pi\pi} + F_\pi g_A^{-1} \lambda_1 \tilde{\lambda}_{\pi\rho A}\,, \qquad 2g_A m_A^{-2} b_3 + F_\pi b_2 = F_\pi g_\rho^{-1} \tilde{\lambda}_{\sigma\pi A} - F_\pi g_A^{-1} \lambda_1 \lambda_{\rho A A}\,, \tag{C5b}$$

$$g_A m_A^{-2} c_2 + F_\pi c_1 = F_\pi g_\rho^{-1} \mu_{\sigma\pi A}\,, \tag{C5c}$$

$$2g_A m_A^{-2} d_1 + F_\pi d_3 = -F_\pi g_A^{-1} \lambda_1 g_{\sigma A A}\,, \qquad g_A m_A^{-2} d_3 + 2F_\pi d_2 = -F_\pi g_A^{-1} \lambda_1 \tilde{\lambda}_{\sigma\pi A}\,, \tag{C5d}$$

$$g_A m_A^{-2} e_1 + F_\pi e_2 = -F_\pi g_A^{-1} \lambda_1 \mu_{\sigma A A}\,, \tag{C5e}$$

$$g_A m_A^{-2} k_2 + F_\pi k_1 = F_\pi g_\rho^{-1} \mu_{\sigma\pi\pi}\,, \tag{C5f}$$

$$g_A m_A^{-2} l_1 + F_\pi l_2 = -F_\pi g_A^{-1} \lambda_1 \lambda_{\sigma\pi A}\,. \tag{C5g}$$

At this stage in the analysis there are then seven new undetermined coupling constants. This additional ambiguity in $\mathcal{L}_I$ is due to the ambiguity in the definition of $\alpha_{\mu a}$, as is duscussed in Sec. V.

The final part of $\mathcal{L}_{(4)}$, required by PCAC, is

$$\begin{aligned}
\mathcal{L}_{(4)\mathrm{PCAC}} = {}& \epsilon_{abe}\epsilon_{cde}\big[-\tfrac{3}{2}g_\rho(F_\pi g_A)^{-1}\mu_{\rho\rho\rho}\varphi_a H^{\mu\nu}{}_b G_\mu{}^\rho{}_c G_{\rho\nu d} - g_A(F_\pi g_\rho)^{-1}\mu_{\rho AA}\varphi_a G^{\mu\nu}{}_b G_\mu{}^\rho{}_c H_{\rho\nu d} \\
& -\tfrac{1}{2}g_\rho(F_\pi g_A)^{-1}\mu_{\rho AA}\varphi_a H^{\mu\nu}{}_b H_\mu{}^\rho{}_c H_{\rho\nu d}\big] + 4\big[g_A{}^2(m_A{}^2 g_\rho)^{-1}\lambda_{\pi\rho A} - g_\rho m_A{}^{-2}\tilde{\lambda}_{\pi\rho A} - g_A m_A{}^{-2}\lambda_1\mu_{\sigma AA}\big]\varphi_a\beta_{\mu a}H^{\mu\nu}{}_b\beta_{\nu b} \\
& +2\big[F_\pi{}^2 g_\rho{}^{-1}g_{\pi\pi\rho} - F_\pi\lambda_1\mu_{\sigma\pi\pi}\big](\varphi_a\beta^\mu{}_a)^2 - 2\big[F_\pi{}^2 g_\rho{}^{-1}g_{\pi\pi\rho} + \tfrac{1}{2}F_\pi(\lambda_1)^{-1}\lambda_{\sigma\pi\pi}\big](\varphi_a)^2(\beta_{\mu b})^2{}_{\nu A b} \\
& +\big\{-\tfrac{1}{8}(F_\pi\lambda_1)^{-1}g_{\sigma\pi\pi}(\varphi_a{}^2)^2 + \tfrac{1}{2}\big[g_A(F_\pi g_\rho)^{-1}\mu_{\pi\rho A} - \tfrac{1}{4}(F_\pi\lambda_1)^{-1}\lambda_{\sigma\rho\rho}\big](\varphi_a)^2 G^{\mu\nu}{}_b G_\mu - \tfrac{1}{2}\big[\tfrac{1}{4}(F_\pi\lambda_1)^{-1}\lambda_{\sigma A} \\
& +g_\rho(F_\pi g_A)^{-1}\mu_{\pi\rho A}\big](\varphi_a)^2 H^{\mu\nu}{}_b H_{\mu\nu b} + \tfrac{1}{2}g_\rho(F_\pi g_A)^{-1}\mu_{\pi\rho A}(\varphi_a H^{\mu\nu}{}_a)^2 - \tfrac{1}{2}g_A(F_\pi g_\rho)^{-1}\mu_{\pi\rho A}(\varphi_a G^{\mu\nu}{}_a)^2\big\} \\
& +\big\{\tfrac{1}{2}\big[F_\pi{}^{-1}\lambda_1 g_{\sigma\pi\pi} - 3(F_\pi\lambda_1)^{-1}g_{\sigma\sigma\sigma}\big](\varphi_a)^2\sigma^2 + \tfrac{1}{2}\big[(F_\pi\lambda_1)^{-1}\mu_{\sigma\pi\pi} - g_A{}^{-1}\lambda_1\lambda_{\sigma\pi A} - (F_\pi\lambda_1)^{-1}\lambda_{\sigma\sigma\sigma}\big](\varphi_a)^2\sigma^\mu\sigma_\mu\big\} \\
& +\big[F_\pi{}^{-1}\lambda_1\lambda_{\pi\rho A} + \tfrac{1}{2}g_A(F_\pi g_\rho)^{-1}\lambda_{\sigma AA} - \tfrac{1}{2}g_\rho(F_\pi g_A)^{-1}\lambda_{\sigma\rho\rho}\big]\epsilon_{abc}\varphi_a G_{\mu\nu b}H^{\mu\nu}{}_c\sigma\,,
\end{aligned} \tag{C6}$$

