data than the previous older prediction because of the change of sign of the ratio $(\nu B/A)_{A_2}$.

Other sum rules. We have considered generalized Schwarz sum rules which evaluate the "off l-shell" amplitudes in the Khuri plane. We find the background to be small in general, so that these relations are satisfied with Regge-pole parameters alone. The t dependence of these relations implies that the background amplitude has cuts, however, and this limits the applications, since further parameters to describe the background will then be needed.

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Analyticity and Broken $SL(2,C)$ Symmetry for Regge Families*

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Explicit constraints on the mass dependence of daughter Regge trajectories, near zero mass, are obtained for fermion trajectories contributing to πN scattering. Both the analyticity and the group-theoretic approaches are investigated. We find agreement between these two methods, but disagreement between our constraints and those previously published. For the dependence on the mass W of the kth daughter trajectory with parity designation $\pm$, we find that $\alpha_k^{(\pm)}(W)=\sigma-k\pm A(\sigma-k+\frac{1}{2})W+[B_1+B_2(\sigma-k)(\sigma-k+1)+A^2(\sigma-k+\frac{1}{2})]W^2\pm\cdots$, where σ, A, B_1, and B_2 are constants over the family. For each of the two methods, we stress the assumptions leading to the MacDowell symmetry evident above.

I. INTRODUCTION

In a recent paper[1] it has been pointed out that two different approaches to daughter Regge trajectories, analyticity and group-theoretic, lead to the same results for the scattering of spinless particles. Mathematically the equivalence of these two approaches has been established.[2] Namely, in order to make the analycity requirement for scattering amplitudes compatible with Lorentz invariance and Regge behavior, it is necessary and sufficient to classify singularities according to the irreducible representations of the homogeneous Lorentz group $SL(2,C)$. However, at the practical level, the ways by which these approaches lead to a given result differ considerably. At present their relationship is by no means trivial.[3] In this paper we compare these approaches for fermion trajectories, with particular emphasis on the mass formula that they yield. Even though the two methods agree, we find that each of the methods seems to have some advantages over the other. We reserve a more detailed discussion of this point for later. The mass formula that we obtain does not agree completely with that obtained previously by Domokos and Surányi,[4] hereafter referred to as DS, using their group-theoretic method. In order to facilitate comparison, our group-theoretic approach closely parallels that of DS. In our approach this disagreement is resolved by recognizing some subtleties associated with the use of wave functions having nonphysical angular momentum values.

In Sec. II we examine the implication of analyticity on the πN scattering amplitude near $u=0$ (u is the square of the momentum transfer for exchange scattering) in some detail, using the method of Ref. 1. In Sec. III we use our apparently modified version of the perturbation theory developed in DS to reproduce the results of Sec. II. Section IV contains some discussion concerning the relative merit of the two approaches and the degree to which the daughters are determined by experiment.

II. ANALYTICITY APPROACH TO πN SCATTERING AMPLITUDE

The πN scattering is dominated in the backward region by the exchange of fermion trajectories. For this reason we go to the u channel and define the invariant

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[1] J. B. Bronzan, C. E. Jones, and P. K. Kuo, Phys. Rev. **175**, 2200 (1968).

[2] G. Domokos and G. L. Tindle, Phys. Rev. **165**, 1906 (1968).

[3] L. Jones and H. K. Shepard, Phys. Rev. **175**, 2117 (1968).

[4] G. Domokos and P. Surányi, in *Proceedings of the Topical Conference on High-Energy Collision of Hadrons* (CERN, Geneva, 1968), Vol. 1, p. 494.

amplitudes A and B by writing the transition matrix element as[5]

$$\bar{u}(p_2)[A+B\gamma\cdot Q]u(p_1).$$

It is convenient to introduce new amplitudes $\mathfrak{M}$ and $\mathfrak{N}$:

$$\begin{aligned}\mathfrak{M}&=WE(A-MB)+W^2MB,\\ \mathfrak{N}&=M(A-MB)+WEB,\end{aligned}\tag{2.1}$$

which are closely akin to helicity amplitudes and are simply related to the partial-wave amplitudes:

$$\begin{aligned}\mathfrak{M}&=W^2\sum_{J=1/2}^{\infty}(f_J^{(+)}-f_J^{(-)})e_{-\frac12\frac12}{}^J(z),\\ \mathfrak{N}&=W\sum_{J=1/2}^{\infty}(f_J^{(+)}+f_J^{(-)})e_{\frac12\frac12}{}^J(z),\end{aligned}\tag{2.2}$$

where $f_J^{(\pm)}$ is the scattering amplitude in the state of total angular momentum J and parity $-(-1)^{J\mp1/2}$. The function $e_{\lambda\mu}{}^J$ is just the $d_{\lambda\mu}{}^J$ of Wigner with half-angle factors removed,[6]

$$e_{\pm\frac12\frac12}{}^J(z)=\tfrac12\sqrt{2}(J+\tfrac12)^{-1}[P_{J+1/2}'(z)\mp P_{J-1/2}'(z)]\tag{2.3}$$

and

$$z\equiv\cos\theta_u=1+2ut/[(M^2-\mu^2)^2-2u(M^2+\mu^2)+u^2].\tag{2.4}$$

The "Reggeizing" process[6] essentially amounts to the statement that $f_J^{(\pm)}(W)$ can be continued as a meromorphic function in the right-half complex J plane. For simplicity we assume that the boundary of meromorphy can be pushed far to the left and that each $f_J^{(\pm)}(J,W)$ has only a single family of poles.

