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Feynman Rules for the Gravitational Field from the Coordinate-Independent Field-Theoretic Formalism

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By using the method developed in the preceding paper, the Feynman-DeWitt perturbation expansion for the gravitational S matrix is shown to follow from the field-theoretic formalism. Again our methd is to express the path-dependent Green's functions in terms of auxiliary, path-independent Green's functions, in such a way that the path-dependence equation is automatically satisfied. The formula relating the path-dependent to the path-independent Green's functions will be similar to the classical formula relating the path-dependent Riemann tensor to the metric tensor. The equations for the auxiliary Green's functions are found and solved in a perturbation series. If the result is expressed as a sum of Feymann diagrams, one obtains the expected vertices, together with closed loops of fictitious vector particles.

I. INTRODUCTION

IN this paper we wish to use the method of the preceding paper[1] to derive the Feynman rules for the gravitational field. Our prescription will agree with that found by Feynman[2] and DeWitt[3] from an application of the tree theorem to the S matrix.

We shall take as our basis the path-dependent field-theoretic formalism which we proposed in an earlier paper.[4] The fundamental principle of that paper was to work entirely in terms of variables which were independent of the choice of the coordinate system. Such variables had necessarily to depend on a path, and were measured in a local coordinate system constructed in the neighborhood of the path.

The details of the theory to be used in this paper will be slightly different from those of Ref. 4. In the latter paper, we devoted considerable effort to overcoming the difficulty that the time ordering of two points in a curved space was not always determined from the characteristics of the paths leading to them. As a result we had to modify the statement of the equal-time commutation relations. For the purpose of obtaining Feynman rules we shall ignore this difficulty, since it does not occur in perturbation theory. We shall start from the coordinate-independent theory of the classical gravitational field developed in Sec. II of Ref. 4, and shall quantize it by making the usual correspondence between commutators and Poisson brackets. We shall then expand the theory in a perturbation series. Finally, we shall return to the fundamental equations and shall attempt to reformulate them without relying on perturbation theory. In this reformulation the concept of a time-ordered product will play a fundamental role, and we shall assume that such a product is defined even if the time ordering of the relevant points is not known.

As in the previous paper, we shall obtain the perturbation series for our path-dependent Green's functions by expressing them in terms of new auxiliary path-independent Green's functions. For the electromagnetic and Yang-Mills fields, we started from the formula which expressed the gauge-independent, path-dependent fields in terms of the potentials. We then expressed our path-dependent Green's functions in terms of the auxiliary Green's functions by similar formulas. We shall use an analogous procedure for the gravitational field, with the metric tensor replacing the potentials.

In a curved space where the metric tensor $g_{\mu\nu}$ is a given function of the coordinates x, we can calculate the path-dependent Riemann tensor as a power series in the g's. The main difficulty in such a calculation is that the path-dependent variables are defined in a local Euclidean system constructed in the neighborhood of the path, whereas the g's are defined in a non-Euclidean coordinate system. Nevertheless, once the g's are given, we can in principle obtain the coordinates in the non-Euclidean system as a function of the coordinates in the local Euclidean system. The relation between the coordinates in the two systems has been written to lowest order in the g's in Sec. II of Ref. 4, and we shall give equations for finding the relation to arbitrary order in the present paper. Once we have found a formula for the path-dependent Riemann tensor as a function of the coordinate-dependent g's ,we shall express the path-dependent Green's functions in terms of the auxiliary Green's functions by similar formulas. We shall then justify the formulas by showing that the path-dependent Green's functions automatically satisfy the path-dependence equation as a consequence of their definition in terms of the auxiliary Green's functions.

We next find the field equations which the auxiliary Green's functions should satisfy in order that the path dependent Green's functions satisfy the correct equations. The solution of the equations can be expanded in a perturbation series and, as with the Yang-Mills field, we obtain certain terms besides those given by the naive Feynman rules. The extra terms correspond to closed loops of fictitious particles which must be included

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[1] S. Mandelstam, preceding paper, Phys. Rev. **175**, 1580 (1968).
[2] R. P. Feynman, Acta Phys. Polon. **24**, 697 (1963).
[3] B. S. DeWitt, Phys. Rev. **162**, 1195 (1967); **162**, 1239 (1967); **171**, 1834(E) (1968).
[4] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 25 (1962).

in the Feynman diagrams. Now, however, the fictitious particles are vector particles and not scalar particles as they were with the Yang-Mills field.

We shall introduce a notation similar to that of the previous paper for expressing the infinite set of Green's-function equations as a single equation in a linear space. The equations for the path-dependent Green's functions will appear as equations for operators $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ in our linear space, while the equations for auxiliary Green's function will appear as equations for operators $\tilde{g}_{\mu\nu}(x')$. Such operators correspond to the variables $R_{\alpha\beta\gamma\delta}(x,P)$ and $g_{\mu\nu}(x')$ of the classical theory. With the electromagnetic and Yang-Mills fields, the formula for the path-dependent operators $\tilde{\Phi}(x,P)$ and $\tilde{F}_{\mu\nu}(x,P)$ as functions of the gauge-dependent operators $\tilde{\phi}(x)$ and $\tilde{A}_\mu(x)$ was the same as the formula for the classical field variables $\Phi(x,P)$ and $F_{\mu\nu}(x,P)$ in terms of the classical field variables $\phi(x)$ and $A_\mu(x)$. Similarly, for the gravitational field, the formula for the path-dependent Riemann tensor $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ in terms of the metric tensor $\tilde{g}_{\mu\nu}(x')$ will be the same as the formula for the classical path-dependent Riemann tensor $R_{\alpha\beta\gamma\delta}(x,P)$ in terms of the classical metric tensor $g_{\mu\nu}(x')$.

The field equations for $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ will be similar to the classical field equations for $R_{\alpha\beta\gamma\delta}(x,P)$, and the resulting field equations for $\tilde{g}_{\mu\nu}(x')$ will be similar to the classical field equations for $g_{\mu\nu}(x')$. The field equations for $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ and $\tilde{g}_{\mu\nu}(x')$ will not be *identical* to the classical field equations for $R_{\alpha\beta\gamma\delta}(x,P)$ and $g_{\mu\nu}(x')$, because the former equations will contain terms which involve the operator η and which correspond to the δ-function terms in the Green's-function equations.

In Sec. II we shall treat the fundamental equations of the theory, basing our approach on the classical theory of Sec. II of Ref. 4. We shall define covariant time-ordered products, which differ by four-dimensional δ functions from ordinary time-ordered products, since the commutators between two path-dependent variables contain derivatives of δ functions. As with the electromagnetic and Yang-Mills fields, we shall show that the covariant time-ordered products obey simpler path-dependence equations than the ordinary time-ordered products. In Sec. III we shall introduce our short-hand notation and, in Secs. IV and V, we shall use this notation to express our path-dependent Green's functions in terms of auxiliary Green's functions. In Sec. VI we shall obtain the field equations for the auxiliary Green's functions. We shall solve these equations as a perturbation series in Sec. VII and shall derive the rules of Feynman and DeWitt. Finally, in Sec. VIII, we shall return to the problem of formulating the equations of the theory in a nonperturbative approach, where the time-ordering of two points is not always determined from the characteristics of the paths leading to them.

II. EQUATIONS SATISFIED BY THE COVARIANT TIME-ORDERED PRODUCTS

In the present section we shall define the path-dependent Green's functions and shall write the equations which they satisfy. We shall take as our starting point the coordinate independent theory of the classical gravitational field, developed in Sec. II of Ref. 4. When quantizing the field we shall depart slightly from the remainder of that paper, which was strongly oriented towards the difficulty that the spacelike or timelike separation between two points is not necessarily defined by the characteristics of the path joining them. This difficulty is not present in perturbation theory, and we shall not concern ourselves with it at the moment.

We shall treat a gravitational field in interaction with itself but with no other field, since such a system possesses all the essential complications of the problem The field equations and path-dependence equations will be taken from Sec. II of Ref. 4, while the commutation relations will be taken to be identical to the Poisson-bracket relations given in that section.

We next turn to the question of the covariance of time-ordered products. As in electromagnetism, the commutators between path-dependent variables contain derivatives of three-dimensional δ functions. It is therefore necessary to add a four-dimensional δ function to a time-ordered product in order to obtain a covariant quantity. The prescription is as follows: If the commutator between two variables $A(x)$ and $B(y)$ contains a term $\partial/\partial y_\alpha \delta^3(x-y) = -\partial/\partial x_\alpha \delta^3(x-y)$, one must define the covariant time-ordered product by the formula

$$T'\{A(x),B(y)\} = T\{A(x)B(y)\} - \delta_{\alpha 0}\delta^4(x-y). \quad (2.1)$$

The commutators between gravitational field variables will also contain terms involving $\partial/\partial y_\alpha \delta^3(\xi-y)$, $\partial/\partial \eta_\alpha \delta^3(x-\eta)$, or $\partial/\partial_\alpha \delta^3(\xi-\eta)$, where ξ and η are the variables of integration associated with the paths leading to x and y, respectively. We must then insert terms $-\delta_{\alpha 0}\delta^4(\xi-y)$, $-\delta_{\alpha 0}\delta^4(x-\eta)$, and $-\delta_{\alpha 0}\delta^4(\xi-\eta)$ into the definitions of the covariant time-ordered product. We shall also encounter a term $\delta_{\alpha 0}\partial/\partial\eta_\beta \delta^3(\xi-\eta) - \delta_{\beta 0}\partial/\partial\xi_\alpha \times\delta^3(\xi-\eta)$ in the commutator; such a term gives rise to a term $-\delta_{\alpha 0}\delta_{\beta 0}\delta^4(\xi-\eta)$ in the definition of the covariant time-ordered product.

Adopting the foregoing prescription, and taking the commutators between two field variables to be given by Eqs. (2.28), (2.30), and (2.31) of Ref. 4, we define the covariant time-ordered product of two path-dependent variables as follows[5]:

$$\begin{aligned} T'\{R_{\alpha\beta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\} &= T\{R_{\alpha\beta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\} \\ &\quad + \sum_{r=1}^{4} t^{(r)}{}_{\alpha\beta\gamma\delta,\epsilon\zeta\eta\theta}(x,P,y,P'), \quad (2.2a) \end{aligned}$$

[5] The sign of the second term in Eq. (2.28b) of Ref. 4 is in error. We have corrected this error in writing Eq. (2.2d).

where

$$t^{(1)}{}_{\alpha\beta\gamma\delta,\epsilon\zeta\eta\theta}=0 \text{ unless at least two of the subscripts } \alpha\cdots\theta \text{ are zero}, \quad (2.2b)$$

$$t^{(1)}{}_{0\beta\gamma\delta,0\zeta\eta\theta}=\tfrac{1}{2}i\kappa \underset{\gamma\leftrightarrow\delta,\eta\leftrightarrow\theta}{A} \underset{\beta\leftrightarrow\delta}{S}\, \partial_\gamma(x)\partial_\eta(y) \times[\delta_{\beta\zeta}\delta_{\delta\theta}\delta^4(x-y)], \quad \gamma,\delta,\eta,\theta\neq 0, \quad (2.2c)$$

$$t^{(1)}{}_{0\beta0\delta,\epsilon\zeta\eta\theta}=-\tfrac{1}{2}i\kappa \underset{\epsilon\leftrightarrow\zeta,\eta\leftrightarrow\theta}{A} \underset{\beta\leftrightarrow\delta}{S}\, \partial_\eta(y)\partial_\epsilon(y) \times[\delta_{\beta\zeta}\delta_{\delta\theta}\delta^4(x-y)] +\tfrac{1}{2}i\kappa \underset{\eta\leftrightarrow\theta}{A} \underset{\beta\leftrightarrow\delta}{S}\, R_{\epsilon\zeta\eta\iota}(y,P')\delta_{\beta\iota}\delta_{\delta\theta}\delta^4(x-y), \quad (2.2d)$$

$$t^{(2)}{}_{\alpha\beta\gamma\ ,\epsilon\zeta\eta\theta}=0 \text{ unless at least one of the subscripts } \alpha,\beta,\gamma,\delta \text{ is zero}, \quad (2.2e)$$

$$t^{(2)}{}_{0\beta\gamma\delta,\epsilon\zeta\eta\theta}=\tfrac{1}{2}\kappa \underset{\gamma\leftrightarrow\delta}{A} \underset{\beta\leftrightarrow\delta}{S} \int_{P'} d\eta_\iota \partial_\gamma(x)[\delta_{\beta\iota}\delta_{\delta\kappa}\delta^4(x-\eta)] \times[J_{\kappa 0}(x),R_{\epsilon\zeta\eta\theta}(y,P')], \quad \gamma,\delta\neq 0, \quad (2.2f)$$

$$t^{(2)}{}_{0\beta0\delta,\epsilon\zeta\eta\theta}=\tfrac{1}{2}\kappa \underset{\beta\leftrightarrow\delta}{S} \int_{P'} d\eta_\iota\{\partial_\kappa(\eta)[\delta_{\beta\iota}\delta_{\delta\lambda}\delta^4(x-\eta)] \times[J_{\kappa\lambda}(\eta),R_{\epsilon\zeta\eta\theta}(y,P')]+i\delta_{\beta\iota}\delta_{\delta\lambda} \times\delta^4(x-\eta)\partial_\lambda(\eta)R_{\epsilon\zeta\eta\theta}(y,P')\}, \quad (2.2g)$$

$$t^{(3)}{}_{\alpha\beta\gamma\delta,\epsilon\zeta\eta\theta}(x,P,y,P')=t^{(2)}{}_{\epsilon\zeta\eta\theta,\alpha\beta\gamma\delta}(y,P',x,P), \quad (2.2h)$$

$$t^{(4)}{}_{\alpha\beta\gamma\delta,\epsilon\zeta\eta\theta}=-\tfrac{1}{2}i\kappa\delta_{\iota\lambda}\delta_{\kappa\mu} \underset{\lambda\leftrightarrow\mu}{S} \int_P d\xi_\iota \int_{P'} d\eta_\lambda \delta^4(\xi-\eta) \times[J_{\kappa 0}(\xi),R_{\alpha\beta\gamma\delta}(x,P)][J_{0\mu}(\eta),R_{\epsilon\zeta\eta\theta}(y,P')]. \quad (2.2i)$$

The notation is similar to that of Ref. 4. All components are measured in the local Euclidean system. The symbol $\epsilon\partial_\lambda(z)R$ means the difference between $R(x,P_1)$ and $R(x,P)$, where the path P_1 is identical to the path P except for an additional element ϵ in the λ direction at the point z [Fig. 1(a)]. The symbol $-i\epsilon[J_{\mu\nu}(z),R]$ means the difference between $R(x,P_2)$ and $R(x,P)$, where the path P_2 is identical to P, except that the local Euclidean system defining it is rotated by an amount ϵ in the $\mu\nu$ plane at the point z [Fig. 1(b)]. We have adopted this notation since, in a flat space, J would correspond to the angular momentum about z but, from our present point of view, the symbol $-i[J_{\mu\nu}(z),R]$ is defined as an entity and should not be regarded as the commutator between two operators.[6] The symbols A and S are defined as follows:

$$\underset{\alpha\leftrightarrow\beta}{A}\, f_{\alpha\beta}=f_{\alpha\beta}-f_{\beta\alpha}, \quad (2.3a)$$

$$\underset{\alpha\leftrightarrow\beta}{S}\, f_{\alpha\beta}=f_{\alpha\beta}+f_{\beta\gamma}-\delta_{\alpha\beta}f_{\gamma\gamma}. \quad (2.3b)$$

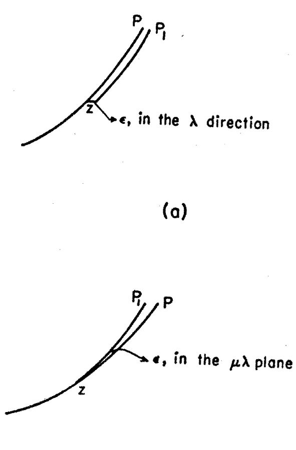

FIG. 1(a) The meaning of the symbol $\partial_\lambda(z)R$. (b) The meaning of the symbol $-i[J_{\mu\nu}\times(z),R]$.

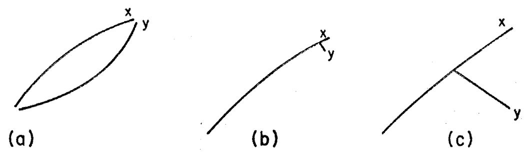

FIG. 2. Paths which are allowed when defining the δ function.

