Derivation of the Mass and Spin Spectrum for Mesons and Baryons*

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The problem of classification and mass spectrum of elementary particles is considered in the framework of the algebraic approach. It is shown that mesons as well as baryons can be described by one and the same algebra, which is essentially a combination of the Majorana representation and the de Sitter model, and that the spectrum can be derived with the use of only one new constant.

I. INTRODUCTION

INFINITE multiplets have been used extensively for the classification and description of the internal structure of hadrons,[1] and there exists a variety of approaches to the derivation of dynamical properties. The use of an algebraic structure for the description of hadron properties is one of the proposed ways. In this approach, the algebraic structure is defined by algebraic relations and the infinite multiplet is connected with an irreducible representation space of the algebraic structure. As is well known from the O'Raifeartaigh[2] theorem, the infinite multiplet cannot be a unitary irreducible representation of a noncompact group containing the Poincaré group with its usual physical interpretation. This means that the defining relations of the algebra of operators in the irreducible representation space of the infinite multiplet can not be Lie-algebraic (i.e., can not be of the form $X_i \bullet X_j - X_j \bullet X_i = c_{ij}{}^k X_k$, where $\bullet$ is the operator product).

One way of introducing such a non-Lie-algebraic relation is in the form of an (infinite component) wave equation,[3] which gives a prescription of symmetry breaking of the "relativistic symmetries.[1,4] A special example of a "relativistic symmetry" is $\mathfrak{S} = \mathcal{P}_{L_{\mu\nu},P_\mu} \vdash SO(3,2)_{S_{\mu\nu},\Gamma_\nu}$.[5] Clearly, $P_\mu P^\mu$ is an invariant operator of $\mathfrak{S}$, and the mass can have only one value in an irreducible representation. The advantage of using these relativistic symmetries is, however, that the spin spectrum is (in general) nontrivial and depends upon the choice of the representation of $SO(3,2)$.

A special representation of $\mathfrak{S}$, the Dirac representation $\mathfrak{S}^{(\mathrm{Dirac})}$, is very well known. This is the representation in which the generators of $SO(3,2)$, $S_{\mu\nu}$ and Γ_μ, are represented by the usual $\frac{1}{2}\sigma_{\mu\nu}$ and $\frac{1}{2}\gamma_\mu$, and it is obtained if one requires, in addition to the commutation relations of $SO(3,2)$ [relations (6)–(8) in Sec. II], the further condition $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$ (Pauli's fundamental theorem). In the Dirac representation of $\mathfrak{S}$, the wave equation $(\gamma^\mu P_\mu - \kappa) = 0$ (Dirac equation) is automatically fulfilled[6] and does not constitute an additional condition. This is connected with the fact that this representation contains only the $s=\frac{1}{2}$ representation of $SO(3)_{S_{ij}}$.

The simplest infinite-dimensional representations of $\mathfrak{S}$ appear to be the (unitary) Majorana representations $\mathfrak{S}^{(\mathrm{Majorana})}$. The four Majorana representations of $\mathfrak{S}$ are obtained if, in addition to the commutation relation [Eqs. (6)–(8) of Sec. II], one requires the further condition (9) of Sec. II. In contrast to the Dirac representation, the Majorana representations contain all integer ($s=0, 1, 2, \cdots$) or all half-integer ($s=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$) representations of $SO(3)_{S_{ij}}$. In these representations the wave equation $(\Gamma^\mu P_\mu - \kappa) = 0$ is not automatically fulfilled. If one defines $\mathfrak{S}_b{}^{(\mathrm{Majorana})}$ to be the algebraic structure which is again generated by P_μ, $L_{\mu\nu} = M_{\mu\nu} + s_{\mu\nu}$, Γ_μ but which, in addition to the commutation and representation relations of $\mathfrak{S}^{(\mathrm{Majorana})}$ [Eqs. (1), (2), (5), and (7)–(9) of Sec. II], also obeys the further relation $P_\mu \Gamma^\mu = \kappa = $ constant, then $\mathfrak{S}_b{}^{(\mathrm{Majorana})}$ and $\mathfrak{S}^{(\mathrm{Majorana})}$ are different from each other, whereas, in the Dirac case, the corresponding $\mathfrak{S}_b{}^{(\mathrm{Dirac})}$ and $\mathfrak{S}^{(\mathrm{Dirac})}$ are equivalent. Because of this, the mass operator $P_\mu P^\mu$ has, in an irreducible representation space of $\mathfrak{S}_b{}^{(\mathrm{Majorana})}$, a nontrivial discrete spectrum: $m^2 = [\kappa/(s+\frac{1}{2})]^2$. As has been shown in Ref. 3, the wave equation can be replaced by an algebraic relation [Eq. (51) of Ref. 3], which leads in an irreducible representation space to the same mass spectrum. So we have an example of an algebraic structure, containing the Poincaré group and the spectrum-generating group $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$, which has irreducible representation

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[1] Y. Nambu, *Proceedings of the 1967 International Conference on Particles and Fields* (Interscience Publishers Inc., New York, 1968), and references therein. A. O. Barut, *Lectures in Theoretical Physics* (Gordon and Breach Science Publishers, Inc., New York, 1968), Vol. 10B, p. 377, which gives an introduction to the approach of Barut and his school and a summary of their numerous results. H. M. Kleinert, Fortschr. Phys. **16**, 1 (1968).

[2] L. O'Raifeartaigh, Phys. Rev. **161**, 1571 (1967), and references therein.

[3] A. Böhm, *Lectures in Theoretical Physics* (Gordon and Breach Science Publishers, Inc., New York, 1968), Vol. 10B, p. 483.

[4] P. Budini and C. Fronsdal, Phys. Rev. Letters **14**, 968 (1965).