where $\beta_{\mu a}$ is defined in Eq. (5.1a). In addition, four of the coupling constants of Eq. (C5) are determined by the PCAC condition:

$$k_1 = -g_A(F_\pi g_\rho)^{-1}\mu_{\sigma\pi A} + (F_\pi\lambda_1)^{-1}\lambda_{\pi\pi\rho} + g_A{}^{-1}\lambda_1\tilde{\lambda}_{\pi\rho A}\,, \tag{C7a}$$

$$k_2 = -g_A(F_\pi g_\rho)^{-1}\mu_{\sigma AA} - (F_\pi\lambda_1)^{-1}\tilde{\lambda}_{\pi\rho A} - g_A{}^{-1}\lambda_1\lambda_{\rho AA}\,, \tag{C7b}$$

$$l_1 = (F_\pi\lambda_1)^{-1}\tilde{\lambda}_{\sigma\pi A} + F_\pi{}^{-1}\lambda_1\lambda_{\sigma\pi A} - g_A{}^{-1}\lambda_1 g_{\sigma AA} + 2g_A{}^{-1}m_A{}^2\lambda_1\lambda_{\sigma\sigma\sigma}\,, \tag{C8a}$$

$$l_2 = (F_\pi\lambda_1)^{-1}\lambda_{\sigma\pi\pi} + F_\pi{}^{-1}\lambda_1\mu_{\sigma\pi\pi} - g_A{}^{-1}\lambda_1\tilde{\lambda}_{\sigma\pi A} - 2F_\pi{}^{-1}\lambda_1\lambda_{\sigma\sigma\sigma}\,. \tag{C8b}$$

Equations (C7) and (C8) may be seen to be consistent with Eqs. (C5f) and (C5g) [upon using the coupling-constant conditions Eqs. (4.14)–(4.17) and (4.21)]. Thus, there remain only five undetermined constants in $\mathcal{L}_{(4)}$ (aside from those arising from $\mathcal{L}_{(3)}$).

Equations (C2)–(C8) represent the "minimal" $\mathcal{L}_{(4)}$ consistent with the current commutation relations, CVC, and PCAC. To this one is free to add any quartic function of $\beta_{\mu a}$, $G_{\mu\nu a}$, $H_{\mu\nu a}$, σ, and σ_μ. While even the minimal interaction Lagrangian is quite complicated, it should be remembered that the four-point functions which $\mathcal{L}_{(3)}+\mathcal{L}_{(4)}$ describe, govern meson-meson scattering through 19 channels.