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Coherent Soft-Photon States and Infrared Divergences. IV. The Scattering Operator*

T. W. B. KIBBLE†
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

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In Paper III of this series, the asymptotic states of quantum electrodynamics were defined in terms of the mass-shell singularity structure of the Green's functions. In this paper the reduction formulas obtained are used to derive simple expressions for the matrix elements of the scattering operator. This operator is defined on the space of asymptotic states, which is the direct product of the Fock space of the particles (massive particles and hard photons) with the nonseparable Hilbert space, defined in Paper I, which is spanned by the soft-photon coherent states. It is shown that the scattering operator so defined is gauge-invariant, Lorentz-invariant, unitary, crossing-symmetric, and independent of the choice of the small parameter that defines the separation between hard and soft photons. For a given initial state, the only nonvanishing scattering matrix elements are those to final states in a specific equivalence class, and conditions for states to be equivalent in this sense are obtained. The relationship between these matrix elements and physically measurable cross sections is discussed. In this way, results obtained by conventional methods are reproduced, but in addition questions inaccessible to such methods, such as the effect of an infinite number of soft photons in the initial state, may be investigated.

1. INTRODUCTION

THE conventional treatment of quantum electrodynamics gives it a special role among renormalizable field theories, because of the way the infrared divergences are handled. Whereas, in other theories one computes scattering matrix elements that are finite after renormalization, in quantum electrodynamics they are infrared-divergent, and one computes only finite scattering probabilities, obtained by summing over the number of emitted soft photons. In the present series of papers[1] our aim has been to show that this special treatment is unnecessary and that quantum electrodynamics can be treated like any other renormalizable field theory. To do this we must drop the invalid assumption,[2] implicit in the conventional perturbation calculations, that the asymptotic states belong to the Fock space, and instead allow the theory itself to dedetermine the structure of the space of asymptotic states.

In I we discussed the case of interaction with an external classical current distribution. We defined a set of generalized coherent states of the radiation field which can contain infinitely many soft photons and which span a nonseparable Hilbert space $\mathcal{H}_{\rm IR}$. We also showed that on this space it is possible to define a unitary scattering operator, all of whose matrix elements are finite. The extension to the fully quantized theory was begun in II, where we investigated the mass-shell singularities of the Green's functions, which are branch points rather than simple poles. Then in III we studied the nature of the asymptotic states implied by this structure. The states were defined by appropriate weak limits. They span a space that is the direct product of the Fock space for the particles (by which we mean massive particles and hard photons) with the Hilbert space $\mathcal{H}^s{}_{\rm IR}$ spanned by the soft-photon coherent states. We obtained reduction formulas that permit matrix elements between these asymptotic states to be extracted from the Green's functions.

In this paper we shall examine the scattering matrix elements obtained by application of these reduction formulas. We begin in Sec. 2 with a critical discussion of the conventional treatment of the infarared-divergence problem. Then in Sec. 3 we obtain a simple general expression for the scattering matrix elements in a manifestly gauge-invariant form. Section 4 is devoted to establishing the independence of these matrix elements on the parameter K that fixes the conventional separation between hard and soft photons. In Sec. 5 we show that for a fixed initial state the only nonvanishing scattering matrix elements are to final states belonging to a certain "equivalence class" and obtain criteria for states to be equivalent in this sense. A specific class of soft-photon coherent states, namely, those that could be produced from the vacuum by the classical current of a point particle accelerated to given momentum, plays an important role. These states are investigated in Sec. 6. In particular, a formula for the momentum spectrum of such a state is obtained. The relationship between the scattering matrix elements and physically measurable cross sections is discussed in Sec. 7. Here we reproduce the results of conventional calculations. However, we are also able to discuss problems that are hard to treat by conventional methods, such as the influence of an infinite number of soft photons in the

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† On leave of absence from Department of Physics, Imperial College, London, England.

[1] Preceding papers in this series were T. W. B. Kibble, J. Math. Phys. **9**, 315 (1968) (referred to as I); Phys. Rev. **173**, 1527 (1968) (II); *ibid.* **174**, 1882 (1968) (III).

[2] In this connection, see K. O. Friedrichs, *Mathematical Aspects of the Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1953), Secs. 14 and 19. The use of coherent states to describe soft photons has also been proposed by V. Chung, Phys. Rev. **140**, B1110 (1965); J. K. Storrow, Nuovo Cimento **54**, 15 (1968).

initial state. The unitarity of the scattering operator is demonstrated in Sec. 8. The conclusions are summarized and discussed in Sec. 9.

2. DISCUSSION OF CONVENTIONAL TREATMENT

It will be useful to begin by recalling certain features of the conventional method of handling the infrared divergences, developed by Yennie, Frautschi, and Suura,[3] by Eriksson,[4] and by others.[5–8] This may be summarized as follows.

First, we give the photon a small fictitious mass λ, so as to remove the infrared divergences from individual Feynman diagrams. Then, for any given basic process, we compute, for each value of n, the probability $P_n(\lambda,\Delta E)$ for this process accompanied by the emission of n soft photons with total energy less than ΔE. When we evaluate the sum

$$P(\lambda,\Delta E)=\sum_{n=0}^{\infty} P_n(\lambda,\Delta E)\,, \tag{2.1}$$

we find that all the terms proportional to $\ln\lambda$ cancel in each order of perturbation theory. Thus, finally, we may take the limit of vanishing photon mass, and obtain a finite result

$$P(\Delta E)=\lim_{\lambda\to 0} P(\lambda,\Delta E)\,. \tag{2.2}$$

This is then the observable probability for the process under experimental conditions in which the energy loss due to soft-photon emission can just be detected when it reaches the value ΔE.

In general, each term of the series (2.1) tends to zero in the limit of vanishing photon mass

$$\lim_{\lambda\to 0} P_n(\lambda,\Delta E)=0\,, \tag{2.3}$$

although the sum remains finite. [In any given order of perturbation theory, $P_n(\lambda,\Delta E)$ diverges logarithmically. However, it can be shown that these divergent terms sum to a form like $e^{\alpha \ln\lambda}$.] As λ approaches zero, the number of terms that must be included in the series increases, and in the limit the expectation value of the total number of photons emitted tends to infinity:

$$\lim_{\lambda\to 0} \sum_{n=0}^{\infty} nP_n(\lambda,\Delta E)=\infty\,. \tag{2.4}$$

Thus the probability that the given process occurs with the emission of any finite number of photons is zero. This is an indication of the fact that when the initial state belongs to the Fock space, the final state in general does not, since it contains infinitely many soft photons.

If all that we want is a set of rules from which observable scattering probabilities can be calculated, then this conventional procedure is adequate. From a logical point of view, however, it has serious drawbacks.

First, it may be remarked that in the procedure as outlined above we never actually solve the equations of quantum electrodynamics itself. Instead, we consider a vector-meson theory with a very small vector-meson mass λ, and assume that taking the limit $\lambda\to 0$ at the end of the calculation reproduces the results of quantum electrodynamics. Physically, it is, of course, an extremely plausible assumption that we would be unable to distinguish electrodynamics from a theory in which the photon had a finite mass, provided that this mass were chosen small enough. However, it would be better to be able to prove this assumption, and this could only be done by providing some alternative method of calculation that works directly with quantum electrodynamics itself, and avoids the necessity of introducing a photon mass.

At least a partial answer to this objection has in fact been provided by the elegant technique for performing the sum in (2.1) developed by Mahanthappa.[7] This technique uses Schwinger's[9] method for calculating directly the expectation value of appropriate final-state projection operators. Instead of evaluating each term in the sum separately, one writes the corresponding expression in terms of operators,

$$\sum_n \langle i|S^*|n\rangle\langle n|S|i\rangle\,, \tag{2.5}$$

where S is represented by a sum of time-ordered products, and S^* by a sum of anti-time-ordered products, and replaces the sum over final states by a projection operator. Because the summation is carried out before the diagrams are evaluated, no infrared divergences appear, and one avoids the need to introduce a photon mass.

However, this answer is not really complete, because the difficulties have merely been transferred to the problem of defining the projection operators involved. What is required is the projection operator P on the subspace of states containing designated charged particles together with any number of soft photons. Formally this operator is defined by

$$P=\sum_n |n\rangle\langle n|\,, \tag{2.6}$$

but as long as the sum runs only over states containing finite numbers of photons, this equation is not really correct, as may be seen from the fact that the expectation value of P is nonzero, while the expectation value

[3] D. Yennie, S. Frautschi, and H. Suura, Ann. Phys. (N. Y.) 13, 379 (1961). Many earlier references are quoted in this paper.
[4] K. E. Ericksson, Nuovo Cimento 19, 1010 (1961).
[5] J. M. Jauch and F. Rohrlich, Helv. Phys. Acta 27, 613 (1954).
[6] E. L. Lomon, Nucl. Phys. 1, 101 (1956); Phys. Rev. 113, 726 (1959); E. L. Lomon and A. Shaw, quoted in Ref. 6; R. Perrin and E. L. Lomon, Ann. Phys. (N. Y.) 33, 328 (1965).
[7] K. T. Mahanthappa, Phys. Rev. 126, 329 (1961).
[8] J. Tarski, J. Math. Phys. 7, 560 (1966).
[9] J. Schwinger, J. Math. Phys. 2, 407 (1961).

of each term in the sum (2.6) is zero. This is again an indication of the need to include states outside the Fock space.

The second objection that may be raised against the conventional procedure is its lack of symmetry between initial and final states. We normally assume that the initial state contains only a finite number of photons, and compute the sum of the scattering probabilities to all possible final states. In general, to obtain a finite answer one must include final states with infinitely many photons. However, in practice we have no way of knowing that the initial state does not already contain infinitely many soft photons, with finite total energy, and for completeness we should allow for this possibility. In fact, in a multiple scattering process in which the initial state before the first stage belongs to the Fock space, the initial state for the second stage will not, in general. It can of course be argued that the infinitely many soft photons emitted in the first stage of the process will have escaped from the region of interest and, in any case, that the presence of undetectable soft photons in the initial state should not substantially affect the scattering probabilities. But this again is something that should be proved and not merely assumed. The conventional procedure would allow us to compute the scattering probabilities from an initial state containing any finite number of soft photons, but it cannot accommodate an infinite number.

Probably the most serious objection to the conventional procedure, however, is simply that it involves a description of the asymptotic states that does not correspond to reality. According to the generally accepted ideas of quantum-mechanical scattering theory, if the initial state is a well-defined pure state, represented by a definite vector in the Hilbert space of asymptotic states, then the final state is another well-defined pure state, which it should be possible to characterize completely in some way. We know that if the initial state belongs to the Fock space, then the final state does not, and it is therefore impossible to characterize it by giving its components in a basis in the Fock space. All of these are actually zero. (It should be remarked that the conventional procedure provides only an incomplete description of the final state, since it is concerned only with the particle labels and not with those characterizing the final soft-photon state.) Consequently, if we want to retain the usual structure of scattering theory, we must extend our space of asymptotic states, and work with a space that can accommodate well-defined states in which the expectation value of the total number of soft photons is infinite.

In the Fock space, we normally choose our basis states to contain definite numbers of photons, each labeled by its momentum and polarization. It is clear, however, that this choice cannot easily be generalized to allow for the possibility of an infinite number of soft photons. A much better choice of basis consists of the coherent states of the radiation field that can be so generalized, as was shown in I. It is rather natural that the coherent states, which have quasiclassical properties,[10] should be a convenient tool in discussing the infrared-divergence problem, since the soft-photon emission process is essentially classical in nature. Indeed, it is clear from the work of Bloch and Nordsieck[11] that the soft photons emitted in a scattering process must be in what is essentially a coherent state,[12] although no precise definition of such states for the case of infinite photon number was in fact given.