We also remind the reader about one point in the definition of the δ function. An expression such as $\delta(x-y)$, where x and y are the endpoints of the paths P and P', will of course depend on the paths P and P' themselves. If P and P' have the form sketched in Fig. 2(a), one will require knowledge of the Riemann tensor in the space between the paths in order to determine whether their end points coincide. The δ function will therefore be a complicated function of the paths and of the Riemann tensor, and we shall avoid defining the δ function for such cases. On the other hand, there is no difficulty in defining the δ function $\delta(x-y)$ for paths such as Fig. 2(b). We shall therefore restrict ourselves to pairs of paths such as those in Fig. 2(b) or, more generally, those in Fig. 2(c). The paths may coincide over the initial portion of their lengths but, once they have begun to separate, they must be completely disjoint. We can restrict ourselves to such paths without any essential loss of generality. In a space with finite curvature one cannot be sure whether two paths have no point in common but, at the moment, we are only interested in obtaining a perturbation series, and this difficulty does not occur.

We shall also assume that an individual path does not turn back and cross itself. Thus different points of the same path will not coincide.

Equations (2.2) follow unambiguously from our prescription only to the extent that they do not involve

[6] In Ref. 4, we used the symbol $i[J_{\mu\nu}(z),R]$ instead of $-i[J_{\mu\nu}(z),R]$. We have changed our notation in the present paper, since the symbol J as presently defined corresponds to the angular momentum. All terms in the Poisson brackets of Ref. 4 must therefore be reversed in sign before being taken over into the present paper.

time derivatives of the δ function. For instance, it is not immediately evident that we can drop the restriction $\epsilon, \zeta, \eta, \theta \neq 0$ in Eq. (2.2d). We regard Eqs. (2.2) as the *definition* of our modified time-ordered products. When we develop the field equations we shall show that they are covariant, so that the definition is suitable.

For time-ordered products of more than two variables one adopts definitions similar to Eqs. (2.2). The covariant time-ordered product will consist of a number of terms in which the operators are paired in all possible ways, as in Wick's theorem. In any particular term the paired operators are represented by the sum of the expressions (2.2b)–(2.2i), while the time-ordered product of the unpaired terms is taken.

We can now write the path-dependence equations and equations of motion for the covariant time-ordered products. We shall begin with the path-dependence equations, which will be derived from the fundamental path-dependence equation

$$\delta_z R_{\alpha\beta\gamma\delta}(x,P) = \tfrac{1}{4} i T'\{R_{\iota\kappa\lambda\mu}(z,P') \times [J_{\iota\kappa}(z), R_{\alpha\beta\gamma\delta}(x,P)]\}\sigma_{\lambda\mu}, \quad (2.4)$$

where, as usual, δ_z represents the change in R caused by an infinitesimal change in the path P at the point z by an amount $\sigma_{\lambda\mu}$. The variation of the path is shown in Fig. 3, where the solid curve represents the path before the variation, the dashed curve the path after the variation. The area $\sigma_{\mu\nu}$ is the area between the solid and the dashed curve, and z is any point within this area.

The classical theory of Ref. 4, Sec. II, does not determine the order of the factors on the right of (2.4). In writing down this equation we have made the simplest hypothesis, namely, that the covariant time-ordered product is to be taken. The question of the factor ordering in (2.4) is complicated by the singular nature of the product of two operators whose paths coincide, and we shall not attempt to deal rigorously with such questions in this paper. If we proceed formally from Eq. (2.4) we shall encounter no difficulties.

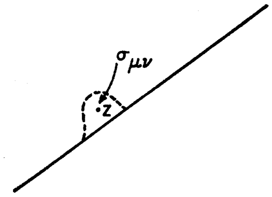

FIG. 3. A variation of the path by an infinitesimal area $\sigma_{\mu\nu}$ at the point z.

We now use Eq. (2.4) to find the change in the time-ordered product

$$T\{R_{\alpha\beta\gamma\delta}(x,P) R_{\epsilon\zeta\eta\theta}(y,P')\}. \quad (2.5)$$

Our calculation parallels the analogous calculation in electrodynamics, which was given in the previous paper. The change in the time-ordered product will consist of two parts, which we shall call $\delta^{(1)}$ and $\delta^{(2)}$. The contribution $\delta^{(1)}$ is obtained by straightforward application of (2.4) to (2.5);

$$\delta_z^{(1)} T\{R_{\alpha\beta\gamma\delta}(x,P) R_{\epsilon\zeta\eta\theta}(y,P')\} = \tfrac{1}{4} i T(T'\{R_{\iota\kappa\lambda\mu}(z,P'')[J_{\iota\kappa}(z), R_{\alpha\beta\gamma\delta}(x,P)]\} \times R_{\epsilon\zeta\eta\theta}(y,P'))\sigma_{\lambda\mu}, \quad (2.6)$$

where P'' represents the portion of P leading to the point z.

The second contribution to the change in the time-ordered product arises from the fact that the change in the path may alter the time ordering of the two operators in (2.5). We can calculate this change in the time-ordered product by arguments analogous to those leading to Eq. (3.10b) of the previous paper. The prescription is to write that part of the commutator between the two operators in (2.5) which consists of an integral $\int_P d\xi_\iota$ over the path P, and then to replace this integral by $\int d\sigma_{\iota 0}\delta_0(z-y)$, σ being the area between the paths and z a point within this area. We can divide the result into two parts, which arise from the contributions of Eqs. (2.30) and (2.31) of Ref. 4 to the commutator in question. Thus

$$\delta_z^{(2)} T = \delta_z^{(2a)} T + \delta_z^{(2b)} T, \quad (2.7a)$$

where

$$\begin{aligned}
&\delta_z^{(2a)} T\{R_{\alpha\beta\gamma\delta}(x,P) R_{\epsilon\zeta\eta\theta}(y,P')\} \\
&= \tfrac{1}{2}\kappa \mathop{A}_{\epsilon\leftrightarrow\zeta,\eta\leftrightarrow\theta} \mathop{S}_{\zeta\leftrightarrow\theta} \int d\sigma_{\iota 0}\partial_\eta(y)\partial_\epsilon(y)[\delta_{\zeta\iota}\delta_{\eta\kappa}\delta^4(y-z)][J_{\kappa 0}(z), R_{\alpha\beta\gamma\delta}(x,P)] \\
&\quad + \tfrac{1}{2}\kappa \mathop{A}_{\eta\leftrightarrow\theta} \int d\sigma_{\iota 0} T\{R_{\epsilon\zeta\eta\lambda}(y,P')(\mathop{S}_{\lambda\leftrightarrow\theta} \delta_{\lambda\iota}\delta_{\theta\kappa})\delta^4(y-z)[J_{\kappa 0}(z), R_{\alpha\beta\gamma\delta}(x,P)]\}\epsilon, \zeta, \eta, \theta \neq 0, \quad (2.7b)
\end{aligned}$$

$$\begin{aligned}
&\delta_z^{(2a)} T\{R_{\alpha\beta\gamma\delta}(x,P) R_{0\zeta\eta\theta}(y,P')\} \\
&= \tfrac{1}{2}\kappa \mathop{A}_{\eta\leftrightarrow\theta} \mathop{S}_{\zeta\leftrightarrow\theta} \int d\sigma_{\iota 0}(1-\delta_{\eta 0})\partial_\eta(y)\partial_\kappa(z)[\delta_{\zeta\iota}\delta_{\theta\lambda}\delta^4(y-z)][J_{\kappa\lambda}(z), R_{\alpha\beta\gamma\delta}(x,P)] \\
&\quad + \tfrac{1}{2}\kappa \mathop{A}_{\eta\leftrightarrow\theta} \int d\sigma_{\iota 0} T\{R_{0\zeta\eta\lambda}(y,P') \mathop{S}_{\lambda\leftrightarrow\theta} \delta_{\lambda\iota}\delta_{\theta\kappa}\delta^4(y-z)[J_{\kappa 0}(z), R_{\alpha\beta\gamma\delta}(x,P)]\} \\
&\quad - \tfrac{1}{2}\kappa \mathop{A}_{\eta\leftrightarrow\theta} \mathop{S}_{\zeta\leftrightarrow\theta} \int d\sigma_{\iota 0}(1-\delta_{\eta 0})\partial_\eta(y)\partial_0(z)\{\delta_{\zeta\iota}\delta_{\theta\lambda}\delta^4(y-z)[J_{0\lambda}(z), R_{\alpha\beta\gamma\delta}(x,P)]\}, \quad (2.7c)
\end{aligned}$$

$$\delta^{(2b)}T\{R_{\alpha\beta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\}$$

$$=\tfrac{1}{2}i\kappa\delta_{\iota\lambda}\delta_{\kappa\mu}\underset{\lambda\leftrightarrow\mu}{S}\int d\sigma_{\iota 0}\int_{P'} d\eta_\lambda T\{\partial_\nu(\eta)\delta^4(z-\eta)[J_{\kappa 0}(z),R_{\alpha\beta\gamma\delta}(x,P)][J_{\nu\mu}(\eta),R_{\epsilon\zeta\eta\theta}(y,P')]$$
$$-(1-\delta_{\nu 0})\partial_\nu(z)\delta^4(z-\eta)[J_{\nu\kappa}(z),R_{\alpha\beta\gamma\delta}(x,P)][J_{\mu 0}(\eta),R_{\epsilon\zeta\eta\theta}(y,P')]$$
$$+i\delta^4(z-\eta)[J_{\kappa 0}(z),R_{\alpha\beta\gamma\delta}(x,P)]\partial_\mu(\eta)R_{\epsilon\zeta\eta\theta}(y,P')\}$$
$$+\tfrac{1}{2}i\kappa\delta_{\iota\lambda}\delta_{\kappa\mu}\underset{\lambda\leftrightarrow\mu}{S}\int d\sigma_{\iota 0}\int d\eta_\lambda T(\partial_0(z)\{\delta^4(z-\eta)[J_{0\kappa}(z),R_{\alpha\beta\gamma\delta}(x,P)][J_{\mu 0}(\eta),R_{\epsilon\zeta\eta\theta}(y,P')]\}). \quad (2.7\text{d})$$

In writing (2.7c) and (2.7d), we have made use of the identity

$$\partial_0(z)[J_{0\kappa}(z),R_{\alpha\beta\gamma\delta}(x,P)]=i\partial_\kappa(z)R_{\alpha\beta\gamma\delta}(x,P). \quad (2.8)$$

The ordering of the factors on the right of (2.7) is not determined from the classical theory, and we make the assumption that the time-ordered product is to be taken.

Our next step is to use Eqs. (2.2), (2.6), and (2.7) to find the path-dependence equations for the covariant time-ordered products T'. After a certain amount of algebra, we find that the extra terms obtained when replacing the ordinary time-ordered products by the covariant products cancel against the extra terms (2.7) in the path-dependence equation for T, so that the path-dependence equation for T' is given by a simple equation of the type (2.6):

$$\delta T'\{R_{\alpha\beta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\}=\tfrac{1}{4}iT'\{R_{\iota\kappa\lambda\mu}(z,P'')[J_{\iota\kappa}(z),R_{\alpha\beta\gamma\delta}(x,P)]R_{\epsilon\zeta\eta\theta}(y,P')\}\sigma_{\lambda\mu}. \quad (2.9)$$

The situation with the gravitational field is thus similar to what it was with the electromagnetic and Yang-Mills Fields. The covariant time-ordered products obey simple path-dependence equations.

We now turn to the field equations for the time-ordered products. We take as our starting point the Einstein equations for the field variables:

$$R_{\gamma\alpha\gamma\beta}(x,P)-\tfrac{1}{2}\delta_{\alpha\beta}R_{\gamma\delta\gamma\delta}(x,P)=0. \quad (2.10)$$

From (2.10) we can at once write the Einstein equations for the time-ordered products:

$$T\{R_{\gamma\alpha\gamma\beta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\}-\tfrac{1}{2}\delta_{\alpha\beta}T\{R_{\gamma\delta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\}=0. \quad (2.11)$$

The Einstein equations for the covariant time-ordered products will be slightly more complicated. From (2.2) and (2.11), we easily find that

$$T'\{R_{\gamma\alpha\gamma\beta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\}-\tfrac{1}{2}\delta_{\alpha\beta}T'\{R_{\gamma\delta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\}$$
$$=-\tfrac{1}{2}i\kappa\underset{\epsilon\leftrightarrow\zeta,\,\eta\leftrightarrow\theta}{A}\partial_\eta(y)\partial_\epsilon(y)[(\delta_{\beta\zeta}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\delta\zeta})\delta^4(x-y)]+\tfrac{1}{2}i\kappa\underset{\eta\leftrightarrow\theta}{A}R_{\epsilon\zeta\eta\iota}(y,P')(\delta_{\beta\iota}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\theta\iota})\delta^4(x-y)$$
$$+\tfrac{1}{2}\kappa\int_{P'} d\eta_\iota(\delta_{\beta\iota}\delta_{\delta\lambda}+\delta_{\beta\lambda}\delta_{\delta\iota})\{\partial_\kappa(\eta)\delta^4(x-\eta)[J_{\kappa\lambda}(\eta),R_{\epsilon\zeta\eta\theta}(y,P')]+i\delta^4(x-\eta)\partial_\lambda(\eta)R_{\epsilon\zeta\eta\theta}(y,P')\}. \quad (2.12)$$

We observe that the covariant time-ordered products obey simpler path-dependence equations but more complicated field equations than the ordinary time-ordered products. However, the terms on the right of (2.12) are precisely analogous to the δ-function terms in the Green's-function equations of other theories, e.g., the δ-function terms in Eqs. (2.4) or (3.8) of the previous paper. Time-ordered products of field variables will obey equations similar to (2.12). There will be one term corresponding to the right side of (2.12) for each field variable in the product (except the variable $R_{\gamma\alpha\gamma\beta}-\frac{1}{2}\delta_{\alpha\beta}R_{\gamma\delta\gamma\delta}$ itself), and the terms will be multiplied by the covariant time-ordered product of the remaining $n-2$ variables.

Having defined our time-ordered products, we can define the Green's functions as the vacuum-expectation values of such products in the usual way. Thus

$$G_{\alpha\beta\gamma\delta,\epsilon\zeta\eta\theta}(x,P,y,P')=\langle 0|T'\{R_{\alpha\beta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\}|0\rangle. \quad (2.13)$$

Green's functions of more than two variables can be similarly defined. They will satisfy equations such as (2.9) and (2.12), and it is hardly necessary to write them explicitly.

III. CONDENSED NOTATION

We can express the equations of motion in the condensed notation developed in the preceding paper. We shall go very quickly through the establishment of the notation, since it is the exact analog of that of the preceding paper, and no new problems arise.