Our input assumptions shall be that (i) A and B are analytic functions in $u=W^2$ near $u=0$ and (ii) A and B have Regge asymptotic behavior continuable to $u=0$. Since $WE=\frac12(u+M^2-\mu^2)$ is also analytic in u, it follows from (2.1) that these statements apply equally to $\mathfrak{M}$ and $\mathfrak{N}$. We write the contribution to $\mathfrak{M}$ and $\mathfrak{N}$ due to these poles as

$$\begin{aligned}\mathfrak{M}&=\sum_{k=0}^{\infty}[\beta_k^{(+)}(W)\hat{e}_{-\frac12\frac12}{}^{\alpha_k^{(+)}(W)}(z)\\&\qquad-\beta_k^{(-)}(W)\hat{e}_{-\frac12\frac12}{}^{\alpha_k^{(-)}(W)}(z)],\\ W\mathfrak{N}&=\sum_{k=0}^{\infty}[\beta_k^{(+)}(W)\hat{e}_{\frac12\frac12}{}^{\alpha_k^{(+)}(W)}(z)\\&\qquad+\beta_k^{(-)}(W)\hat{e}_{\frac12\frac12}{}^{\alpha_k^{(-)}(W)}(z)],\end{aligned}\tag{2.5}$$

where $\alpha_k^{(\pm)}(W)$ gives the position of the kth pole of $f_J^{(\pm)}(W)$ and $\beta_k^{(\pm)}(W)/W^2$ is its residue.[7] The function $\hat{e}_{\lambda\mu}{}^J$ bears the same relation to $e_{\lambda\mu}{}^J$ as Q_{-l-1} to P_l. At this point the machinery of Ref. 1 can be turned on to derive constraint equations for the α's and β's. We sketch the steps here. One first replaces the $\hat{e}$ functions by their asymptotic expansions,[8]

$$\begin{aligned}\hat{e}_{\pm\frac12\frac12}{}^{\alpha}(z)&=\frac{(-1)^{\alpha+\frac12}\pi}{\tan[(\alpha+\frac12)\pi]}\tfrac12\sqrt{2}\left(\frac{1-z}{2}\right)^{\alpha-\frac12}\\&\quad\times\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac12)\Gamma(\alpha+\frac32)}G^{(\pm)}\left(\alpha,\frac{1}{1-z}\right),\end{aligned}\tag{2.6}$$

where $G^{(\pm)}$ is a hypergeometric function:

$$\begin{aligned}G^{(\pm)}(\alpha,x)&=F(-\alpha\pm\tfrac12,\,-\alpha+\tfrac12;\,-2\alpha;\,2x)\\&=\sum_{r=0}^{\infty}g_r^{(\pm)}(\alpha)x^r,\\ g_r^{(+)}(\alpha)&=\frac{(-2)^r}{r!}\left(\frac{\Gamma(\alpha+\frac12)}{\Gamma(\alpha-r+\frac12)}\right)^2\frac{\Gamma(2\alpha-r+1)}{\Gamma(2\alpha+1)},\\ g_r^{(-)}(\alpha)&=\frac{\alpha+\frac12}{\alpha-r+\frac12}g_r^{(+)}(\alpha).\end{aligned}\tag{2.7}$$

Note that the coefficient of $G^{(\pm)}$ in (2.6) is the same for both $\hat{e}$'s. One then redefines the residue functions and rewrites (2.5) as

$$\begin{aligned}\mathfrak{M}(W^2,s)&\approx\sum_{k=0}^{\infty}\left[\gamma_k^{(+)}t^{\alpha_k^{(+)}-\frac12}G^{(-)}\left(\alpha_k^{(+)},\frac{1}{1-z}\right)\right.\\&\qquad\left.+\gamma_k^{(-)}t^{\alpha_k^{(-)}-\frac12}G^{(-)}\left(\alpha_k^{(-)},\frac{1}{1-z}\right)\right]Y^k,\\ W\mathfrak{N}(W^2,s)&\approx\sum_{k=0}^{\infty}\left[\gamma_k^{(+)}t^{\alpha_k^{(+)}-\frac12}G^{(+)}\left(\alpha_k^{(+)},\frac{1}{1-z}\right)\right.\\&\qquad\left.-\gamma_k^{(-)}t^{\alpha_k^{(-)}-\frac12}G^{(+)}\left(\alpha_k^{(-)},\frac{1}{1-z}\right)\right]Y^k,\end{aligned}\tag{2.8}$$

where the reduced residue function $\gamma_k^{(\pm)}$ is related to the old $\beta_k^{(\pm)}$ by

$$\begin{aligned}\gamma_k^{(\pm)}&=\pm\beta_k^{(\pm)}(W)\frac{(-1)^{\alpha_k^{(\pm)}+\frac12}\pi}{\tan[(\alpha_k^{(\pm)}+\frac12)\pi]}2^{-\alpha_k^{(\pm)}}\\&\quad\times\frac{\Gamma(2\alpha_k^{(\pm)}+1)}{\Gamma(\alpha_k^{(\pm)}+\frac12)\Gamma(\alpha_k^{(\pm)}+\frac32)}Y^{\frac12-\alpha_k^{(\pm)}-k}\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}Y&=(-1/2u)[(M^2-\mu^2)^2-2u(M^2+\mu^2)+u^2]\\&=t/(1-z).\end{aligned}\tag{2.10}$$

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[5] Our notation is such that p_1, q_1 (p_2,q_2) are the initial (final) momenta of the nucleon and pion. $Q=\frac12(q_1+q_2)$; μ and M are the pion and nucleon masses. The c.m. variables W and E are the total energy and nucleon energy. The Mandelstam variables are $s=(p_1-q_2)^2$, $t=(q_1-q_2)^2$, and $u=(p_1+q_1)^2=W^2$. Isospin labels are suppressed throughout. Our partial-wave amplitudes do not have the conventional normalization.

[6] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

[7] We have combined various factors, including the signature, into $\beta_k^{(\pm)}(W)$.

[8] *Bateman Manuscript Project*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I.

The last factor in (2.9) is tacitly chosen so that the singularity in $\beta_k^{(\pm)}$ is cancelled and all M, μ dependences factor out of the constraint equations for $\alpha_k^{(\pm)}$ and $\gamma_k^{(\pm)}$. That we have made the correct choice is shown by the fact that $\gamma_k^{(\pm)}$, as determined below, is analytic in W and has no zero at $W=0$. Furthermore, Y is factorizable into factors depending on masses in initial and final states separately. This leads to the desired property that $\beta_k^{(\pm)}$ will be factorizable.

