charge-exchange polarization, the πp total cross section, the real forward amplitude $D^{(-)}$, and the non-spin-flip superconvergence relations. They found that it is not necessary to introduce a conspiring ρ' trajectory, but that it is possible to satisfy the superconvergence relations and the polarization data with a nonconspiring ρ' ($j_0=0$). Moreover, one of the two solutions of their model gives $\alpha_{\rho'}(0)=-0.5$, $\alpha_\rho(0)=0.57$, in very good agreement with the prediction $\alpha_{\rho'}=\alpha_\rho-1$.

Finally, Högaasen and Fischer,[7] in their attempt to fit the experimental data on nucleon-nucleon charge-exchange scattering, found for the intercept of the ρ' trajectory the value $\alpha_{\rho'}(0)=-0.63$, again in good agreement with the results of our model.

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Intrinsic Invariance Groups, Mass Operators, and Linear Wave Equations*

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Relativity theory permits motions of a free-spinning particle in which the instantaneous velocity and linear momentum are not collinear. Therefore, it is necessary to specify two invariance groups in order to completely describe the space-time symmetry properties of a particle with intrinsic spin. If the quantal description of such a particle is given by a covariant linear field equation, then the *external* space-time symmetry is specified by the 10-element Poincaré group generated by the linear four-momentum and total angular momentum operators. However, invariance of the field equation under external inhomogeneous Lorentz transformations does not complete the algebra of the 10-element group generated by the instantaneous four-velocity and spin angular-momentum operators. Several formulations of linear field equations admitting a mass-spin spectrum are based on different choices for this latter, *intrinsic*, space-time symmetry group. We begin with the simplest choice, namely, intrinsic Poincaré invariance, and establish the formal connection between these several formulations by constructing a unitary transformation which generates an infinite sequence of linear wave equations describing ascendingly more complex intrinsic space-time symmetry, but linked by a common mass operator. Of special interest is the infinitesimal transformation which generates the familiar intrinsic DeSitter group. Finally, some kinematical properties of this transformation are discussed for various proposed mass operators.

I. INTRODUCTION

THEORETICAL attempts to explain the proliferation of elementary particles have, of late, led to extensive investigation of the class of Lorentz-covariant wave equations of the form,[1]

$$(i\Gamma_\mu P_\mu+Mc)\psi=0. \tag{1}$$

If $x_\mu\equiv(\mathbf{x},ict)$ is the instantaneous position four-vector, then the linear four-momentum is given by $P_\mu=-i\hbar\partial_\mu$, so that

$$(x_\mu,P_\nu)=i\hbar\delta_{\mu\nu} \tag{2}$$

specifies the conjugate relation between the position and momentum. Unlike the momentum, however, the remaining two operators appearing in (1), Mc and Γ_μ, do not possess unique representations although their physical interpretation is understood as follows.

First, the Lorentz frame defined by $P_\mu\equiv(0,iE/c)$ is the "momentum rest" frame, and is usually referred to as the rest frame. In this frame, Eq. (1) becomes

$$\Gamma_4E\psi=Mc^2\psi\,, \tag{3}$$

so that Mc is a mass operator, whose specification should lead, via the eigenvalue equation (3), to the spectrum of rest energies admitted by (1). Explicit representation of the Lorentz-invariant mass operator Mc depends on the dynamical model studied, if one exists, and on the desired algebraic properties. It is most generally represented by a finite- or infinite-dimensional matrix, which may possess more structure than constant multiple of the unit matrix.[1]

Second, the covariant wave equation, (1), defines the proper-time Hamiltonian operator,[2]

$$H=i\Gamma_\mu P_\mu+Mc\,, \tag{4}$$

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[1] (a) H. C. Corben, Proc. Natl. Acad. Sci. U. S. **48**, 1559 (1962); **48**, 1746 (1962); Phys. Rev. Letters **15**, 268 (1965); Y. Nambu, Progr. Theoret. Phys. (Kyoto) **37**, 368 (1966); Phys. Rev. **160**, 1171 (1967); L. Castell, Nuovo Cimento **50**, 945 (1967). (b) For additional references see H. C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden-Day Publishing Co., San Francisco, 1968), Chap. 4.