As before, we begin by constructing the linear space of the totality of all functions $\hat{C}_{\alpha\beta\gamma\delta,\epsilon\zeta\eta\theta\cdots}(x_1,P_1,x_2,P_2,\cdots)$. We next construct the dual space of vectors $(\hat{H}_{\alpha\beta\gamma\delta,\epsilon\zeta\eta\theta}(x_1,P_1,x_2,P_2,\cdots)|$, which are defined in the same way as in the preceding paper. We define the operator $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ in our linear space as follows:

$$(\hat{H}_{\epsilon\zeta\eta\theta}(x_1,P_1,\cdots)|\tilde{R}_{\alpha\beta\gamma\delta}(x,P)=(\hat{H}_{\alpha\beta\gamma\delta,\epsilon\zeta\eta\theta}(x,P,x_1,P_1,\cdots)|\,. \tag{3.1}$$

Proceeding in the usual way, we now define an operator $U(x,P)$ in our linear space to correspond to the right side of (2.12). The definition is as follows:

$$\begin{aligned}
&(\hat{H}_{\mu\nu\xi\pi,\cdots}(x_1,P_1,\cdots)|U_{\beta\delta}(x,P)\\
&=\tfrac{1}{2}\kappa\sum_r \underset{\epsilon\leftrightarrow\zeta,\eta\leftrightarrow\theta}{A}(\hat{H}_{\mu\nu\xi\pi,\cdots,[\epsilon\zeta\eta\theta],\cdots}(x_1,P_1,\cdots,[x_r,P_r],\cdots)|\partial_\eta(x_r)\partial_\epsilon(x_r)[(\delta_{\beta\zeta}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\delta\zeta})\delta^4(x-x_r)]\\
&\quad-\tfrac{1}{2}\kappa\sum_r \underset{\eta\leftrightarrow\theta}{A}(\hat{H}_{\mu\nu\xi\pi,\cdots,\epsilon\zeta\eta[\theta]\iota,\cdots}(x_1,P_1,\cdots,x_r,P_r,\cdots)|(\delta_{\beta\iota}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\delta\iota})\delta^4(x-x_r)\\
&\quad+\tfrac{1}{2}i\kappa\sum_r\int_{P_r}d\eta_\iota(\delta_{\beta\iota}\delta_{\delta\lambda}+\delta_{\beta\lambda}\delta_{\delta\iota})\{[J_{\kappa\lambda}(\eta),(\hat{H}_{\mu\nu\xi\pi,\cdots,\epsilon\zeta\eta\theta^2\cdots}(x_1,P_1,\cdots,x_r,P_r,\cdots)|]\partial_\kappa(\eta)\delta^4(x-\eta)\\
&\qquad\qquad-i\delta^4(x-\eta)\partial_\lambda(\eta)(\hat{H}_{\mu\nu\xi\pi,\cdots,\epsilon\zeta\eta\theta,\cdots}(x_1,P_1,\cdots,x_r,P_r,\cdots)|\}\,.
\end{aligned}\tag{3.2}$$

In Eq. (3.2), the subscripts $\epsilon\zeta\eta\theta$ correspond to the coordinates x_r, P_r, and the subscripts $\epsilon\zeta\eta[\theta]\iota$ on the second term indicate that θ is to be replaced by ι. The operator U may also be defined by its commutation relations:

$$\begin{aligned}
&[U_{\beta\delta}(x,P),\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')]\\
&=-\tfrac{1}{2}\kappa\underset{\epsilon\leftrightarrow\zeta,\eta\leftrightarrow\theta}{A}\partial_\eta(y)\partial_\epsilon(y)[(\delta_{\beta\zeta}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\delta\zeta})\delta^4(x-y)]+\tfrac{1}{2}\kappa\underset{\eta\leftrightarrow\theta}{A}\tilde{R}_{\epsilon\zeta\eta\iota}(y,P')(\delta_{\beta\iota}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\delta\iota})\delta^4(x-y)\\
&\quad-\tfrac{1}{2}i\kappa\int_{P'}d\eta_\iota(\delta_{\beta\iota}\delta_{\delta\lambda}+\delta_{\beta\lambda}\delta_{\delta\iota})\{[J_{\kappa\lambda}(\eta),\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')]\partial_\kappa(\eta)\delta^4(x-\eta)+i\delta^4(x-\eta)\partial_\lambda(\eta)\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')\}\,.
\end{aligned}\tag{3.3}$$

The definition (3.3) of U is completed by the equation

$$(H_0|U_{\beta\delta}(x,P)=0\,. \tag{3.4}$$

In our condensed notation, the field equations (2.12) take the form

$$[\tilde{R}_{\gamma\alpha\gamma\beta}(x,P)-\tfrac{1}{2}\delta_{\alpha\beta}\tilde{R}_{\gamma\delta\gamma\delta}(x,P)-iU_{\beta\delta}(x,P)]|\hat{G})=0\,. \tag{3.5}$$

The path-dependence equation (2.9) has the simple form,

$$\delta\tilde{R}_{\alpha\beta\gamma\delta}(x,P)=\tfrac{1}{4}i\tilde{R}_{\epsilon\zeta\eta\theta}(z,P')[J_{\epsilon\zeta}(z),\tilde{R}_{\alpha\beta\gamma\delta}(x,P)]\sigma_{\eta\theta}\,. \tag{3.6}$$

As usual, P' represents the portion of P leading to the point z.

IV. AUXILIARY VARIABLES

We now attempt to express the operators $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ in our linear space as functions of path-independent variables $\tilde{g}_{\mu\nu}(x')$. This step in our method corresponds precisely to that of the previous paper, where we expressed the path-dependent variables $\tilde{\Phi}$ and $\tilde{F}$ as functions of path-independent variables $\tilde{\phi}$ and $\tilde{A}$.

In writing down the connection between the variables $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ and $\tilde{g}_{\mu\nu}(x')$ we shall be *guided* by the classical theory. The *justification* of our formulas will of course be the fact, which we shall demonstrate explicitly, that the path-dependence equation (3.6) is a consequence of our definitions.

To begin, we define the Christoffel three-index symbols $\tilde{\Gamma}_{\mu\nu}{}^\lambda$, and the path-independent Riemann tensor $\tilde{r}^\lambda{}_{\mu\nu\rho}$, by the same formulas as in the classical theory:

$$\tilde{\Gamma}_{\mu\nu}{}^\lambda(x')=\tilde{g}^{\lambda\rho}(x')\tilde{\Gamma}_{\rho,\mu\nu}(x')\,, \tag{4.1a}$$

$$\tilde{\Gamma}_{\lambda,\mu\nu}(x')=\frac{1}{2}\left(\frac{\partial\tilde{g}_{\lambda\mu}(x')}{\partial x'^\nu}+\frac{\partial\tilde{g}_{\lambda\nu}(x')}{\partial x'^\mu}-\frac{\partial\tilde{g}_{\mu\nu}(x')}{\partial x'^\lambda}\right), \tag{4.1b}$$

$$\begin{aligned}
\tilde{r}_{\lambda\mu\nu\rho}(x')=&\frac{1}{2}\left(\frac{\partial^2\tilde{g}_{\lambda\nu}(x')}{\partial x'^\mu\partial x'^\rho}+\frac{\partial^2\tilde{g}_{\mu\rho}(x')}{\partial x'^\lambda\partial x'^\nu}-\frac{\partial^2\tilde{g}_{\lambda\rho}(x')}{\partial x'^\mu\partial x'^\nu}-\frac{\partial^2\tilde{g}_{\mu\nu}(x')}{\partial x'^\lambda\partial x'^\rho}\right)\\
&+\tilde{g}^{\sigma\tau}[\tilde{\Gamma}_{\sigma,\lambda\nu}(x')\tilde{\Gamma}_{\tau,\mu\rho}(x')\\
&\qquad-\tilde{\Gamma}_{\sigma,\lambda\rho}(x')\tilde{\Gamma}_{\tau,\mu\nu}(x')]\,.
\end{aligned}\tag{4.2a}$$

$$\tilde{r}^\lambda{}_{\mu\nu\rho}(x)=\tilde{g}^{\lambda\sigma}(x')\tilde{r}_{\sigma\mu\nu\rho}(x') \tag{4.2b}$$

$$=\frac{\partial}{\partial x'^\rho}\tilde{\Gamma}_{\mu\nu}{}^\lambda(x')-\frac{\partial}{\partial x'^\nu}\tilde{\Gamma}_{\mu\rho}{}^\lambda(x')+\tilde{\Gamma}_{\sigma\rho}{}^\lambda(x')\tilde{\Gamma}_{\mu\nu}{}^\sigma(x')-\tilde{\Gamma}_{\sigma\nu}{}^\lambda(x')\tilde{\Gamma}_{\mu\rho}{}^\sigma(x')\,. \tag{4.2c}$$

The variables $\tilde{g}^{\mu\nu}$ are, as usual, the elements of the reciprocal of the matrix $\tilde{g}_{\mu\nu}$. In perturbation theory, we define the variables

$$(2\kappa)^{1/2}\tilde{\phi}_{\mu\nu}=\tilde{g}_{\mu\nu}-\delta_{\mu\nu}\,. \tag{4.3}$$

We can expand $\tilde{g}^{\mu\nu}$ in terms of the $\tilde{\phi}_{\mu\nu}$'s to any desired order.

It is still necessary to express the elements $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ of the path-dependent Riemann tensor in terms of the elements $\tilde{r}_{\lambda\mu\nu\rho}(x')$ of the path-independent Riemann tensor. The difference between the variables $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ and $\tilde{r}_{\lambda\mu\nu\rho}(x')$ lies in the coordinates in terms of which they are expressed. The variable $\tilde{r}_{\lambda\mu\nu\rho}(x')$ is expressed as a function of non-Euclidean coordinates, whereas the variable $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ is expressed in terms of a Euclidean coordinate system constructed along the path P. For the moment let us denote the non-Euclidean coordinate by X^λ, the Euclidean coordinate by x_α. Furthermore, let us take a unit vector in the α direction at the point x, and let us denote its contravariant coordinates in the non-Euclidean system by the symbol $V_\alpha{}^\lambda(x,P)$. The four vectors $V_\alpha{}^\lambda(x,P)$, $\alpha=1,\cdots,4$, thus form a tetrad which is moved parallel to itself along the path. These vectors form the axes of our Euclidean coordinate system, and the coordinates x are measured in this system. The shape of the path itself is defined by giving the four coordinates x_α ($\alpha=1,\cdots,4$) as a function of some parameter.

In general, we shall use subscripts or superscripts from the middle of the Greek alphabet to denote coordinates in the non-Euclidean system. We distinguish between contravariant components, represented by superscripts, and covariant components, represented by subscripts. We shall use subscripts from the beginning of the Greek alphabet to denote coordinates in the path-dependent local Euclidean system. There is no distinction between covariant and contravariant components, (besides the trivial distinction associated with the Lorentz metric), and we shall only use subscripts for these components. In one case, Eq. (4.6), we shall be unable to use consistently the convention just outlined. This equation will express the coordinates in the local Euclidean system as a Taylor series in the coordinates in the non-Euclidean system, and we shall have to use subscripts or superscripts from the middle of the Greek alphabet for all coordinates.

Once the shape of the path and the metric tensor $\tilde{g}_{\mu\nu}$ are known, we must be able to calculate the variables X^λ and $V_\alpha{}^\lambda$ as functions of x, and we now construct equations for doing so. First, since the contravariant coordinates of a unit vector in the α direction are $V_\alpha{}^\lambda(x,P)$, the non-Euclidean coordinates of a vector with Euclidean coordinates dx_α will be $V_\alpha{}^\lambda(x,P)dx_\alpha$. On the other hand, the non-Euclidean coordinates of such a vector will be dX^λ by definition, so that we may write[7]

$$dX^\lambda=V_\alpha{}^\lambda(x,P)dx_\alpha$$

or

$$V_\alpha{}^\lambda(x,P)=\partial X^\lambda(x,P)/\partial x_\alpha. \tag{4.4}$$

[7] Strictly speaking, we should write (4.4) in the form $V_\alpha{}^\lambda(x,P)=\partial_\alpha(x)X^\lambda(x,P)$, since X is a path-dependent quantity. In the remainder of the paper we shall use the usual differential notation $\partial f(x,P)/\partial x_\alpha$ for $\partial_\alpha(x)f(x,P)$, where f is any path-dependent function and $dx_\alpha\partial_\alpha$ is the change of f caused by the addition of an element dx_α at the *end* of the path. We continue to use the notation $\partial_\alpha(\xi)f(x,P)$ for the change in f due to the addition of a path element at some arbitrary point ξ on the path.

In Eq. (4.4), the coordinates X^λ in the non-Euclidean system are regarded as functions of the coordinates x_α of the Euclidean system.

To find a second equation between $V_\alpha{}^\lambda$ and X^μ, we use the fact that the variables $V_\alpha{}^\lambda(x)$ represent the coordinates of a vector which undergoes parallel displacement as x is varied. Thus, by the fundamental formula for parallel displacements;

$$\begin{aligned} dV_\alpha{}^\lambda&=-\tilde{\Gamma}_{\mu\nu}{}^\lambda V_\alpha{}^\mu dX^\nu\\ &=-\tilde{\Gamma}_{\mu\nu}{}^\lambda V_\alpha{}^\mu V_\beta{}^\nu dx_\beta, \quad \text{[from (4.4)]}, \end{aligned}$$

or

$$\partial V_\alpha{}^\lambda(x,P)/\partial x_\beta=-\tilde{\Gamma}_{\mu\nu}{}^\lambda(X)V_\alpha{}^\mu(x,P)V_\beta{}^\nu(x,P). \tag{4.5}$$

In Eq. (4.5), the argument X of the variable $\tilde{\Gamma}_{\mu\nu}{}^\lambda(X)$ is to be regarded as a function of x (and P). Equations (4.4) and (4.5) enable the functions $V_\alpha{}^\lambda(x,P)$ and $X^\lambda(x,P)$ to be calculated, and it is a straightforward matter to calculate them to any order of perturbation theory.

In obtaining relations between path-dependent and path-independent quantities, we shall frequently require to calculate a function $f(X)$ in terms of the coordinates x, where X is given in terms of x by (4.4). The function $\tilde{\Gamma}_{\mu\nu}{}^\lambda(X)$ in (4.5) is an example. One can perform this calculation to any order of perturbation theory by making a Taylor expansion. Thus

$$f(X)=\sum_{r_\lambda=1}^{\infty}\prod_{\lambda=1}^{4}\frac{1}{r_\lambda!}[X^\lambda(x,P)-x_\lambda]^{r_\lambda}\frac{\partial^{r_\lambda}}{(\partial x^\lambda)^{r_\lambda}}f(x). \tag{4.6}$$

We may write (4.6) in the form

$$f(X)=\int dx^4W(x,P,x')f(x'), \tag{4.7}$$

where

$$W(x,P,x')=\sum_{r_\lambda=1}^{\infty}\prod_{\lambda=1}^{4}\frac{(-1)^{r_\lambda}}{r_\lambda!}[X^\lambda(x,P)-x_\lambda]^{r_\lambda}\times\frac{\partial^{r_\lambda}}{(\partial x'^\lambda)^{r_\lambda}}\delta(x-x'). \tag{4.8}$$

The function W is defined by the equations

$$\frac{\partial}{\partial x_\alpha}W(x,P,x')=-V_\alpha{}^\lambda(x,P)\frac{\partial}{\partial x'^\lambda}W(x,P,x'), \tag{4.9a}$$

$$W(x,P,x')=\delta(x-x') \quad \text{at the beginning of the path}. \tag{4.9b}$$

From (4.7), we may rewrite (4.5) in the form

$$\frac{\partial V_\alpha{}^\lambda(x,P)}{\partial x_\beta}=-\int d^4x'V_\alpha{}^\mu(x,P)V_\beta{}^\nu(x,P)\times W(x,P,x')\tilde{\Gamma}_{\mu\nu}{}^\lambda(x'). \tag{4.10}$$

Equations (4.9) and (4.10) may be used instead of (4.4) and (4.5) to define the functions V and W, and we shall use these equations in our subsequent work. The primed coordinates now represent coordinates in the non-Euclidean system, the unprimed coordinates those in the local Euclidean system.

From (4.8) we can easily derive the following useful formula:

$$\int d^4x'd^4x''W(x,P,x')W(x,P,x'')f_1(x')f_2(x'') = \int d^4x'W(x,P,x')f_1(x')f_2(x'). \quad (4.11)$$

It will sometimes be convenient to use the covariant components of the vector V. They are defined in the usual way:

$$V_{\lambda\alpha}(x,P)=\tilde{g}_{\lambda\mu}(X)V_\alpha{}^\mu(x,P) = \int d^4x'V_\alpha{}^\mu(x,P)W(x,P,x')\tilde{g}_{\lambda\mu}(x'). \quad (4.12)$$

From (4.9) and (4.10), one can show that $V_{\lambda\alpha}$ obeys the equation

$$\frac{\partial V_{\lambda\alpha}(x,P)}{\partial x_\beta}=\int d^4x'V_{\mu\alpha}(x,P)V_\beta{}^\nu(x,P) \times W(x,P,x')\tilde{\Gamma}_{\lambda\nu}{}^\mu(x'). \quad (4.13)$$

From (4.8) and (4.13) one can then show that

$$\frac{\partial}{\partial x^\gamma}[V_\alpha{}^\lambda(x,P)V_{\lambda\beta}(x,P)]=0.$$

If we adopt the boundary conditions $V_\alpha{}^\lambda(k,P)V_{\lambda\beta}(x,P) =\delta_{\alpha\beta}$ at the beginning of the path, we conclude that

$$V_\alpha{}^\lambda(x,P)V_{\lambda\beta}(x,P)=\delta_{\alpha\beta}, \quad (4.14a)$$

quite generally. Equation (4.10) is of course a consequence of the fact that $V_\alpha{}^\lambda$ and $V_{\lambda\beta}$ represent the coordinates of two unit vectors which are perpendicular if $\alpha\neq\beta$, and the development leading to (4.14) shows directly that this equation follows from the equations used to define the V's.

The following equations are immediate consequences of (4.11), (4.12), and (4.14):

$$V_\alpha{}^\lambda(x,P)V_\alpha{}^\mu(x,P)=\int d^4x'W(x,P,x')\tilde{g}^{\lambda\mu}(x'), \quad (4.14b)$$

$$V_\alpha{}^\lambda(x,P)V_{\mu\alpha}(x,P)=\delta_\mu{}^\lambda, \quad (4.14c)$$

$$V_{\lambda\alpha}(x,P)V_{\mu\alpha}(x,P)=\int d^4x'W(x,P,x')\tilde{g}_{\lambda\mu}(x'). \quad (4.14d)$$

We are now equipped to define the path-dependent variables $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ in terms of the path-independent variables $\tilde{r}_{\lambda\mu\nu\rho}(x')$. Since the components of R are measured in the local Euclidean system and those of $\tilde{r}$ are measured in the non-Euclidean system, we require a factor $V_\alpha{}^\lambda$ for each subscript to convert from one system to the other. Furthermore, the argument of the variable $\tilde{R}$ is the coordinate in the local Euclidean system, while that of the variable $\tilde{r}$ is the coordinate in the non-Euclidean system. We therefore have to apply Eq. (3.7) to reexpress $\tilde{r}$ as a function of the local Euclidean coordinates. The relation between the variables $\tilde{R}$ and $\tilde{r}$ is thus given by

$$\tilde{R}_{\alpha\beta\gamma\delta}(x,P)=\int d^4x'V_\alpha{}^\lambda(x,P)V_\beta{}^\mu(x,P)V_\gamma{}^\nu(x,P) \times V_\delta{}^\rho(x,P)W(x,P,x')\tilde{r}_{\lambda\mu\nu\rho}(x'). \quad (4.15)$$

Finally, then, the Eqs. (4.1), (4.2), (4.9), (4.10), and (4.15) can be used to calculate the components of the path-dependent Riemann tensor $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ in terms of the components $\tilde{g}_{\lambda\mu}(x')$ of the metric tensor, to any order of perturbation theory.