[5] The subscripts X_i on the symbol for the group G_{X_i} indicate that X_i are the generators of G. This notation is necessary to allow us to distinguish between mathematically isomorphic groups, which have different physical interpretation, i.e., whose generators represent different physical observables. $\mathcal{P}_{L_{\mu\nu},P_\mu} \vdash SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ means the semidirect product of the Poincaré group $\mathcal{P}$ with the generators P_μ and $L_{\mu\nu} = M_{\mu\nu} + S_{\mu\nu}$ and $SO(3,2)$ with the generators Γ_μ and $S_{\mu\nu}$. P_μ represents the momenta, $L_{\mu\nu}$ the angular momenta; $M_{\mu\nu}$ is the "orbital part" of the angular momentum and $S_{\mu\nu}$ is its "spin part." The new vector operator Γ_μ is a generalization of the Dirac matrices, and its physical meaning will become clearer from the following.

[6] E.g., H. Joos, Fortschr. Physik **10**, 65 (1962).

spaces with a nontrivial discrete mass spectrum (Majorana spectrum). However, this mass spectrum contradicts our experience, and we should try to find an algebraic structure which gives an experimentally correct mass spectrum and perhaps further predictions. This is done in the following, where an algebraic structure A_1 is described, which is, in a certain way, a combination of the Majorana representation $\mathfrak{S}^{(\text{Majorana})}$ of the relativistic symmetry $\mathfrak{S}=\mathcal{P}\vdash SO(3,2)$ with the algebra of the de Sitter model.[7]

The general idea of this scheme is the following: One starts with an algebraic structure (in our case A_1). A particular physical system is described by one irreducible representation, where all the states of this physical system are elements of the irreducible representation space. Different physical systems are described in the same way by different (inequivalent) irreducible representation spaces of the same algebraic structure. The distinction between different physical systems and different states of the same physical system depends upon the algebraic structure one uses. In our present case, intrinsic properties like isospin, hypercharge, $SU(3)$ quantum numbers, etc., are not contained in A_1: Therefore, resonances with different intrinsic quantum numbers belong to different physical systems and are thus described by different irreducible representations of A_1. However, operators corresponding to the observables, spin and mass, are contained in A_1. Therefore, resonances with the same intrinsic quantum numbers and different spins and masses are described by the same irreducible representation, and are considered different states of the same physical system.

The physical observables are represented by elements of the algebraic structure; the physical interpretation, i.e., the coordination between a physical observable and an operator $\in A_1$, which is to represent this observable in the irreducible representation space, usually follows simply from the construction of the algebraic structure. After this coordination has been done, the properties of the physical observable are determined by the properties of its representing operator (spectrum, value of matrix elements) which in turn is completely fixed by the defining relations and the choice of the representation.

Thus, the actual task is to find the right algebraic structure with its irreducible representations suitable to the physical system. This can, of course, only be done by looking into the experimental data and trying out various algebraic substructures and algebraic relations until agreement with these data is reached. This is the way in which A_1 was conjectured, but in the following we present this in the reverse order: From the definition of A_1 we derive properties of the observables and compare them with experimental findings.

In Sec. II, we give the definition of the mathematical image of our model, the algebraic structure A_1, and point out some immediate consequences of it. In Sec. III, we find irreducible representations of A_1. Properties of the irreducible representations will be studied in Sec. IV. The comparison of these properties of A_1 with the experimental data is done in Sec. V.

II. THE UNDERLYING ALGEBRAIC STRUCTURE A_1

The algebraic structure A_1 is defined by its generators and their relation for multiplication. The generators of A_1 are

$$P_\mu,\quad L_{\mu\nu}=M_{\mu\nu}+S_{\mu\nu},\quad (P_\mu P^\mu)^{1/2}=M,\quad \Gamma_\mu,\quad \mu=0,1,2,3$$

and the defining relations[8]:

$$[P_\mu,P_\nu]=0\,, \tag{1}$$

$$[L_{\mu\nu},P_\rho]=i(g_{\nu\rho}P_\mu-g_{\mu\rho}P_\nu)\,, \tag{2}$$

$$[L_{\mu\nu},L_{\rho\sigma}]=-i(g_{\mu\rho}L_{\nu\sigma}+g_{\nu\sigma}L_{\mu\rho}-g_{\mu\sigma}L_{\nu\rho}-g_{\nu\rho}L_{\mu\sigma})\,,\quad (g_{ii}=-1,\quad i=1,2,3,\quad g_{00}=1) \tag{3}$$

$$[M_{\mu\nu},S_{\rho\sigma}]=0\,, \tag{4}$$

$$\tfrac{1}{2}\mathcal{E}_{\mu\nu\rho\sigma}P^\gamma M^{\rho\sigma}=0\,, \tag{5}$$

$$[S_{\mu\nu},S_{\rho\sigma}]=-i(g_{\mu\rho}S_{\nu\sigma}+g_{\sigma 5}S_{\mu\rho}-g_{\mu\sigma}S_{\nu\rho}-g_{\nu\rho}S_{\mu\sigma})\,, \tag{6}$$

$$[L_{\rho\sigma},\Gamma_\mu]=[S_{\rho\sigma},\Gamma_\mu]=i(g_{\sigma\mu}\Gamma_\rho-g_{\rho\mu}\Gamma_\sigma)\,, \tag{7}$$

$$[\Gamma_\rho,\Gamma_\sigma]=-iS_{\rho\sigma}\,, \tag{8}$$

$$\{\Gamma_\rho,\Gamma_\sigma\}+\{S_{\rho\mu},S_\sigma{}^\mu\}=-g_{\rho\sigma}\,, \tag{9}$$

$$[MP_\nu,\Gamma_\lambda M]=-i\lambda^2P^\rho P^\sigma M^{-2}\{S_{\rho\lambda},\Gamma_\sigma\}P_\nu\,,\quad \text{(see Ref. 8.)} \tag{10}$$

$$[P_\lambda M^{-1},\Gamma_\rho]=0. \tag{11}$$

From the above definition, we see that A_1 contains the (enveloping algebra of the) Poincaré group generated by P_μ and $L_{\mu\nu}$ as a substructure, and it also contains the whole "relativistic symmetry" $\mathfrak{S}=\mathcal{P}_{P_\mu,L_{\mu\nu}}\vdash SO(3,2)_{S_{\mu\nu}\Gamma_\nu}$.[3] The relation (9) is a relation which determines the representation of $\mathfrak{S}$ or $SO(3,2)$; it ensures that only the four Majorana representations of $SO(3,2)$ can appear.[9]