[2] The role of proper time in quantum mechanics, and its application to the temporal evolution of the wave packet, is rigorously treated in the literature. See, for example, S. Shanmugadhasan, Can. J. Phys. **29**, 593 (1951); G. Szamosi, Nuovo Cimento, **20**, 1090 (1961); R. Schiller, Phys. Rev. **125**, 1116; **128**, 1402 (1962); G. N. Fleming, J. Math. Phys. **7**, 1959 (1966).

so that in the Heisenberg picture the evolution of an operator O is governed by the equation of motion

$$i\hbar\frac{dO}{d\tau}=(H,O)\,, \tag{5}$$

where τ is the proper time for the particle. Thus, with Mc represented by some matrix, so that

$$(x_\mu,Mc)=0\,, \tag{6}$$

it follows from (2), (4) and (5) that

$$\dot{x}_\mu=i\Gamma_\mu \tag{7}$$

is the instantaneous four-velocity operator, the dot denoting differentiation with respect to the proper time τ.

It has long been known that if a particle possessing an intrinsic spin is described within the framework of special-relativistic classical dynamics, then complex motions are permitted where in the instantaneous particle velocity and the linear momentum are not collinear.[3] Indeed, the difference between these two classical variables is a measure of the effect of the spin on the trajectory and expresses the fact that *Zitterbewegung* is an integral part of classical theories, as well as quantum theories of spinning particles.

Our concern here is with the quantum theory of spinning particles as described by the field equation (1). However, the well-established correspondence between the operator dynamics in the Heisenberg picture, based on (4), and the classical relativistic dynamics of spinning particles,[3] permits, or rather demands, some degree of language overlap. Thus, if the particle described by (1) has an intrinsic spin angular momentum represented by the operator $S_{\mu\nu}=-S_{\nu\mu}$, the total angular momentum operator is given by

$$M_{\mu\nu}=L_{\mu\nu}+S_{\mu\nu}\,, \tag{8}$$

where

$$L_{\mu\nu}=x_\mu P_\nu-x_\nu P_\mu \tag{9}$$

is the orbital angular momentum operator, and the instantaneous four-velocity operator (7) need not be proportional to P_μ. Indeed, the lack of proportionality is the measure of the effect of the spin on the trajectory (*Zitterbewegung*),[4] and their proportionality reduces (1) to a Klein-Gordon (KG) equation with separately conserved spin and orbital angular momenta. In the more general case, Γ_μ is represented by a set of four finite- or infinite-dimensional matrices.

The goal here is to establish an interpretable connection between some of the various explicit formulations of (1). To this end, in Sec. II, we briefly review the algebraic properties imposed on Γ_μ and Mc by the requirement that the field equation (1) be invariant under *external* inhomogeneous Lorentz transformations generated by P_μ and $M_{\mu\nu}$. This is of course equivalent to the statement

$$\mathcal{G}_e=\mathcal{P}_e:\{P_\mu;\,M_{\mu\nu}\}\,, \tag{10}$$

that the external space-time invariance group is a Poincaré group, or that the linear and total angular momentum must be conserved for the free particle. This requirement, (10), does not exhaust the algebraic properties of Γ_μ, nor does it completely specify Mc, the latter simply saying that there remains some freedom in the choice of a dynamical model. Requirement (10) leaves undetermined the commutation relations, (Γ_μ,Γ_ν), which means that the group structure of the ten elements, Γ_μ and $S_{\mu\nu}$, is not completely specified. We shall refer to this group as the *intrinsic* space-time invariance group,

$$\mathcal{G}_i:\{\Gamma_\mu,S_{\mu\nu}\}\,. \tag{11}$$

In Sec. III, we develop a unitary transformation scheme which establishes a connection between various choices for $\mathcal{G}_i$ while leaving the form of the field equation (1) unchanged. We begin with the simplest choice, namely, an intrinsic Poincaré group

$$\mathcal{G}_i=\mathcal{P}_i:\{\Gamma_\mu,S_{\mu\nu}\}\,, \tag{12}$$

which is a consequence of choosing

$$(\Gamma_\mu,\Gamma_\nu)=0\,. \tag{13}$$

This simplest choice, (12), is the basis of a new formulation due to Corben,[5] in which the mass operator Mc is also specified, since it is deduced from the dynamical model of a symmetric top. Whether an acceptable rest-energy spectrum emerges from this formulation remains to be seen. However, it is worth noting its special position as the simplest of an infinite sequence of previously unrelated formulations.