We showed in I that, for the electromagnetic field interacting only with an external current, the space of asymptotic states is the space $\mathcal{H}_{\mathrm{IR}}$ spanned by the generalized coherent states $|f,\lambda\rangle$. That is to say, for any current $J^\mu(x)$ (of the class specified), a unitary scattering operator $S(J)$ is defined on $\mathcal{H}_{\mathrm{IR}}$. Thus, if the initial state belongs to $\mathcal{H}_{\mathrm{IR}}$, then does so the final state, whatever the current chosen may be. No subspace of $\mathcal{H}_{\mathrm{IR}}$ has this property, and in fact the generalized coherent states $|f,\lambda\rangle$ that span $\mathcal{H}_{\mathrm{IR}}$ may all be produced from the vacuum by the action of some suitably chosen external current.

It will be convenient to recall here certain properties of these coherent states. The label f stands for a photon wave function $f^\mu(\mathbf{k})$, while $\lambda(\mathbf{k})$ is a real function whose (possibly divergent) integral determines the generalized phase of the state

$$e^{i\lambda}=\exp i\int\frac{d\mathbf{k}}{(2\pi)^3 2k^0}\lambda(\mathbf{k}).$$

We frequently use the notation

$$f^*g=\int\frac{d\mathbf{k}}{(2\pi)^3 2k^0}f_\mu{}^*(\mathbf{k})g^\mu(\mathbf{k}),$$

in terms of which the scalar product of two generalized coherent states $|f,\lambda\rangle$ and $|g,\mu\rangle$ is, formally,

$$\langle f,\lambda|g,\mu\rangle=\exp(f^*g-\tfrac{1}{2}f^*f-\tfrac{1}{2}g^*g-i\lambda+i\mu). \quad (2.7)$$

This integral may diverge, but we adopt the same convention as in I and III: All terms in the exponent are to be combined *before* doing the integration, and if the resulting sum is a divergent integral, its exponential is zero by definition. Unitary operators $U(f)V(\lambda)$ that create these coherent states from the vacuum may be defined by

$$U(f)V(\lambda)|g,\mu\rangle=|f+g,\,\lambda+\mu+\tfrac{1}{2}i(f^*g-g^*f)\rangle. \quad (2.8)$$

The scattering operator $S(J)$ in the presence of a classical external current J may be expressed in terms of these operators by the relation (65) of I, namely,

$$S(J)=U(iJ)V(\sigma). \quad (2.9)$$

[10] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[11] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).
[12] R. J. Glauber, Phys. Rev. **84**, 395 (1951).

In the argument of U, J signifies the Fourier transform of $J^\mu(x)$, restricted to the positive-energy mass shell, $k^2=0$, $k^0>0$. The phase σ is a quadratic function of J, and is physically irrelevant.

We also note a special case of the relation (2.8) of III which may easily be derived from (2.7) and (2.8) above, namely,

$$\langle f,\lambda | U(h_1)U(h_2) | g,\mu\rangle = \langle f,\lambda | g,\mu\rangle \exp[f^*(h_1+h_2)-(h_1+h_2)^*g - \tfrac{1}{2}h_1^*h_1 - h_1^*h_2 - \tfrac{1}{2}h_2^*h_2]. \quad (2.10)$$

The exponents here and in (2.7) must, of course, be combined before applying the interpretational rules given above.

Finally, it will be useful to recall the notation for a translated wave function,

$$(x)f^\mu(\mathbf{k}) = f^\mu(\mathbf{k})e^{-ik\cdot x}. \quad (2.11)$$

It is reasonable to expect that even in the fully interacting theory the space of asymptotic photon states will again be $\mathcal{H}_{\rm IR}$. However, we do not assume this *a priori*, but seek to deduce it from the structure of the Green's functions. We have already shown, in III, that it is possible to define asymptotic soft-photon coherent states. What we have not yet shown is that the asymptotic states so defined form a complete set, or, in other words, that the scattering operator defined by the matrix elements between them is unitary. This we shall prove in Sec. 8.

3. SCATTERING MATRIX ELEMENTS

The scattering matrix elements

$$\langle \mathbf{l}_1\cdots\mathbf{l}_s; f,\lambda; \text{out} | -\mathbf{l}_{s+1}\cdots -\mathbf{l}_n; g,\mu; \text{in}\rangle \quad (3.1)$$

define a scattering operator S on the space spanned by the states $|\mathbf{l}_1\cdots\mathbf{l}_n; f,\lambda\rangle$. This is the direct-product space $\mathcal{H}^h{}_0\otimes\mathcal{H}^s{}_{\rm IR}$, where $\mathcal{H}^h{}_0$ is the Fock space of massive particles and hard photons, and $\mathcal{H}^s{}_{\rm IR}$ is the nonseparable Hilbert space spanned by the soft-photon coherent states $|f,\lambda\rangle$. It is the properties of this operator S that we shall study in this paper.

The matrix element (3.1) is a special case of the matrix elements discussed in III, one in which no field operators remain between the asymptotic states and in which the soft-photon external current J is set equal to zero. Let us begin by recalling the rules given in III for calculating such matrix elements. We may start with the corresponding formula for the n-point Green's function $G(p_1\cdots p_n)$. This function is represented by a sum of terms, each representing core diagrams of one particular connectivity structure. In each term, the contribution of the core diagrams is represented by a product of factors $M^h{}_\alpha$, one for each connected piece. The soft-photon contribution is entirely contained in a single function Δ^s depending on the external momenta p_j and on the momentum variables q_j that label the inner ends of the external lines and appear in the functions $M^h{}_\alpha$. To pass from the Green's function to the matrix element (3.1) we must replace each p_j by a mass-shell momentum l_j, drop the renormalization functions $[Z^h{}_j(p_j)]^{1/2}$, replace the spin matrices $\Lambda_j(p_j)$ by spin-wave functions $\bar{u}(l_j)$ or $\bar{u}^c(-l_j)$, and modify the function Δ^s to take account of the fact that the momenta l_j are on their mass shells.

The most general function Δ^s considered in III had r lines with both ends on the mass shell, $m-r$ lines with one end on the mass shell, and $n-m$ off-mass-shell lines. For our present purposes, we need no off-mass-shell lines, since all the external momenta are on their mass shells. Thus we may set $m=n$. The expression (10.3) of III for Δ^s then simplifies to

$$\Delta^s{}_{f\lambda,g\mu}(l_1\cdots l_n;\, l_1'\cdots l_r',\, q_{r+1}\cdots q_n)\prod_{j=1}^{r}2\pi\delta(m_j^2+l_j'^2)=\int dy_1\cdots dy_n \exp\Big(-i\sum_{j=1}^{r}(l_j-l_j')\cdot y_j - i\sum_{j=r+1}^{n}(l_j-q_j)\Big)y_j\cdot$$

$$\times\langle f,\lambda|g,\mu\rangle \exp\left(i(f^*I+I^*g)+\tfrac{1}{2}i\int_{\Omega^s}'\frac{dk}{(2\pi)^4}I^\mu(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu(k)\right), \quad (3.2)$$

where

$$I^\mu(k)=\sum_{j=1}^{r}e_jl_j{}^\mu 2\pi\delta(l_j\cdot k)\exp(-ik\cdot y_j)-i\sum_{j=r+1}^{n}\frac{e_jl_j{}^\mu}{l_j\cdot k-i\epsilon}\exp(-ik\cdot y_j). \quad (3.3)$$

We recall also the significance of the prime on the integral in (3.2). It indicates the special treatment that is required for the terms in which the same term of (3.3) contributes both to I and I^* in (3.2). For these a special rule is required to make the integral well defined: We must, in fact, omit the contribution of the double pole at $l_j\cdot k=0$ and retain only the contribution of the pole at $k^2=0$. [See (6.8) of III.]

We now wish to obtain a more convenient expression for the matrix element (3.1) than the one yielded by direct application of the rules discussed above. In particular, we shall seek a form that is manifestly gauge-invariant. The original one is not, because of the appearance of the gauge function $\gamma_{\mu\nu}$ in the integral in (3.2).

For simplicity, let us first assume that in the region of interest no core diagrams with straight-through lines can contribute significantly to the matrix element (3.1). This condition will be satisfied if no pair of the external momenta has a sum l_i+l_j close to zero. In this case, the application of the rules yields the expression

$$\langle \mathbf{l}_1\cdots\mathbf{l}_s;\, f,\lambda;\, \text{out}\,|-\mathbf{l}_{s+1}\cdots-\mathbf{l}_n;\, g,\,\mu;\, \text{in}\rangle$$
$$=\prod_{j=1}^{s}\bar{u}(\mathbf{l}_j)\prod_{j=s+1}^{n}\bar{u}^c(-\mathbf{l}_j)\int\frac{dq_1}{(2\pi)^4}\cdots\frac{dq_n}{(2\pi)^4}$$
$$\times\Delta^s{}_{f\lambda,g\mu}(l_1\cdots l_n;\, q_1\cdots q_n)\sum_{\{A\}}\prod_{\alpha=1}^{N}(2\pi)^4\delta(\sum_{j\in A_\alpha}q_j)$$
$$\times M^h{}_\alpha(\{q_j|\,j\in A_\alpha\}),\quad(3.4)$$

where the summation indicates a sum over all partitions of $(1\cdots n)$ into sets A_α, that is, over all classes of diagrams with different connectivity structures.

Now, since $M^h{}_\alpha$ is a slowly varying function of its arguments, we may there replace q_j by l_j. Then for each δ function we introduce a Fourier representation

$$(2\pi)^4\delta(\sum_{j\in A_\alpha}q_j)=\int dx_\alpha\exp(-i\sum_{j\in A_\alpha}q_j\cdot x_\alpha).$$

When we substitute (3.2) into (3.4), we then find that the effect of each q_j integration is to set the corresponding $y_j=x_\alpha$. Then the function $I^\mu(k)$ given by (3.3) may be written in the form

$$I^\mu(k)=\sum_{\alpha=1}^{N}I_\alpha{}^\mu(k),\qquad(3.5)$$

where

$$I_\alpha{}^\mu(k)=-i\sum_{j\in A_\alpha}\frac{e_jl_j{}^\mu}{l_j\cdot k-i\epsilon}\exp(-ik\cdot x_\alpha).\qquad(3.6)$$

But charge is conserved in each connected piece of the core diagrams,

$$\sum_{j\in A_\alpha}e_j=0,\qquad(3.7)$$

and it follows that each of the currents I_α is conserved:

$$k_\mu I_\alpha{}^\mu(k)=0.\qquad(3.8)$$

Indeed, $I_\alpha{}^\mu(k)$ is the classical current corresponding to a point-scattering process occurring at x_α, where for each j either a particle of charge e_j and momentum l_j is emitted or else a particle of charge $-e_j$ and momentum $-l_j$ is absorbed.