The only difference between $\mathfrak{S}_b{}^{(\text{Majorana})}$ and A_1 lies in the relations (10) and (11) between P_μ and M and the

[7] A. O. Barut and A. Böhm, Phys. Rev. **139**, B1107 (1965); A. Böhm, *ibid.* **145**, 1212 (1966). A uniform presentation of this problem is given in A. Böhm, *Lectures in Theoretical Physics* (Gordon and Breach Science Publishers, Inc., New York, 1967), Vol. 9B, p. 327. (Note the difference in the notation; Γ_μ there is denoted here by w_μ.)

[8] Instead of relation (10), we could use the relation (10′): $[MP_\nu,\Gamma_\lambda M]=\lambda^2(P^\rho P^\sigma/M^2)[\Gamma_\rho\Gamma_\sigma,\Gamma_\lambda]P_\gamma$, which can be shown, with relation (8), to be equivalent with (10).

[9] Cf. Ref. 3. It is the equivalent of the relation (*): $\sigma_i\sigma_j=\delta_{ij}+i\epsilon_{ijk}\sigma_k$ for $SU(2)$; the commutation relation $[\sigma_i,\sigma_k]=i\epsilon_{ikl}\sigma_l$ only states that one has one of the representations of $SU(2)$ with arbitrary j, the additional relation (*) makes sure that $\{\sigma_i\}$ is the 2-dimensional representation of $SU(2)$. The corresponding relation for the Dirac representation of $\mathfrak{S}$ is $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}$. As we know from the Dirac case, and as we shall see in the following, it is very useful for the calculation to know the representation relation.

spin-changing operators Γ_μ. For $\mathfrak{S}^{(\text{Majorana})}$, we have the relations (1)–(9) and the symmetry-breaking relation[3] $[P_\rho,\Gamma_\lambda]=-2iP_\sigma S^\sigma\cdot\lambda\Gamma_\rho$ (which is, in an irreducible representation characterized by κ, equivalent to the wave equation $P_\mu\Gamma^\mu=\kappa$). The symmetry-breaking relation for A_1, Eq. (10), will therefore lead to a mass spectrum that is different from the Majorana spectrum.

The relation (11) (Werle relation) ensures that the Γ_i changes the mass and momentum only in such a way that they leave the velocities (direction of momentum) constant. This might not be true, and condition (11) might have to be relaxed. We want to keep it for our present model because it makes the calculation much easier,[10] and it will not affect the results of classification, mass spectrum, and spin spectrum.

One of the features of this model given by the algebraic structure A_1 is the appearance of the constant λ. As we shall see later [cf. Eq. (40)], the limit $\lambda\to 0$ corresponds to the case of no mass splitting. In this limit one can show, from the basic relations Eqs. (10) and (11), that P_μ and Γ_ν commute and Γ_i only changes the spin.

To understand the significance of this constant λ, we first note a further substructure of A_1, which—as we shall see later—is the central part of our model. If we define

$$B_\mu=P_\mu+\tfrac{1}{2}\lambda(P_\mu P^\mu)^{-1/2}\{P^\rho,L_{\rho\mu}\}\,, \qquad (12)$$

then we can verify the following relations[7] for B_μ:

$$[B_\mu,B_\nu]=i\lambda^2L_{\mu\nu}\,, \qquad (13)$$

$$[L_{\mu\nu},B_\rho]=i(g_{\nu\rho}B_\mu-g_{\mu\rho}B_\nu)\,. \qquad (14)$$

From (13), (14), and (3), we see that B_μ and $L_{\mu\nu}$ generate a (4+1) de Sitter group $SO(4,1)_{B_\mu,L_{\mu\nu}}$. Thus, the enveloping algebra $\mathcal{E}(SO(4,1))$ is a subalgebra of our algebraic structure A_1. The central role that $SO(4,1)_{B_\mu,L_{\mu\nu}}$ plays in our model comes from the fact that the second-order operator of $SO(4,1)_{B_\mu,L_{\mu\nu}}$,

$$\lambda^2Q=B_\mu B^\mu-\tfrac{1}{2}\lambda^2L_{\mu\nu}L^{\mu\nu}\,, \qquad (15)$$

commutes with every element of A_1 (as we shall show in Sec. III) and is therefore also an invariant operator of A_1. Thus the irreducible representations are characterized by the eigenvalues α^2 of the $SO(4,1)$ Casimir operator Q.

Now the mathematical significance of the constant λ has become clear. λ is the contraction constant in the Inönü-Wigner group contraction from $SO(4,1)_{B_\mu,L_{\mu\nu}}\to\mathcal{P}_{P_\mu,L_{\mu\nu}}$.[11] From the relation (13), one can see immediately that when $\lambda\to 0$ the commutation relations of $SO(4,1)$ go into the commutation relations of $\mathcal{P}$, and from the relation (12) one sees that (the representation dependence of the contraction process has been chosen such that) $B_\mu\to P_\mu$.

The physical meaning of the constant λ is that of a scale factor relating the units of P_μ and B_μ (MeV) to the units of angular momentum (1 in the units with $\hbar=c=1$). As we shall see from the resulting mass formula, the constant λ gives, in our model, the connection between a mass measurement and a spin measurement (in the same way as, in quantum mechanics, h connects a frequency measurement with an energy measurement) and its introduction appears inevitable from dimensional considerations. The value of λ will be determined from the experimental mass spectrum to be $\lambda^2=0.28$ BeV^2.