The next step is to examine (2.8) for large s, or, equivalently, large $-t$, and demand that all terms with inverse powers of u cancel. The justification of such practice is found in Ref. 1. Some conclusions can be drawn immediately: Unless $\gamma_0^{(+)}(0)=\gamma_0^{(-)}(0)=0$, it is necessary to have $\alpha_0^{(+)}(0)=\alpha_0^{(-)}(0)=\sigma$ and $\alpha_k^{(\pm)}(0)$ $\sigma-k$. This establishes the existence of two conspiring families of trajectories, one of each parity. Furthermore, assuming complete knowledge of $\mathfrak{M}(W^2,s)$ and $\mathfrak{N}(W^2,s)$, (2.8) then determines $\alpha_k^{(\pm)}(W)$ and $\gamma_k^{(\pm)}(W)$ uniquely. The solution satisfies the MacDowell symmetry,

$$\alpha_k^{(+)}(W)=\alpha_k^{(-)}(-W),\quad \gamma_k^{(+)}(W)=\gamma_k^{(-)}(-W). \tag{2.11}$$

By looking at the coefficient of $t^{\sigma-n}(\ln t)^r$ in (2.8) one obtains the following infinite set of constraint equations:

$$\sum_{k=0}^{n}\{g_{n-k}^{(-)}(\alpha_k^{(+)}(W))\gamma_k^{(+)}(W)[\delta_k^{(+)}(W)]^r + g_{n-k}^{(-)}(\alpha_k^{(-)}(W))\gamma_k^{(-)}(W)[\delta_k^{(-)}(W)]^r\} = O(W^{2n-r}), \tag{2.12a}$$

$$\sum_{k=0}^{n}\{g_{n-k}^{(+)}(\alpha_k^{(+)}(W))\gamma_k^{(+)}(W)[\delta_k^{(+)}(W)]^r - g_{n-k}^{(+)}(\alpha_k^{(-)}(W))\gamma_k^{(-)}(W)[\delta_k^{(-)}(W)]^r\} = O(W^{2n-r+1}), \tag{2.12b}$$

with

$$\delta_k^{(\pm)}(W)=W^{-1}[\alpha_k^{(\pm)}(W)-\alpha_k^{(\pm)}(0)]. \tag{2.13}$$

The details of solving these equations are found in the Appendix. The results are expressed in expansion parameters, defined with (2.11) taken into account:

$$\begin{aligned}\alpha_k^{(\pm)}(W)&=\alpha_k\pm\alpha_k'W+\tfrac{1}{2}\alpha_k''W^2\pm\cdots,\\ \gamma_k^{(\pm)}(W)&=\gamma_k\pm\gamma_k'W+\tfrac{1}{2}\gamma_k''W^2\pm\cdots.\end{aligned} \tag{2.14}$$

We find

$$\begin{aligned}\gamma_k&=\gamma_0\frac{2^k}{k!}\frac{\Gamma(\sigma+\frac{1}{2})\Gamma(\sigma+\frac{3}{2})\Gamma(2\sigma-2k+2)}{\Gamma(\sigma-k+\frac{1}{2})\Gamma(\sigma-k+\frac{3}{2})\Gamma(2\sigma-k+2)},\\ \gamma_k'&=\frac{2^k}{k!}\left(\gamma_0'+\gamma_0\alpha_0'\frac{d}{d\sigma}\right)\left[\frac{\Gamma(\sigma+\frac{1}{2})}{\Gamma(\sigma-k+\frac{1}{2})}\right]^2\frac{\Gamma(2\sigma-2k+2)}{\Gamma(2\sigma-k+2)}\end{aligned} \tag{2.15}$$

and[9]

$$\begin{aligned}\alpha_k'&=A(\sigma-k+\tfrac{1}{2}),\\ \tfrac{1}{2}\alpha_k''&=B_1+B_2(\sigma-k)(\sigma-k+1)+A^2(\sigma-k+\tfrac{1}{2}),\end{aligned} \tag{2.16}$$

where A, B_1, and B_2 are arbitrary constants and can be solved in favor of α_0', α_0'', and α_1''. The last expression disagrees with the mass formula of DS.

We summarize results obtained by the analyticity approach.

(i) Regge poles again appear in families with the same properties as those appearing in the spinless case.[1]

(ii) If the leading pole does not decouple at $u=0$, it is necessary to have two families of poles with opposite parity conspiring at $u=0$.

(iii) The two families are images of each other under MacDowell symmetry. This follows from our assumption that the invariant amplitudes A and B are analytic in u near $u=0$.

(iv) At $u\neq0$ the higher symmetry is broken, so that poles no longer have to be integrally spaced. However, the most general solution is the mass formula (2.16), and the residues behave according to (2.15).

III. GROUP-THEORETIC APPROACH

The classification of families of Regge trajectories according to irreducible representations of the group $SL(2,C)$ has been considered by several authors over the past few years.[2,4,10] In particular, we follow closely the work of DS in order to see where the previously indicated discrepancy arises. The basis of this technique may be found in the original DS article,[11] where enlightening discussion on the use of "pseudostates" for "particles" with nonphysical angular momentum is given. Most of that discussion will not be reproduced here. Our primary aim in this section will be to obtain the formula relating the mass and the spin of these pseudostates.

A particular irreducible representation of the $SL(2,C)$ algebra is specified by the eigenvalues of the Casimir operators. Since this algebra of six elements can be expressed as the direct sum of two $SU(2)$ algebras, $\mathbf{J}_1$ and $\mathbf{J}_2$, the irreducible representations of $SL(2,C)$ may be specified by (j_1,j_2), where $j_i(j_i+1)$ is the eigenvalue of of $\mathbf{J}_i^2$, $i=1$, 2. An equivalent specification is (n,j_0), where

$$n=j_1+j_2,\quad j_0=j_1-j_2. \tag{3.1}$$

For integer or half-integer values of j_1 and j_2 the representation (j_1,j_2) has dimension $(2j_1+1)(2j_2+1)$. The specification of the states within the representation is given by j, m_j, the angular momentum quantum numbers associated with $\mathbf{J}^2$ and J_z, where $\mathbf{J}=\mathbf{J}_1+\mathbf{J}_2$. That this is an appropriate specification for this problem will be seen later. For the finite-dimensional representation given above, the ranges of j and m_j are

$$\begin{aligned}j&=n,\ n-1,\ n-2,\ \cdots,\ |j_0|,\\ m_j&=j,\ j-1,\ j-2,\ \cdots,\ -j.\end{aligned} \tag{3.2}$$

[9] This mass formula has been derived independently by J. B. Bronzan and C. E. Jones by a similar analyticity method (private communication).