It follows that the factor $\gamma_{\mu\nu}(k)$ in the last term of the exponent in (3.2) is redundant. (It is easy to check that the prime on the integral makes no difference to this argument.) Hence we may write

$$\langle \mathbf{l}_1\cdots\mathbf{l}_s;\, f,\lambda;\, \text{out}\,|-\mathbf{l}_{s+1}\cdots-\mathbf{l}_n;\, g,\mu;\, \text{in}\rangle$$
$$=\prod_{j=1}^{s}\bar{u}(\mathbf{l}_j)\prod_{j=s+1}^{n}\bar{u}^c(-\mathbf{l}_j)\sum_{\{A\}}\prod_{\alpha=1}^{N}M^h{}_\alpha(\{l_j|\,j\in A_\alpha\})$$
$$\times\int dx_1\cdots dx_N\exp(-i\sum_{\alpha=1}^{N}\sum_{j\in A_\alpha}l_j\cdot x_\alpha)\langle f,\lambda|g,\mu\rangle$$
$$\times\exp\left(i(f^*I+I^*g)+\tfrac{1}{2}i\int'_{\Omega^\bullet}\frac{dk}{(2\pi)^4}\frac{I^\mu(k)^*I_\mu(k)}{k^2-i\epsilon}\right).\qquad(3.9)$$

Since $\gamma_{\mu\nu}(k)$ no longer appears anywhere in this formula, it is manifestly gauge-invariant, provided, of course, that the core diagrams themselves are so.

In this discussion we have omitted the diagrams with straight-through lines, but it is easy to see that they too can be accommodated in the general formula (3.9). The dependence of the integrand of (3.2) on the variables of a straight-through line is already of precisely the form indicated in (3.9), if we identify the corresponding y_j with x_α and take

$$I_\alpha{}^\mu(k)=e_jl_j{}^\mu2\pi\delta(l_j\cdot k)\exp(-ik\cdot x_\alpha).\qquad(3.10)$$

The only real difference is the appearance of the δ-function factor on the left of (3.2). Hence (3.9) is again valid if we define the corresponding function $M^h{}_\alpha(l_il_j)$ by the formal relation

$$M^h{}_\alpha(l_il_j)2\pi\delta(m_j{}^2+l_j{}^2)=N_i{}^{-1}C_{ij}.\qquad(3.11)$$

The soft-photon factor in the integrand of (3.9) may be recognized as being essentially the scattering matrix element

$$\langle f,\lambda;\, \text{out}\,|g,\mu;\, \text{in}\rangle_I=\langle f,\lambda|S(I)|g,\mu\rangle\qquad(3.12)$$

in the presence of the classical external current $I^\mu(k)$, as given by (2.9) or by (59) of I. The only difference is the prime on the integral in (3.9). This prime is required to give the integral a meaning, even within the context of the interpretational rules given in Sec. 2 above and, in more detail, in I and III. For, without it, the k^0 integral would diverge even for $\mathbf{k}\neq\mathbf{0}$. In fact, the current $I^\mu(k)$ does not belong to the class of currents considered in I. (Physically, this is because of the artificial choice of sharp momentum states. We could obtain a current that does belong to that class by integrating over a small range of each l_j.)

Despite this difference we may define an operator $S'(I)$ on the space $\mathcal{H}^s{}_{\text{IR}}$ and write (3.9) more briefly as

$$\langle \mathbf{l}_1\cdots\mathbf{l}_s;\, f,\lambda;\, \text{out}\,|-\mathbf{l}_{s+1}\cdots-\mathbf{l}_n;\, g,\mu;\, \text{in}\rangle$$
$$=\prod_{j=1}^{s}\bar{u}(\mathbf{l}_j)\prod_{j=s+1}^{n}\bar{u}^c(-\mathbf{l}_j)\sum_{\{A\}}\prod_{\alpha=1}^{N}M^h{}_\alpha(\{l_j|\,j\in A_\alpha\})$$
$$\times\int dx_1\cdots dx_N\exp(-i\sum_{\alpha=1}^{N}\sum_{j\in A_\alpha}l_j\cdot x_\alpha)$$
$$\times\langle f,\lambda|S'(I)|g,\mu\rangle.\qquad(3.13)$$

It can easily be seen that, like $S(J)$, $S'(I)$ is unitary. Indeed, it may be expressed in terms of the unitary operators introduced in I and in Sec. 2 of this paper in a form similar to (2.9), namely,

$$S'(I)=U(iI)V(\sigma'), \tag{3.14}$$

where σ' is given formally by the principal-value integral

$$\sigma'=\frac{1}{2}\int'_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{I^\mu(k)^*I_\mu(k)}{k^2}. \tag{3.15}$$

Here the significance of the prime is simply that the diagonal terms with $i=j$ are to be dropped. [Note that in I we wrote $U(ij)$, where j denoted the positive-energy mass-shell restriction of J. Here this restriction is implicit.]

The translational invariance of the scattering operator is easy to verify from (3.13). For using the notation (2.11) for translated wave functions, we easily see that

$$\langle \mathbf{l}_1\cdots\mathbf{l}_s;(x)f,\lambda;\text{out}|-\mathbf{l}_{s+1}\cdots-\mathbf{l}_n;(x)g,\mu;\text{in}\rangle\exp(i\sum_{j=1}^n l_j\cdot x)=\prod_{j=1}^s\bar{u}(\mathbf{l}_j)\prod_{j=s+1}^n\bar{u}^c(-\mathbf{l}_j)\sum_{\{A\}}\prod_{\alpha=1}^N M^h{}_\alpha(\{l_j|j\in A_\alpha\})$$
$$\times\int dx_1\cdots dx_n(-i\sum_{\alpha=1}^N\sum_{j\in A_\alpha}l_j\cdot(x_\alpha-x))\langle(x)f,\lambda|S'(I)|(x)g,\mu\rangle. \tag{3.16}$$

If we then make the substitutions $x_\alpha\to x_\alpha+x$, we find that the soft-photon matrix element becomes

$$\langle(x)f,\lambda|S'[(x)I]|(x)g,\mu\rangle=\langle f,\lambda|S'(I)|g,\mu\rangle, \tag{3.17}$$

so that the matrix element (3.16) is equal to (3.13). Correspondingly, if we were to project out from the soft-photon states components with definite momentum, we should obtain a matrix element containing an over-all energy-momentum-conserving δ function.

The invariance under infinitesimal Lorentz transformations may be established similarly, taking account of the transformation properties of the spin functions and the M^h functions (which we assume) and using the relation

$$\langle(\Lambda)f,\lambda|S'[(\Lambda)I]|(\Lambda)g,\mu\rangle=\langle f,\lambda|S'(I)|g,\mu\rangle, \tag{3.18}$$

which is valid for any *infinitesimal* Lorentz transformation Λ.

4. DEPENDENCE ON THE SOFT-PHOTON CUTOFF

We now turn to the problem of proving that the scattering matrix elements computed from (3.9) are independent of the choice of the small constant K that defines the separation between hard and soft photons. We shall consider a small addition $\delta\Omega$ to the soft-photon region Ω^s, and seek to show that we get the same answers whether we work with Ω^s or with the enlarged region $\Omega'^s=\Omega^s+\delta\Omega$. There are really two distinct problems here: one concerned with the photons in the asymptotic states and one with the contributions of internal photon lines. We begin with the external lines.

Let us consider a state $|f,\lambda\rangle'$ that, relative to Ω'^s, is a pure soft-photon coherent state. Since the part of $\lambda(\mathbf{k})$ within $\delta\Omega$ contributes only a finite phase factor, there is no real loss of generality in setting $\lambda(\mathbf{k})=0$ in this region. Now, in terms of Ω^s this state $|f,\lambda\rangle'$ is not a pure soft-photon state, but has a small component containing one hard photon. In general, there must, of course, also be components containing two and more hard photons, but their contributions are evidently of second order in $\delta\Omega$ and so we neglect them. Since the norm of the component with one hard photon is of order $(\delta\Omega)^{1/2}$, there is also to first order a change in normalization, so that we may write

$$|f,\lambda\rangle'=|f,\lambda\rangle+\delta|f,\lambda\rangle$$
$$=|f,\lambda\rangle\left(1-\frac{1}{2}\int_{\delta\Omega}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}f_\mu(\mathbf{k})^*f^\mu(\mathbf{k})\right)$$
$$+\int_{\delta\Omega}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}|\mathbf{k}_\mu;f,\lambda\rangle f^\mu(\mathbf{k}), \tag{4.1}$$

in which the subscript μ on $\mathbf{k}$ denotes the polarization index.

Let us consider the matrix element (3.1) with the soft-photon states $|f,\lambda\rangle$ and $|g,\mu\rangle$ replaced by $|f,\lambda\rangle'$ and $|g,\mu\rangle'$. In terms of Ω'^s we see that the contribution to the integrand of (3.9) from the parts of f and g within the small region $\delta\Omega$ is

$$\int_{\delta\Omega}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}[f_\mu(\mathbf{k})^*g^\mu(\mathbf{k})-\tfrac{1}{2}f_\mu(\mathbf{k})^*f^\mu(\mathbf{k})-\tfrac{1}{2}g_\mu(\mathbf{k})^*g^\mu(\mathbf{k})$$
$$+if_\mu(\mathbf{k})^*I^\mu(\mathbf{k})+iI_\mu(\mathbf{k})^*g^\mu(\mathbf{k})]. \tag{4.2}$$

What we have to prove is that the same result is obtained by working in terms of Ω^s.

Now, in terms of Ω^s the difference between the primed and unprimed matrix elements is, according to (4.1),

$$\delta\langle \mathbf{l}_1\cdots\mathbf{l}_s;\, f,\lambda;\, \text{out}\,|\, -\mathbf{l}_{s+1}\cdots -\mathbf{l}_n;\, g,\mu;\, \text{in}\rangle = -\frac{1}{2}\int_{\delta\Omega}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}[f_\mu(\mathbf{k})^* f^\mu(\mathbf{k}) + g_\mu(\mathbf{k})^* g^\mu(\mathbf{k})]$$

$$\times\langle \mathbf{l}_1\cdots\mathbf{l}_s;\, f,\lambda;\, \text{out}\,|\, -\mathbf{l}_{s+1}\cdots -\mathbf{l}_n;\, g,\mu;\, \text{in}\rangle + \int_{\delta\Omega}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}[f^\mu(\mathbf{k})^*\langle \mathbf{l}_1\cdots\mathbf{l}_s\mathbf{k}_\mu;\, f,\lambda;\, \text{out}\,|\, -\mathbf{l}_{s+1}\cdots -\mathbf{l}_n;\, g,\mu;\, \text{in}\rangle$$

$$+\langle \mathbf{l}_1\cdots\mathbf{l}_s;\, f,\lambda;\, \text{out}\,|\, -\mathbf{l}_{s+1}\cdots -\mathbf{l}_n\mathbf{k}_\mu;\, g,\mu;\, \text{in}\rangle g^\mu(\mathbf{k})] + \int_{\delta\Omega}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}\int_{\delta\Omega}\frac{d\mathbf{k}'}{(2\pi)^3 2k^{0\prime}} f^\mu(\mathbf{k})^*$$

$$\times\langle \mathbf{l}_1\cdots\mathbf{l}_s\mathbf{k}_\mu;\, f,\lambda;\, \text{out}\,|\, -\mathbf{l}_{s+1}\cdots -\mathbf{l}_n\mathbf{k}'_\nu;\, g,\mu;\, \text{in}\rangle g^\nu(\mathbf{k}'). \quad (4.3)$$

In the last term the only contribution of first order in $\delta\Omega$ is that from diagrams with a straight-through line joining k to k'. The contribution from such a straight-through line is proportional to

$$\langle \mathbf{k}_\mu | \mathbf{k}'_\nu\rangle = (2\pi)^3 2k^0 \gamma_{\mu\nu}(k)\delta(\mathbf{k}-\mathbf{k}').$$

Hence this term and the quadratic terms in f and g are together precisely equal to the first three terms of (4.2).