We can also give a geometrical interpretation of the constant λ; this is obtained if one considers $\mathcal{P}_{P_\mu,L_{\mu\nu}}$ as the group of motion in the Minkowski space. Then $SO(4,1)_{B_\mu,L_{\mu\nu}}$ is the group of motion in the (4+1) de Sitter space, the most symmetrical curved space with infinite time and finite space extension; and $R=1/\lambda$ is the radius of this de Sitter space.[12]

III. IRREDUCIBLE REPRESENTATIONS OF THE ALGEBRAIC STRUCTURE A_1

We give now the description of the irreducible representation spaces of A_1 [13]; for this purpose we use results given in Refs. 3 and 7.

We first prove that the second-order Casimir operator of $SO(4,1)_{B,L}$, namely,

$$\lambda^2Q=B_\mu B^\mu-\tfrac{1}{2}\lambda^2L_{\mu\nu}L^{\mu\nu}\,, \qquad (15)$$

commutes with every element of A_1. To show this, we use the fact that λ^2Q can be written as

$$\lambda^2Q=P_\mu P^\mu+(9/4)\lambda^2-\lambda^2(P_\mu P^\mu)^{-1}W \qquad (16)$$

which follows from the relations (1)–(3), and (12)–(14).[7] Here, W is the usual spin operator

$$W=-w_\mu w^\mu,\quad w_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu L^{\rho\sigma}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu S^{\rho\sigma}\,, \qquad (17)$$

which can also be written

$$W=\tfrac{1}{2}P_\rho P^\rho S_{\mu\nu}S^{\mu\nu}-P^\rho P^\sigma S_{\rho\mu}S_\sigma{}^\mu$$
$$=P^\rho P^\sigma\Gamma_\rho\Gamma_\sigma-\tfrac{1}{4}P_\mu P^\mu. \qquad (18)$$

For the derivation of the second part of (18), one has to use the Majorana representation relation (9).[3]

From (15) and (16) it follows immediately that B_μ, $L_{\mu\nu}$, and P_μ commute with λ^2Q, and it only remains to show that

$$[\lambda^2Q,\Gamma_\mu]=0. \qquad (19)$$

Inserting (18) into (16), we obtain

$$\lambda^2Q=M^2+\tfrac{5}{2}\lambda^2-\lambda^2M^{-2}P^\rho P^\sigma\Gamma_\rho\Gamma_\sigma, \qquad (20)$$

where $M^2=P_\mu P^\mu$ so that

$$[\lambda^2Q,\Gamma_\lambda]=[M^2,\Gamma_\lambda]-\lambda^2(P^\rho P^\sigma/M^2)[\Gamma_\rho\Gamma_\sigma,\Gamma_\lambda]. \qquad (21)$$

[10] Only for the case that we have relation (11) do we know how to construct the measure μ of the Dirac spectral theorem (Ref. 17).

[11] Reference 7 and references therein. E. Inönü and E. P. Wigner, Proc. Nat. Acad. Sci. U. S. 39, 510 (1953).

[12] C. Fronsdal, Ref. Mod. Phys. 37, 221 (1965).

[13] We restrict ourselves to "Hermitian" representations of A_1, i.e., to those for which the generators and all symmetric elements of A_1 are "Hermitian" operators. (For the precise statement, cf. Ref. 3, p. 41, footnote**, and references given there.)

In the second term on the right-hand side, we have used (11). From (10), (11), and (8), we obtain

$$[M^2,\Gamma_\lambda]=-i\lambda^2(P^\rho P^\sigma/M^2)\{S_{\rho\lambda},\Gamma_\sigma\}$$
$$=\lambda^2(P^\rho P^\sigma/M^2)[\Gamma_\rho\Gamma_\sigma,\Gamma_\lambda]. \quad (22)$$

Inserting this into (21) gives (19). So $\lambda^2 Q$ is an invariant operator of A_1 and the irreducible representation spaces of A_1 are characterized by the eigenvalues α^2 of Q.

An irreducible representation space of A_1 is further characterized by the irreducible representation of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ which it contains. Proof: It has been shown in Ref. 3 that, as a consequence of relation (9),[14] only four irreducible representation spaces of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ are possible; these are the ones with the following properties (Majorana representations):

$$\mathcal{H}^{SO(3,2)(\frac{1}{2},\pm)} \underset{SO(3,1)}{=} \mathcal{H}^{SO(3,1)}(k_0=\tfrac{1}{2},\, c=0)$$
$$\underset{SO(3)}{=} \sum_{s=\frac{1}{2},\frac{3}{2},\cdots}^{\infty} \oplus\mathfrak{M}_s{}^{SO(3)S_{ij}}, \quad (23)$$

$$\mathcal{H}^{SO(3,2)(0,\pm)} \underset{SO(3,1)}{=} \mathcal{H}^{SO(3,1)}(k_0=0,\, c=\tfrac{1}{2})$$
$$\underset{SO(3)}{=} \sum_{s=0,1,\cdots}^{\infty} \oplus\mathfrak{M}_s{}^{SO(3)S_{ij}}. \quad (24)$$

($=_G$ means reduction with respect to the subgroup G). In $\mathcal{H}^{(\frac{1}{2},+)}$, the spectrum of Γ_0 is positive:

$$\text{spectrum } \Gamma_0=\mu=s+\tfrac{1}{2}=1,\ 2,\ 3,\ \cdots; \quad (25)$$

and in $\mathcal{H}^{(\frac{1}{2},-)}$ the spectrum of Γ_0 is negative:

$$\text{spectrum } \Gamma_0=\mu=-s-\tfrac{1}{2}=-1,\ -2,\ -3,\ \cdots; \quad (26)$$

and correspondingly for $\mathcal{H}^{(0,\pm)}$.

The question arises whether all these four representations of $SO(3,2)$ appear in an irreducible representation of A_1 or only one. It can be seen that there is no operator in A_1 which changes s by a half integer, and also no operator which transforms between eigenstates of Γ_0 with negative and positive eigenvalues. Consequently, in an irreducible representation space of A_1 there can be only one of the irreducible representations $(\frac{1}{2},+)$, $(\frac{1}{2},-)$, $(0,+)$, $(0,-)$, so that this irreducible representation space of A_1 is characterized by the irreducible representation of $SO(3,2)_{\Gamma_\mu,S_\nu}$ that it contains.