[10] M. Toller, Nuovo Cimento **37**, 63 (1965); M. Toller, *ibid.* **54**, 295 (1968), and references therein; D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

[11] G. Domokos and P. Surányi, Nucl. Phys. **54**, 529 (1964).

If j_1 and j_2 are not integer or half-integer, the sequence for j in (3.2) does not terminate at $|j_0|$, and the irreducible representation is not finite-dimensional. In any case, the basis states spanning the representation (j_1,j_2) are denoted by

$$|(j_1,j_2);\, j,m_j\rangle\,, \tag{3.3}$$

or, equivalently,

$$|n,j_0;\, j,m_j\rangle;\quad j=n-k,\quad k=0,1,2,\cdots. \tag{3.4}$$

The Wigner-Eckart theorem for the matrix elements of an irreducible tensor $T_{j,m}{}^{(j_1,j_2)}$ has been given by DS:

$$\begin{aligned}\langle(j_1',j_2'),\, j',m'\,|T_{j'',m''}{}^{(j_1'',j_2'')}|\,(j_1,j_2);\, j,m\rangle\\ =C(jj''j';\, mm''m')\langle(j_1',j_2')\|T^{(j_1'',j_2'')}\|(j_1,j_2)\rangle\\ \times\langle(j_1',j_2')j'\|(j_1,j_2)j;\,(j_1'',j_2'')j''\rangle.\end{aligned} \tag{3.5}$$

Equation (3.5) defines the reduced matrix elements of $T^{(j_1,j_2)}$ in terms of a Clebsch-Gordan coefficient and a coefficient related to the 9-j symbol,

$$\begin{aligned}\langle(j_1',j_2')j'\|(j_1,j_2)j;\,(j_1'',j_2'')j''\rangle\\ =[(2j+1)(2j''+1)(2j_1'+1)(2j_2'+1)]^{1/2}\\ \times\begin{Bmatrix} j_1 & j_1'' & j_1' \\ j_2 & j_2'' & j_2' \\ j & j'' & j' \end{Bmatrix}.\end{aligned} \tag{3.6}$$

In the Bethe-Salpeter (BS) model as presented by DS, corresponding to each Regge trajectory $j=\alpha(W)$, there is a pseudostate $\psi(W;\alpha)$ that is an eigenstate of the BS equation

$$H(W)\psi(W;\alpha)=0\,, \tag{3.7}$$

$$H=G_0^{-1}+I\,, \tag{3.8}$$

where G_0^{-1} is the inverse of the free two-particle propagator and I is the sum of all T-matrix graphs having no intermediate states identical to the two free particles. For example, for the u-channel πN state in the c.m. frame, G_0^{-1} in the momentum representation is given by[5]

$$G_0^{-1}(p,W)=(\tfrac{1}{2}\gamma_0 W+\gamma\cdot p-M)[(\tfrac{1}{2}W-p)^2-\mu^2]\,, \tag{3.9}$$

where

$$p=\tfrac{1}{2}(p_1-q_1)\,.$$

The operator $\mathbf{J}$, which occurs in the $SL(2,C)$ algebra, is to be associated with the total spin of the BS psuedostate. This includes intrinsic spin as well as the angular momentum associated with the relative momentum, p. The basis states (3.4) can be represented in terms of the intrinsic spin coordinates and the angular coordinates (including the boost angle) of p.[12]

At $W\neq 0$ the BS operator H is invariant under homogeneous Lorentz transformations of p, the four-vector $\Delta=p_1+q_1$, of which W is the fourth component in the c.m. frame, and the intrinsic spin.[13] However, for $\Delta^\mu=0$ ($W=0$ in the c.m. frame), H transforms as an invariant under the $SL(2,C)$ transformation of p and the intrinsic spin only. Therefore the solutions to (3.7) with $W=0$ fall into degenerate families corresponding to irreducible representations of $SL(2,C)$. We consider one such family of pseudostates corresponding to a particular representation at $W=0$,

$$n=\sigma,\quad j_0,\quad j\equiv\alpha_k=\sigma-k;\quad k=0,1,2,\cdots. \tag{3.10}$$

Here α_k is the generally nonintegral eigenvalue for (3.7).

If I in (3.8) contains only contributions from the strong interactions, H is invariant under parity. However, under parity,[14] $\mathbf{J}_1\to\mathbf{J}_2$, so that

$$n\to n,\quad j_0\to -j_0,\quad j\to j\,. \tag{3.11}$$

It follows that for every solution to (3.7) with designation (3.10) there is another with j_0 replaced by $-j_0$. Definite-parity linear combinations of these states may be expressed in terms of the definite-parity ($\pi=\pm$) basis states[15]

$$\begin{aligned}|n,M;\, j,\pm\rangle=\tfrac{1}{2}\sqrt{2}(|n,M;\, j,m_j\rangle\\ \pm|n,-M;\, j,m_j\rangle),\quad M\equiv|j_0|\,.\end{aligned} \tag{3.12}$$

The definite-parity pseudostates belonging to the family (3.10) are

$$\psi_{\sigma,M}(0;\, k,\pi)=R(\sigma,M)\,|\sigma,M;\,\sigma-k,\,\pi\rangle\,. \tag{3.13}$$

Because of the degeneracy of the family, the "radial" function $R(\sigma,M)$ depends only on the quantum numbers of the family.