Next we must consider the matrix elements in (4.3) that involve a single extra photon in the initial or final state. Clearly, this photon still has small momentum, even though we choose to classify it as hard. Therefore the same approximations that we used in computing the soft-photon contributions in Sec. 3 are still valid. When we insert a photon-emission vertex into the core diagrams, we need only consider insertions in the external lines. The effect of inserting such a vertex in a mass-shell line of charge e and momentum l is given by the mass-shell form of (3.15) of III, that is, it is to multiply it by a factor $el^\mu/l\cdot k$. When we consider all possible insertions in a particular connected piece of the core diagrams, we must sum these factors for all its external lines. Also, of course, the momentum k must appear in the corresponding δ function in (3.4), which means that in (3.9) there must be an extra factor $\exp(-ik\cdot x_\alpha)$ in the integrand. Therefore the net effect of making all possible insertions in this connected piece of the diagram is to introduce an extra factor

$$\sum_{j\in A_\alpha}\frac{e_j l_j{}^\mu}{l_j\cdot k}\exp(-ik\cdot x_\alpha) = iI_\alpha{}^\mu(k).$$

The corresponding expression for a photon-absorption vertex is obtained by changing the sign of k, and is therefore $iI_\alpha{}^\mu(k)^*$. Hence, summing over all connected pieces of the diagrams, we find that the linear terms in f and g in (4.3) are given by (3.9), but with an extra factor in the integrand equal to

$$i\int_{\delta\Omega}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}[f_\mu(\mathbf{k})^* I^\mu(k) + I_\mu(k)^* g^\mu(\mathbf{k})]. \quad (4.4)$$

This agrees precisely with (4.2).

A very similar argument may be used to deal with the internal photon lines, and we merely sketch it briefly. In terms of $\Omega^{s\prime}$ the contribution to the integrand of (3.9) coming from the region $\delta\Omega$ is simply

$$\tfrac{1}{2}i\int'_{\delta\Omega}\frac{dk}{(2\pi)^4}\frac{I^\mu(k)^* I_\mu(k)}{k^2-i\epsilon}. \quad (4.5)$$

On the other hand, in terms of Ω^s this contribution must be a part of the core diagrams. Indeed, it is clear that this is the kind of structure that one would obtain by inserting a single internal photon with momentum restricted to $\delta\Omega$ in all possible ways in the external lines of the core diagrams. Note that this insertion may change the connectivity structure of the core diagrams. Two pieces that are disconnected in terms of the separation defined by $\Omega^{s\prime}$ become in terms of Ω^s parts of a single connected piece when joined by a line with momentum in $\delta\Omega$. However, it is not difficult to verify in detail that the structure (4.5) is indeed what we obtain in this way. The prime on the integral comes from the fact that when both ends of the internal photon line are attached to the same external line of the core diagram, we are required to subtract out the contribution to the mass renormalization constant. In defining the function M^h, we removed a factor of $[Z^h{}_j(p_j)]^{1/2}$ for each external line. Thus we should retain one-half of the self-energy parts on each line, the other half being the contribution to this factor. This is actually what the formula (4.5) yields, because of the explicit factor of $\frac{1}{2}$ outside it. It may be remarked that if we were considering the change not in M^h but in the Green's function itself, we should get the full contribution, because this factor of $\frac{1}{2}$ would be compensated by the factor of 2 in the formula (4.20) of III for X_{ii}.

5. EQUIVALENCE CLASSES

Just as in I, we may separate the *in* or *out* basis states $|\mathbf{l}_1\cdots\mathbf{l}_n;\, f,\lambda\rangle$ into equivalence classes, each of which spans a separable subspace of $\mathcal{H}^h{}_0\otimes\mathcal{H}^s{}_{\text{IR}}$ in such a way that if the initial state belongs to a given equivalence class of *in* states, then the only nonvanishing scattering matrix elements are those to final states in some one definite equivalence class of *out* states. We shall find, however, that in the present case the two decompositions of $\mathcal{H}^h{}_0 \otimes \mathcal{H}^s{}_{\mathrm{IR}}$ are not the same. That is, if two *in* states are equivalent in this sense, it need not follow that the two *out* states with the same labels are also equivalent.

Since scattering matrix elements are zero in any case if the initial and final states have different total charge, we may restrict our considerations to states with some definite given value of the charge (and also of other absolutely conserved discrete quantum numbers such as baryon number).

We must investigate the conditions under which the matrix element (3.9), interpreted according to the rules given, is nonvanishing. Now, any integral involving the quantity $(x)I-I$ will be convergent because of the appearance of the factor $e^{-ik\cdot x}-1$, which vanishes at $k=0$, in the integrand. Hence the matrix element in the integrand of (3.9) will be finite and nonzero for all values of x_α if it is so when each x_α is set equal to zero. (No infrared divergence can arise from the integrations over x_α.)

Let us consider the matrix element

$$\langle \mathbf{l}_1'\cdots\mathbf{l}_s';f,\lambda;\mathrm{out}|\mathbf{l}_1\cdots\mathbf{l}_t;g,\mu;\mathrm{in}\rangle. \quad (5.1)$$

We write the corresponding current I evaluated for $x_\alpha=0$ as the sum of two terms:

$$I^\mu(k)=I^\mu{}_{\mathrm{out}}(k)+I^\mu{}_{\mathrm{in}}(k)\,, \quad (5.2)$$

where

$$I^\mu{}_{\mathrm{out}}(k)=-i\sum_{j=1}^{s}\frac{e_j'l_j'^\mu}{l_j'\cdot k-i\epsilon} \quad (5.3)$$

and

$$I^\mu{}_{\mathrm{in}}(k)=i\sum_{j=1}^{t}\frac{e_jl_j^\mu}{l_j\cdot k+i\epsilon}. \quad (5.4)$$

These currents are, of course, no longer individually conserved, so that when we separate them, we must reinsert the factor $\gamma_{\mu\nu}(k)$ in the integrand.

Now the cross term in the exponent of (3.9) between these two currents is

$$-i\sum_{i=1}^{s}\sum_{j=1}^{t}\int_{\Omega^*}\frac{dk}{(2\pi)^4}\frac{e_i'l_i'^\mu}{l_i'\cdot k+i\epsilon}\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}\frac{e_jl_j^\nu}{l_j\cdot k+i\epsilon}. \quad (5.5)$$

In this integral, both the poles at $l_i'\cdot k=0$ and $l_j\cdot k=0$ lie above the real axis. Hence, closing the contour in the lower half k^0 plane, we obtain only the contribution from the pole at $k^0=|\mathbf{k}|$,

$$\sum_{i=1}^{s}\sum_{j=1}^{t}s_i'^*s_j\,, \quad (5.6)$$

where, as before, s_j denotes the photon wave function

$$s_{j\mu}(k)=\gamma_{\mu\nu}(k)e_jl_j^\nu/l_j\cdot k\,. \quad (5.7)$$

On the other hand, the term quadratic in I_{out} is

$$\tfrac{1}{2}i\sum_{i=1}^{s}\sum_{j=1}^{s}\int_{\Omega^*}'\frac{dk}{(2\pi)^4}\frac{e_i'l_i'^\mu}{l_i'\cdot k+i\epsilon}\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}\frac{e_j'l_j'^\nu}{l_j'\cdot k-i\epsilon}. \quad (5.8)$$

Here, when we close the contour in the lower half k^0 plane, we obtain not only a contribution from the pole at $k^0=|\mathbf{k}|$,

$$-\tfrac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{t}s_i'^*s_j'\,, \quad (5.9)$$

but also a contribution from the pole at $l_j'\cdot k=0$, namely,

$$-\sum_{i<j=1}^{s}\int_{\Omega^*}\frac{dk}{(2\pi)^4}\frac{e_i'l_i'^\mu}{l_i'\cdot k+i\epsilon}\frac{\gamma_{\mu\nu}(k)}{k^2}e_{j'}l_j'^\nu 2\pi\delta(l_j'\cdot k).$$

By the symmetry of the integrand, it is clear that only the δ-function part of the first denominator will contribute, so that we obtain

$$i\sum_{i<j=1}^{s}\sigma_{ij}'\,,$$

where

$$\sigma_{ij}'=\tfrac{1}{2}e_i'e_j'l_i'\cdot l_j'\int_{\Omega^*}\frac{dk}{(2\pi)^4}\frac{2\pi\delta(l_i'\cdot k)2\pi\delta(l_j'\cdot k)}{k^2}. \quad (5.10)$$

Adding all these contributions, we find that the soft-photon function in the integrand of (3.9) is

$$\langle f,\lambda|S'(I)|g,\mu\rangle=\langle f,\lambda|g,\mu\rangle$$
$$\times\exp\Big(\sum_{j=1}^{s}(f^*s_j'-s_j'^*g)-\sum_{j=1}^{t}(f^*s_j-s_j^*g)$$
$$-\tfrac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{s}s_i'^*s_j'+\sum_{i=1}^{s}\sum_{j=1}^{t}s_i'^*s_j-\tfrac{1}{2}\sum_{i=1}^{t}\sum_{j=1}^{t}s_i^*s_j$$
$$+i\sum_{i<j=1}^{s}\sigma_{ij}'+i\sum_{i<j=1}^{t}\sigma_{ij}\Big). \quad (5.11)$$

Using the identity (2.10), with $h_1=\sum s_j'$ and $h_2=-\sum s_j$, we then find that (5.11) may be written in the form

$$\langle f,\lambda|S'(I)|g,\mu\rangle=\langle f,\lambda|S'(I_{\mathrm{out}})S'(I_{\mathrm{in}})|g,\mu\rangle\,, \quad (5.12)$$

where we have defined

$$S'(I_{\mathrm{out}})=U\Big(\sum_{j=1}^{s}s_j'\Big)V\Big(\sum_{i<j=1}^{s}\sigma_{ij}'\Big) \quad (5.13)$$

and

$$S'(I_{\mathrm{in}})=U\Big(-\sum_{j=1}^{t}s_j\Big)V\Big(\sum_{i<j=1}^{t}\sigma_{ij}\Big). \quad (5.14)$$

Note that (5.13) and (5.14) represent an extension of our previous definition of $S'(I)$ in as much as the currents here are not conserved. Correspondingly, the arguments of the operators U are not gauge-invariant. Under a gauge transformation in which the function $l^\mu(k)$ of (2.2) of II changes by an amount $\delta l^\mu(k)$, so that

$$\delta\gamma_{\mu\nu}(k)=-k_\mu\delta l_\nu(k)-k_\nu\delta l_\mu(k)\,,$$

we find that

$$\delta\sum_{j=1}^{s} s_j'^\mu(k)=-Q\delta l^\mu(k)=\delta\sum_{j=1}^{t} s_j^\mu(k)\,, \qquad (5.15)$$

where

$$Q=\sum_{j=1}^{s} e_j'=\sum_{j=1}^{t} e_j. \qquad (5.16)$$

We find then that the condition for the matrix element (5.1) to be nonzero is that (5.12) should be nonzero, or, in terms of the definition of equivalence for coherent states given in I, that

$$S'^{-1}(I_{\rm out})|f,\lambda\rangle\sim S'(I_{\rm in})|g,\mu\rangle \qquad (5.17)$$

(We recall that for two coherent states $|\alpha\rangle\sim|\beta\rangle$ if and only if $\langle\alpha|\beta\rangle\neq 0$.) Hence we may define the equivalence classes of the states in (5.1) to be those of these corresponding soft-photon states. The correspondence is given explicitly by

$$|\mathbf{l}_1'\cdots\mathbf{l}_s';f,\lambda;\text{out}\rangle\to S'^{-1}(I_{\rm out})|f,\lambda\rangle$$
$$=U(-\sum_{j=1}^{s} s_j')V(-\sum_{i<j=1}^{s}\sigma_{ij}')|f,\lambda\rangle \qquad (5.18)$$

or

$$|\mathbf{l}_1\cdots\mathbf{l}_t;g,\mu;\text{in}\rangle\to S'(I_{\rm in})|g,\mu\rangle$$
$$=U(-\sum_{j=1}^{t} s_j)V(\sum_{i<j=1}^{t}\sigma_{ij})|g,\mu\rangle. \qquad (5.19)$$

We may regard the operator U as removing the "soft-photon clouds" associated with the charged particles, and V as removing the Coulomb phase factors associated with pairs of particles. We note in particular that one equivalence class contains all the states of the form

$$|\mathbf{l}_1'\cdots\mathbf{l}_s';\sum_{j=1}^{s} s_j',\ \sum_{i<j=1}^{s}\sigma_{ij}';\text{out}\rangle \qquad (5.20)$$

and

$$|\mathbf{l}_1\cdots\mathbf{l}_t;\sum_{j=1}^{t} s_j,\ -\sum_{i<j=1}^{t}\sigma_{ij};\text{in}\rangle\,, \qquad (5.21)$$

in which each particle is accompanied by its appropriate soft-photon cloud, and each pair of particles by their Coulomb phase factor.