It appears that there is no further independent invariant operator of A_1, and we denote the irreducible representation spaces by $\mathcal{H}(\alpha,(0,\pm))$ and $\mathcal{H}(\alpha,(\frac{1}{2},\pm))$. The value of α is determined by the irreducible representations of $SO(4,1)_{B,L}$ which are contained in the irreducible representation space $\mathcal{H}(\alpha,(\cdots))$ of A_1. We shall restrict ourselves here to the following values of α^2.

For $\mathcal{H}(\alpha,(0,\pm))$ α^2 can be any real number $\alpha^2>9/4$,

and (27)

for $\mathcal{H}(\alpha,(\frac{1}{2},\pm))$ α^2 can be any real number $\alpha^2>\frac{3}{2}$.

This restricts us to representations of A_1 which contain only class-I and class-III representations of $SO(4,1)_{BL}$,[15] which in turn ensures that $M^2=P_\mu P^\mu$ is a positive-definite operator.[16] This property has already been used in the definition of A_1, where we have assumed that the inverse and the square root of $P_\mu P^\mu$ can always be defined.

In the following, we want to restrict ourselves to the spaces $\mathcal{H}(\alpha,(0,+))$ and $\mathcal{H}(\alpha,(\frac{1}{2},+))$; the spaces $\mathcal{H}(\alpha,(\cdots))$ can be treated analogously. As we shall see later, the spaces $\mathcal{H}(\alpha,(0,+))$ will describe the mesons and meson resonances, and α will be different for mesons with different internal quantum numbers like isospin, hypercharge etc., in the same way, the various $\mathcal{H}(\alpha,(\frac{1}{2},+))$ will describe baryons.

IV. MASS AND SPIN SPECTRUM IN AN IRREDUCIBLE REPRESENTATION SPACE

We now want to study the properties of the irreducible representations $(\alpha,(\frac{1}{2},+))$ and $(\alpha,(0,+))$ in more detail; in particular, we want to investigate the spectrum of the mass and the spin operator in the representation spaces $\mathcal{H}(\alpha(\cdots,+))$ and see how they are connected. First, we shall introduce a ("generalized") basis in $\mathcal{H}(\alpha,(\frac{1}{2},+))$. This is most easily done by choosing a complete set of commuting operators and using the "Dirac spectral theorem".[17] One can show that

$$P_i,\quad M^2=P_\mu P^\mu,\quad w_3,\quad W=-w_\mu w^\mu,$$
$$\text{and the invariants of } A_1 \quad (28)$$

commute. From (16) we can see that (28) is already an overcomplete system, because M^2 and W are not independent. Thus, we can introduce the canonical basis

$$|p_i,s,s_3;(\alpha,(\tfrac{1}{2},+))\rangle, \quad (29)$$

with

$$P_i|p_i,s_1,s_3;(\alpha,(\tfrac{1}{2},+))\rangle=p_i|p_i,s,s_3;(\alpha,(\tfrac{1}{2},+))\rangle,$$
$$M^2|p_i,s,s_3;(\alpha,(\tfrac{1}{2},+))\rangle=m^2|p_i;s,s_3;(\alpha,(\tfrac{1}{2},+))\rangle,$$
$$W|p_i,s,s_3;(\alpha,(\tfrac{1}{2},+))\rangle \quad (29')$$
$$=m^2s(s+1)|p_i,s,s_3;(\alpha,(\tfrac{1}{2},+))\rangle,$$
$$w_3|p_i,s,s_3;(\alpha,(\tfrac{1}{2},+))\rangle=ms_3|p_i,s,s_3;(\alpha,(\tfrac{1}{2},+))\rangle,$$

which is well known from the representation of the Poincaré group.

[14] From relation (9) and the commutation relations [(6)–(8)] of $SO(3,2)$, one can show that $\{S_{ab},S_c{}^b\}=-g_{ac}$, where $a, b, c,=0, 1, 2, 3, 5$, $g_{55}=1$, and $S_{\mu5}=\Gamma_\mu$. This relation has been used in Ref. 3 to derive the following results.

[15] J. Dixmier, Bull. Soc. Math., France **89**, 9 (1961). T. D. Newton, Ann. Math. **51**, 730 (1950) and references therein.

[16] This is a consequence of (16) and will become obvious later. It has also been shown in Ref. 7, where the relation between $SO(4,1)_{B,L}$ and $\mathcal{P}_{P,L}$ has been described in detail.

[17] A. Böhm, in *Lectures in Theoretical Physics* (Gordon and Breach Science Publishers, Inc., New York, 1967), Vol. 9A, p. 255.

However, from (22) we can see that M^2 is not an invariant operator, and its spectrum is therefore nontrivial. Then it follows from (16) that also the spectrum of $M^{-2}W$ is nontrivial because the eigenvalue $\lambda^2\alpha^2$ of $\lambda^2 Q$ is a constant in $\mathcal{H}(\alpha,(\frac{1}{2},+))$. Further, it follows from (16) that the mass spectrum is determined by the spin spectrum. We shall therefore first find the spin spectrum in the irreducible representation space $\mathcal{H}(\alpha,(\frac{1}{2},+))$. In doing this, we shall also give an explicit construction of the generalized basis vectors (29).