Of special interest is the infinitesimal part of the unitary transformation which yields a field equation of the form (1) possessing an intrinsic space-time DeSitter group, $\mathcal{G}_i=\mathcal{S}_i$. It is this formulation, particularly with Mc a constant multiple of the unit matrix, that has, up to now, received the most attention.[6] There are now known to be difficulties inherent in such a formulation such as a rest energy that varies inversely with the spin, space-like solutions (which prevent saturation of the current algebra by single-particle, time-like states), CPT and spin-statistics violations, etc.[7] Some of these difficulties depend on

[3] J. Frenkel, Z. Physik **37**, 243 (1926); H. J. Bhabha, Proc. Indian Acad. Sci. **A11**, 247 (1940); **A11**, 467 (1940); H. C. Corben, Phys. Rev. **121**, 1833 (1961); K. Rafanelli, *ibid.* **155**, 1420 (1967); J. Math. Phys. **8**, 1440 (1967).

[4] Kerson Huang, Am. J. Phys. **20**, 479 (1952); M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1961), Sec. 18.

[5] H. C. Corben, see Ref. 1(b); Boulder Lecture Series, 1968, (unpublished); TRW Report No. 00875-6002-R000.

[6] P. A. M. Dirac, Proc. Roy. Soc. (London) **A117**, 610 (1928); **A118**, 351 (1928); N. Kemmer, *ibid.* **A592**, 91 (1939); E. Majorana, Nuovo Cimento **9**, 335 (1932).

[7] E. Abers, I. T. Grodsky, and R. Norton, Phys. Rev. **159**, 1222 (1967).

dimensionality of the chosen representation, while others pervade both the finite- and infinite-dimensional representations.[8] It is also conjectured that with any choice of mass operator such formulations must encounter at least some of these troubles.[9] We shall not dwell further on the subject of difficulties since that is not the purpose of this paper. However, it is well to bear in mind the possible difficulties of any $\mathcal{G}_i$.

Finally, in Sec. IV we present a kinematical interpretation of the constructed transformation theory as it applies to suggested models.

II. EXTERNAL INVARIANCE REQUIREMENTS

Since Eq. (1) describes a particle on which no external forces or torques are acting, the linear and total angular momentum must be conserved. Let us now see to what extent these conservation laws restrict the possible commutation relations among the operators appearing in (1).

Specification of mass values in the rest frame and conservation of linear four-momentum are obtained, from (4), if

$$(P_\mu,Mc)=(P_\mu,\Gamma_\nu)=0. \tag{14}$$

Writing

$$S_{\mu\nu}=-i\hbar\Gamma_{\mu\nu}, \tag{15}$$

with the $\Gamma_{\mu\nu}$ represented by a set of six matrices, of as yet unspecified dimensionality, satisfying

$$(P_\mu,\Gamma_{\rho\sigma})=(x_\mu,\Gamma_{\rho\sigma})=0 \tag{16}$$

and

$$(\Gamma_{\mu\nu},\Gamma_{\rho\sigma})=-(\Gamma_{\mu\rho}\delta_{\nu\sigma}+\Gamma_{\nu\sigma}\delta_{\mu\rho}-\Gamma_{\mu\sigma}\delta_{\nu\rho}-\Gamma_{\nu\rho}\delta_{\mu\sigma}), \tag{17}$$

then Eqs. (2), (16), and (17) ensure that

$$(M_{\mu\nu},M_{\rho\sigma})=i\hbar(M_{\mu\rho}\delta_{\nu\sigma}+M_{\nu\sigma}\delta_{\mu\rho}-M_{\mu\sigma}\delta_{\nu\rho}-M_{\nu\rho}\delta_{\mu\sigma}), \tag{18}$$

i.e., that the six elements of the total angular momentum tensor constitute the external homogeneous Lorentz group $SL(2,c)$. Thus, conservation of total angular momentum is assured if

$$(\Gamma_{\mu\nu},Mc)=0 \tag{19}$$

and either

$$(\Gamma_{\mu\nu},\Gamma_\sigma)=(\Gamma_\mu\delta_{\nu\sigma}-\Gamma_\nu\delta_{\mu\sigma}) \tag{20}$$

if $(x_\mu,\Gamma_\nu)=0$, or

$$(\Gamma_{\mu\nu},\Gamma_\sigma)=0 \tag{21}$$

if Γ_μ is proportional to P_μ. In this latter case the orbital and spin angular momenta are separately conserved. Relations (2), (8), (9), (18), and of course $(P_\mu,P_\nu)=0$ are equivalent to the statement that the Poincaré group $IO(3,1)$ is the external space-time invariance group, symbolically stated in (10).