We note that because the Coulomb phases enter with opposite sign for the *in* and *out* states, the decomposition of $\mathcal{H}^h{}_0\otimes\mathcal{H}^s{}_{\rm IR}$ into equivalence classes is different for the two cases.

The definition of equivalence that follows from the correspondences (5.18) and (5.19) is gauge-independent, as long as we stick to states with a given total charge, since under a gauge transformation the states on the right sides in both cases undergo a unitary transformation induced by the operator $U(Q\delta l)$. We may also regard these correspondences as defining the relation of equivalence for states with different total charge, but the relation is not then independent of the choice of gauge. A physical interpretation of this extended definition is provided by the observation that any time-ordered product of field operators has matrix elements only between states that are equivalent in this sense. To see this, let us consider the matrix element of some time-ordered product between the asymptotic states of (5.1). We may recall that no infrared divergence is associated with off-mass-shell lines. Correspondingly, in the general function Δ^s, given by (10.3) of III which, gives the soft-photon contribution to this matrix element, the off-mass-shell lines contribute only a finite factor. Hence the infrared-divergent part depends only on the *in* and *out* states, but not on the operators that appear between them, and is again given by (5.11). Thus the condition for such a matrix element to be nonvanishing is precisely (5.17). In view of this interpretation, it is natural that the definition of equivalence should in general be gauge-dependent, since the operators in the time-ordered product are so. We note in particular that the equivalence class containing the states (5.20) and (5.21) consists of those states that can be produced from the vacuum by the action of a finite number of field operators.

We may also note that this interpretation is consistent with the general formula (10.1) of III, which gives the matrix element of a field operator ϕ between *out* states containing n and $n+1$ particles:

$$\langle\mathbf{l}_1\cdots\mathbf{l}_n;\alpha;\text{out}|\phi_{n+1}(x)|\mathbf{l}_1'\cdots\mathbf{l}_n'\mathbf{l};\beta;\text{out}\rangle$$
$$=[Z^h{}_{n+1}(l)]^{1/2}u(\mathbf{l})\langle\alpha|U[-(x)s_l]|\beta\rangle e^{il\cdot x}$$
$$\times\prod_{j=1}^{n}\{N_j^{-1}[\bar{u}(\mathbf{l}_j)u(\mathbf{l}_j')]\Delta^s(\mathbf{l}_j';\mathbf{l}_j|I_{(n+1)})^*\}. \qquad (5.22)$$

As far as its infrared divergence is concerned, $U[-(x)s_l]$ is equivalent to $U[-s_l]$. Moreover, the infrared-divergent part of the function $\Delta^s(l_j';l_j|I_{(n+1)})$ may be obtained from the formula for it [(8.10) of III] by setting $y=0$ in the integrand, and is

$$\exp\left(\int_{\Omega^s}\frac{dk}{(2\pi)^4}e_jl_j{}^\mu 2\pi\delta(l_j\cdot k)\frac{\gamma_{\mu\nu}(k)}{k^2}\frac{e_{n+1}l^\nu}{l\cdot k+i\epsilon}\right)$$
$$=\exp(-i\sigma_{j,n+1}).$$

Thus (5.22) is nonvanishing if and only if the soft-photon matrix element

$$\langle\alpha|U(-s_{n+1})V(-\sum_{j=1}^{n}\sigma_{j,n+1})|\beta\rangle \qquad (5.23)$$

is nonzero. But, according to the rule of correspondence

(5.18), this is also precisely the condition for equivalence of the states $|l_1\cdots l_n;\alpha;\text{out}\rangle$ and $|l_1\cdots l_n l_{n+1};\beta;\text{out}\rangle$.

6. MOMENTUM SPECTRUM OF SOFT-PHOTON COHERENT STATES

It will be convenient at this point to investigate certain properties of the special class of coherent states that, as we have seen, plays an important role.

Let us consider the state

$$|s,0\rangle=|s_1+\cdots+s_n,0\rangle\,, \tag{6.1}$$

where s_j is the wave function defined in (5.7). We note that this state is explicitly gauge-dependent *except* in the special case where the total charge is zero, $\sum e_j=0$. [In that case, a change in the gauge function $\gamma_{\mu\nu}(k)$ changes s^μ only by terms proportional to k^μ.]

The function (5.7) appears in all treatments of the infrared-divergence problem, and has a simple physical interpretation. To be specific, let us choose the radiation gauge, and consider the process of accelerating a particle from rest to momentum p^μ. The hard-photon emission during this process depends on the details of the acceleration mechanism, but the soft-photon emission does not, except in one specific respect to be noted below. It is essentially that produced by the classical current

$$J^\mu(x)=\int_{-\infty}^{0}d\tau\; en^\mu\delta(x-n\tau)+\int_{0}^{\infty}d\tau\; ev^\mu\delta(x-v\tau)\,, \tag{6.2}$$

where $v^\mu=p^\mu/m$ and, as usual, $n^\mu=(1,0)$. This represents a point particle accelerated at $x=0$ from rest to velocity v^μ. According to the results of I, the photons emitted by this classical current are in a coherent state whose wave function is given by i times the mass-shell value of the Fourier transform of (6.2), namely,

$$e(p^\mu/p\cdot k-n^\mu/n\cdot k)\,,$$

which in the radiation gauge yields (5.7).

The reason for the explicit gauge dependence is now clear. The state $|s,0\rangle$ represents the photons emitted in accelerating the particles to momenta p_j^μ from a particular canonical state, in this case the state of rest, and the choice of the canonical state depends on the choice of gauge. We may regard the photons in the state $|s_j,0\rangle$ as belonging to a "soft-photon cloud" associated with the particle of momentum p_j^μ, but the separation of soft photons into ones belonging to these clouds and the remainder is to some extent arbitrary, and gauge-dependent.

Note the special role played by the origin in (6.2) as the point at which the acceleration occurs. We could equally well have chosen any other point x, and would then have obtained in place of s a translated wave function $(x)s$. To this extent only, the soft-photon state produced by the acceleration does depend on the way in which the acceleration occurs.

We now wish to investigate the momentum spectrum of the state (6.1).

In general, the momentum spectrum of a coherent state $|f,\lambda\rangle$ is given by the distribution

$$\rho(k)=\int dx\; e^{-ik\cdot x}\langle f,\lambda|e^{iP\cdot x}|f,\lambda\rangle$$
$$=\int dx\; e^{-ik\cdot x}\langle (x)f,\lambda|f,\lambda\rangle. \tag{6.3}$$

Using the scalar-product formula given in Sec. 2, we may write $\rho(k)$ in the form

$$\rho(k)=\int dx\; e^{-ik\cdot x}e^{L(x)}\,, \tag{6.4}$$

with

$$L(x)=(x)f^*f-\tfrac{1}{2}(x)f^*(x)f-\tfrac{1}{2}f^*f$$
$$=\int\frac{d\mathbf{k}}{(2\pi)^3 2k^0}f_\mu{}^*(\mathbf{k})f^\mu(\mathbf{k})(e^{ik\cdot x}-1). \tag{6.5}$$

Note that, because the last factor vanishes at $\mathbf{k}=0$, this integral is always convergent. It is clear that $\rho(k)$ has its support in the forward light cone, $k^2\leqslant 0$, $k^0\geqslant 0$, is non-negative, and satisfies the normalization condition

$$\int\frac{dk}{(2\pi)^4}\rho(k)=1. \tag{6.6}$$

The coherent states may be decomposed into states with definite energy and momentum, according to the relations

$$|f,\lambda[k]\rangle=\int dx\; e^{ik\cdot x}|(x)f,\lambda\rangle \tag{6.7}$$

and

$$|(x)f,\lambda\rangle=\int\frac{dk}{(2\pi)^4}e^{-ik\cdot x}|f,\lambda[k]\rangle. \tag{6.8}$$

These states are normalized according to the relation

$$\langle f,\lambda[k]|f,\lambda[k']\rangle=(2\pi)^4\delta(k-k')\rho(k). \tag{6.9}$$

Now, substituting s for f in (6.5) we find that the exponent $L(x)$ has the form

$$L(x)=\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}(x)\,, \tag{6.10}$$

where

$$L_{ij}(x)=\epsilon_i e_j\int\frac{d\mathbf{k}}{(2\pi)^3 2k^0}\frac{p_i{}^\mu\gamma_{\mu\nu}(k)p_j{}^\nu}{(p_i\cdot k)(p_j\cdot k)}(e^{ik\cdot x}-1). \tag{6.11}$$

To find the nature of the spectrum near $k=0$, we must determine the asymptotic behavior of the function $L(x)$ for large values of x. But $L_{ij}(x)$ is precisely the function (2.9) of II whose asymptotic behavior we studied in

II. We found it to be given by

$$L_{ij}(x)\approx\tfrac{1}{2}\xi_{ij}\ln(K^2x^2)+f_{ij}(\mathbf{x}/x^0)\,, \qquad (6.12)$$

where f_{ij}, as indicated, is a function only of the direction of x, not its magnitude. In the radiation gauge, the parameter ξ_{ij} is a function of the relative velocity

$$u_{ij}=[1-m_i^2m_j^2/(p_j\cdot p_j)^2]^{1/2}$$

and the velocities $u_i=|\mathbf{p}_i/p_i^0|$, given by

$$\xi_{ij}^R=(e_ie_j/4\pi^2)[\Phi(u_{ij})-\Phi(u_i)-\Phi(u_j)+1]\,,$$

where $\Phi(u)$ is the function

$$\Phi(u)=\frac{1}{2u}\ln\frac{1+u}{1-u}.$$

Thus the asymptotic behavior of the function $L(x)$ is described by

$$L(x)\approx\tfrac{1}{2}\xi\ln(K^2x^2)+f(\mathbf{x}/x^0)\,, \qquad (6.13)$$

where

$$\xi=\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}. \qquad (6.14)$$

This result is sufficient to determine the nature of the singularity in the momentum spectrum $\rho(k)$ at $k=0$. A straightforward calculation shows that as $k\to0$ from any direction within the forward light cone,

$$\rho(k)\approx(-k^2)^{-2}(-k^2/K^2)^{-\xi/2}g(\mathbf{k}/k^0)\,, \qquad (6.15)$$

where again g is a function only of the direction of k.