We start with a basis of $SO(3,2)_{S_{\mu\nu},\Gamma_\nu}$. From (23), we know that the irreducible representation space $\mathcal{H}^{SO(3,2)(\frac{1}{2},+)}$ contains only the irreducible representation ($k_0=\frac{1}{2}$, $c=0$) of its $SO(3,1)$ subgroup. There are two algebraically equivalent (but physically different) $SO(3,1)$ subgroups of $SO(3,2)$; the one generated by $S_{\mu\nu}$, $SO(3,1)_{S_{\mu\nu}}$ and the one generated by Γ_i, S_{ij} (i, $j=1$, 2, 3), $SO(3,1)_{\Gamma_i,S_{ij}}$. We shall choose here a basis in which the following chain of subgroups appears completely reduced:

$$SO(3,2)_{\Gamma_\mu,S_{\mu\nu}}\supset SO(3,1)_{\Gamma_i,S_{ij}}\supset SO(3)_{S_{ij}}\supset SO(2)_{S_{12}}. \quad (30)$$

As only the representation ($k_0=\frac{1}{2}$, $c=0$) of $SO(3,1)_{\Gamma_i,S_{ij}}$ is contained in $\mathcal{H}^{SO(3,2)(\frac{1}{2},+)}$, the well-known[18] basis $|s,s_3\rangle$ of $\mathcal{H}^{SO(3,1)\Gamma_i S_{ij}}(k_0=\frac{1}{2}$, $c=0)$ with

$$\tfrac{1}{2}S_{ij}S^{ij}|s,s_3\rangle=s(s+1)|s,s_3\rangle, \quad (31a)$$

$$S_{12}|s,s_3\rangle=s_3|s,s_3\rangle, \quad (31b)$$

$$\Gamma_0|s,s_3\rangle=(s+\tfrac{1}{2})|s,s_3\rangle, \quad (31c)$$

$$\Gamma_3|s,s_3\rangle=[(s-s_3)(s+s_3)]^{1/2}C_s|s-1,\,s_3\rangle -[(s+s_3+1)(s-s_3+1)]^{1/2}C_{s+1}|s+1,\,s_3\rangle, \quad (31d)$$

is already a basis of $\mathcal{H}^{SO(3,2)(\frac{1}{2},+)}$. From the well-known reduction of $\mathcal{H}(k_0=\frac{1}{2}$, $c=0)$ with respect to $SO(3)_{S_{ij}}$, (23), we know that the spectrum of s is

$$s=\tfrac{1}{2},\,\tfrac{3}{2},\,\tfrac{5}{2},\,\cdots. \quad (32)$$

To give the explicit construction of the generalized basis vectors (29), we consider the algebra $\mathcal{E}(\mathcal{P}^\times)$ which is defined by its generators P_μ, $L_{\mu\nu}$ and the defining relations $W_\mu{}^\times=\frac{1}{2}\mathcal{E}_{\mu\nu\rho\sigma}P^\nu L^{\rho\sigma}=0$ ("orbital Poincarè group"). Let $\phi(p)$ denote the canonical basis of the representation space of $\mathcal{E}(\mathcal{P}^\times)$ and $\phi(p_R)$ be the state with

$$P_\mu\phi(p_R)=p_\mu{}^R\phi(p_R),\quad p_\mu{}^R+=+(m,0,0,0). \quad (33)$$

Then, we define

$$|p_i,s,s_3\rangle=\mathcal{U}^{-1}(L(p))|p_R,s,s_3\rangle =\mathcal{U}^{-1}(L(p))(\phi(p_R)\otimes|s,s_3\rangle), \quad (34)$$

where $L(p)$ is the Lorentz transformation that transforms p_μ into rest: $L_\nu{}^\mu p_\mu=p_\nu{}^R=(m,0,0,0)$. It can be shown[3] that the states defined by (34) have all the properties of the canonical basis states (29) given by (29′) and the well-known transformation properties

$$\mathcal{U}(\Lambda)|p_i,s,s_3\rangle=\sum_{s_3'}|(\Lambda p)_i,s,s_3'\rangle D_{s_3's_3}{}^{(s)}(R),\quad R=L(\Lambda p)\Lambda L^{-1}(p). \quad (35)$$

From (31c), (31d), (34), and (35), it follows further that the action of Γ_μ on these states is:

$$P_\mu\Gamma^\mu|p_i,s,s_3\rangle=m(s+\tfrac{1}{2})|p_i,s,s_3\rangle, \quad (36a)$$

$$\mathcal{U}^{-1}(L)\Gamma_3\mathcal{U}(L)|p_i,s,s_3\rangle=[(s+s_3)(s-s_3)]^{1/2}\times C_s|p_i,\,s-1,\,s_3\rangle -[(s+s_3+1)(s-s_3+1)]^{1/2}C_{s+1}|p_i,\,s+1,\,s_3\rangle, \quad (36b)$$

and correspondingly for the other $\mathcal{U}^{-1}(L)\Gamma_i\mathcal{U}(L)$.

Therewith we have found the spectrum of the spin-operator $M^{-2}W$ in the irreducible representation spaces $\mathcal{H}(\alpha,(\frac{1}{2},+))$; from (29′) and (32) it follows that

$$\text{spectrum }(M^{-2}W)=s(s+1)\text{ with }s=\tfrac{1}{2},\,\tfrac{3}{2},\,\tfrac{5}{2},\,\cdots, \quad (37)$$

so that s is the spin.[19]

The only difference between $\mathcal{H}(\alpha,(\frac{1}{2},+))$ and $\mathcal{H}(\alpha,(0,+))$ is the spin spectrum. As follows from (24) in the representation spaces $\mathcal{H}(\alpha,(0,+))$ the spin spectrum is

$$\text{spectrum }(M^{-2}W)=s(s+1)\text{ with }s=0,\,1,\,2,\,3,\,\cdots. \quad (38)$$