As W is increased from zero, the degeneracy of the states (3.13) is broken because H is not invariant under Lorentz transformations with W fixed. However, H is invariant under the $SU(2)$ algebra $\mathbf{J}$ and under parity. This is the reason that the j,m_j classification of the $W=0$ degeneracy is proper. The j,m_j states go continuously into eigensolutions for $W\neq 0$ where the higher symmetry is broken. We now ask what constraints are placed on the Regge trajectory $\alpha_k{}^\pi(W)$, originating at $W=0$ with a particular daughter number k. This pseudostate, for $W\neq 0$, can be expanded in terms of the definite-parity $SL(2,C)$ basis states (3.12). However, a complete set of irreducible representations must be summed over, since H is no longer an invariant. Because of the $SU(2)$ invariance, only those states with $j=\alpha_k{}^\pi(W)$ for a particular value of W need be included. We have

$$\begin{aligned}\psi_{\sigma,M}(W;\, k,\pi)=\sum_{k''M''}R_{k''M''}(W;\,\sigma,M,k,\pi)\\ \times|\alpha_k{}^\pi(W)+k'',M'';\,\alpha_k{}^\pi(W),\pi\rangle.\end{aligned} \tag{3.14}$$

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[12] G. Domokos, Phys. Rev. **159**, 1387 (1967), especially Sec. 2 B, which gives these representations explicitly for the case of no intrinsic spin.

[13] One must be careful to distinguish an invariant from a scalar in the sense of M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), p. 81.

[14] Stephen Gasiorowicz, *Elementary Particle Physics* (John Wiley & Sons, Inc., New York, 1966), p. 84.

[15] We do not explicitly carry the m_j quantum number any further.

Comparing (3.14) and (3.13), we see that

$$R_{k''M''}(W;\sigma,M,k,\pi)\overset{W\to 0}{\longrightarrow}\delta_{k''k}\delta_{M''M}R(\sigma,M)\,,\qquad \alpha_k^{\pi}(W)\overset{W\to 0}{\longrightarrow}\sigma-k\,. \tag{3.15}$$

The expansion (3.14) must be considered unconventional, to say the least. Not only is it based on a noninteger j value, but the j value changes with W, and is in fact the eigenvalue to be determined in the problem. We shall assume that the expansion is justifiable by considering the differential operators for $\mathbf{J}_1^2$, $\mathbf{J}_2^2$, and $\mathbf{J}^2$ with appropriate boundary conditions to allow the particular j value that we desire. In general, this boundary condition will not be the physical condition of single-valuedness.[16] One might ask how the j value can change as a rotationally invariant perturbation W is turned on. The answer lies in the fact that we permit the boundary conditions satisfied by the pseudostate to violate rotational symmetry, and to change as j changes with W.

We now obtain a perturbation expansion in powers of W, being careful to include the W dependence of the basis states. This is most easily done by working with radial equations for the functions $R_{k''M''}(W;\sigma,M,k,\pi)$ of (3.14).[17] Using (3.7), we have

$$\sum_{k''M''} H_{k'M';\,k''M''}(W;\alpha_k^{\pi}(W),\pi)\times R_{k''M''}(W;\sigma,M,k,\pi)=0\,, \tag{3.16}$$

where, in terms of the states (3.12),

$$H_{k'M';\,k''M''}(W;j,\pi)=\langle j+k',\,M';j,\pi|H(W)\times|j+k'',\,M'';j,\pi\rangle. \tag{3.17}$$

A restatement of the notation of (3.16) is perhaps in order. The radial functions $R_{k''M''}(W;\sigma M,k,\pi)$ are the $n=j+k''$, $|j_0|=M''$, components of the eigenstate of (3.16) with eigenvalue $j=\alpha_k^{\pi}(W)$, which goes as $W\to 0$ to the $n=\sigma$, $|j_0|=M$, $j=\sigma-k$ pseudostate. Everything is to be expanded in powers of W:

$$H(W)=H^{(0)}+H^{(1)}W+\tfrac{1}{2}H^{(2)}W^2+\cdots\,, \tag{3.18}$$

$$\alpha_k^{\pi}(W)=\sigma-k+\alpha_k^{\pi\prime}W+\tfrac{1}{2}\alpha_k^{\pi\prime\prime}W^2+\cdots\,, \tag{3.19}$$

$$R_{k''M''}(W;\sigma,M,k,\pi)=\delta_{k''k}\delta_{M''M}R(\sigma,M)+R_{k''M''}{}'W+\tfrac{1}{2}R_{k''M''}{}''W^2+\cdots\,, \tag{3.20}$$

where the leading terms in (3.19) and (3.20) have been chosen to satisfy (3.15). If we define

$$H^{(i)}{}_{k'M';\,k''M''}(j,\pi)=\langle j+k',\,M';\,j,\,\pi|H^{(i)}|j+k'',\,M'';\,j,\,\pi\rangle\,, \tag{3.21}$$

and we recall that $H(0)$ is an $SL(2,C)$ invariant, i.e.,

$$H^{(0)}{}_{k'M';\,k''M''}(j,\pi)=\delta_{k'k''}\delta_{M'M''}H^{(0)}(j+k,'\,M')\,, \tag{3.22}$$

we find from (3.18) and (3.19) that

$$\begin{aligned}H_{k'M';\,k''M''}(W,\alpha_k^{\pi}(W))&=\delta_{k'k''}\delta_{M'M''}H^{(0)}(\sigma-k+k',\,M')\\&+\Big[H^{(1)}{}_{k'M';\,k''M''}(j,\pi)+\delta_{k'k''}\delta_{M'M''}\alpha_k^{\pi\prime}\frac{\partial}{\partial j}H^{(0)}(j+k',\,M')\Big]_{j=\sigma-k}W\\&+\Big\{H^{(2)}{}_{k'M';\,k''M''}(j,\pi)+2\alpha_k^{\pi\prime}\frac{\partial}{\partial j}H^{(1)}{}_{k'M';\,k''M''}(j,\pi)+\delta_{k'k''}\delta_{M'M''}\\&\times\Big[\alpha_k^{\pi\prime\prime}\frac{\partial}{\partial j}H^{(0)}(j+k',\,M')+(\alpha_k^{\pi\prime})^2\frac{\partial^2}{\partial j^2}H^{(0)}(j+k',\,M')\Big]\Big\}_{j=\sigma-k}\tfrac{1}{2}W^2+\cdots.\end{aligned} \tag{3.23}$$

Now (3.20) and (3.23) give the perturbation expansion of (3.16). We shall look at the first three equations obtained in this series.