It is now necessary to show directly that the above definitions of $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ in terms of $\tilde{g}_{\lambda\mu}(x')$ do lead to the path-dependence Eq. (3.6). Such a result can certainly be anticipated, since the definitions are valid in the classical theory, where the path-dependence equation is true. We have carried out the proof in the Appendix for those readers who wish to see an explicit demonstration. We are thus justified in relating the path-dependent operator $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ to the path-independent operators $\tilde{g}_{\lambda\mu}(x')$ in the manner outlined above.

The operator $\eta^{\lambda\mu}(x)$ is defined in the same way as the operators η in electrodynamics and the Yang-Mills field:

$$[\eta^{\lambda\mu}(x'),\tilde{g}_{\rho\nu}(y')] =-(2\kappa)^{1/2}(\delta_\nu{}^\lambda\delta_\rho{}^\mu+\delta_\rho{}^\lambda\delta_\nu{}^\mu)\delta^4(x'-y'), \quad (4.16a)$$

$$(H_0|\eta^{\lambda\mu}(x')=0. \quad (4.16b)$$

We shall require an expression for the path-dependent operator $U_{\beta\delta}(x,P)$, defined by (3.3), in terms of $\eta^{\lambda\mu}(x')$ and $\tilde{g}_{\lambda\mu}(x')$, before we can write the equations of motion in terms of the auxiliary variables. In the Appendix we shall show that

$$2^{-1/2}\kappa^{1/2}\int d^4x'V_{\lambda\alpha}(x,P)V_{\mu\beta}(x,P)W(x,P,x') \times[\tilde{g}(x')]^{-1/2}\eta^{\lambda\mu}(x')\approx U_{\beta\delta}(x,P), \quad (4.17)$$

in the sense that both sides of (4.17) have the same commutation relations (3.3) with the operators, $\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')$. Equation (4.17) is precisely analogous to a result in the previous paper, where it was shown that the operator $V_{\alpha\gamma}(x,P)\eta_\nu{}^\gamma(x)$ was a possible choice for the operator $U_\nu{}^\alpha(x,P)$ of the Yang-Mills field. The proof is carried through in the same way, though the algebra is somewhat long.

We have emphasized that all formulas in this section are to be expanded as a perturbation series in the $\tilde{\phi}$'s. They can then be rewritten as formulas for the auxiliary Green's functions. The method is simply to apply the formula to one of the vectors $(H|$ in the dual space and then to take the scalar product with the vector $|G)$. We have given examples in the previous paper [Eqs. (3.25)–(3.29)] and we need not repeat them here. The perturbation formula for the $\tilde{R}$'s in terms of the $\tilde{\phi}$'s is thus simply a shorthand for the formulas expressing the path-dependent Green's functions in terms of the auxiliary Green's functions. It remains to find the field equations which the $\tilde{g}$'s must satisfy in order that the $\tilde{R}$'s should satisfy the field equations (3.5). This is equivalent to finding the equations which the auxiliary Green's functions must satisfy in order to ensure that the path-dependent Green's functions satisfy the required equations.

V. GAUGE TRANSFORMATIONS

One can define gauge transformations for our auxiliary variables; such transformations are analogous to the gauge transformations of the electromagnetic and Yang-Mills fields. They are identical in form to the general coordinate transformation of the classical gravitational field.

The definition of the gauge transformations is as follows:

$$\tilde{g}_{\lambda\mu}(x') \to \tilde{g}_{\lambda\mu}(x') + \lambda\left(-\tilde{g}_{\lambda\nu}(x') \frac{\partial \chi^\nu(x')}{\partial x'^\mu} - \tilde{g}_{\nu\mu}(x') \times \frac{\partial \chi^\nu(x')}{\partial x'^\lambda} - \frac{\partial \tilde{g}_{\lambda\mu}(x')}{\partial x'^\nu} \chi^\nu(x') \right). \quad (5.1)$$

The functions $[\tilde{g}(x')]^{-1/2}$ and $\tilde{\Gamma}(x')$ will then undergo the following transformations:

$$[\tilde{g}(x')]^{-1/2} \to [\tilde{g}(x')]^{-1/2} + \lambda [\tilde{g}(x')]^{-1/2} \times \frac{\partial \chi^\nu(x')}{\partial x'^\nu} - \lambda\left(\frac{\partial}{\partial x'^\nu} [\tilde{g}(x')]^{-1/2} \right) \chi^\nu(x'), \quad (5.2)$$

$$\tilde{\Gamma}_{\lambda\mu}{}^\kappa(x') \to \tilde{\Gamma}_{\lambda\mu}{}^\kappa(x') + \lambda\left(-\tilde{\Gamma}_{\lambda\nu}{}^\kappa(x') \frac{\partial \chi^\nu(x')}{\partial x'^\mu} - \tilde{\Gamma}_{\nu\mu}{}^\kappa(x') \frac{\partial \chi^\nu(x')}{\partial x'^\lambda} + \tilde{\Gamma}_{\lambda\mu}{}^\nu(x') \frac{\partial \chi^\kappa(x')}{\partial x'^\nu} - \left(\frac{\partial}{\partial x'^\nu} \tilde{\Gamma}_{\lambda\mu}{}^\kappa(x') \right) \chi^\nu(x') - \frac{\partial^2 \chi^\kappa(x')}{\partial x'^\lambda \partial x'^\mu} \right), \quad (5.3)$$

and the path-independent Riemann tensor transforms as a tensor:

$$\tilde{r}_{\lambda\mu\nu\rho}(x') \to \tilde{r}_{\lambda\mu\nu\rho}(x') + \lambda\left(-\tilde{r}_{\sigma\mu\nu\rho}(x') \frac{\partial}{\partial x'^\lambda} - \tilde{r}_{\lambda\sigma\nu\rho}(x') \frac{\partial}{\partial x'^\mu} - \tilde{r}_{\lambda\mu\sigma\rho}(x') \frac{\partial}{\partial x'^\nu} - \tilde{r}_{\lambda\mu\nu\sigma}(x') \frac{\partial}{\partial x'^\rho} \right) \chi^\sigma(x') - \lambda\left(\frac{\partial}{\partial x'^\sigma} \tilde{r}_{\lambda\mu\nu\rho}(x') \right) \chi^\sigma(x'). \quad (5.4)$$

The functions V and W will transform as follows under the transformation (5.1):

$$V_\alpha{}^\lambda(x,P) \to V_\alpha{}^\lambda(x,P) + \int dx' V_\alpha{}^\nu(x,P) \times W(x,P,x') \frac{\partial \chi^\lambda(x')}{\partial x'^\nu}, \quad (5.5)$$

$$W(x,P,x') \to W(x,P,x') - \frac{\partial}{\partial x'^\nu} (W(x,P,x') \chi^\nu(x')). \quad (5.6)$$

Equations (5.5) and (5.6) are proved by substituting them into Eqs. (4.9) and (4.10) which define the functions V and W. When V and W undergo the transformations (5.5) and (5.6), and $\tilde{\Gamma}$ undergoes the transformation (5.3), Eqs. (4.9) and (4.10) remain valid.

From Eqs. (4.15), (5.4), (5.5), and (5.6) one can conclude that the path-dependent Riemann tensor $\tilde{R}_{\alpha\beta\gamma\delta}$ remains unchanged under the transformation (5.1). The transformation does therefore possess the physical significance of a gauge transformation.

We shall also require to find the effect of a gauge transformation on the function $V_{\lambda\alpha}(x,P)$. From (5.5) and (4.14) it follows at once that

$$V_{\lambda\alpha}(x,P) \to V_{\lambda\alpha}(x,P) - \int dx' V_{\nu\alpha}(x,P) \times W(x,P,x') \frac{\partial \chi^\nu(x')}{\partial x'^\lambda}. \quad (5.7)$$

The operator in our linear space which effects the transformation (5.1) is

$$Y_\nu(y') = -(2\kappa)^{-1/2} \frac{\partial}{\partial y'^\lambda} [\eta^{\lambda\mu}(y') \tilde{g}_{\nu\mu}(y')] + \tfrac{1}{2}(2\kappa)^{-1/2} \eta^{\lambda\mu}(y') \partial \tilde{g}_{\lambda\mu}(y') / \partial y'^\nu. \quad (5.8)$$

When the integral $\lambda \int dy' Y_\nu(y') \chi^\nu(y')$ is commuted with $\tilde{g}_{\lambda\mu}(x')$, the result is equal to the right side of (4.1). It follows that

$$\left[\int d^4y' Y_\nu(y') \chi^\nu(y'), [\tilde{g}(x')]^{-1/2} \right] = [\tilde{g}(x')]^{-1/2} \frac{\partial \chi^\nu(x')}{\partial x'^\nu} - \left(\frac{\partial}{\partial x'^\nu} [\tilde{g}(x')]^{-1/2} \right) \chi^\nu(x'), \quad (5.9)$$

$$\left[\int d^4y' Y_\nu(y')\chi^\nu(y'), V_{\lambda\alpha}(x,P)\right] = -\int d^4x' V_{\nu\alpha}(x,P)W(x,P,x')\frac{\partial\chi^\nu(x')}{\partial x'^\lambda}, \quad (5.10)$$

$$\left[\int d^4y' Y_\nu(y')\chi^\nu(y'), W(x,P,x')\right] = -\frac{\partial}{\partial x'^\nu}[W(x,P,x')\chi^\nu(x')], \quad (5.11)$$

$$\left[\int d^4y' Y_\nu(y')\chi^\nu(y'), \tilde{R}_{\alpha\beta\gamma\delta}(x,P)\right] = 0. \quad (5.12)$$

VI. FIELD EQUATIONS FOR AUXILIARY VARIABLES

We can now rewrite the field equations (3.5) as equations for our auxiliary path-independent variables $\tilde{g}_{\lambda\mu}(x')$. According to (4.15) and (4.14) we may rewrite (3.5) in the form

$$\left(\int d^4x' V_{\lambda\beta}(x,P)V_{\mu\delta}(x,P)V_{\mu\delta}(x,P)W(x,P,x') \times\tilde{m}^{\lambda\mu}(x') - iU_{\beta\delta}(x,P)\right)|G) = 0, \quad (6.1)$$

where $\tilde{m}^{\lambda\mu}$ is the path-independent Einstein tensor

$$\tilde{m}^{\lambda\mu}(x') = \tilde{r}^{\nu\lambda}{}_\nu{}^\mu(x') - \tfrac{1}{2}\tilde{g}^{\lambda\mu}(x')\tilde{r}^{\nu\rho}{}_{\nu\rho}(x'). \quad (6.2)$$

The function $U_{\beta\delta}$ is defined by its commutation relations (3.3). We have seen (Eq. 4.17) that the operator

$$2^{-1/2}\kappa^{1/2}\int d^4x' V_{\lambda\beta}(x,P)V_{\mu\delta}(x,P) \times W(x,P,x')[\tilde{g}(x')]^{-1/2}\eta^{\lambda\mu}(x')$$

satisfied those commutation relations. However, as with the electromagnetic and Yang-Mills fields, the commutation relations do not define $U(x)$ uniquely in the enlarged linear space of our auxiliary variables. In particular, the quantity $\int d^4y' Y_\nu(y')\chi^\nu(y')$ commutes with $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$, and we may add such a quantity to the function U without changing its commutation relations with the elements of the path-dependent Riemann tensor. We shall see below that it is necessary to add such a quantity in order to satisfy a consistency condition.

We therefore write the following formula for $U(x,P)$:

$$\begin{aligned}U_{\beta\delta}(x,P) = {}& 2^{-1/2}\kappa^{1/2}\int d^4x' V_{\lambda\beta}(x,P)V_{\mu\delta}(x,P) \\ &\times W(x,P,x')[\tilde{g}(x')]^{-1/2}\eta^{\lambda\mu}(x') \\ &+2^{-1/2}\kappa^{1/2}\int d^4x' \int d^4y' Y_\nu(y')V_{\lambda\beta}(x,P) \\ &\times V_{\mu\delta}(x,P)W(x,P,x') \\ &\times[\tilde{g}(x')]^{-1/2}\chi^{\lambda\mu,\nu}(x',y'), \quad (6.3)\end{aligned}$$

where the function $\chi^{\lambda\mu,\nu}(x',y')$ is at the moment an arbitrary function of the $\tilde{g}$'s. The right side of (6.3) satisfies both conditions (3.3) which define the operator U. We should like to commute the operator $Y_\nu(y')$ in the second term of (6.3) through the operators $V_{\lambda\beta}(x,P)$, $V_{\mu\delta}(x,P)$, $W(x,P,x')$, and $[\tilde{g}(x')]^{-1/2}$, since all terms of (6.1) would then have the factors V and W in front of the other factors. We can easily do so by using (5.9)–(5.11), and we obtain

$$U_{\beta\delta}(x,P) = 2^{-1/2}\kappa^{1/2}\int d^4x' V_{\lambda\beta}(x,P)V_{\mu\delta}(x,P) \times W(x,P,x')[\tilde{g}(x')]^{-1/2}\theta^{\lambda\mu}(x'), \quad (6.4)$$

where

$$\theta^{\lambda\mu}(x') = \eta^{\lambda\mu}(x') + \int d^4y' Y_\nu(y')\chi^{\lambda\mu,\nu}(x',y') + \left(-\frac{\partial\chi^{\nu\mu,\lambda}(x',y')}{\partial y'^\nu} - \frac{\partial\chi^{\lambda\nu,\mu}(x',y')}{\partial y'^\nu} + \frac{\partial\chi^{\lambda\mu,\nu}(x',y')}{\partial y'^\nu} + \frac{\partial\chi^{\lambda\mu,\nu}(x',y')}{\partial x'^\nu}\right)\Bigg|_{x'=y'}. \quad (6.5)$$

We can now substitute (6.4) in (6.1) to give

$$\int dx' V_{\lambda\beta}(x,P)V_{\mu\delta}(x,P)W(x,P,x')[\tilde{g}(x')]^{-1/2} \times\{[\tilde{g}(x')]^{1/2}\tilde{m}^{\lambda\mu}(x') - i2^{-1/2}\kappa^{1/2}\theta^{\lambda\mu}(x')\}|G) = 0, \quad (6.6)$$

and therefore

$$\{[\tilde{g}(x')]^{1/2}\tilde{m}^{\lambda\mu}(x') - i2^{-1/2}\kappa^{1/2}\theta^{\lambda\mu}(x')\}|G) = 0. \quad (6.7)$$

Equation (6.7) is a sufficient condition for (6.6) and, if the integral operator $\int dx' V_{\lambda\beta}(x,P)V_{\mu\delta}(x,P)W(x,P,x') \times[\tilde{g}(x')]^{-1/2}$ has a reciprocal, as it does in perturbation theory, it is also a necessary condition. We shall therefore adopt (6.7) as our field equations. The function θ is given by (6.5), with χ still to be determined. In Eq. (6.7) we have finally eliminated the path-dependent variables.

As in all gauge theories, the function $\theta^{\lambda\mu}(x')$ in (6.7) is limited by a consistency condition. From the fact that the covariant divergence of the Einstein tensor vanishes identically, we can show that

$$\frac{\partial}{\partial x'^\lambda}[\tilde{g}_{\mu\nu}(x')\theta^{\lambda\nu}(x')] - \frac{1}{2}\frac{\tilde{g}_{\lambda\nu}(x')}{\partial x'^\mu}\theta^{\lambda\nu}(x') = 0. \quad (6.8)$$

We may rewrite (6.8) in the following form, which bears a resemblance to the corresponding equation for the Yang-Mills field:

$$\delta_{\mu\nu}\frac{\partial\theta^{\lambda\nu}(x')}{\partial x'^\lambda} + (2\kappa)^{1/2}\frac{\partial}{\partial x'^\lambda}[\tilde{\phi}_{\mu\nu}(x')\theta^{\lambda\nu}(x')] - \tfrac{1}{2}(2\kappa)^{1/2}\frac{\partial\tilde{\phi}_{\mu\nu}(x')}{\partial x'^\mu}\theta^{\lambda\nu}(x') = 0. \quad (6.9)$$

The new field variables $\tilde{\phi}$ are defined by (4.3). We have to determine the function χ in (6.5) so that (6.9) is satisfied.