[8] K. Rafanelli, J. Math. Phys. **9**, 1425 (1968).

[9] I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968); L. O'Raifeartaigh and S. Chang, Phys. Rev. **171**, 1587 (1968).

Clearly then, the above considerations are not exhaustive enough to specify the group property of the ten elements Γ_μ and $\Gamma_{\mu\nu}$ which we have denoted as $\mathcal{G}_i$ [except for Γ_μ proportional to P_μ, in which case (10) exhausts the space-time group properties]. In order to specify $\mathcal{G}_i$, we must, in addition to (17) and (20), close the algebra by giving (Γ_μ,Γ_ν).

III. INTRINSIC INVARIANCE GROUPS

The algebra of the Γ_μ and $\Gamma_{\mu\nu}$ is most simply closed by choosing (13), which, together with (17) and (20), gives (12), or equivalently, $\mathcal{G}_i=\mathcal{P}_i:\{\Gamma_\mu,\Gamma_{\mu\nu}\}$. In this case, Eq. (1) describes a system with external and intrinsic space-time invariance under inhomogeneous Lorentz transformations for which the spaces coincide only if the velocity and momentum are proportional. For consistency of notation, we hereafter reserve the symbol Γ_μ, for case (13). We now want to study the extension of the group (12) to other, more complex structures, not because there is an *a priori* preference for any particular $\mathcal{G}_i$, but rather to demonstrate the equivalence of seemingly unrelated points of view.

A technique for extending an $IO(3,1)$ algebra to one appropriate to $O(4,1)$ has appeared previously[10]; the discussion is usually confined, however, to the external invariance group simply because it has become customary to associate $P_\mu P_\mu$ with the mass operator. On the other hand, the assertion here is that the physical mass spectrum emerges as the solution of Eq. (3). The mass operator is therefore not tied to a Casimir operator of the external space-time invariance group, and O'Raifeartaigh's theorem does not prevent mass splitting.[11] The extension technique, $\mathcal{P}_e\rightarrow\mathcal{S}_e$, consists simply in observing that the four-vector

$$P_\mu'=\frac{1}{2(\sqrt{P_\mu P_\mu})}(M_{\mu\nu},P_\nu)_+ \tag{22}$$

has the property

$$(P_\mu',P_\nu')=i\hbar M_{\mu\nu}, \tag{23}$$

as does $P_\mu+P_\mu'$.

Since we have chosen $\mathcal{P}_i:\{\Gamma_\mu;\Gamma_{\mu\nu}\}$, a similar extension technique applies, that is, the four-vector[12]

$$\psi_\mu\equiv\tfrac{1}{2}(\Gamma_{\mu\sigma}\Gamma_\sigma+\Gamma_\sigma\Gamma_{\mu\sigma}) \tag{24}$$

has the property

$$(\psi_\mu,\psi_\nu)=C_0{}^P\Gamma_{\mu\nu}, \tag{25}$$

where $C_0{}^P=\Gamma_\mu\Gamma_\mu$ is a Casimir operator of $\mathcal{P}_i$. Thus, with $b\equiv(C_0{}^P)^{-1/2}$, we have

$$(b\psi_\mu,b\psi_\nu)=\Gamma_{\mu\nu}. \tag{26}$$

Therefore, the ten elements $\{b\psi_\mu,\Gamma_{\mu\nu}\}$ constitute the DeSitter group, $O(4,1)$. More generally, if

$$\Gamma_\mu'\equiv\Gamma_\mu+b\psi_\mu, \tag{27}$$

[10] A. Sankaranarayanan, J. Math. Phys. **9**, 611 (1968).

[11] L. O'Raifeartaigh; Phys. Rev. **139**, B1052 (1965).

[12] The notation adopted here is consistent with that of Ref. 1(b).

then

$$(\Gamma_\mu',\Gamma_\nu')=\Gamma_{\mu\nu} \tag{28}$$

and the ten elements $\{\Gamma_\mu';\Gamma_{\mu\nu}\}$ also constitute $O(4,1)$. It is this latter extension, $\mathcal{P}_i:\{\Gamma_\mu,\Gamma_{\mu\nu}\}\rightarrow \mathcal{S}_i:\{\Gamma_\mu',\Gamma_{\mu\nu}\}$, that we wish to generalize and incorporate into a larger transformation theory.