The zeroth-order equation

$$H^{(0)}(\sigma,M)R(\sigma,M)=0 \tag{3.24}$$

gives the radial wave function for the family and, as an eigenvalue, the position σ of the parent pole.

The first-order equation

$$H^{(0)}(\sigma-k+k',\,M')R_{k',M'}{}'+\Big[H^{(1)}{}_{k'M',kM}(j,\pi)+\delta_{k'k}\delta_{M'M}\alpha_k^{\pi\prime}\frac{\partial}{\partial j}H^{(0)}(j+k,\,M)\Big]_{j=\sigma-k}R(\sigma,M)=0 \tag{3.25}$$

may be solved for $\alpha_k^{\pi\prime}$ and $R_{k'M'}{}'$, $(k',M')\neq(k,M)$, by taking appropriate matrix elements and using (3.24). We introduce the notation for the radial matrix element

[16] We thank K. Johnson for discussion related to this point.

[17] If one does perturbation theory based on the full four-dimensional equation (3.7), the W-dependent boundary conditions on the angular part of ψ force one to include the boundary perturbations whenever the Hermiticity property of H is used.

of an operator $\mathcal{O}$:

$$\langle\mathcal{O}\rangle\equiv(R(\sigma,M),\mathcal{O}R(\sigma,M)), \quad (3.26)$$

$$\alpha_k^{\pi\prime}=-\langle H^{(1)}{}_{kM;kM}(\sigma-k,\,\pi)\rangle\Big/\left\langle\frac{\partial}{\partial\sigma}H^{(0)}(\sigma,M)\right\rangle, \quad (3.27)$$

$$R_{k'M'}{}'=-\frac{1}{H^{(0)}(\sigma-k+k',\,M')}\times H^{(1)}{}_{k'M';kM}(\sigma-k,\,\pi)R(\sigma,M), \quad (k',M')\neq(k,M). \quad (3.28)$$

The one missing first-order result $R_{kM}{}'$ is seen to be zero by expanding the normalization condition for the pseudostate. We use

$$(\psi_{\sigma,M}(W;\,k,\pi),\psi_{\sigma,M}(W;\,k,\pi))=1. \quad (3.29)$$

In (3.28) the existence of a radial Green's function has been assumed.[11]

Finally, from the coefficient of W^2 in (3.16), we only want the result for $\alpha_k^{\pi\prime\prime}$, which the reader may verify, using (3.24) and (3.28), to be

$$\alpha_k^{\pi\prime\prime}=\left\langle\frac{\partial H^{(0)}(\sigma,M)}{\partial\sigma}\right\rangle^{-1}\Bigg[2\sum_{(k',M')\neq(k,M)}\left\langle H^{(1)}{}_{kM,k'M'}(\sigma-k,\,\pi)\frac{1}{H^{(0)}(\sigma-k+k',\,M')}H^{(1)}{}_{k'M',kM}(\sigma-k,\,\pi)\right\rangle$$
$$-\langle H^{(2)}{}_{kM,kM}(\sigma-k,\,\pi)\rangle-(\alpha_k^{\pi\prime})^2\left\langle\frac{\partial^2}{\partial\sigma^2}H^{(0)}(\sigma,M)\right\rangle-2\alpha_k^{\pi\prime}\left\langle\frac{\partial}{\partial j}H^{(1)}{}_{kM,kM}(j,\pi)\Big|_{j=\sigma-k}\right\rangle\Bigg]. \quad (3.30)$$

In order to decide how the results (3.27) and (3.30) depend on the daughter index k, we must use the known transformation properties of $H^{(i)}$ under $SL(2,C)$ together with (3.5).

Since $H(W)$ is invariant under Lorentz transformations of p, Δ, and the intrinsic spin variables, $H^{(0)}$ is invariant under $SL(2,C)$ transformations of p and intrinsic spin, while $H^{(1)}$ and $H^{(2)}$ must transform like W and W^2, respectively. Since W is the timelike component of a four-vector, it transforms according to $(j_1,j_2)j=(\frac{1}{2},\frac{1}{2})0$. Therefore, we have the $SL(2,C)$ irreducible tensor representations[18]

$$H^{(1)}\colon(\tfrac{1}{2},\tfrac{1}{2})0,\qquad H^{(2)}\colon[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})]0=(0,0)0\oplus(1,1)0. \quad (3.31)$$

The selection rules corresponding to (3.31) may be obtained from (3.5) and are tabulated in DS. The $j=\sigma-k$ dependence of the matrix elements (3.21) between states given by (3.12) is obtained by using the irreducible tensor properties (3.31) and the Wigner-Eckart theorem (3.5). We give the explicit results for the πN system taking $M=\frac{1}{2}$:

$$H^{(1)}{}_{k\frac{1}{2},k\frac{1}{2}}(j,\pm)=\pm(j+\tfrac{1}{2})h(j+k), \quad (3.32)$$

$$2\sum_{(k',M')\neq(k,\frac{1}{2})}H^{(1)}{}_{k\frac{1}{2},k'M'}(j,\pi)\frac{1}{H^{(0)}(j+k',\,M')}\times H^{(1)}{}_{k'M',k\frac{1}{2}}(j,\pi)-H^{(2)}{}_{k\frac{1}{2},k\frac{1}{2}}(j,\pi)=g_1(j+k)+j(j+1)g_2(j+k). \quad (3.33)$$

The source of the $\pm$ sign appearing in (3.32) is discussed in more detail at the end of this section when we consider the MacDowell symmetry. The two terms on the left-hand side of (3.33) actually each separately have the j dependence of the right-hand side. We see by (3.30) that we also need[19]

$$\frac{\partial}{\partial j}H^{(1)}{}_{k\frac{1}{2},k\frac{1}{2}}(j,\pm)=\pm[h(j+k)+(j+\tfrac{1}{2})h'(j+k)]. \quad (3.34)$$