We begin by looking for a function $\theta_1{}^{\lambda\mu}$ which has the form $\eta^{\lambda\mu}+\theta'^{\lambda\mu}$ and which satisfies (6.9). The function $\theta'^{\lambda\mu}$ will not have the form of the second term of (6.5), but we shall then be able to modify it to bring it into this form. By analogy with the development for the electromagnetic and Yang-Mills fields, we might try a function $\theta_1{}^{\lambda\mu}$ of the form $\eta^{\lambda\mu}-(\partial/\partial x'^\lambda)h^\mu$. Such a trial function is not symmetric in μ and ν, however, and we replace it by the trial function

$$\theta_1{}^{\lambda\mu}(x')=\eta^{\lambda\mu}(x') -\left(\frac{\partial}{\partial x_\lambda'}\delta_\nu{}^\mu+\frac{\partial}{\partial x_\mu'}\delta_\nu{}^\lambda-\frac{\partial}{\partial x'^\nu}\delta^{\lambda\mu}\right)H^\nu(x'). \quad (6.10)$$

We adopt the notation $\delta^{\lambda\mu}=\delta_\mu{}^\lambda$, $x_\nu'=x'^\nu$ simply to keep our formulas conventional with regard to upper and lower indices. [If we were using a Lorentz metric with real time, the Kronecker delta δ^{ij} would be equal to $-\delta_j{}^i$, while x_i' would be equal to $-x'^i$ ($i=1,2,3$).] The consistency condition (6.9) when applied to (6.10) leads to

$$\left[\frac{\partial^2}{\partial x'^\lambda\partial x_\lambda'}\delta_{\mu\nu}+(2\kappa)^{1/2}\left(\frac{\partial}{\partial x'^\lambda}\tilde{\phi}_{\mu\nu}\frac{\partial}{\partial x_\lambda'}+\frac{\partial}{\partial x'^\nu}\tilde{\phi}_{\mu\lambda}\frac{\partial}{\partial x'_\lambda} -\frac{\partial}{\partial x_\lambda'}\tilde{\phi}_{\mu\lambda}\frac{\partial}{\partial x'^\nu}-\frac{\partial\phi_{\lambda\nu}}{\partial x'^\mu}\frac{\partial}{\partial x_\lambda'}+\tfrac{1}{2}\delta^{\lambda\rho}\frac{\partial\tilde{\phi}_{\lambda\rho}}{\partial x'^\mu}\frac{\partial}{\partial x'^\nu}\right)\right]H^\nu(x')$$
$$=\left(\delta_{\mu\nu}\frac{\partial}{\partial x'^\lambda}+(2\kappa)^{1/2}\frac{\partial}{\partial x'^\lambda}\tilde{\phi}_{\mu\nu} -\tfrac{1}{2}(2\kappa)^{1/2}\frac{\partial\tilde{\phi}_{\lambda\nu}}{\partial x'^\mu}\right)\eta^{\lambda\nu}(x'). \quad (6.11)$$

We shall therefore define a function $O^{\nu\rho}(x',y')$ by

$$\left(\frac{\partial^2}{\partial x'^\lambda\partial x_\lambda'}\delta_{\mu\nu}+(2\kappa)^{1/2}\left\{\frac{\partial}{\partial x'^\lambda}\tilde{\phi}_{\mu\nu}\frac{\partial}{\partial x_\lambda'}+\frac{\partial}{\partial x'^\nu}\tilde{\phi}_{\mu\lambda}\frac{\partial}{\partial x_\lambda'} -\frac{\partial}{\partial x_\lambda'}\tilde{\phi}_{\mu\lambda}\frac{\partial}{\partial x'^\nu}-\frac{\partial\tilde{\phi}_{\lambda\nu}}{\partial x'^\mu}\frac{\partial}{\partial x_\lambda'}+\tfrac{1}{2}\delta^{\lambda\sigma}\frac{\partial\tilde{\phi}_{\lambda\sigma}}{\partial x'^\mu}\frac{\partial}{\partial x'^\nu}\right\}\right) \times O^{\nu\rho}(x',y')=\delta_\mu{}^\rho\delta^4(x'-y'). \quad (6.12)$$

From (6.11) and (6.12), the function H^ν will be given by

$$H^\nu(x')=\int dy'O^{\nu\rho}(x',y')\left(\delta_{\rho\tau}\frac{\partial}{\partial y'^\sigma}+(2\kappa)^{1/2}\frac{\partial}{\partial y'^\sigma}\tilde{\phi}_{\rho\tau}(y') -\tfrac{1}{2}(2\kappa)^{1/2}\frac{\partial\tilde{\phi}_{\sigma\tau}(y')}{\partial y'^\tau}\right)\eta^{\sigma\tau}(y'). \quad (6.13)$$

Thus, from (6.10),

$$\theta_1{}^{\lambda\mu}(x')=\eta^{\lambda\mu}(x')-\left(\frac{\partial}{\partial x_\lambda'}\delta_\nu{}^\mu+\frac{\partial}{\partial x_\mu'}\delta_\nu{}^\lambda-\frac{\partial}{\partial x'^\nu}\delta^{\lambda\mu}\right) \times\int d^4y'O^{\nu\rho}(x',y')\left(\delta_{\rho\tau}\frac{\partial}{\partial y'^\sigma}+(2\kappa)^{1/2}\frac{\partial}{\partial y'^\sigma}\tilde{\phi}_{\rho\tau}(y') -\tfrac{1}{2}(2\kappa)^{1/2}\frac{\partial\tilde{\phi}_{\sigma\tau}(y')}{\partial y'^\rho}\right)\eta^{\sigma\tau}(y'). \quad (6.14)$$

The choice (6.14) for $\theta_1{}^{\lambda\mu}(x')$ satisfies the consistency condition (6.9), but it does not quite have the required form (6.5). The last factor of (6.14) is just equal to $-(2\kappa)^{1/2}$ times the operator $Y_\sigma(y')$, apart from factor ordering. The difference between the right sides of (6.14) and (6.5) is, first that the factor $\eta^{\sigma\tau}(y)$ in (6.14) is ordered to the right of the other factors instead of to the left, and second that the last term of (6.5) is absent. We therefore take the following choice for $\theta^{\lambda\mu}(x')$, which is of the form (6.5):

$$\theta^{\lambda\mu}(x')=\eta^{\lambda\mu}(x')+\int d^4y'\left(\delta_{\rho\tau}\frac{\partial\eta^{\sigma\tau}(y')}{\partial y'^\sigma}+(2\kappa)^{1/2}\frac{\partial}{\partial y'^\sigma}[\eta^{\sigma\tau}(y')\tilde{\phi}_{\rho\tau}(y')]-\tfrac{1}{2}(2\kappa)^{1/2}\eta^{\sigma\tau}(y')\frac{\partial\tilde{\phi}_{\sigma\tau}(y')}{\partial y'^\rho}\right)\chi^{\lambda\mu,\rho}(x',y') +(2\kappa)^{1/2}\left(-\frac{\partial\chi^{\rho\mu,\lambda}(x',y')}{\partial y'^\rho}-\frac{\partial\chi^{\lambda\rho,\mu}(x',y')}{\partial\lambda'^\rho}+\frac{\partial\chi^{\lambda\mu,\rho}(x',y')}{\partial y'^\rho}+\frac{\partial\chi^{\lambda\mu,\rho}(x',y')}{\partial x'^\rho}\right)\Bigg|_{y'=x'}, \quad (6.15a)$$

where

$$\chi^{\lambda\mu,\rho}(x',y')=-\left(\frac{\partial}{\partial x_\lambda'}\delta_\nu{}^\mu+\frac{\partial}{\partial x_\mu'}\delta_\nu{}^\lambda-\frac{\partial}{\partial x'^\nu}\delta^{\lambda\mu}\right)O^{\nu\rho}(x',y'). \quad (6.15b)$$

Having made this choice for θ, it is necessary to test the self-consistency condition (6.9). If we substitute (6.15) in (6.9) we find, after some straightforward but tedious algebra,

$$\delta_{\mu\nu}\frac{\partial\theta^{\lambda\nu}(x')}{\partial x'^\lambda}+(2\kappa)^{1/2}\frac{\partial}{\partial x'^\lambda}[\tilde{\phi}_{\mu\nu}(x')\theta^{\lambda\nu}(x')]-\tfrac{1}{2}(2\kappa)^{1/2}\frac{\partial\tilde{\phi}_{\mu\nu}(x')}{\partial x'^\mu}\theta^{\lambda\nu}(x') =4(2\kappa)^{1/2}\frac{\partial}{\partial x'^\mu}\delta^4(x'-y')\,|_{x'=y'}-4(2\kappa)^{1/2}\frac{\partial}{\partial x'^\mu}\delta(x'-x'). \quad (6.16)$$

The right side of (6.16) is of course undefined, but if we subtract infinities in the usual perturbation-theory manner we shall get zero. In momentum space to the right side of (6.16) would have been $4i(2\kappa)^{1/2}\int pd^4p - 4i(2\kappa)^{1/2}\int pd^4p$ and we would normally have set the result equal to zero. Since we are only attempting to obtain results within the heuristic framework of Feynman-diagram perturbation theory, we can set the right side of (6.16) equal to zero, and the consistency condition (6.9) is proved.

Finally, therefore, we can take (6.7) as our field equations, with the function θ given by (6.15) and the function $O^{\nu\rho}(x,'y')$ defined in turn by (6.12). These equations are self-consistent and they imply the validity of our path-dependent field equations (3.5).

VII. FEYNMAN RULES FOR GRAVITATIONAL FIELD

To determine the Feynman rules we must separate (6.7) into those terms which contain powers of $\kappa^{1/2}$ and those which do not. Thus,

$$\left(\tfrac{1}{2}(\delta^{\lambda\nu}\delta^{\mu\rho}+\delta^{\lambda\rho}\delta^{\mu\nu}-\delta^{\lambda\mu}\delta^{\nu\rho})\delta^{\sigma\tau}\underset{\nu\leftrightarrow\sigma,\rho\leftrightarrow\tau}{A}\frac{\partial^2\tilde{\phi}_{\nu\rho}}{\partial x'^\sigma\partial x'^\tau}+(2\kappa)^{1/2}\tilde{t}^{\lambda\mu}-i\theta^{\lambda\mu}\right)|G)=0, \tag{7.1}$$

where $\tilde{t}^{\lambda\mu}$ is κ^{-1} times that part of the Einstein tensor density $\tilde{g}^{1/2}\tilde{m}^{\lambda\mu}$ which contains at least one power of κ. Equation (7.1) can be integrated to give

$$\Big[\tilde{\phi}_{\lambda\mu}(x')-\tfrac{1}{2}i(2\kappa)^{1/2}(\delta_{\lambda\sigma}\delta_{\mu\tau}+\delta_{\lambda\tau}\delta_{\mu\sigma}-\delta_{\lambda\mu}\delta_{\sigma\tau})\int dx''\tfrac{1}{2}\Delta_F(x'-x'')\tilde{t}^{\sigma\tau}(x'')$$

$$-\tfrac{1}{2}(\delta_{\lambda\sigma}\delta_{\mu\tau}+\delta_{\lambda\tau}\delta_{\mu\sigma}-\delta_{\lambda\mu}\delta_{\sigma\tau})\int dx''\tfrac{1}{2}\Delta_F(x'-x'')\eta^{\sigma\tau}(x'')-\tfrac{1}{2}(2\kappa)^{1/2}(\delta_{\lambda\sigma}\delta_{\mu\tau}+\delta_{\lambda\tau}\delta_{\mu\sigma}-\delta_{\lambda\mu}\delta_{\sigma\tau})\int dx''\tfrac{1}{2}\Delta_F(x'-x'')$$

$$\times\left(-\frac{\partial\chi^{\rho\tau,\sigma}(x'',y')}{\partial y'^\rho}-\frac{\partial\chi^{\sigma\rho,\tau}(x'',y')}{\partial y'^\rho}+\frac{\partial\chi^{\sigma\tau,\rho}(x'',y')}{\partial y'^\rho}+\frac{\partial\chi^{\sigma\tau,\rho}(x'',y')}{\partial x''^\rho}\right)_{x''=y'}\Big]|G)=0. \tag{7.2}$$

In writing (7.2) we have used Eq. (6.15) for θ and have omitted the second term on the right of (6.15a), since the contribution to the left side of (7.2) would be a pure divergence. One can obtain more general gauges by making the replacement

$$\delta_{\lambda\sigma}\delta_{\mu\tau}+\delta_{\lambda\tau}\delta_{\mu\sigma}-\delta_{\lambda\mu}\delta_{\sigma\tau}\rightarrow\delta_{\lambda\sigma}\delta_{\mu\tau}+\delta_{\lambda\tau}\delta_{\mu\sigma}-\delta_{\lambda\mu}\delta_{\sigma\tau}$$

$$-a\left\{\frac{\partial^2}{\partial x'^\lambda\partial x'^\sigma}\delta_{\mu\tau}+\frac{\partial^2}{\partial x'^\mu\partial x'^\tau}\delta_{\lambda\sigma}+\frac{\partial^2}{\partial x'^\lambda\partial x'^\tau}\delta_{\mu\sigma}+\frac{\partial^2}{\partial x'^\mu\partial x'^\sigma}\delta_{\lambda\tau}\right\}\Box^{-2}+b\frac{\partial^4}{\partial x'^\sigma\partial x'^\mu\partial x'^\lambda\partial x'^\tau}\Box^{-2}\Box^{-2}. \tag{7.3}$$

The first three terms on the right of (7.2) are the terms we would have obtained by an uncritical use of the analog of the Lorentz gauge. In this method one writes the Einstein Lagrangian in the form

$$\mathcal{L}=-\tfrac{1}{8}(\delta^{\lambda\nu}\delta^{\mu\rho}+\delta^{\lambda\rho}\delta^{\mu\nu}-\delta^{\lambda\mu}\delta^{\nu\rho})\times\delta^{\sigma\tau}\underset{\nu\leftrightarrow\sigma,\rho\leftrightarrow\tau}{A}\left(\frac{\partial\phi_{\lambda\mu}}{\partial x^\sigma}\right)\left(\frac{\partial\phi_{\nu\rho}}{\partial x^\tau}\right)+\mathcal{L}_{\text{int}}, \tag{7.4}$$

where $\mathcal{L}_{\text{int}}$ represents all terms in the expanded Lagrangian which contain at least one factor of $\kappa^{1/2}$. One then uses the supplementary condition

$$(\delta^{\lambda\nu}\delta^{\mu\rho}+\delta^{\lambda\rho}\delta^{\mu\nu}-\delta^{\lambda\mu}\delta^{\nu\rho})\partial\phi_{\lambda\mu}/\partial x^\nu=0. \tag{7.5}$$

to rewrite the Lagrangian in the form

$$\mathcal{L}=-\tfrac{1}{8}(\delta^{\lambda\nu}\delta^{\mu\rho}+\delta^{\lambda\rho}\delta^{\mu\nu}-\delta^{\lambda\mu}\delta^{\nu\rho})\delta^{\sigma\tau}\times(\partial\phi_{\lambda\mu}/\partial x^\sigma)(\partial\phi_{\nu\rho}/\partial x^\tau)+\mathcal{L}_{\text{int}}. \tag{7.6}$$

From the Lagrangian (7.6) one can construct Feynman rules by following the standard procedure. The function $\mathcal{L}_{\text{int}}$ is an infinite power series in the variables $\phi_{\mu\nu}$, so that there exist vertices with an arbitrary large number of external lines. It is in principle straightforward to find the factors associated with the m-point vertex, but the algebraic complexity of the result increases rapidly with n. We shall not carry through the algebra here; it has been done in Ref. 3 for $n=3$ and $n=4$. The graviton propagator associated with the zero-order Lagrangian in (7.6) is simply

$$G_{\lambda\mu,\sigma\tau}(p)=\frac{i}{(2\pi)^4}(\delta_{\lambda\sigma}\delta_{\mu\tau}+\delta_{\lambda\tau}\delta_{\mu\sigma}-\delta_{\lambda\mu}\delta_{\sigma\tau})\frac{1}{-p^2+i\epsilon}. \tag{7.7}$$

It is thus equal to the coefficient of $\tilde{t}^{\sigma\tau}$ on the left of (7.2).