Consider the transformation

$$\psi'=e^{gC_0^L}\psi \tag{29}$$

generated by the intrinsic $SL(2,c)$ Casimir operator, $C_0^L=-\frac{1}{2}\Gamma_{\mu\nu}\Gamma_{\mu\nu}$. Since C_0^L is Hermitian, the transformation (29) is unitary if the parameter g is pure imaginary. Under (29), an operator A transforms as

$$A'=e^{gC_0^L}Ae^{-gC_0^L}. \tag{30}$$

Expansion of (30) gives

$$A'=\sum_{m=0}^{\infty}\frac{1}{m!}A^{(m)}=A+A^{(1)}+\frac{1}{2!}A^{(2)}+\frac{1}{3!}A^{(3)}+\cdots, \tag{31}$$

where

$$\begin{aligned}
&A^{(1)}=(gC_0^L,A),\\
&A^{(2)}=[gC_0^L,(gC_0^L,A)]=(gC_0^L,A^{(1)}),\\
&A^{(3)}=\{gC_0^L,[gC_0^L,(gC_0^L,A)]\}=(gC_0^L,A^{(2)}),
\end{aligned} \tag{32}$$

etc.

Since Mc satisfies (19), then

$$(Mc,C_0^L)=0 \tag{33}$$

so that the covariant Hamiltonian operator (4) transforms as

$$H'=e^{gC_0^L}He^{-gC_0^L}=i\Gamma_\mu'P_\mu+Mc, \tag{34}$$

where

$$\Gamma_\mu'=\Gamma_\mu+\Gamma_\mu^{(1)}+\frac{1}{2!}\Gamma_\mu^{(2)}+\cdots, \tag{35}$$

with

$$\begin{aligned}
&\Gamma_\mu^{(1)}=2g\psi_\mu,\\
&\Gamma_\mu^{(2)}=4g^2\chi_\mu,
\end{aligned} \tag{36}$$

etc., where ψ_μ is given by (24), and

$$\chi_\mu\equiv\tfrac{1}{2}(\Gamma_{\mu\sigma}\psi_\sigma+\psi_\sigma\Gamma_{\mu\sigma}). \tag{37}$$

The commutation relations among the terms generated in (35) get rapidly complicated, i.e.,

$$\begin{aligned}
(\Gamma_\mu^{(1)},\Gamma_\nu^{(1)})&=4g^2C_0^P\Gamma_{\mu\nu},\\
(\Gamma_\mu^{(2)},\Gamma_\nu^{(2)})&=4g^4(C_0^P)^2[-\Gamma_{\mu\nu}+2(\Gamma_{\mu\nu}\psi_\sigma\psi_\sigma\\
&\quad+\psi_\sigma\psi_\sigma\Gamma_{\mu\nu})+\Gamma_{\mu\sigma}(\Gamma_{\nu\rho}\Gamma_{\sigma\rho}+\Gamma_{\sigma\rho}\Gamma_{\nu\rho})\\
&\quad+(\Gamma_{\nu\rho}\Gamma_{\sigma\rho}+\Gamma_{\sigma\rho}\Gamma_{\nu\rho})\Gamma_{\mu\sigma}],
\end{aligned} \tag{38}$$

etc. Thus, we may write

$$H'=H+H^{(1)}+H^{(2)}+\cdots \tag{39}$$

as a sequence of iterated wave operators, possessing ascendingly complicated intrinsic space-time invariance groups, generated from $\mathcal{P}_i$ by a Casimir operator of the intrinsic $SL(2,c)$, and satisfying the field equation

$$H'\psi'=(i\Gamma_\mu'P_\mu+Mc)\psi'=0. \tag{40}$$

Owing largely to the success of Dirac's equation, little, if any, consideration has been given to intrinsic space-time structure other than $\mathcal{G}_i=\mathcal{S}_i$. This is certainly understandable considering the algebraic complexity of the second of Eqs. (38). However, in light of the above analysis we see that $\mathcal{G}_i=\mathcal{S}_i$ is related to the simpler case, $\mathcal{G}_i=\mathcal{P}_i$, on which rests an alternative approach to the question of mass spectra.[5] Therefore, it seems reasonable to examine, in greater detail, the special case of the above transformation theory, $\mathcal{P}_i\rightarrow\mathcal{S}_i$. Explicitly, the infinitesimal part of the unitary transformation (29),

$$\psi'=(1+gC_0^L)\psi, \tag{41}$$

which is then also unitary, terminates the sequence (35) at

$$\Gamma_\mu'=\Gamma_\mu+2g\psi_\mu \tag{42}$$

and we have

$$(\Gamma_\mu',\Gamma_\nu')=\pm\Gamma_{\mu\nu} \tag{43}$$

corresponding to the two cases

$$\begin{aligned}
&g=\tfrac{1}{2}b, \quad \text{for} \quad C_0^P<0\\
&g=\tfrac{1}{2}ib, \quad \text{for} \quad C_0^P>0.
\end{aligned} \tag{44}$$

Thus, the adoption of (1) (based on $\mathcal{P}_i$) leads, via the infinitesimal unitary transformation (41), to the field equation (40) (based on $\mathcal{S}_i$) of identical form and based on the same dynamical model, in the sense that Mc is unchanged.