Here h, g_1, and g_2 are radial operators depending only on the family designation (σ, $M=\frac{1}{2}$). Substituting (3.32)–(3.34) into (3.27) and (3.30) gives results agreeing with the analyticity results (2.16), with the MacDowell symmetry explicitly exhibited:

$$\alpha_k^{(+)\prime}=-\alpha_k^{(-)\prime}=A(\sigma-k+\tfrac{1}{2}),\qquad \alpha_k^{(\pm)\prime\prime}=B_1+B_2(\sigma-k)(\sigma-k+1)+A^2(\sigma-k+\tfrac{1}{2}). \quad (3.35)$$

The constants A, B_1, and B_2 may be obtained in terms of the matrix elements of h, g_1, and g_2. For instance,

$$A=-\langle h(\sigma)\rangle\Big/\left\langle\frac{\partial}{\partial\sigma}H^{(0)}(\sigma,\tfrac{1}{2})\right\rangle. \quad (3.36)$$

As we have indicated, (3.35) is consistent with the MacDowell symmetry (2.11). This fact arises solely from our observation that the BS operator H is invariant under parity. This led to the expansion of the positive-(negative-) parity pseudostates in terms of the symmetrized (antisymmetrized) basis states (3.12). The selection rules for $H^{(1)}$, and for the coefficients of all odd powers of W in $H(W)$, are such as to couple the state (σ, $j_0=\frac{1}{2}$) to the state (σ, $j_0=-\frac{1}{2}$), but not to itself. In this way the $\pm$ sign of (3.12) appears in (3.32) and finally as a sign difference between $\alpha_k^{(+)\prime}$ and $\alpha_k^{(-)\prime}$.

[18] Our $H^{(1)}$ and $H^{(2)}$ correspond to H_W and H_{WW} of DS. We do not have to consider transformation properties of H_σ, $H_{\sigma W}$, etc., as in DS, since we differentiate with respect to j only after the angular matrix elements have been evaluated.

[19] The discrepancy between DS and our results arises because of a difference in the term equivalent to (3.33). DS uses $H_{\sigma W}$: $(\frac{1}{2},\frac{1}{2})0$, whereas we find an additional (0,0)0 contribution in our corresponding term $\partial H^{(1)}/\partial j$.

TABLE I. Comparison of the predicted resonances with their experimentally determined values.[a] The choice $\sigma=-0.37$ is made arbitrarily.

Resonances	k	Theor. (MeV)	Expt. (MeV)[a]
P_{11}	0	940[b]	940
F_{15}	0	1692[b]	1692
D_{15}	0	1650	1678
P_{13}	1	1960	1863
D_{13}	1	1660	1526
G_{17}	1	2210	2260
$I_{1,11}$	1	2650[b]	2650
$K_{1,15}$	1	3030	3030
P_{11}	2	1770	1750

[a] C. Lovelace, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Interscience Publishers, Inc., New York, 1968), p. 79; A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).
[b] Input data.

IV. DISCUSSION

We have seen how the two approaches give rise to the same results in very different manners. The analyticity approach is most efficient in deriving mass formulas for the boson and fermion trajectories and the corresponding residue functions. However, it is not clear that the determination of the former is always independent of the latter, although it is found to be true in the two cases that have been studied. Another point is the identification of the M quantum number. The analyticity approach has so far been applied to the cases of scalar-scalar and scalar-spinor scatterings, where all integral or half-integral M-valued trajectories can participate. Without working out cases with general spin it is not clear to us how one should attach the M quantum number to trajectories. For this reason the restriction imposed by factorization when a given trajectory contributes to two or more helicity amplitudes is also not obvious. The group-theoretic approach, on the other hand, remedies all these shortcomings at the expense of having to use a BS equation model and slightly more cumbersome algebra. We do not believe that the advantages of the group-theoretic approach have been fully exploited. More investigation is needed in this direction.

Finally, a remark about comparison with available experimental data is in order. In DS an excellent fit is obtained for the case of the $I=\frac{3}{2}$, Δ family, and a less successful fit for the $I=\frac{1}{2}$, N family. Since $A\neq 0$ only for the N family, we need only reexamine this case. It should be pointed out from the outset that the behavior of the higher daughter trajectories depend on the parameters σ, A, B_1, and B_2 far more sensitively than that of the parent. One can change these parameters slightly so as to retain a reasonable fit for the parent while causing wild fluctuations of the daughter trajectories. For instance, using the same input data as in DS, we find[20] $\sigma=-0.33$, $A=0.65$ BeV^{-1}, $B_1=1.06$ BeV^{-2}, and $B_2=0.33$ BeV^{-2}. The crucial difference here is that B_2 has changed sign from the value -0.33 in DS, so that all the higher daughters, instead of avoiding the physical region, crowd the region of low-lying resonances. It is possible to fit most of the resonances on various trajectories, but always at the expense of introducing more unobserved resonances. To give an example, we arbitrarily choose $\sigma=-0.37$. By forcing the $k=0$ trajectory through $P_{11}(940)$, the nucleon, and $F_{15}(1690)$, and the $k=1$ trajectory through $I_{1,11}(2650)$, we find that $A=0.31$, $B_1=1.04$, and $B_2=-0.08$. We favor this fit because it gives small values of A and B_2, which measure the strength of symmetry breaking at $W\neq 0$. Table I shows the comparison of the predicted resonances with their experimentally determined values. We emphasize, however, that such fitting should not be taken seriously, since the above is but one example out of many other possible fittings, all having various degrees of success. Our belief is that the presently available data are not accurate enough to give a convincing test of the mass formula (2.16).

[20] The parent trajectory data are taken from a recent fit by Y. Noirit, M. Rimpault, and Y. Saillard, Phys. Letters **26B**, 454 (1968), and the first daughter is determined by forcing the $k=1$ trajectory through $D_{13}(1526)$.