The mass spectrum in the irreducible representation spaces $\mathcal{H}(\alpha,(\frac{1}{2},+))$ and $\mathcal{H}(\alpha,(0,+))$ is now easily obtained. We call $m_s{}^2$ the eigenvalue of $P_\mu P^\mu$ in the states (29)

$$P_\mu P^\mu|p_i,s,s_3;\,(\alpha,(\cdots,+))\rangle =m_s{}^2|p_i,s_3;\,(\alpha_1(\cdots,+))\rangle, \quad (39)$$

and obtain, from (16),

$$m_s{}^2=\lambda^2\alpha^2-(9/4)\lambda^2+\lambda^2 s(s+1). \quad (40)$$

Summarizing, we have found that the irreducible representation spaces $\mathcal{H}(\alpha,(\frac{1}{2},+))$ of the algebraic structure A_1 reduce with respect to the Poincaré group $\mathcal{P}$ in the following way[20]:

$$\mathcal{H}(\alpha,(\tfrac{1}{2},+))\underset{\mathcal{P}}{=}\sum_{s\to\frac{1}{2},\frac{3}{2},\cdots}^{\infty}\oplus\mathcal{H}^{\mathcal{P}}(m_s,s) \quad (41)$$

and correspondingly, for $\mathcal{H}(\alpha,(0,+))$,

$$\mathcal{H}(\alpha,(0,+))\underset{\mathcal{P}}{=}\sum_{s=0,1,2,\cdots}^{\infty}\oplus\mathcal{H}^{\mathcal{P}}(m_s,s), \quad (42)$$

where m_s is given by the mass formula (40).

[18] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon Press, Inc., New York, 1964).

[19] This does not, however, mean that S_{ij}, the so-called "spin part" of the angular momentum operator, is the spin operator. Only for the states at rest is S_{ij} the spin operator. The basis consisting of eigenvectors of $\frac{1}{2}S_{ij}S^{ij}$ and S_{12} is different from (29), and is called the spinor basis; cf., e.g., Refs. 3 and 6.

[20] We remark that $\epsilon=p_0/|p_0|$ is an invariant in the whole representation space $\mathcal{H}(\alpha,(\frac{1}{2}+))$ and can be chosen $\epsilon=+1$. However, if we then repeat the same construction for $\mathcal{H}(\alpha,(\frac{1}{2},-))$ using also here the $\epsilon^\times=+1$ representation of $\mathcal{P}^\times$ in (33) and (34), then it follows from (36a) that $\epsilon=-1$ in $\mathcal{H}(\alpha,(\cdots,-))$, so that $\mathcal{H}(\alpha,(\cdots,-))$ offers itself as the representation space for the antiparticles, after a suitable definition of the conjugation operation.

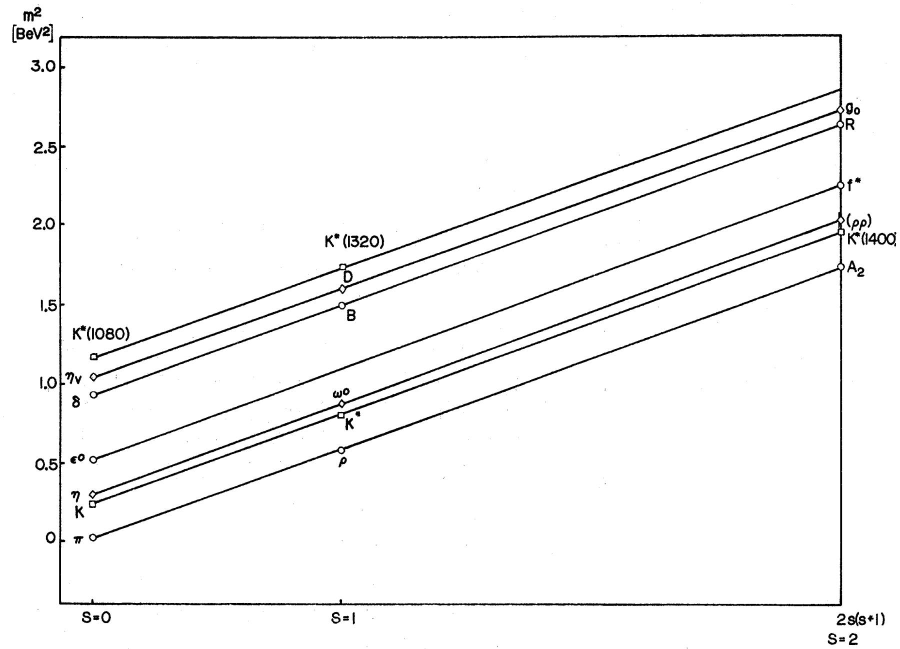

FIG. 1. The mass squared of the meson resonances, plotted versus $s(s+1)$, with s the spin of the resonance. (ω^0 is the $I=Y=0$ member of the octet.)

V. COMPARISON WITH THE EXPERIMENTAL PARTICLE SPECTRUM

To compare the results of our model with experiment, we have to assign the various particles and resonances of the same kind to one of the representation spaces. We first consider the meson resonances that have to be assigned to the irreducible representation spaces $\mathcal{H}(\alpha,(0,+))$ with appropriately chosen α. It is well known that the mass formula (40) is well satisfied in this case, the best examples being the $I=1$, $Y=0$ meson tower which starts with π, ρ, A_2, and the $Y=1$ meson tower starting with K, K^*, $K^*(1400)$. According to the general idea of this program, these two towers have to belong to representation spaces differing in $\alpha=\alpha(I,Y,\cdots)$; however, λ^2 is a "universal" constant and must be the same for all towers characterized by different α. If m^2 is plotted versus $s(s+1)$ as done in Fig. 1, the mesons should lie on parallel straight lines,