The presence of the last term on the left of (7.2) shows that the simple Feynman prescription is not correct and that additional vertices are necessary. We begin by expanding the function $O^{\nu\rho}(x',y')$, which is defined by (6.12) and which occurs in the function $\chi^{\lambda\mu,\rho}(x',y')$, as a perturbation series:

$$O^{\nu\rho}(x',y')=i\sum_{n=1}^{\infty}\int dx_1'\cdots dx_n'\tfrac{1}{2}\Delta_F(x'-x_1')$$
$$\times\delta^{\nu\rho_1}i(2\kappa)^{1/2}D^{\lambda_1\mu_1}{}_{\rho_1\nu_1}\tilde{\phi}_{\lambda_1\mu_1}(x_1')\tfrac{1}{2}\Delta_F(x_1'-x_2')$$
$$\times\delta^{\nu_1\rho_2}i(2\kappa)^{1/2}D^{\lambda_2\mu_2}{}_{\rho_2\nu_2}\tilde{\phi}_{\lambda_2\mu_2}(x_2')\cdots$$
$$\tfrac{1}{2}\Delta_F(x_n'-y')\delta^{\nu_n\rho},\quad(7.8\text{a})$$

where

$$D^{\lambda\mu}{}_{\rho\nu}\tilde{\phi}_{\lambda\mu}=\delta_\rho{}^\lambda\delta_\nu{}^\mu\frac{\partial}{\partial x'^\sigma}\tilde{\phi}_{\lambda\mu}(x')\frac{\partial}{\partial x'_\sigma}+\delta_\rho{}^\lambda\frac{\partial}{\partial x'^\nu}\tilde{\phi}_{\lambda\mu}(x')\frac{\partial}{\partial x_\mu'}$$
$$-\delta_\rho{}^\lambda\frac{\partial}{\partial x_\mu'}\tilde{\phi}_{\lambda\mu}(x')\frac{\partial}{\partial x'^\nu}-\delta_\nu{}^\mu\frac{\partial\tilde{\phi}_{\lambda\mu}(x')}{\partial x'^\rho}\frac{\partial}{\partial x_\lambda'}$$
$$+\tfrac{1}{2}\delta^{\lambda\mu}\frac{\partial\tilde{\phi}_{\lambda\mu}(x')}{\partial x'^\rho}\frac{\partial}{\partial x_\nu'}.\quad(7.8\text{b})$$

The expression (7.8) can now be substituted in (6.15b), which in turn can be substituted in the last term of (7.2) to give the result

$$\tfrac{1}{2}i(2\kappa)^{1/2}(\delta_{\lambda\sigma}\delta_{\mu\tau}+\delta_{\lambda\tau}\delta_{\mu\sigma}-\delta_{\lambda\mu}\delta_{\sigma\tau})\sum_{n=1}^{\infty}\int dx''dx_1'\cdots$$
$$dx_n'dy'\tfrac{1}{2}\Delta_F(x'-x'')\delta^4(x''-y')(\bar{D}^{\sigma\tau}{}_{\rho\nu}+\bar{D}^{\tau\sigma}{}_{\rho\nu})$$
$$\times\tfrac{1}{2}\Delta_F(x''-x_1')\delta^{\nu\rho_1}i(2\kappa)^{1/2}D^{\lambda_1\mu_1}{}_{\rho_1\nu_1}\tilde{\phi}_{\lambda_1\mu_1}(x_1')$$
$$\times\tfrac{1}{2}\Delta_F(x_1'-x_2')\delta^{\nu_1\rho_2}i(2\kappa)^{1/2}D^{\lambda_2\mu_2}{}_{\rho_2\nu_2}\tilde{\phi}_{\lambda_2\mu_2}(x_2')\cdots$$
$$\tfrac{1}{2}\Delta_F(x_n'-y')\delta^{\nu_n\rho},\quad(7.9)$$

where $D^{\lambda\mu}{}_{\rho\nu}$ is given by (7.8b) and

$$\bar{D}^{\lambda\mu}{}_{\rho\nu}=-\delta_\rho{}^\lambda\delta_\nu{}^\mu\frac{\partial}{\partial y'^\sigma}\frac{\partial}{\partial x_\sigma'}-\delta_\rho{}^\lambda\frac{\partial}{\partial y'^\nu}\frac{\partial}{\partial x_\mu'}+\delta_\rho{}^\lambda\frac{\partial}{\partial y_\mu'}\frac{\partial}{\partial x'^\nu}$$
$$+\delta_\nu{}^\mu\left(\frac{\partial}{\partial x'^\rho}+\frac{\partial}{\partial y'^\rho}\right)\frac{\partial}{\partial x_\lambda'}-\tfrac{1}{2}\delta^{\lambda\mu}\left(\frac{\partial}{\partial x'^\rho}+\frac{\partial}{\partial y'^\rho}\right)\frac{\partial}{\partial x_\nu'}.\quad(7.10)$$

The expression (7.9) is similar in form to the analogous term in the equation for the Yang-Mills field. Again, one can represent the term diagramatically by an $(n+1)$-sided polygon (Fig. 4), which is inserted into the Feynman diagram in all possible ways. The solid lines represent the gravitons, while the dashed lines are associated with the factor $\tfrac{1}{2}\Delta_F(x_r'-x_{r+1}')\delta^{\nu_r\rho_{r+1}}$ in (7.9).

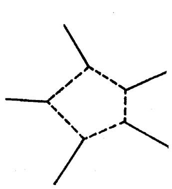

FIG. 4. A Feynman vertex corresponding to the last terms of (8.2.)

Thus, in momentum space,

$$\text{dashed-line factor}=\frac{i}{(2\pi)^4}\frac{\delta^{\nu\rho}}{-p^2+i\epsilon}.\quad(7.11\text{a})$$

The vertex factor is obtained by transcribing the factor $i(2\kappa)^{1/2}D^{\lambda\mu}{}_{\rho\nu}$, or $i(2\kappa)^{1/2}\bar{D}^{\lambda\mu}{}_{\rho\nu}$, in (7.9) into momentum space. Thus the vertex factor is given by

$$w(p_1\rho,p_2\lambda\mu,p_3\nu)=i(2\pi)^4(2\kappa)^{1/2}(\delta_\rho{}^\lambda\delta_\nu{}^\mu p_1\cdot p_3+\delta_\rho{}^\lambda p_{1\nu}p_3{}^\mu$$
$$-\delta_\rho{}^\lambda p_1{}^\mu p_{3\nu}+\delta_\nu{}^\mu p_{2\rho}p_3{}^\lambda-\tfrac{1}{2}\delta^{\lambda\mu}p_{2\rho}p_3{}^\nu).\quad(7.11\text{b})$$

In (7.11b) the quantities $p_1\rho$ and $p_3\nu$ are associated with the dashed lines, and the quantities $p_2\lambda\mu$ with the solid line, meeting at the vertex. If we wish we may symmetrize the vertex factor in λ and μ. Finally, from (7.2) and (7.9), we find

$$\text{an over-all factor }-1.\quad(7.11\text{c})$$

As a mnemonic device we can associate the dashed lines in Fig. 4 with fictitious particles. The presence of the indices ν and ρ in (7.11) shows that these particles must be vector particles. The propagator associated with the fictitious vector particles is given in (7.11b), while the factor associated with a vertex involving two fictitious vector particles and one graviton is given in (7.11c). In addition, we must include a factor (-1) for each closed loop of vector particles; such particles may therefore be regarded as (very) fictitious "vector fermions." The vertices involving gravitions alone are obtained by expanding the interaction term in the Einstein Lagrangian, and we have observed that there exist such vertices with an arbitrary large number of lines. The only vertex involving the fictitious vector particles, however, is the vertex with two fictitious-particle lines and one graviton line.

VIII. NONPERTURBATIVE FORMULATION OF THE THEORY OF THE QUANTIZED GRAVITATIONAL FIELD

Our work in the body of the paper, and indeed the formulation of the equations of the theory given in Sec. I, has been within the framework of perturbation theory (to arbitrary order). We shall now attempt to reinterpret the contents of Secs. II–VII without assuming perturbation theory. We are not concerned here with methods of obtaining a nonperturbative solution of the field equations (7.2). Since we do not have a reliable method of solving the equations of other field theories, it would probably be premature to investigate nonperturbative approximations to the solution of (7.2) at the present time. We wish simply to show that the development leading to Eq. (7.2) can be understood without perturbation theory.

The first point which must be examined is the definition of a time-ordered product. In theories of fields other than the gravitational field, or in the perturbation theory of the gravitational field, the definition of a time-ordered product is unambigious. In the general theory of the gravitational field, however, one is faced with the problem that one does not necessarily know the time ordering of two points from the characteristics of the paths leading to them. We shall assume that there exists an operator $T'\{R_{\alpha\beta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\}$, even if the time ordering of the points x and y is not known. Functions identical to time-ordered products have sometimes been defined in quantized field theories without explicit use of time ordering. For instance, the Green's function, instead of being defined as the vacuum-expectation value of a time-ordered product, is sometimes defined as the multiple derivative of the S matrix with respect to changes of the external source. The assumption that the operator $T'\{R_{\alpha\beta\gamma\delta}(x,P)\times R_{\epsilon\zeta\eta\theta}(y,P')\}$ has a meaning will be taken as a fundamental assumption of the theory.

The operator $T'\{R_{\alpha\beta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(y,P')\}$ will no longer be a product of two operators. We shall assume that it is equal to a product of two operators, time-ordered in the usual way, when the paths P and P' are far apart.[8] We shall also assume that the covariant time-ordered product satisfies the path-dependence equation (2.4) and the field equations (2.12). Time-ordered products of more than two operators are similarly defined. We have shown that Eqs. (2.9) and (2.10) are sufficient to calculate the time-ordered product. Hence the assumption that there exist operators which satisfy (2.9) and (2.10), and which tend asymptotically to ordinary time-ordered products, is sufficient both to determine the theory and to define the time-ordered products.

Next it is necessary to obtain a definition of the path-dependent δ function, which occurs in the field equations (2.12). The definition of this function is not trivial, since we do not in general know whether two paths P and P', whose endpoints are x and y, lead to the same point. We shall assume that there exists an operator $\delta^4(x-y)$, which depends on the paths P and P', and which has the properties specified below. We shall further assume that there exist covariant time-ordered products such as

$$T'\{R_{\alpha\beta\gamma\delta}(z,P'')\, R_{\epsilon\zeta\eta\theta}(w,P''')\delta(x-y)\}\,, \qquad (8.1)$$

and similar time-ordered products with any number of R's. The time-ordered product (8.1) must satisfy the field equations (2.12) in z and w, and the path-dependence equations (2.9) for variations of the paths P'' and P'''. The definition of the δ function is then completed by the following three requirements:

(i) If P and P' are coincident along their entire lengths except for infinitesimal portions at their ends [Fig. 2(b)], the δ function is defined as in Ref. 4.

(ii) The covariant time-ordered product (8.1) satisfies a path-dependence equation analogous to (2.9) for variations in the paths P or P'.

[8] We require this assumption in order to apply Feynman boundary conditions, and also to use the reduction formulas.

(iii) The covariant time-ordered product satisfies the equations

$$T'\{R_{\alpha\beta\gamma\delta}(x,P)R_{\epsilon\zeta\eta\theta}(w,P''')\delta(x-y)\} = T'\{R_{\alpha\beta\gamma\delta}(y,P')R_{\epsilon\zeta\eta\theta}(w,P''')\delta(x-y)\}\,,$$

$$T'\{R_{\alpha\beta\gamma\delta}(z,P'')R_{\epsilon\zeta\eta\theta}(x,P)\delta(x-y)\} = T'\{R_{\alpha\beta\gamma\delta}(z,P'')R_{\epsilon\zeta\eta\theta}(y,P')\delta(x-y)\}\,.$$

Requirement (i) ensures that the δ function has the appropriate properties when two paths end at or near the same point, while requirement (iii) ensures that the δ function is zero if the paths do not lead to the same point.

The condensed notation introduced in Sec. III made no use of perturbation theory, and we can take that section over into our nonperturbation treatment. In Sec. IV, however, where we defined the path-dependent variables in terms of auxiliary variables, we must modify our approach. In perturbation theory, all formulas were to be expanded in powers of the operator $\tilde{\phi}$, and they were equivalent to formulas involving the Green's functions. Now, however, the formulas are to be regarded as genuine formulas involving operators in a linear space and not as a shorthand for perturbation theory. The operators $\tilde{g}$ are known from their definitions, and we then have to find operators $V(x,P)$, $W(x,P,x')$, and $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ which satisfy the equations of Sec. IV. Once the operator $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ and the vector $|G)$ are known, we can write the matrix elements $(H_0|\tilde{R}_{\alpha\beta\gamma\delta}\times(x,P)|G)$, $(H_0|\tilde{R}_{\alpha\beta\gamma\delta}(x,P)\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')|G)$, etc. These matrix elements will be the Green's functions $G_{\alpha\beta\gamma\delta}\times(x,P)$, $G_{\alpha\beta\gamma\delta,\epsilon\zeta\eta\theta}(y,P')$, etc. To calculate the path-dependent Green's functions from the auxiliary Green's functions is thus a nontrivial problem involving operators in our linear space.

Another point to be verified in Sec. IV (actually in the Appendix) is that the path-dependent δ function, defined in terms of the path-independent δ function by (A5), does satisfy conditions (i)–(iii) above. It is not difficult to show that the conditions are in fact satisfied.

The remarks which we have just made in connection with the formulas of Sec. IV apply equally to the field equations (6.7). The function $g^{1/2}\tilde{m}^{\lambda\mu}$, when expanded in powers of ϕ, gives rise to an infinite series. The equation thus connects an infinite number of Green's functions or, equivalently, there exist vertices with an arbitrarily large number of lines. If we are not using perturbation theory, we therefore have to regard (6.7) as a genuine operator equation in our linear space. Given the operators $\tilde{g}$ and η, we have to find a vector $|G)$ satisfying (6.7).

By making use of our linear space, we can therefore express the gravitational field equations as equations for path-independent quantities. We have given the perturbation solution to these equations. Any more adequate treatment would involve at least all the difficulties of quantum field theory. One can of course sum subsets of perturbation diagrams in the usual way and can construct approximation schemes such as the Bethe-Salpeter scheme. It has frequently been suggested that an adequate treatment of the quantized gravitational field might remove the divergence difficulties both from the gravitational field equations themselves and from the equations of other field theories. If one could construct an approximation scheme with these features, either by starting from Eq. (6.7) or by summing subsets of perturbation diagrams, one would have achieved major progress. At the moment, however, one does not have any indication of how to proceed.

Note added in manuscript: Faddeev and Popov (unpublished) have extended their method of Ref. 3 of the preceding paper to the gravitational field. Their results are the same as those of Feynman and DeWitt and those of the present paper, though they differ somewhat in appearance since the variables used are different [$(-g)^{1/2}g^{\mu\nu}$ instead of $g_{\mu\nu}$]. With such variables one can replace the fictitious vector particles by fictitious scalar particles if and only if one uses the Landau gauge. For the gravitational field, Faddeev and Popov do not relate their ansatz to a field theory.