Nothing has been said so far about particular representations for either $\mathcal{P}_i$ or $\mathcal{S}_i$. It is, therefore, worth noting at this point that for those representations of $\mathcal{S}_i$ with vanishing χ_μ, the sequence (35) generated by (29) is exactly terminated as in (42), all $\Gamma_\mu^{(m)}$ vanishing identically for $m\geqslant 2$. Therefore, for this class of representations we have (1) $\rightarrow$ (40) with $\mathcal{P}_i\rightarrow\mathcal{S}_i$, via the finite unitary transformation (29). We defer a detailed discussion of representations until a later study and base the ensuing analysis on the infinitesimal unitary transformation (41) effecting $\mathcal{P}_i\rightarrow\mathcal{S}_i$.

IV. MASS OPERATORS AND DYNAMICAL MODELS

Physical interpretation of the transformation (41) requires the choice of dynamical model, or mass operator. Clearly, the simplest choice is Mc = constant parameter, or Casimir operator of $\mathcal{P}_i$. A consequence of this choice is that either $(Mc,\Gamma_\mu)=0$, or $\dot{x}_\mu$ = const, and the particle described by (1) undergoes **no** *Zitterbewegung*. Hence, the velocity and momentum must be proportional. The orbital and spin angular momenta are separately conserved, the spaces of $\mathcal{P}_i$ and $\mathcal{P}_e$ coincide, and it is consistent to think of (1) as the

KG, or second-order equation corresponding to (40). This is not a very interesting case, however, since it is reasonably well established that if $Mc=\text{const}$ the mass spectrum is not indicative of the elementary particles.[1,6,8]

In order to obtain a theory, based on (1), admitting *Zitterbewegung*, we must have $(\Gamma_\mu, Mc)\neq 0$. There is discussion in the literature on possible choices for Mc, consistent with the external invariance requirements.[1] The discussion is usually in conjunction with Eq. (40). However, since we now recognize that (1) and (40) are identical formulations, modulo-a unitary transformation of intrinsic space-time symmetry, the discussion applies to both cases.

In the remainder of this section we examine the transformation $\mathcal{P}_i \to \mathcal{S}_i$, with the choice

$$Mc=mc-\tfrac{1}{2}aC_0{}^L=mc+\tfrac{1}{4}a\Gamma_{\mu\nu}\Gamma_{\mu\nu}\,, \tag{45}$$

where, for purposes of this discussion, we leave mc and a as unspecified constant parameters. The selection of (45) corresponds, in Corben's formulation, to an operator representing the rotational levels of a spherical top.[5] It also follows that, since

$$C_0{}^S=-\tfrac{1}{2}\Gamma_{ab}\Gamma_{ab}=\Gamma_\mu'\Gamma_\mu'+C_0{}^L;\quad a,b=1,\cdots,5 \tag{46}$$

is a Casimir operator of $\mathcal{S}_i$, other discussed choices for Mc, such as $\Gamma_\mu'\Gamma_\mu'$,[1] can differ from (45) by at most a constant parameter, and that the model studied by Sutton[13] is equivalent for those representations of $\mathcal{S}_i$ with $(\Gamma_\mu', C_0{}^L)=0$. So, while the model leading to (45) is quite specific, the mass operator itself has received a broader appeal.

With (45), Eq. (1) is explicitly

$$H\psi=(i\Gamma_\mu P_\mu+mc-\tfrac{1}{2}aC_0{}^L)\psi=0\,. \tag{47}$$

Thus, the infinitesimal transformation (41) is equivalent to

$$\psi'=[1+(2g/a)(mc+i\Gamma_\mu P_\mu)]\psi \tag{48}$$

for solutions of (47).