We may note that in the radiation gauge (or any physical gauge) the parameter ξ is never positive. This may be seen most directly from (6.10) and (6.11), by noting that the non-negativity of $\gamma_{\mu\nu}(k)$ implies that $\mathrm{Re}L(x)\leqslant0$. This inequality must remain valid in the asymptotic region, and thus requires $\xi\leqslant0$. In this case $\rho(k)$ has a singularity at $k=0$ which is integrable in the usual sense. If we define a function $\rho(k)$ in the Lorentz gauge by (6.4), (6.10), and (6.11), with $\gamma_{\mu\nu}(k)=g_{\mu\nu}$, then we find that although this $\rho(k)$ is well defined as a distribution, it is no longer an integrable function. Nor, of course, is it any longer non-negative, or directly related to physical quantities. (It might be given a physical interpretation by defining coherent states in a space of indefinite metric, but we shall not attempt to do so here.)

An explicit formula can be found for the energy spectrum. Using the method introduced in a closely related context by Lomon and Shaw,[6] it is easy to show that the probability $P(E)$ that the energy is less than E is given, for all $E\leqslant K$, by

$$P(E)=\int\frac{dk}{(2\pi)^4}\theta(E-k^0)\rho(k)=\frac{e^{\gamma\xi}}{(-\xi)!}\left(\frac{E}{K}\right)^{-\xi}, \qquad (6.16)$$

where γ is Euler's constant and we use the notation $(-\xi)!=\Gamma(1-\xi)$. In a physical gauge, where $\xi\leqslant0$, this probability rises rapidly from zero to a value close to unity.

7. RELATIONSHIP TO PHYSICAL MEASUREMENTS

We now turn to the important question of extracting the values of physically observable quantities such as cross sections from the scattering matrix elements that we have obtained.

For simplicity, we shall suppose that we are interested in a region of momenta in which only completely connected core diagrams can contribute significantly, so that we may set $N=1$ in (3.9).

Let us introduce the total momenta

$$l^\mu=\sum_{j=1}^{t}l_j^\mu,\quad l'^\mu=\sum_{j=1}^{s}l_j'^\mu \qquad (7.1)$$

and write the integration over the final-state phase space in the form

$$\prod_{j=1}^{s}\frac{d\mathbf{l}_j'}{(2\pi)^3 2l_j'^0}=\frac{dl'}{(2\pi)^4}d\Omega'\,, \qquad (7.2)$$

where $d\Omega'$ represents the integration over the $3s-4$ variables other than the total momentum. (We assume that $s\geqslant2$, since the cases $s=0$, 1 are trivial.) Since $l'-l$ is necessarily small, we may write the contribution M^h of the core diagrams in a form independent of l', as a function only of Ω' and the initial-state variables.

If we were to remove all the soft-photon contributions, we should obtain a scattering matrix element of the form

$$\langle\mathbf{l}_1'\cdots\mathbf{l}_s';\text{out}|\mathbf{l}_1\cdots\mathbf{l}_t;\text{in}\rangle^h =(2\pi)^4\delta(l'-l)\langle\Omega'|S^h(l)|\Omega\rangle, \qquad (7.3)$$

where

$$\langle\Omega'|S^h(l)|\Omega\rangle=\prod_{j=1}^{s}\bar{u}(\mathbf{l}_j')\prod_{j=1}^{t}\bar{u}^c(\mathbf{l}_j)M^h(\{l_j',l_j\}). \qquad (7.4)$$

The corresponding transition probability per unit space-time volume into a set of final states defined by a region Ω' would then be

$$w(\Omega')=\int_{\Omega'}d\Omega'|\langle\Omega'|S^h(l)|\Omega\rangle|^2. \qquad (7.5)$$

In the actual theory, with soft-photon contributions, (7.3) is replaced by (3.9), which may be written in the form

$$\langle\mathbf{l}_1'\cdots\mathbf{l}_s';f,\lambda;\text{out}|\mathbf{l}_1\cdots\mathbf{l}_t;g,\mu;\text{in}\rangle$$

$$=\langle\Omega'|S^h(l)|\Omega\rangle\int dx\,e^{-i(l'-l)\cdot x}\langle f,\lambda|S'[(x)I]|g,\mu\rangle, \qquad (7.6)$$

where we have chosen to exhibit the dependence of the current on x explicitly, using the notation (2.11), and I is given by (5.2).

Now let us ask for the transition probability to a set of final states specified by given ranges L' and Ω' of the total momentum l' and the remaining final-state variables, regardless of what final soft-photon state is produced. This is obtained by taking the squared modulus of (7.6) and summing over a complete set of soft-photon states:

$$\int_{\Omega'} d\Omega' |\langle\Omega'|S^h(l)|\Omega\rangle|^2 \int_{L'} \frac{dl'}{(2\pi)^4} \times\sum_\alpha \left|\int dx\, e^{-i(l'-l)\cdot x}\langle\alpha|S'[(x)I]|g,\mu\rangle\right|^2$$
$$=\int_{\Omega'} d\Omega' |\langle\Omega'|S^h(l)|\Omega\rangle|^2 \int_{L'} \frac{dl'}{(2\pi)^4} \times\int dx'\, dx\, e^{i(l'-l)\cdot(x'-x)}\langle g,\mu|S'^{-1}[(x')I] \times S'[(x)I]|g,\mu\rangle. \quad (7.7)$$

It is clear that significant contributions can come only from the region where $l'-l$ is small. Let us suppose first that no further restriction is placed on l' or, physically, that we measure only the variables Ω' and not the total momentum l' of the final-state particles. Then the integral over l' has the effect of setting $x'=x$, so that the two operators S' cancel. The remaining integral over x may be interpreted as usual as the space-time volume of the interaction region. Hence we obtain a scattering probability per unit space-time volume

$$w(\Omega')=\int_{\Omega'} d\Omega' |\langle\Omega'|S^h(l)|\Omega\rangle|^2, \quad (7.8)$$

which is exactly equal to (7.5). Thus we see that, independent of the initial soft-photon state, the probability of transition to all final states in a given range Ω' is equal to the corresponding expression obtained by dropping all soft-photon contributions, provided that we do not attempt to measure the total momentum of the final-state particles.

Now let us consider what happens if we do impose a restriction on the final-state total momentum l'. Clearly, in that case we can no longer expect in general to obtain a result independent of the initial soft-photon state. For example, for some special soft-photon states there might be a net transfer of energy from the soft photons to the other particles, but this is clearly a property not shared by all soft-photon states. We shall come back to this general case in a moment, but first let us assume that the initial soft-photon state is the vacuum, and ask for the transition probability if the momentum transfer $k=l-l'$ is limited to a small region Δ. For this case the soft-photon matrix element in (7.7) is a function only of $x'-x=y$, so that there is still an integral over x that yields a space-time-volume factor. The transition probability per unit space-time volume is now

$$w(\Omega',\Delta)=w(\Omega')\int_\Delta \frac{dk}{(2\pi)^4}\rho(k), \quad (7.9)$$

where

$$\rho(k)=\int dy\, e^{-ik\cdot y}\langle 0|S'^{-1}[(y)I]S'[I]|0\rangle. \quad (7.10)$$

This is the momentum spectrum, as given by (6.3), of the soft-photon coherent state $S'[I]|0\rangle$, whose photon wave function is

$$s=\sum_{j=1}^{s} s_j'-\sum_{j=1}^{t} s_j. \quad (7.11)$$

This is, of course, the function whose structure is described by (6.15). We note that the parameter ξ of (6.14) is now necessarily gauge-invariant, because of the gauge invariance of s that follows from charge conservation. This parameter ξ is therefore always negative.

In particular, if the region Δ is defined by a limit E on the total energy transfer to the soft photons, then, according to (6.16), we obtain (for $E<K$)

$$w(\Omega',E)=w(\Omega')[e^{\gamma\xi}/(-\xi)!](E/K)^{-\xi}, \quad (7.12)$$

a result that has been obtained by many other authors.[3–7] (It is valid, of course, only when $E<K\ll m$, and must be modified if $E\approx m$.)

Now let us consider the general case in which the initial soft-photon state is not the vacuum. Then the soft-photon matrix element in (7.7) is no longer a function only of $x'-x$, and so we do not obtain a factor of the space-time volume. This is hardly surprising, for clearly we should no longer expect to get a transition probability per unit space-time volume independent of position. The transition probability may be expected to depend on whether the field $G^\mu(x)$ associated with the soft-photon wave function $g^\mu(\mathbf{k})$ is large in the interaction region or not.

To handle this problem, therefore, we must adopt a slightly more sophisticated approach. For simplicity, let us suppose that the initial state contains two spinless particles described by wave functions ψ_1 and ψ_2:

$$|\psi_1\psi_2;\, g,\mu;\, \text{in}\rangle=\int \frac{d\mathbf{l}_1}{(2\pi)^3 2l_1^0}\frac{d\mathbf{l}_2}{(2\pi)^3 2l_2^0} \times|\mathbf{l}_1\mathbf{l}_2;\, g,\mu;\, \text{in}\rangle\psi_1(\mathbf{l}_1)\psi_2(\mathbf{l}_2). \quad (7.13)$$

These functions are normalized according to

$$\int \frac{d\mathbf{l}}{(2\pi)^3 2l^0}|\psi_j(\mathbf{l})|^2=1.$$

We shall assume that the momentum spread in each of the functions ψ_1 and ψ_2 is small compared to the size of the region L' to which we are going to limit the final momentum.