APPENDIX

In this Appendix we shall give the proofs of two results quoted in Sec. IV. The first is that the definition of $\tilde{R}_{\alpha\beta\gamma\delta}(x,P)$ in terms of $\tilde{g}_{\lambda\mu}(x')$ does lead to the path-dependence equation (3.6). We begin by examining the function $V(x,P)$ and calculating the change which occurs when the path P is varied by a small area $\sigma_{\alpha\beta}$ at the point z (Fig. 3). Equation (4.10) for V can be written in integral form as follows:

$$V_\gamma{}^\lambda(x,P)=-\int_P^x dz_\beta V_\gamma{}^\mu(z,P')V_\beta{}^\nu(z,P')\times\int d^4z'W(z,P',z')\tilde{\Gamma}_{\mu\nu}{}^\lambda(z').$$

If the path P is deformed by an amount $\sigma_{\alpha\beta}$ at the point z, we can use Stokes's theorem to find the change of V just beyond the point z. Thus

$$\begin{aligned}\delta V_\gamma{}^\lambda(z+,P')&=-\tfrac{1}{2}\underset{\alpha\leftrightarrow\beta}{A}\frac{\partial}{\partial z_\alpha}\left\{V_\gamma{}^\mu(z,P')V_\beta{}^\nu(z,P')\int d^4z'W(z,P',z')\tilde{\Gamma}_{\mu\nu}{}^\lambda(z')\right\}\sigma_{\alpha\beta}\\&=-\tfrac{1}{2}\underset{\alpha\leftrightarrow\beta}{A}\Bigg(V_\gamma{}^\mu(z,P')V_\beta{}^\nu(z,P')\frac{\partial}{\partial z_\alpha}\left[\int d^4z'W(z,P',z')\tilde{\Gamma}_{\mu\nu}{}^\lambda(z')\right]\\&\qquad-V_\gamma{}^\rho(z,P')V_\alpha{}^\sigma(z,P')V_\beta{}^\nu(z,P')\int d^4z'W(z,P',z')\tilde{\Gamma}_{\mu\nu}{}^\lambda(z')\tilde{\Gamma}_{\rho\sigma}{}^\mu(z')\\&\qquad-V_\beta{}^\rho(z,P')V_\alpha{}^\sigma(z,P')V_\gamma{}^\mu(z,P')\int d^4z'W(z,P',z')\tilde{\Gamma}_{\mu\nu}{}^\lambda(z')\tilde{\Gamma}_{\rho\sigma}(z')\Bigg)\sigma_{\alpha\beta}.\end{aligned}$$

In the last two terms of this equation we have used (4.10) and (4.11) to express derivatives of V in terms of the V's themselves. We may use Eq. (4.9) to rewrite the derivative $(\partial/\partial z_\alpha)W(z,P',z')$ as $-V_\alpha{}^\rho(z,P')(\alpha/\partial z'^\rho)W(z',P',z')$. On doing so and rearranging some of the dummy indices λ, μ, ν, and ρ, we find that

$$\begin{aligned}\delta V_\gamma{}^\lambda(z+,P')&=-\tfrac{1}{2}V_\gamma{}^\mu(z,P')V_\beta{}^\nu(z,P')V_\alpha{}^\rho(z,P')\int d^4z'W(z,P',z')\\&\quad\times\left(\frac{\partial}{\partial z'^\rho}\tilde{\Gamma}_{\mu\nu}{}^\lambda(z')-\frac{\partial}{\partial z'^\nu}\tilde{\Gamma}_{\mu\rho}{}^\lambda(z')+\tilde{\Gamma}_{\sigma\rho}{}^\lambda(z')\Gamma_{\mu\nu}{}^\sigma(z')-\tilde{\Gamma}_{\sigma\nu}{}^\lambda(z')\tilde{\Gamma}_{\mu\nu}{}^\sigma(z')\right)\sigma_{\alpha\beta}\\&=-\tfrac{1}{2}V_\gamma{}^\mu(z,P)'V_\beta{}^\nu(z,P')V_\alpha{}^\rho(z,P')\int d^4z'W(z,P',z')\tilde{r}^\lambda{}_{\mu\nu\rho}(z')\sigma_{\alpha\beta}\\&=-\tfrac{1}{2}V_\delta{}^\lambda(z,P')\tilde{R}_{\delta\gamma\beta\alpha}(z,P')\sigma_{\alpha\beta}\quad\text{[from (4.15) and (4.14)]}\\&=-\tfrac{1}{2}\partial_\delta{}^\lambda(z,P')\tilde{R}_{\gamma\delta\alpha\beta}(z,P')\sigma_{\alpha\beta}.\end{aligned}$$

This equation may be brought into a form resembling (3.6) by noting that

$$i[J_{\gamma\delta}(z),V_\epsilon{}^\lambda(z,P)]=-\delta_{\gamma\epsilon}V_\delta{}^\lambda(z,P)+\delta_{\delta\epsilon}V_\gamma{}^\lambda(z,P).$$

Thus

$$\delta V_\epsilon{}^\lambda(z+,P')=\tfrac{1}{4}i\tilde{R}_{\gamma\delta\alpha\beta}(z,P')[J_{\gamma\delta}(z),V_\epsilon{}^\lambda(z,P)]\sigma_{\alpha\beta}.\tag{A1}$$

We have thus far only proved (A1) when the argument $z+$ is a point just beyond that at which the path was deformed. However, we can now readily extend (A1a) to arbitrary arguments:

$$\delta V_\epsilon{}^\lambda(x,P)=\tfrac{1}{4}i\tilde{R}_{\gamma\delta\alpha\beta}(z,P')[J_{\gamma\delta}(z),V_\epsilon{}^\lambda(x,P)]\sigma_{\alpha\beta},\tag{A2a}$$

$$\delta W(x,P,x')=\tfrac{1}{4}i\tilde{R}_{\gamma\delta\alpha\beta}(z,P')[J_{\gamma\delta}(z),W(x,P,x')]\sigma_{\alpha\beta}.\tag{A2b}$$

Equations (A2) are proved by examining Eqs. (4.9) and (4.10), which define the functions V and W. To begin with Eq. (4.10), the two sides undergo the following changes when V undergoes the change (A2a):

$$\delta\frac{\partial V_\alpha{}^\lambda(x,P)}{\partial x_\beta}=\tfrac{1}{4}i\tilde{R}_{\gamma\delta\epsilon\zeta}(z,P')\left[J_{\gamma\delta}(z),\frac{\partial V_\alpha{}^\lambda(x,P)}{\partial x_\beta}\right]\sigma_{\epsilon\zeta}, \tag{A3a}$$

$$\delta\Big(-\int d^4x' V_\alpha{}^\mu(x,P)V_\beta{}^\nu(x,P)W(x,P,x')\tilde{\Gamma}_{\mu\nu}{}^\lambda(x')\Big)$$
$$=\tfrac{1}{4}i\tilde{R}_{\gamma\delta\epsilon\zeta}(x,P')\left[J_{\gamma\delta}(z),\ -\int dx' V_\alpha{}^\mu(x,P)V_\beta{}^\nu(x,P)W(x,P,x')\tilde{\Gamma}_{\mu\nu}{}^\lambda(x')\right]. \tag{A3b}$$

On the other hand, it is permissible to apply the operator $J_{\gamma\delta}(z)$ to Eq. (4.10) at all points except the point z, since (4.10) must hold for the original and for the rotated path. Hence

$$\left[J_{\gamma\delta}(z),\frac{\partial V_\alpha{}^\lambda(x,P)}{\partial x_\beta}\right]=\left[J_{\gamma\delta}(z),\ -\int d^4x' V_\alpha{}^\mu(x,P)V_\beta{}^\nu(x,P)W(x,P,x')\tilde{\Gamma}_{\mu\nu}{}^\lambda(x')\right],\quad x\neq z. \tag{A4}$$

From (A3) and (A4), we observe that the changes of the two sides of (4.10) are the same when V and W undergo the transformation (A2), except possibly at the point $x=z$. One can prove in the same way that the changes of the two sides of (4.9) are the same. Hence, since V and W satisfy (4.9) and (4.10), we can conclude that $V+\delta V$ and $W+\delta W$ satisfy these equations, except possibly Eq. (4.10) at the point $x=z$. The development leading to (A1) showed that $V+\delta V$ also satisfies (4.10) along the new path near $x=z$, so that the functions $V+\delta V$ and $W+\delta W$ satisfy (4.9) and (4.10) along the entire new path. Since Eq. (4.9) and (4.10) are the definitions of V and W, we have proved Eqs. (A2).

It now follows at once from (4.15) and (A2) that

$$\delta\tilde{R}_{\epsilon\zeta\eta\theta}(x,P)=\tfrac{1}{4}i\tilde{R}_{\gamma\delta\alpha\beta}(z,P')[J_{\gamma\delta}(z),\tilde{R}_{\epsilon\zeta\eta\theta}(x,P)]\sigma_{\alpha\beta}.$$

This is the required path-dependence equation (3.6), which has thus been shown to follow from our definitions.

Before we prove our next result, Eq. (4.11), we shall have to derive one further formula, a formula relating the path-dependent and path-independent δ functions. We shall show that

$$\delta^4(x-y)=\int d^4x'd^4y' W(x,P,x')W(y,P,y')[\tilde{g}(x')]^{-1/2}\delta^4(x'-y'), \tag{A5}$$

where $\tilde{g}$ is the determinant of the matrix $\tilde{g}_{\lambda\mu}$. We shall assume this relation to be true at one value of x (for all y) and, at that value of x, we shall show that the derivative of the right side with respect to x_α is equal to $\partial\delta^4(x-y)/\partial x_\alpha$. Since the equation is true at the beginning of the path from our boundary conditions, we shall thereby have proved it in general. Proceeding in this manner, we find

$$\frac{\partial}{\partial x_\alpha}\Big(\int d^4x'd^4y' W(x,P,x')W(y,P,y')[\tilde{g}(x')]^{-1/2}\delta^4(x'-y')\Big)$$
$$=\int d^4x'd^4y' V_\alpha{}^\lambda(x,P)W(x,P,x')W(y,P,y')\frac{\partial}{\partial x'^\lambda}\{[\tilde{g}(x')]^{-1/2}\delta^4(x'-y')\}\,[\text{from (4.9)}]$$
$$=-\int d^4x'd^4y' V_\alpha{}^\lambda(x,P)W(x,P,x')W(y,P,y')\tilde{\Gamma}_{\lambda\mu}{}^\mu(x')[\tilde{g}(x')]^{-1/2}\delta^4(x'-y')$$
$$+\int d^4x'd^4y' V_\alpha{}^\lambda(x,P)W(x,P,x')W(y,P,y')[\tilde{g}(x')]^{-1/2}\frac{\partial}{\partial x'^\lambda}\delta^4(x'-y')$$
$$=\int d^4x'd^4y' V_\alpha{}^\lambda(x,P)V_{\lambda\gamma}(x,P)V_{\mu\beta}(x,P)\frac{\partial V_\gamma{}^\mu(x,P)}{\partial x_\beta}W(x,P,x')W(y,P,y')[\tilde{g}(x')]^{-1/2}\delta^4(x'-y')$$
$$-\int d^4x'd^4y' V_\alpha{}^\lambda(x,P)V_{\lambda\beta}(y,P)\frac{\partial}{\partial y_\beta}\{W(x,P,x')W(y,P,Y')[\tilde{g}(x)]^{-1/2}\delta^4(x-y)\},$$

from (4.9)–(4.11). Since we are assuming (A5) at the value of x under consideration, for all y, we can now deduce that

$$\frac{\partial}{\partial x_\alpha}\left(\int d^4x' d^4y' W(x,P,x')W(y,P,y')[\bar{g}(x)]^{-1/2}\delta^4(x-y)\right)$$

$$=V_{\mu\beta}(x,P)\frac{\partial V_\alpha{}^\mu(x,P)}{\partial x_\beta}\delta^4(x-y)-V_\alpha{}^\lambda(x,P)V_{\lambda\beta}(y,P)\frac{\partial}{\partial y_\beta}\delta^4(x-y)$$

$$=-V_\alpha{}^\lambda(y,P)V_{\lambda\beta}(y,P)\frac{\partial}{\partial\lambda_\beta}\delta^4(x-y)=-\frac{\partial}{\partial y_\alpha}\delta^4(x-y)=-\frac{\partial}{\partial x_\alpha}\delta^4(x-y).$$

This is the equation that we wished to prove. The physical meaning of Eq. (A5) is that the path-dependent δ function is $\bar{g}^{-1/2}$ times the path-independent, coordinate-dependent δ function. Such a result is not surprising; the factor $\bar{g}^{-1/2}$ is the reciprocal of the volume element in the non-Euclidean coordinate system.

We now prove that (4.17) is a possible expression for $U_{\beta\delta}(x,P)$, in the sense that they both have the same commutation relations with the operator $\bar{R}_{\epsilon\zeta\eta\theta}(y,P')$. From (4.16a), we obtain

$$[\eta^{\lambda\mu}(x'),f(\bar{g}(y'))]=-2(2\kappa)^{1/2}\frac{\delta f(g(y'))}{\delta g_{\lambda\mu}(x')}. \tag{A6}$$

Hence, in order to find the commutator of $\eta^{\lambda\mu}(x')$ with any function of $\bar{g}$, we require to know the derivative of that function with respect to $\bar{g}$. We shall therefore begin by supposing that $\bar{g}_{\lambda\mu}(y')$ undergoes a change

$$\delta\bar{g}_{\lambda\mu}(y')=h_{\lambda\mu}\delta^4(x'-y'), \tag{A7}$$

and shall find the corresponding changes in the functions $V_\alpha{}^\lambda(y,P')$, $W(y,P',y')$, and $\bar{r}_{\lambda\mu\nu\rho}(y')$.

First let us find the change in the variable $\bar{\Gamma}_{\gamma\rho}{}^\kappa$ when $\bar{g}_{\lambda\mu}$ undergoes the change (A7). It follows at once from the definition of Γ that

$$\delta\bar{\Gamma}_{\nu\rho}{}^\kappa(y')=-\tfrac{1}{2}\bar{g}^{\kappa\sigma}(y')(\delta_\sigma{}^\lambda\Gamma_{\nu\rho}{}^\mu(y')+\delta_\nu{}^\lambda\delta_\rho{}^\mu\partial/\partial y'^\sigma-\delta_\sigma{}^\lambda\delta_\rho{}^\mu\partial/\partial y'^\nu-\delta_\nu{}^\lambda\delta_\sigma{}^\mu\partial/\partial y'^\rho)\delta(x'-y')h_{\mu\nu}. \tag{A8a}$$

From (A8a) we can calculate the integral $-\int dy' V_\epsilon{}^\nu(y,P')V_\zeta{}^\rho(y,P')W(y,P',y')\delta\bar{\Gamma}_{\nu\rho}{}^\kappa(y')$, which we shall need when treating Eq. (4.10). Thus multiplying (A8a) by the factor $-V_\epsilon{}^\nu(y,P')V_\zeta{}^\rho(y,P')W(y,P',y')$, using (4.14) for the factor $\bar{g}^{\kappa\sigma}(y)$, and rearranging some dummy indices, we find that

$$-\int d^4y' V_\epsilon{}^\nu(y,P')V_\zeta{}^\rho(y,P')W(y,P',y')\delta\bar{\Gamma}_{\nu\rho}{}^\kappa(y')$$

$$=\tfrac{1}{2}\int d^4y' V_\eta{}^\kappa(y,P')V_\eta{}^\lambda(y,P')V_\epsilon{}^\nu(y,P')V_\zeta{}^\rho(y,P')W(y,P',y')\bar{\Gamma}_{\nu\rho}{}^\mu(y')\delta^4(x'-y')h_{\lambda\mu}$$

$$+\tfrac{1}{2}\int d^4y' V_\eta{}^\kappa(y,P')V_\beta{}^\lambda(y,P')V_\delta{}^\mu(y,P')\{\delta_{\beta\epsilon}\delta_{\delta\zeta}V_\eta{}^\nu(y,P')$$

$$-\delta_{\beta\eta}\delta_{\delta\zeta}V_\epsilon{}^\nu(y,P')-\delta_{\beta\epsilon}\delta_{\delta\eta}V_\zeta{}^\nu(y,P')\}W(y,P',y')\frac{\partial}{\partial y'^\nu}\delta^4(x'-y')h_{\lambda\mu}$$

$$=\tfrac{1}{2}\int d^4y' V_\eta{}^\kappa(y,P')V_\eta{}^\lambda(y,P')V_\epsilon{}^\nu(y,P')V_\zeta{}^\rho(y,P')W(y,P',y')\bar{\Gamma}_{\nu\rho}{}^\mu(y')\delta^4(x'-y')h_{\lambda\mu}$$

$$+\tfrac{1}{2}\int d^4y' V_\eta{}^\kappa(y,P')V_\beta{}^\lambda(y,P')V_\delta{}^\mu(y,P')(\delta_{\beta\epsilon}\delta_{\delta\zeta}\partial/\partial y_\eta-\delta_{\beta\eta}\delta_{\delta\zeta}\partial/\partial y_\epsilon-\delta_{\beta\epsilon}\delta_{\delta\eta}\partial/\partial y_\zeta)W(y,P',y')\delta^4(x'-y')h_{\lambda\mu}, \tag{A8b}$$

from (4.9). We can simplify (A8b) by taking the factors $V_\gamma{}^\lambda(x,P)$ and $V_\delta{}^\kappa(y,P)$ in the second term to the right of the differential operators. Using (4.10), we find that the terms obtained by doing so cancel the first term of (A8b).