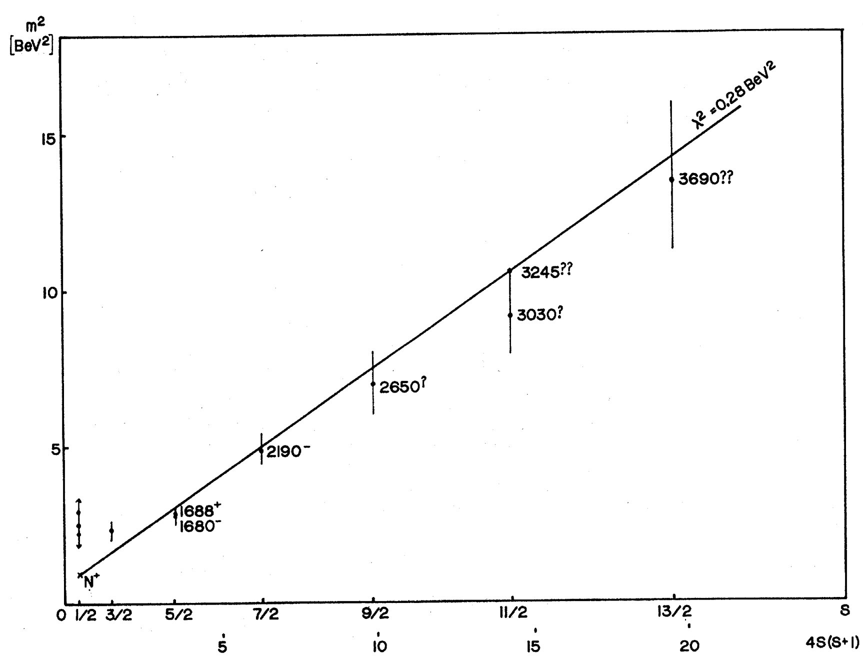

FIG. 2. The mass squared of the nucleon resonances is plotted versus $s(s+1)$, with s the spin of the resonances. The ? indicates that the spin of that resonance is not known, and ?? indicates that the existence is not conclusively established. The slope of the straight line λ is the same as in Fig. 1 for the mesons.

and the mesons of the same kind, i.e., with the same value of $\alpha^2=\alpha^2(I,Y,\cdots)$, should lie on the same line, every line corresponding to one irreducible representation space $\mathcal{H}(\alpha(I,Y,\cdots)(\frac{1}{2},+))$. That this is very well satisfied can be seen from Fig. 1, where, besides the towers π, ρ, A_2, $\cdots$ and K, K^*, $K^*(1400)\cdots$, some less-well-established towers have been indicated. From the slope of these straight lines, we obtain the value of the constant λ

$$\lambda^2=0.285 \text{ BeV}^2. \tag{43}$$

From this, the radius of the de Sitter space, in which the $SO(4,1)_{B,L}$ subgroup of A_1 is the group of motion, is obtained to be $R=1/\lambda=0.36\times10^{-13}$ c.m.[21]

The uninteresting value of the quantum number $\alpha^2(I,Y,\cdots)$ for the π tower is then $\alpha^2(I=1, Y=0, \cdots) =[m_\pi^2+(9/4)\lambda^2]/\lambda^2$ and similarly for the other towers.

Of the predictions this model makes for higher meson resonances we mention only those of the π tower. From (42) and (40) we expect $I=1$, $Y=0$ resonances with ($s=3$, $m\approx1850$ MeV) and ($s=4$, $m\approx2380$ MeV). At these mass values indications of $I=1$, $Y=0$ resonances have been observed (R_4 and U).

The baryons have to be assigned to the irreducible representation spaces $\mathcal{H}(\alpha,(\frac{1}{2},+))$. Though one would be very well prepared to accept the mass formula (40) for mesons, one would spurn it for the baryons, because of the strong belief in the usual Regge-trajectory idea. We have therefore plotted in Fig. 2 the $I=\frac{1}{2}$ nucleon resonances; the straight line is given by (40). As the value of λ is already given from the meson spectrum, the only free parameter is α, which is determined by the nucleon mass $\alpha_N^2=[m_N^2+(9/4)\lambda^2]/\lambda^2$. The agreement of (40) with the experimental data is certainly comparable with that of the Regge trajectories and, in view of the fact that the only parameter that enters here is the nucleon mass, is probably more remarkable. The major defect is the absence of an $I=\frac{1}{2}$, $\frac{3}{2}^+$ resonance with a mass equal to that of the 33 resonance. For the other baryon resonances the situation looks very much the same.

VI. CONCLUSION

The main purpose of this work was not to give a further scheme for the classification of a maximum number of particles. For this the model is probably too simple. In particular, the choice of the Majorana representations [which follows from the representation relation (9)] for the spectrum-generating subgroup $SO(3,2)$ is certainly not sophisticated enough. With one of the more complicated (singleton) representations of $SO(3,2)$ (which would require the introduction of a new "space-time" quantum number n) known resonances, which now lie outside the scheme (or in a different representation), could certainly be accommodated in the same irreducible representation space with the above ones. Furthermore, the intrinsic quantum numbers have not been incorporated in this model. This shortcoming would become more apparent if there is an interrelation between space-time and intrinsic symmetries (like that one given by the usual $SU(6)$). Then, models like the present one can only describe a subclass of particles (namely, those which have recurrences in all intrinsic symmetry multiplets). The purpose of this work is rather to show how meson as well as baryon properties can be derived from the same algebraic structure, and how, in the framework of this concept, an explanation of the particle spectrum can be given. The main result of this model is not that a mass formula can be derived that does not disagree with experimental data, but that this can be done using only one new constant λ and that this constant appears in a natural way within the framework of this algebraic approach.

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[21] If one assumes that, in the small domain in which the strong interaction processes are relevant, the space is to be viewed as being curved (with a curvature much greater than the one caused by gravitation; because of the much stronger interaction), then the group of motion in this small curved domain is no longer the Poincaré group. If one further assumes that in a certain state of equilibrium this space is of a highly symmetrical form and may be approximated by a de Sitter space with a finite space and infinite time extension, then the group of motion in this domain will be the above-mentioned $SO(4,1)_{B,L}$. Thus, R may be compared with the radius of the region of interaction obtained from completely different considerations: J. Pašut, M. Roos, CERN Report No. Th. 885, 1968 (unpublished), p. 28. Comparing their results in Sec. 5.5 with our value for R, we see that their parametrization (30) of the width $\Gamma(\omega)$ appears to give the best agreement. Cf. also H. Schopper, CERN Nucl. Phys. Division Report No. 67-3, 1967 (unpublished).