Note added in proof. Two articles bearing on the mass formula discrepancy have come to our attention since submission of this article. A brief account of a group-theoretic derivation of a mass formula in agreement with our (2.16) has been given by G. Domokos, S. Kövesi-Domokos and P. Suranyi, Nuovo Cimento **56A**, 233 (1968). A mass formula in agreement with Ref. 4 has been obtained by N. W. McFadyen, Phys. Rev. **171**, 1691 (1968).

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APPENDIX

Equations (2.12) can be solved iteratively by substituting (2.14) and collecting powers of W. However, such a process is laborious; we present here a more systematic solution.

The first set of equations that we treat are obtained by setting $W=0$ in (2.12):

$$\sum_{k=0}^{n} g_{n-k}^{(-)}(\sigma-k)\gamma_k\alpha_k'^{2l}=0,\quad n\geqslant l+1 \qquad \text{(A1)}$$

$$\sum_{k=0}^{n} g_{n-k}^{(+)}(\sigma-k)\gamma_k\alpha_k'^{2l+1}=0,\quad n\geqslant l+1, \qquad \text{(A2)}$$

where we have substituted $\sigma-k$ for α_k and $\pm\alpha_k'$ for $\delta_k^{(\pm)}(0)$. To facilitate solution of these equations we introduce the triangular matrix T by

$$T_{nk}=g_{n-k}^{(-)}(\sigma-k),\quad n\geqslant k$$
$$=0,\quad n<k.$$

Since $T_{nn}=1$, $T-1$ is also triangular and has zero diagonal elements. It follows that any finite element of T^{-1} can be computed by a finite number of terms in the expansion

$$T^{-1}=1-(T-1)+(T-1)^2-(T-1)^3+\cdots.$$

Thus the solution to the equation

$$\sum_{k=0}^{n} T_{nk}x_k=0, \quad n\geqslant l+1$$

is that x_k be any linear combination of the first $l+1$ columns of T^{-1}. Taking $l=0$ in (A1) and (A2), we have

$$\gamma_k=\gamma_0(T^{-1})_{k0},$$

$$\frac{\gamma_k\alpha_k'}{\sigma-k+\frac{1}{2}}=\frac{\gamma_0\alpha_0'}{\sigma+\frac{1}{2}}(T^{-1})_{k0}. \tag{A3}$$

Comparing these two, we have, without even knowing T^{-1},

$$\alpha_k'=\alpha_0'(\sigma-k+\tfrac{1}{2})/(\sigma+\tfrac{1}{2}). \tag{A4}$$

It is necessary, however, to know T^{-1} in order to compute γ_k.

With the help of the following identities,

$$\sum_{r=0}^{m}\binom{m}{r}(-1)^r\mathrm{B}(p-r,\, q)=(-1)^m\mathrm{B}(p-m,\, q+m),$$

$$\sum_{r=0}^{m}\binom{m}{r}(-1)^r r\mathrm{B}(p-r,\, q)=(-1)^m m\mathrm{B}(p-m,\, q+m-1),$$

where

$$\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}=\int_0^1 x^{p-1}(1-x)^{q-1}dx$$

is the B function, and the definition (2.7), the reader can easily verify that

$$(T^{-1})_{kn}=\frac{2^{k-n}}{(k-n)!}\frac{\Gamma(\sigma-n+\frac{1}{2})}{\Gamma(\sigma-k+\frac{1}{2})}\times\frac{\Gamma(\sigma-n+\frac{3}{2})}{\Gamma(\sigma-k+\frac{3}{2})}\frac{\Gamma(2\sigma-2k+2)}{\Gamma(2\sigma-k-n+2)}. \tag{A5}$$

More equations can be obtained by applying $(d/dW)^m$ to (2.12) before setting $W=0$. We apply, e.g., d/dW to (2.12b):

$$\sum_{k=0}^{n}\{g_{n-k}^{(+)\prime}(\sigma-k)\gamma_k\alpha_k'^{2l+1}+g_{n-k}^{(+)}(\sigma-k)\times[\gamma_k'\alpha_k'^{2l}+l\gamma_k\alpha_k'^{2l-1}\alpha_k'']\}=0, \quad n\geqslant l+1$$

or

$$\frac{d}{d\sigma}\sum_{k=0}^{n} g_{n-k}^{(+)}(\sigma-k)\gamma_k\alpha_k'^{2l+1}+\sum_{k=0}^{n}T_{nk}\frac{\sigma-n+\frac{1}{2}}{\sigma-k+\frac{1}{2}}\times\left[\gamma_k'\alpha_k'^{2l}+l\gamma_k\alpha_k'^{2l-1}\alpha_k''-\frac{d}{d\sigma}\gamma_k\alpha_k'^{2l+1}\right]=0, \quad n\geqslant l+1.$$

The first term vanishes because of (A2). Setting $l=0$ and 1, we have

$$\gamma_k'-\frac{d}{d\sigma}(\gamma_k\alpha_k')=\gamma_0'\frac{\sigma-k+\frac{1}{2}}{\sigma+\frac{1}{2}}(T^{-1})_{k0}, \tag{A6}$$

$$\alpha_k''-2\alpha_k'\frac{d\alpha_k'}{d\sigma}=\text{const}+\text{const}\,\alpha_k'^2. \tag{A7}$$

In deriving (A7) it is not necessary to make use of the explicit form for T^{-1}; instead, we have used a fact implied by (A1), i.e., the most general solution to

$$\sum_{k=0}^{n} T_{nk}\gamma_k x_k=0, \quad n\geqslant l+1$$

is $x_k=\mathcal{P}_l(\alpha_k'^2)$, where $\mathcal{P}_l$ is any polynomial of degree l. Equations (A3), (A4), (A6), and (A7) together with (A5) lead to results (2.15) and (2.16).

We remark that since (A1) and (A2) overdetermine the constants γ_k and α_k', it is necessary to show that (A1) and (A2) with higher values of l are all consistent with the solution determined with $l=0$. The consistency is ensured by the above-mentioned fact and another fact, which the reader can also verify, that the rth column of T^{-1}, $(T^{-1})_{kr-1}$, multiplied by $\alpha_k'^2$, can be expressed as some linear combination of the rth and $(r+1)$th columns.