Thus

$$-\int d^4y' V_\epsilon^{\nu}(y,P')V_\zeta^{\rho}(y,P')W(y,P',y')\delta\tilde{\Gamma}_{\nu\rho}{}^{\kappa}(y')$$
$$=\tfrac{1}{2}\{\delta_{\beta\epsilon}\delta_{\delta\zeta}V_\eta{}^\kappa(y,P')\partial/\partial y_\eta-\delta_{\delta\zeta}V_\beta{}^\kappa(y,P')\partial/\partial y_\epsilon-\delta_{\beta\epsilon}V_\delta{}^\kappa(y,P')\partial/\partial y_\zeta\}d_{\beta\delta}, \quad \text{(A8c)}$$

where

$$d_{\beta\delta}(x',y)=\int d^4y' V_\beta{}^\lambda(y,P')V_\delta{}^\mu(y,P')W(y,P',y')\delta^4(x'-y')h_{\lambda\mu}. \quad \text{(A9a)}$$

The function d is symmetric in β and δ:

$$d_{\beta\delta}(x,y)=d_{\delta\beta}(x,y). \quad \text{(A9b)}$$

Equation (A8c) may be simplified further by noting that

$$\delta_{\beta\epsilon}V_\eta{}^\kappa(y,P')-\delta_{\epsilon\eta}V_\beta{}^\kappa(y,P')=i[J_{\eta\beta}(y),V_\epsilon{}^\kappa(y,P')].$$

Thus

$$-\int d^4y' V_\epsilon^{\nu}(y,P')V_\zeta^{\rho}(y,P')W(y,P',y')\delta\tilde{\Gamma}_{\nu\rho}{}^{\kappa}=\tfrac{1}{2}\{i\delta_{\delta\zeta}[J_{\eta\beta}(y),V_\epsilon{}^\kappa(y,P')]\partial/\partial y_\eta-\delta_{\beta\epsilon}V_\delta{}^\kappa(y,P')\partial/\partial y_\zeta\}d_{\beta\delta}(x',y). \quad \text{(A8d)}$$

We can now investigate the effect of the change of $\tilde{\Gamma}$, given by (A8d), on the change of the functions $V_\epsilon{}^\kappa(y,P')$ and $W(y,P',x')$ defined by (4.9) and (4.10). We shall write the result and shall show that it does satisfy (4.8) and (4.10), in writing the result we were guided by Eq. (3.3) for the function U. The result is as follows:

$$\delta V_\epsilon{}^\kappa(y,P')=\tfrac{1}{2}i\int_{P'}^{y} d\eta_\delta\{[J_{\eta\beta}(\eta),V_\epsilon{}^\kappa(y,P')]\partial_\eta(\eta)d_{\beta\delta}(x',\eta)+id_{\beta\delta}(x',\eta)\partial_\beta(\eta)V_\epsilon{}^\kappa(y,P')\}-\tfrac{1}{2}\delta_{\beta\epsilon}V_\delta{}^\kappa(y,P')d_{\beta\delta}(x',y) \quad \text{(A10)}$$

$$\delta W(y,P',y')=\tfrac{1}{2}i\int_{P'}^{y} d\eta_\delta\{[J_{\eta\beta}(\eta),W(y,P',y')]\partial_\eta(\eta)d_{\beta\delta}(x',\eta)+id_{\beta\delta}(x',\eta)\partial_\beta(\eta)W(y,P',y')\}. \quad \text{(A11)}$$

To verify (A10) and (A11), we calculate the resulting changes of the two sides of (4.9) and (4.10). The change of the left side of (4.10) is

$$\delta\frac{\partial V_\epsilon{}^\kappa(y,P')}{\partial y_\zeta}=\tfrac{1}{2}i\delta_{\delta\zeta}\left([J_{\eta\beta}(y),V_\epsilon{}^\kappa(y,P')]\frac{\partial}{\partial y_\eta}d_{\beta\delta}(x',y)+id_{\beta\delta}(x',y)\frac{\partial V_\epsilon{}^\kappa(y,P')}{\partial y_\beta}\right)$$
$$+\frac{i}{2}\int_P^y d\eta_\delta\left(\left[J_{\eta\beta}(\eta),\frac{\partial V_\epsilon{}^\kappa(y,P')}{\partial y_\zeta}\right]\partial_\eta(\eta)d_{\beta\delta}(x',\eta)+id_{\beta\delta}(x',\eta)d_\beta(\eta)\frac{\partial V_\epsilon{}^\kappa(y,P')}{\partial y_\zeta}\right)$$
$$-\tfrac{1}{2}\delta_{\beta\epsilon}\frac{\partial V_\delta{}^\kappa(y,P')}{\partial y_\zeta}d_{\beta\delta}(x',y)-\tfrac{1}{2}\delta_{\beta\epsilon}V_\delta{}^\kappa(y,P')\frac{\partial}{\partial y_\zeta}d_{\beta\delta}(x',y). \quad \text{(A12)}$$

The change of the right side of (4.10) is equal to the sum of two terms. The first is given by Eq. (A8d), and the second is as follows:

$$-\int d^4y'\delta\{V_\epsilon^{\nu}(y,P')V_\zeta^{\rho}(y,P')W(y,P',y')\}\tilde{\Gamma}_{\nu\rho}{}^\kappa(y')$$
$$=-\tfrac{1}{2}i\int_P^y d\eta_\delta\left\{\left[J_{\eta\beta}(\eta),\int d^4y\; V_\epsilon^{\nu}(y,P')V_\zeta^{\rho}(y,P')W(y,P',y')\tilde{\Gamma}_{\nu\rho}{}^\kappa(y')\right]\partial_\eta(\eta)d_{\beta\delta}(x',\eta)\right.$$
$$\left.+id_{\beta\delta}(x',\eta)\partial_\beta(\eta)\left(\int d^4y' V_\epsilon^{\nu}(y,P')V_\zeta^{\rho}(y,P')W(y,P',y')\tilde{\Gamma}_{\nu\rho}{}^\kappa(y')\right)\right\}$$
$$+\tfrac{1}{2}\int d^4y'[\delta_{\beta\epsilon}V_\delta^{\nu}(y,P')V_\zeta^{\rho}(y,P')+\delta_{\beta\zeta}V_\epsilon^{\nu}(y,P')V_\delta^{\rho}(y,P')]W(y,P',y')\tilde{\Gamma}_{\nu\rho}{}^\kappa(y')d_{\beta\delta}(x',y). \quad \text{(A13)}$$

We can now show that the right side of (A12) is equal to the sum of the right sides of (A8d) and (A13). We first note the equations

$$\left[J_{\eta\beta}(\eta),\frac{\partial V_\epsilon{}^\kappa(y,P')}{\partial y_\zeta}\right]=-\left[J_{\eta\beta}(\eta),\int d^4y'V_\epsilon{}^\nu(y,P')V_\zeta{}^\rho(y,P')W(y,P',y')\tilde{\Gamma}_{\nu\rho}{}^\kappa(y')\right], \tag{A14a}$$

$$\partial_\beta(\eta)\frac{\partial V_\epsilon{}^\kappa(y,P')}{\partial y_\zeta}=-\partial_\beta(\eta)\left(\int d^4y'V_\epsilon{}^\nu(y,P')K_\zeta{}^\rho(y,P')W(y,P')\tilde{\Gamma}_{\nu\rho}{}^\kappa(y)\right). \tag{A14b}$$

Equations (A14) are obtained by applying the operators $J_{\beta\eta}(\eta)$ and $\partial_\beta(\eta)$ to Eq. (4.10); it is permissible to do so, since (4.10) must hold for the original and for the rotated or displaced paths. Using (A14), we observe that the terms on the right of (A12) and (A13) which involve an integral over η_ζ are equal. Using (4.10) and (A9b), we can show that the two terms on the right of (A12) which involve a derivative of V with respect to y are equal to the last two terms of (A13). The right side of (A8d) is equal to the sum of the remaining terms on the right of (A12).

We can similarly show that the changes (A10) and (A11) of V and W are consistent with Eq. (4.9). The changes of the two sides of this equation are as follows:

$$\delta\left(\frac{\partial}{\partial y_\zeta}W(y,P',y')\right)=\tfrac{1}{2}i\delta_{\delta\zeta}[J_{\eta\beta}(y),W(y,P',y')]\frac{\partial}{\partial y_\eta}d_{\beta\delta}(x',\eta)-\tfrac{1}{2}\delta_{\delta\zeta}d_{\beta\delta}(x',y')\frac{\partial}{\partial y_\beta}W(y,P',y')$$
$$+\tfrac{1}{2}i\int_P^y d\eta_\delta\left(\left[J_{\eta\beta}(\eta),\frac{\partial W(y,P',y')}{\partial y_\zeta}\right]\partial_\eta(\eta)d_{\beta\delta}(x',\eta)+id_{\beta\delta}(x',\eta)\partial_\beta(\eta)\frac{\partial W(y,P',y')}{\partial y_\zeta}\right), \tag{A15}$$

$$-\delta\left\{V_\zeta{}^\lambda(y,P')\frac{\partial}{\partial y'^\lambda}W(y,P',y')\right\}=-\tfrac{1}{2}i\int_P^y d\eta_\delta\left(\left[J_{\eta\beta}(\eta),V_\zeta{}^\lambda(y,P')\frac{\partial}{\partial y'^\lambda}W(y,P',y')\right]\frac{\partial}{\partial y_\eta}d_{\beta\delta}(x',\eta)\right.$$
$$\left.-id_{\beta\delta}(x',\eta)\partial_\beta(\eta)\left[V_\zeta{}^\lambda(y,P')\frac{\partial}{\partial y'^\lambda}W(y,P',y')\right]\right)+\tfrac{1}{2}\delta_{\beta\zeta}d_{\beta\delta}(x',y)V_\delta{}^\lambda(y,P')\frac{\partial}{\partial y'^\lambda}W(y,P,y'). \tag{A16}$$

The first term on the right of (A15) is zero. This is because the symbol $i[J_{\eta\beta}(y),W(y,P',y')]$ is the change of W caused by a rotation of the path P about the point y, the end of the path. Such a rotation cannot change a variable which has no tensor indices. The second term on the right of (A15) is equal to the last term on the right of (A16), by (4.10) and (A9b). As in our previous example, the terms on the right of (A15) and (A16) which involve an integral over η_ζ are equal to one another. Hence the right sides of (A15) and (A16) are equal. We have thus verified that (A10) and (A11) satisfy the defining equations for V and W, so that (A10) and (A11) do give the change of V and W when $\tilde{g}$ undergoes the change (A7).

Next we find the change in the path-independent Riemann tensor $\tilde{r}_{\lambda\mu\nu\rho}(y')$, defined by (4.2a), when $\tilde{g}$ undergoes the change (A7). Since our main object is to find the change of the path-dependent Riemann tensor, given by (4.15), we shall actually calculate the product $V_\epsilon{}^\lambda V_\zeta{}^\mu V_\eta{}^\nu V_\theta{}^\rho W\delta\tilde{r}_{\lambda\mu\nu\rho}$. The calculation is similar to the calculation leading to (A8c), and we need not give the details. The result is

$$\int dy'V_\epsilon{}^\lambda(y,P')V_\zeta{}^\mu(y,P')V_\eta{}^\nu(y,P')V_\theta{}^\rho(y,P')W(y,P',y')\delta\tilde{r}_{\lambda\mu\nu\rho}(y')$$
$$=\tfrac{1}{2}\underset{\epsilon\leftrightarrow\zeta,\eta\leftrightarrow\theta}{A}\partial_\eta(y)\partial_\epsilon(y)[\delta_{\beta\zeta}\delta_{\delta\theta}d_{\beta\delta}(x',y)]+\tfrac{1}{2}\underset{\epsilon\leftrightarrow\zeta}{A}\tilde{R}_{\epsilon\iota\eta\theta}(y,P')\delta_{\beta\iota}\delta_{\delta\theta}d_{\beta\delta}(x',y). \tag{A17}$$

We can now calculate the change in the path-dependent Riemann tensor (4.15). Inserting (A10), (A11), and (A17) in (4.15), we find that

$$\delta\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')=\tfrac{1}{2}\underset{\epsilon\leftrightarrow\zeta,\eta\leftrightarrow\theta}{A}\partial_\eta(y)\partial_\epsilon(y)\delta_{\beta\zeta}\delta_{\delta\theta}d_{\beta\delta}(x',y)-\tfrac{1}{2}\underset{\eta\leftrightarrow\theta}{A}\tilde{R}_{\epsilon\zeta\eta\iota}(y,P')\delta_{\beta\iota}\delta_{\delta\theta}d_{\beta\delta}(x,y)$$
$$+\tfrac{1}{2}i\int_{P'}d\eta_\delta\{[J_{\iota\beta}(\eta),\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')]\partial_\iota(\eta)d_{\beta\delta}(x',\eta)+id_{\beta\delta}(x',\eta)\partial_\beta(\eta)\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')\}. \tag{A18}$$

The functional derivative $\delta\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')/\delta\tilde{g}_{\lambda\mu}(x')$ is the coefficient of $h_{\lambda\mu}$ on the right of (A18), symmetrized in λ and

μ. Thus

$$\frac{\delta\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')}{\delta\tilde{g}_{\lambda\mu}(x')}=\tfrac{1}{4}\underset{\epsilon\leftrightarrow\zeta,\eta\leftrightarrow\theta}{A}\partial_\eta(y)\partial_\epsilon(y)[(\delta_{\beta\zeta}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\delta\zeta})d_{\beta\delta}{}^{\lambda\mu}(x',y)]-\tfrac{1}{4}\underset{\eta\leftrightarrow\theta}{A}\tilde{R}_{\epsilon\zeta\eta\iota}(y,P')(\delta_{\beta\iota}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\delta\iota})d_{\beta\delta}{}^{\lambda\mu}(x',y)$$

$$+\tfrac{1}{4}i\int_{P'}d\eta_\iota(\delta_{\beta\iota}\delta_{\delta\lambda}+\delta_{\beta\lambda}\delta_{\delta\iota})\{[J_{\kappa\lambda}(\eta),\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')]\partial_\kappa(\eta)d_{\beta\delta}{}^{\lambda\mu}(x',\eta)+id_{\beta\delta}{}^{\lambda\mu}(x',\eta)\partial_\lambda(\eta)\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')\}\,,\quad\text{(A19)}$$

where $d_{\beta\delta}{}^{\lambda\mu}(x',y)$ is the coefficient of $h_{\lambda\mu}$ in the function $d_{\beta\delta}(x',y)$, i.e.,

$$d_{\beta\delta}{}^{\lambda\mu}(x',y)=\int d^4y'V_\beta{}^\lambda(y,P')V_\delta{}^\mu(y,P')W(y,P',y')\delta^4(x'-y')\,.\quad\text{(A20)}$$

The right side of (A19) resembles that of (3.31), except that the δ functions $\delta^4(x-y)$ [or $\delta^4(x-\eta)$] are here replaced by $d_{\beta\delta}{}^{\lambda\mu}(x',y)$ [or $d_{\beta\delta}{}^{\lambda\mu}(x',\eta)$]. However, from (A20) ,(A5), and (4.14), it follows that

$$\int d^4x'V_{\lambda\beta}(x,P)V_{\mu\delta}(x,P)W(x'P,x')[\tilde{g}(x')]^{-1/2}d_{\alpha\gamma}{}^{\lambda\mu}(x',y)=\delta_{\alpha\gamma}\delta_{\beta\delta}\delta^4(x-y)\,.\quad\text{(A21)}$$

We can therefore multiply both sides of (A19) by the factors $V_{\lambda\beta}(x,P)V_{\mu\delta}(x,P)W(x,P,x')[\tilde{g}(x')]^{1/2}$, and integrate over x', to obtain the final result[9]

$$\int d^4x'V_{\lambda\beta}(x,P)V_{\mu\delta}(x,P)W(x,P,x')[\tilde{g}(x')]^{-1/2}\frac{\delta\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')}{\delta\tilde{g}_{\lambda\mu}(x')}$$

$$=\tfrac{1}{4}\underset{\epsilon\leftrightarrow\zeta,\eta\leftrightarrow\theta}{A}\partial_\eta(y)\partial_\epsilon(y)[(\delta_{\beta\zeta}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\delta\zeta})\delta^4(x-y)]-\tfrac{1}{4}\underset{\eta\leftrightarrow\theta}{A}\tilde{R}_{\epsilon\zeta\eta\iota}(y,P')(\delta_{\beta\iota}\delta_{\delta\theta}+\delta_{\beta\theta}\delta_{\delta\iota})\delta^4(x-y)$$

$$+\tfrac{1}{4}i\int_{P'}d\eta_\iota(\delta_{\beta\iota}\delta_{\delta\lambda}+\delta_{\beta\lambda}\delta_{\delta\iota})\{[J_{\kappa\lambda}(\eta),\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')]\partial_\kappa(\eta)\delta^4(x-\eta)+i\delta^4(x-\eta)\partial_\lambda(\eta)\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')\}\,.\quad\text{(A22)}$$

Since the right side of (A22) is the same as that of (3.3), apart from a factor -2κ, we can conclude from (A6) that

$$2^{-1/2}\kappa^{1/2}\int d^4x'V_{\lambda\alpha}(x,P)V_{\mu\beta}(x,P)W(x,P,x')[\tilde{g}(x')]^{-1/2}\eta^{\lambda\mu}(x')\approx U_{\alpha\beta}(x,P)\,,\quad\text{(A23)}$$

in the sense that both sides of this equation have the same commutators with $\tilde{R}_{\epsilon\zeta\eta\theta}(y,P')$.

[9] In this equation, the "beginning of the Greek alphabet" over-laps into the middle. All subscripts in (5.20) refer to the local Euclidean system and are to be regarded as belonging to the beginning of the alphabet.