The transformation $\mathcal{P}_i\to\mathcal{S}_i$ now admits the following kinematical interpretation. The instantaneous position four-vector x_μ transforms under (48) as

$$x_\mu'=x_\mu+(2g/a)\Gamma_\mu\,, \tag{49}$$

yielding, from (47), the new four-velocity

$$\dot{x}_\mu'=i\Gamma_\mu'=i(\Gamma_\mu+2g\psi_\mu) \tag{50}$$

as given by (42). Therefore, with the mass operator given by (45), $\mathcal{P}_i\to\mathcal{S}_i$ is equivalent to a transformation of coordinate representation. Corresponding to the change of coordinate, the division of the total angular momentum between orbital and spin now becomes

$$M_{\mu\nu}=x_\mu'P_\nu=x_\nu'P_\mu-i\hbar\Gamma_{\mu\nu}'=\Gamma_{\mu\nu}'+S_{\mu\nu}'\,. \tag{51}$$

Using (49), comparison of (51) with

$$M_{\mu\nu}=x_\mu P_\nu-x_\nu P_\mu-i\hbar\Gamma_{\mu\nu}=L_{\mu\nu}+S_{\mu\nu}$$

yields

$$-i\hbar\Gamma_{\mu\nu}'=-i\hbar\Gamma_{\mu\nu}-(2\hbar g/a)(\Gamma_\mu P_\nu-\Gamma_\nu P_\mu)\,; \tag{52}$$

the same result is, of course, obtained by directly transforming $\Gamma_{\mu\nu}$ under (48).[14] It is easily seen that

$$(x_\mu',\Gamma_{\rho\sigma}')=0=(P_\mu,\Gamma_{\rho\sigma}')=(L_{\mu\nu}',\Gamma_{\rho\sigma}')\,, \tag{53}$$

and

$$(\Gamma_{\mu\nu}',\Gamma_{\rho\sigma}')=-(\Gamma_{\mu\rho}'\delta_{\nu\sigma}+\Gamma_{\nu\sigma}'\delta_{\mu\rho}-\Gamma_{\mu\sigma}'\delta_{\nu\rho}-\Gamma_{\nu\rho}'\delta_{\mu\sigma})\,. \tag{54}$$

It also follows that

$$-i\hbar\dot{\Gamma}_{\mu\nu}'=i(P_\mu\Gamma_\nu'-P_\nu\Gamma_\mu') \tag{55}$$

as a consequence of

$$-i\hbar\dot{\Gamma}_{\mu\nu}=i(P_\mu\Gamma_\nu-P_\nu\Gamma_\mu)\,. \tag{56}$$

Thus, it is clear that $\Gamma_{\mu\nu}'$ has those properties of an intrinsic angular momentum with respect to the primed-coordinate system possessed by $\Gamma_{\mu\nu}$ with respect to the unprimed-coordinate system.

It is also straightforward to verify that rewriting the $\mathcal{P}_i\to\mathcal{S}_i$ transformation as in (48), gives rise to the transformed mass operator

$$M'c=Mc+2ig\psi_\mu P_\mu\,. \tag{57}$$

It therefore follows that, if ψ is a solution of (47), then

$$H'=i\Gamma_\mu'P_\mu+Mc \tag{58}$$

and

$$H'=i\Gamma_\mu P_\mu+M'c \tag{59}$$

are equivalent statements with $M'c$ given by (57), and $H'\psi'=0$. While it is true that the momentum dependence in $M'c$ gives it a somewhat different status than Mc, we are merely using the form (59) to illustrate a kinematical equivalent for $\mathcal{P}_i\to\mathcal{S}_i$, the results of which are stated in (58). As such, we can see the origin of the enhanced group structure as follows. From (4) we have $\dot{x}_\mu=i\Gamma_\mu$, and with (45) we have $\ddot{x}_\mu=a/\hbar\psi_\mu$. It then follows from (24) that $\ddot{x}_\mu\dot{x}_\mu+\dot{x}_\mu\ddot{x}_\mu=0$, stating, in Heisenberg operator language, the orthogonality of the instantaneous velocity and acceleration, and reflecting the classical trajectory of a free-spinning particle in which the *Zitterbewegung* is a uniform circular motion superimposed on the average, straight-line path. Thus, the form of (49) indicates that the transformation $\mathcal{P}_i\to\mathcal{S}_i$ is in the rest system, kinematically equivalent to the transformation to a primed set of axes, rotating with respect to the unprimed axes, and the momentum-dependent term in (57) is the spurious contribution of the rotation.