Now we may introduce the corresponding x-space functions

$$\psi_j(x)=\int\frac{d\mathbf{l}}{(2\pi)^3 2l^0}e^{il\cdot x}\psi_j(\mathbf{l}) \qquad (7.14)$$

and regard $|\psi_j(x)|^2$ as providing an invariant measure of the particle density, equal to the density divided by twice the mean energy $n_j/2l_j^0$. The integral

$$A=\int dx\, |\psi_1(x)|^2|\psi_2(x)|^2 \qquad (7.15)$$

then measures the space-time volume V of the region in which the two beams of particles intersect and therefore in which the interaction can take place. We may write $A=Vn_1n_2/4l_1^0l_2^0$. Thus the cross section $\sigma(\Omega')$ is determined by dividing the transition probability by AF, where F is the invariant flux factor

$$F=4l_1^0l_2^0v_{12}=4[(l_1\cdot l_2)^2-m_1^2m_2^2]^{1/2}. \qquad (7.16)$$

The expression (7.7) for the transition probability is now replaced by

$$\int_{\Omega'}d\Omega'\int_{L'}\frac{dl'}{(2\pi)^4}\sum_\alpha\left|\int dx\int\frac{d\mathbf{l}_1}{(2\pi)^3 2l_1^0}\frac{d\mathbf{l}_2}{(2\pi)^3 2l_2^0}\times\langle\Omega'|S^h(l)|\Omega\rangle\psi_1(\mathbf{l}_1)\psi_2(\mathbf{l}_2)e^{-i(l'-l)\cdot x}\langle\alpha|S'[(x)I]|g,\mu\rangle\right|^2.$$

We may again remove the non-soft-photon matrix element from within the integral, since it is slowly varying in the region of interest, and perform the sum over α, obtaining as before

$$\langle g,\mu|S'^{-1}[(x')I]S'[(x)I]|g,\mu\rangle.$$

We then make the substitution $x'=x+y$. The exponential factors become

$$e^{i(l'-l'')\cdot y-i(l''-l)\cdot x}.$$

In the factor depending on y, the change in l'' is small compared to that of l' by assumption, so that we may replace it by its mean value $l_{(0)}$. Going over again to the variable $k=l_{(0)}-l'$, we thus obtain

$$\int_{\Omega'}d\Omega'\,|\langle\Omega|S^h(l_{(0)})|\Omega_{(0)}\rangle|^2\int_\Delta\frac{dk}{(2\pi)^4}\int dy\, e^{-ik\cdot y}\times\int dx\,|\psi_1(x)|^2|\psi_2(x)|^2\langle g_1\mu|S'^{-1}[(x+y)I]\times S'[(x)I]|g,\mu\rangle.$$

Since $|\psi_1(x)|^2|\psi_2(x)|^2$ is slowly varying, the x integral yields essentially A, as given by (7.15), multiplied by the mean value over the interaction volume of the soft-photon matrix element. Thus the cross section $\sigma(\Omega',\Delta)$ is related to $\sigma(\Omega')$ by

$$\sigma(\Omega',\Delta)=\sigma(\Omega')\int_\Delta\frac{dk}{(2\pi)^4}\rho_g(k), \qquad (7.17)$$

where now

$$\rho_g(k)=\int dy\, e^{-ik\cdot y}\frac{1}{V}\int_V dx\,\langle(-x)g,\mu|S'^{-1}[(y)I]\times S'[I]|(-x)g,\mu\rangle \qquad (7.18)$$

or, equivalently,

$$\rho_g(k)=\int dy\, e^{-ik\cdot y}\exp[(y)s^*s-s^*s]\frac{1}{V}\int_V dx\times\exp\{(-x)g^*[(y)s-s]-[(y)s-s]^*(-x)g\}. \qquad (7.19)$$

We note that this function is again real, and normalized in the sense that

$$\int\frac{dk}{(2\pi)^4}\rho_g(k)=1. \qquad (7.20)$$

Equation (7.17) describes the effect in general of the initial soft-photon state. We may note certain special cases. First, it is clear that if we remove the restriction to a finite region Δ, then by (7.20) we obtain simply $\sigma(\Omega')$, and that, on the other hand, if we set $g=0$, we recover (7.9). Thus this formula includes the earlier ones as special cases. Next, we note that it is easy to see that a soft-photon state for which the corresponding field $G^\mu(x)$ is concentrated far from the interaction region will not contribute significantly. For suppose, for example, that $g^\mu(\mathbf{k})$ is real, which means, in general, that $G^\mu(x)$ will have its maximum intensity in the vicinity of the origin. Then if the interaction region is far from the origin, we shall have $Kx\gg1$ within this region. In that case the exponent will be the integral of a rapidly oscillating function of k, and will therefore be very small, so that we again recover (7.9).

At the opposite extreme we may suppose that $g^\mu(\mathbf{k})$ contains such soft components that $G^\mu(x)$ is effectively constant in the interaction region, so that we may replace the x integral in (7.19) by the value of its integrand at $x=0$. We note that the exponent is purely imaginary, so that its main effect is to translate the whole distribution $\rho(k)$ in k space. In fact, when g is confined to very soft components, we may approximate $e^{-ik\cdot y}-1$ by $-ik\cdot y$ and thus obtain

$$\rho_g(k)=\rho(k+\delta k), \qquad (7.21)$$

where

$$\delta k^\mu=\int\frac{d\mathbf{k}'}{(2\pi)^3 2k'^0}k'^\mu 2\,\mathrm{Re}[g_\mu(\mathbf{k}')^*s^\mu(\mathbf{k}')]. \qquad (7.22)$$

It is evident that the sign of δk^μ depends on the phase of g. It is interesting to note that the spread in the total momentum of the final-state particles is not much affected by the presence of the soft-photon state. The main effect is to introduce an additional constant momentum transfer described by (7.22). This result is not perhaps of great practical importance, but it does serve to illustrate the fact that there are questions that can be answered using the methods developed here but not by conventional methods. The same results could, of course, be obtained by treating the initial-state soft photons as a classical external field, which is effectively equivalent to a coherent state, but within the conventional formalism there seems to be little justification for such a treatment, since this is not the way in which the final-state soft photons are handled.

8. UNITARITY

We still must prove that the scattering operator whose matrix elements are defined by (3.9) is unitary, as, of course, it must be if the asymptotic states that we have found constitute a complete set.

What we must prove is that

$$\sum_{s=0}^{\infty}\int\frac{d\mathbf{l}_1'}{(2\pi)^3 2l_1'^0}\cdots\frac{d\mathbf{l}_s'}{(2\pi)^3 2l_s'^0}$$
$$\times\sum_{\alpha}\langle\mathbf{l}_1'\cdots\mathbf{l}_s';\alpha;\mathrm{out}|\mathbf{l}_1''\cdots\mathbf{l}_{t''}'';g'',\mu'';\mathrm{in}\rangle^*$$
$$\times\langle\mathbf{l}_1'\cdots\mathbf{l}_s';\alpha;\mathrm{out}|\mathbf{l}_1\cdots\mathbf{l}_t;g,\mu;\mathrm{in}\rangle$$
$$=\langle\mathbf{l}_1''\cdots\mathbf{l}_{t''}''|\mathbf{l}_1\cdots\mathbf{l}_t\rangle\langle g'',\mu''|g,\mu\rangle,\quad(8.1)$$

where, as usual, $|\alpha\rangle$ denotes a complete set of soft-photon states. Throughout this paper we have assumed that the non-soft-photon contributions have the correct properties, and we shall do so here too. Specifically, we assume that the scattering operator with all soft-photon contributions removed is a unitary operator on $\mathcal{H}^h{}_0$, so that its matrix elements satisfy the corresponding unitarity relation, obtained by discarding all the soft-photon parts in (8.1).

It will be helpful to begin by recalling certain features of the unitarity relation for the theory without soft photons. We may decompose the scattering matrix elements into their connected pieces, and represent the contribution to the unitarity integral from diagrams with a particular connectivity pictorially as in Fig. 1. Here the lines on the right are the $\mathbf{l}_j$ lines and those on the left are the $\mathbf{l}_j''$ lines, while the lines in the middle represent final-state particles, with momenta $\mathbf{l}_j'$. The circles on the right denote connected pieces of the scattering amplitude and those on the left denote connected pieces of its complex conjugate.

The decomposition of the matrix element on the right corresponds to a partitioning of the labels $(1\cdots t, 1'\cdots s')$ into sets A_α, while that of the matrix element on

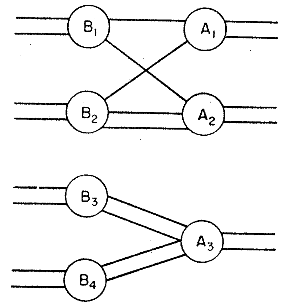

FIG. 1. Example of a contribution to the unitarity integral.

the left corresponds similarly to a partitioning of $(1''\cdots t'',1'\cdots s')$ into sets B_β. Regarding the diagram as a whole, we may introduce a coarser partitioning of these labels into sets C_γ that correspond to the connected pieces of the entire diagram. For example, in Fig. 1 there are three sets A_α and four sets B_β, but only two sets C_γ. The unitarity integral may be expressed as a sum over all classes of diagrams corresponding to the different ways of assigning the labels $(1\cdots t)$ and $(1''\cdots t'')$ to such sets C_γ. The right-hand side of the unitarity integral is equal to the contribution from just one such class, namely, the one in which the diagrams consist entirely of straight-through lines. (Actually, if the particles are identical, there may be several of these.) Thus the sum of all other contributions must vanish. Indeed, it is easy to prove by induction on the number of initial-state particles that the contribution from each class must vanish individually. For example, in the three-particle unitarity equation the contribution from classes with two sets C_γ vanishes in virtue of the two-particle unitarity equation. Hence also the contribution from the connected class with only one set C_γ must vanish. The contribution to the unitarity integral from diagrams of a given class is a product of factors, one corresponding to each C_γ, and each of these factors is zero, unless the corresponding C_γ contains only a single label of each type, as is the case for straight-through lines.

In the case of the theory with soft photons, it is convenient to make the same decomposition of the unitarity integral according to the connectivity structure of the core diagrams that contribute. We shall again try to show that the contribution from each class vanishes individually except for the single class consisting entirely of straight-through lines. However, it is no longer true that its contribution to the unitarity integral can be factorized into factors associated with the individual sets C_γ, since they are connected to each other by soft-photon parts.

Let us consider first the class of diagrams for which there is only one set C_γ. To obtain its contribution we must sum over all possible ways of assigning the labels to sets A_α or B_β, that is, over all connectivity classes of the individual diagrams that preserve the over-all connectedness. Consider the contribution from one particular assignment into sets $A_1\cdots A_p$ and $B_1\cdots B_q$. Each set A_α or B_β corresponds to a particular connected piece of the individual diagrams. We define the corresponding momentum transfer

$$k_\alpha = l_\alpha - l_\alpha' = \sum_{j\in A_\alpha} l_j - \sum_{j'\in A_\alpha} l_j' \tag{8.2}$$

or

$$k_\beta'' = l_\beta'' - l_\beta' = \sum_{j''\in B_\beta} l_j'' - \sum_{j'\in B_\beta} l_j'. \tag{8.3}$$

Clearly, the diagrams in question can only contribute in a restricted region of the final-state phase space in which all of these variables k_α, k_β'' are small. Indeed, in the theory without soft photons the matrix elements contain δ-function factors $\delta(k_\alpha)$ and $\delta(k_\beta'')$.

Now there is precisely one linear relation between the variables k_α and k_β'', namely,

$$\sum_{\alpha=1}^{p} k_\alpha - \sum_{\beta=1}^{q} k_\beta'' = l - l''. \tag{8.4}$$

Thus we can make a transformation of variables in the final-state integration in which these $p+q-1$ independent four-vector variables appear explicitly, leaving an integration $d\Omega'_{\{A,B\}}$ of dimensionality $3s-4(p+q-1)$. Straight-through lines constitute an exceptional case, since the variable k_α corresponding to a set A_α containing only one final-state label j' is restricted to lie on its mass shell. However, we can bring this case within the same framework by writing the corresponding integral as a four-dimensional integral $(2\pi)^{-4}dk_\alpha$ with a δ-function factor $2\pi\delta(m_j'^2+[k_\alpha-l_j]^2)$ in the integrand. This means that we no longer have to remove this factor according to the rule (3.11). It also means that we should add to the dimensionality $3s-4(p+q-1)$ the number of straight-through lines in either matrix element. (Of course, by assumption no lines pass straight through both parts of the diagram.) When this adjustment has been made, the dimensionality of $d\Omega'_{\{A,B\}}$ may be assumed to be non-negative, for otherwise we should be trying to impose more conditions on the final-state momenta than there are independent variables, so that we should not, in general, be able to satisfy all the conditions anywhere in the phase space.