To further illustrate this kinematical equivalence, we examine the relation between the two local position

[13] A. M. Sutton, Phys. Rev. **160**, 1055 (1967).

[14] See also J. A. De Vos and J. Hilgevoord, Nucl. Phys. **B494** (1967).

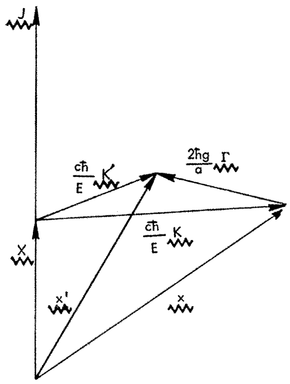

FIG. 1. Rest-frame relationship between particle coordinates.

operators x_μ and x_μ' and the nonlocal position operator X_μ, usually associated with center of mass.[15] Consider

$$X_\mu = x_\mu - i\hbar\Gamma_{\mu\nu}P_\nu/P_\sigma P_\sigma\,, \tag{60}$$

for which $(X_\mu,X_\nu)\neq 0$, and

$$\dot{X}_\mu = (i\Gamma_\sigma P_\sigma/P_\sigma P_\sigma)P_\mu\,. \tag{61}$$

There is another operator with similar traits, namely,

$$X_\mu' = x_\mu' - \frac{i\hbar\Gamma_{\mu\nu}'P_\nu}{P_\sigma P_\sigma}\,, \tag{62}$$

for which $(X_\mu',X_\nu')\neq 0$, and

$$\dot{X}_\mu' = \left(\frac{i\Gamma_\sigma' P_\sigma}{P_\sigma P_\sigma}\right)P_\mu\,. \tag{63}$$

It is not surprising that, as viewed from the two coordinate systems, the center of mass is different; in fact we see that

$$X_\mu' - X_\mu = \frac{2ig\hbar}{a}\left(-\frac{i\Gamma_\sigma P_\sigma}{P_\sigma P_\sigma}\right)P_\mu\,. \tag{64}$$

Introducing the notation

$$\mathbf{J} = (\Gamma_{23},\Gamma_{31},\Gamma_{12})\,,$$
$$\mathbf{K} = (\Gamma_{14},\Gamma_{24},\Gamma_{34})\,, \tag{65}$$

[15] M. H. L. Pryce, Proc. Roy. Soc. (London) **195**, 62 (1948).

in terms of which

$$C_0{}^L = -\tfrac{1}{2}\Gamma_{\mu\nu}\Gamma_{\mu\nu} = -(\mathbf{J}^2+\mathbf{K}^2)\,, \tag{66}$$

and

$$\mathbf{J}' = (\Gamma_{23}',\Gamma_{21}',\Gamma_{12}')\,,$$
$$\mathbf{K}' = (\Gamma_{14}',\Gamma_{24}',\Gamma_{34}')\,, \tag{67}$$

then in the rest frame, $P_\mu = (0,iE/c)$, we have

$$\mathbf{X} = \mathbf{X}' = \mathbf{x} - (c\hbar/E)\mathbf{K} = x' - (c\hbar/E)\mathbf{K}'\,,$$
$$\frac{d\mathbf{X}}{d\tau} = \frac{d\mathbf{X}'}{d\tau} = 0\,, \tag{68}$$
$$\mathbf{x}' - \mathbf{x} = (2\hbar g/a)\mathbf{\Gamma} = (c\hbar/E)(\mathbf{K}-\mathbf{K}')\,,$$
$$X_4' - X_4 = x_4' - x_4 = (2\hbar g/a)\Gamma_4\,,$$

and

$$\mathbf{J} = \mathbf{J}'\,,$$
$$\frac{d\mathbf{J}}{d\tau} = \frac{d\mathbf{J}'}{d\tau} = 0\,. \tag{69}$$

The relations among these vectors in the rest frame are diagrammed in Fig. 1.

V. CONCLUSION

The intrinsic space-time particle symmetry is defined by the ten-element group composed of the four-velocity and spin angular momentum operators. This group is not completely specified by external Poincaré invariance. From the simplest choice, an intrinsic Poincaré group, it is possible to effect a unitary transformation to more complex intrinsic group structures, leaving invariant the form of the linear wave equation describing the particle, and the mass operator.

Choosing a mass operator based on Corben's model of a spherical top, the intrinsic symmetry-group transformation $\mathcal{P}_i \to \mathcal{S}_i$ has as a kinematical equivalent, a transformation from an inertial to a rotating coordinate system.

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