Because the variables k_α and k_β'' are limited to very small ranges, we may write the non-soft-photon parts of the matrix element, and the current I, in a form independent of these variables, in which they are functions only of the remaining variables $\Omega'_{\{A,B\}}$ and of initial-state momenta. Then the only dependence on the $p+q-1$ independent variables among the k_α and k_β'' is in the exponential factors

$$\exp\left(-i\sum_{\beta=1}^{q} k_\beta''\cdot x_\beta'' + i\sum_{\alpha=1}^{p} k_\alpha\cdot x_\alpha\right).$$

Hence the effect of integrating over these variables is to introduce $p+q-1$ δ functions, which set all the $p+q$ variables x_α, x_β'' equal to a single variable x. The contribution to the unitarity integral is thus of the form

$$\int d\Omega'_{\{A,B\}} \prod_{\beta=1}^{q} \langle\Omega'_\beta|S^h(l_\beta'')|\Omega''_\beta\rangle^* \prod_{\alpha=1}^{p} \langle\Omega'_\alpha|S^h(l_\alpha)|\Omega_\alpha\rangle$$
$$\times\int dx\, e^{-i(l''-l)\cdot x}\langle g'',\mu''|S'^{-1}(I'')S'(I)|g,\mu\rangle, \tag{8.5}$$

where the variables denoted by Ω'_β or Ω'_α constitute a subset of the variables $\Omega'_{\{A,B\}}$, and $S^h(l)$ denotes the connected part of the scattering amplitude for total energy momentum l with all soft-photon contributions removed. In the case of a straight-through line, it must be replaced simply by a constant matrix $N_i^{-1}C_{ij}$, according to (3.11). Displaying the x dependence of the current explicitly, we may write

$$\begin{aligned} I &= (x)I'_{\rm out} + (x)I_{\rm in}\,,\\ I'' &= (x)I'_{\rm out} + (x)I''_{\rm in}\,, \end{aligned} \tag{8.6}$$

as in (5.2), where the primes indicate the momenta on which these currents depend, and $I_{\rm out}$ and $I_{\rm in}$ are given by (5.3) and (5.4).

It is at this point that our assumption that there is only a single set C_γ introduces an essential simplification. For when there is only one variable x, the decomposition (5.12) remains valid, so that we may write the soft-photon matrix element in (8.5) as

$$\begin{aligned} &\langle g'',\mu''|S'^{-1}[(x)I''_{\rm in}]S'^{-1}[(x)I'_{\rm out}]\\ &\qquad\times S'[(x)I'_{\rm out}]S'[(x)I_{\rm in}]|g,\mu\rangle\\ &\quad= \langle g'',\mu''|S'^{-1}[(x)I''_{\rm in}]S'[(x)I_{\rm in}]|g,\mu\rangle. \end{aligned} \tag{8.7}$$

This matrix element no longer contains any dependence on the final-state variables $\Omega'_{\{A,B\}}$. But it is clear from the structure of the x integral in (8.5) that this expression is essentially zero unless $l''-l$ is small. Hence the integration over $\Omega'_{\{A,B\}}$ is just a unitarity integral for a theory without soft photons, and so when we perform the integration and sum over all ways of assigning the labels to sets A_α and B_β, the result that we obtain must be zero.

We have proved, therefore, that the contribution to the unitarity integral from the class of diagrams corresponding to a single set C_γ that cannot be separated into disconnected pieces vanishes. It remains to prove the same thing for other classes of diagrams.

If we go through the same analysis as before, we find that in the general case all the variables x_α and x_β'' corresponding to sets A_α and B_β in a single C_γ are set equal by the corresponding k_α or k_β'' integrations. Thus in place of (8.5) we obtain an expression in which an integration dx_γ appears for each set C_γ. In place of

(8.6) we have

$$I=\sum_\gamma[(x_\gamma)I'_{\gamma,\text{out}}+(x_\gamma)I_{\gamma,\text{out}}],$$
$$I''=\sum_\gamma[(x_\gamma)I'_{\gamma,\text{out}}+(x_\gamma)I''_{\gamma,\text{in}}]. \tag{8.8}$$

However, (5.12) is not true in general, so that we cannot pass immediately to (8.7). The reason for the difficulty may be seen by examining (5.5). If we consider the cross term between $(x_{\gamma'})I'_{\gamma',\text{out}}$ and $(x_\gamma)I_{\gamma,\text{in}}$, we find that there is an extra factor $\exp[ik\cdot(x_{\gamma'}-x_\gamma)]$ in the integrand. Hence we can complete the contour in the lower half-plane and obtain the analog of (5.6) only if $x_{\gamma'}{}^0 \geqslant x_\gamma{}^0$. This suggests that to overcome the difficulty we should consider the various sets C_γ in sequence, beginning with the one associated with the largest value of $x_\gamma{}^0$. However, we also must consider the cross term between $(x_{\gamma'})I'_{\gamma',\text{out}}$ and $(x_\gamma)I'_{\gamma,\text{out}}$ that is of the form (5.8). Explicitly it is

$$i\sum_{i'\in C_{\gamma'}}\sum_{'\in C_\gamma}\int_{\Omega^*}\frac{dk}{(2\pi)^4}\frac{e_i'l_i'^\mu}{l_i'\cdot k+i\epsilon}\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}\frac{e_j'l_j'^\nu}{l_j'\cdot k-i\epsilon}\times\exp[ik\cdot(x_{\gamma'}-x_\gamma)].$$

For $x_{\gamma'}{}^0\geqslant x_\gamma{}^0$, we may complete the contour in the lower half-plane and obtain as before two contributions. The one from the pole at $k^0=|\mathbf{k}|$ is just what we want, namely,

$$\sum_{i'\in C_{\gamma'}}\sum_{j'\in C_\gamma}(x_{\gamma'})s_i'^*(x_\gamma)s_j=-(x_{\gamma'})I'_{\gamma',\text{out}}{}^*\times(x_\gamma)I'_{\gamma,\text{out}}. \tag{8.9}$$

However, we also have the contribution from the pole at $l_j'\cdot k=0$, namely,

$$-\sum_{i'\in C_{\gamma'}}\sum_{j'\in C_\gamma}\int\frac{dk}{(2\pi)^4}\frac{e_i'l_i'^\mu}{l_i'\cdot k+i\epsilon}\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}e_j'l_j'^\nu 2\pi\delta(l_j'\cdot k)\times\exp[ik\cdot(x_{\gamma'}-x_\gamma)]. \tag{8.10}$$

Let us consider the $C_{\gamma'}$ with the largest value of $x_{\gamma'}{}^0$. It is no longer true that we can write $S'(I)$ in a form in which $S'[(x_{\gamma'})I'_{\gamma',\text{out}}]$ appears as a factor on the left with no other dependence on the final-state momenta in the set $C_{\gamma'}$, because of the extra terms (8.10). However, these extra terms are purely imaginary and correspond to a phase factor $V[\sigma_{\gamma',\gamma}]$. Since $\sigma_{\gamma',\gamma}$ is a function only of final-state momenta, this factor has the same form for both $S'(I)$ and $S'(I'')$. Hence it is still true that all the factors depending on the final-state variables associated with C_γ cancel, as in (8.7). Then the argument can go through as before. The integration over the final-state phase space $d\Omega'_{\{A,B\}}$ is again part of a non-soft-photon unitarity equation, and when summed over all assignments to sets A_α and B_β, yields zero.

Note that for a line that runs straight through both diagrams there is no dependence of the matrix element on the component of the corresponding x_γ in the direction of l_γ. Thus we can always add an appropriate multiple of l_γ to x_γ so as to arrange that this is *not* the variable with the largest time component. Hence, if there are any sets C that do not correspond to such lines, we can apply the above argument to one of them, and obtain a vanishing contribution.

All that remains is to show that in the special case where all the lines run straight through both diagrams we recover the right-hand side of the unitarity integral. Now, when all lines run straight through, (3.9) reduces to

$$\langle \mathbf{l}_1'\cdots\mathbf{l}_s';f,\lambda;\text{out}|\mathbf{l}_1\cdots\mathbf{l}_s;g,\mu;\text{in}\rangle\prod_{j=1}^s 2\pi\delta(m_j^2+l_j'^2)=\prod_{j=1}^s[N_j^{-1}\bar u(\mathbf{l}_j')u(\mathbf{l}_j)]\langle f,\lambda|g,\mu\rangle$$
$$\times\int dx_1\cdots dx_s\exp\Big(-i\sum_{j=1}^s(l_j-l_j')\cdot x_j\Big)\exp\bigg(i\sum_{i<j=1}^s e_ie_jl_i\cdot l_j\int_{\Omega^*}\frac{dk}{(2\pi)^4}\frac{2\pi\delta(l_i\cdot k)2\pi\delta(l_j\cdot k)}{k^2}\exp[ik\cdot(x_i-x_j)]\bigg). \tag{8.11}$$

(We assume here that the particles are distinguishable. Otherwise we must add all the diagrams with final-state labels permuted. This does not change anything essential.) The spin factor is simply a spin δ symbol, and the soft-photon matrix element is trivial. Thus in order to prove the unitarity relation (8.1), it is only necessary to consider the factor represented by the integral over $x_1\cdots x_s$. If we write the final-state phase-space integrals in the form

$$\int\frac{dl_j'}{(2\pi)^4}2\pi\delta(m_j^2+l_j'^2)$$

and multiply the unitarity integral by

$$\prod_{j=1}^s 2\pi\delta(m_j^2+l_j''^2), \tag{8.12}$$

then both factors on the left-hand side of (8.1) are of the form given by (8.11). The four-dimensional integration over l_j' has, as before, the effect of setting $x_j''=x_j$. The integrand of the x_j integral consists of the product of the exponential factor in (8.11) and its complex conjugate evaluated with l_j replaced by l_j'', together with the factor $\exp[i(l_j''-l_j)\cdot x_j]$. But clearly $l_j''-l_j$ must be small, so that in the slowly varying exponential we may replace l_j'' by l_j. Then the two exponential factors cancel, and we are left with

$$\prod_{j=1}^s(2\pi)^4\delta(l_j''-l_j),$$

which is equal to the factor on the right-hand side of (8.1), multiplied by (8.12).

Finally, therefore, we have established the validity of the unitarity equation (8.1), and thus the completeness of our set of asymptotic states.

9. CONCLUSIONS

We have obtained a general formula (3.9) or (3.13) for the scattering matrix elements between the asymptotic states defined in III. These matrix elements define a scattering operator S on the space $\mathcal{H}^h{}_0 \otimes \mathcal{H}^s{}_{\mathrm{IR}}$ of asymptotic states. We have shown that S has all the expected properties. It is gauge-invariant, Lorentz-invariant, unitary, crossing-symmetric, and independent of the choice of the parameter K that fixes the conventional separation between hard and soft photons. Moreover, when interpreted according to the rules introduced in preceding papers, all its matrix elements are finite. They are nonzero only for states satisfying the relation of equivalence defined by (5.18) and (5.19).

We have also shown how these matrix elements are related to observable quantities. The formalism reproduces the conventional results, embodied in (7.8) or (7.12). However, it can also be used to investigate questions that cannot be studied by conventional methods, such as the dependence of scattering probabilities on the initial soft-photon state, as given by (7.17).

It must be emphasized that all these results are valid only in the limit where the soft-photon cutoff K is very small compared to the particle masses. If the experimental resolution is comparable to the masses, one must, as usual, sum over the probabilities for emission of various number of *hard* photons with total energy less than this experimental limit. However, this is, of course, always a finite sum.

In this series of papers we have shown that quantum electrodynamics may be treated according to the same principles that apply to any other renormalizable field theory, provided that one does not make *a priori* assumptions about the nature of the asymptotic states but determines them from the structure of the Green's functions.

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