Feynman Rules for Electromagnetic and Yang-Mills Fields from the Gauge-Independent Field-Theoretic Formalism*

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The Feynman rules for the Yang-Mills field, originally derived by Feynman and DeWitt from S-matrix theory and the tree theorem, are here derived as a consequence of field theory. Our starting point is the gauge-independent, path-dependent formalism which we proposed earlier. The path-dependent Green's functions in this theory are expressed in terms of auxiliary, path-independent Green's functions in such a way that the path-dependence equation is automatically satisfied. The formula relating the path-dependent to the auxiliary Green's functions is similar to the classical formula relating the path-dependent field variables to the potentials. By using a notation similar but not identical to Schwinger's functional notation, the infinite set of equations satisfied by the Green's function can be replaced by a single equation. When the equation for the auxiliary Green's functions of electromagnetism is solved in a perturbation series, the usual Feynman rules result. For the Yang-Mills field, however, one obtains extra terms; such terms correspond precisely to the closed loops of fictitious scalar particles introduced by Feynman, DeWitt, and Faddeev and Popov.

1. INTRODUCTION

THE discovery of the Feynman rules for the Yang-Mills and gravitational fields by Feynman himself[1] has solved a long-standing problem in relativistic quantum mechanics. Feynman only derived his procedure for diagrams with a single closed loop, but DeWitt[2] has recently extended the procedure to diagrams of arbitrary complexity. Another general proof of the prescription for the Yang-Mills field has been given by Faddeev and Popov,[3] who used a functional integration procedure which is probably equivalent to that of DeWitt.

Feynman and DeWitt obtained their prescription by a somewhat indirect method. From the Feynman rules for nongauge particles they obtained the "tree theorem," which relates the contribution to the S matrix from a closed-loop diagram to the contribution from a diagram where the loop is opened at one point. They then assumed that the tree theorem was valid in theories with gauge particles; they were thus able to derive the Feynman rules for the S matrix. The validity of the tree theorem guarantees that the S matrix is unitarity, and their results can almost certainly be derived from an analyticity-unitarity calculation in perturbation theory.

The question arises whether one can obtain the Feynman rules within the framework of a field theory of the Yang-Mills field or of gravity, and it is the purpose of the present paper to attempt to do so. We shall take as our basis the path-dependent theory of gauge fields which we suggested earlier.[4] The theory was originally formulated for electromagnetism and for gravity, but it can be applied to any gauge field. The path-dependent theory of the Yang-Mills field has been treated by Bialynicki-Birula.[5]

In the present paper we shall rederive the Feynman rules for the electromagnetic field from the path-dependent formalism, and we shall then derive the more complicated Feynman rules for the Yang-Mills field. We shall derive the Feynman rules for the gravitational field in the following paper.

The fundamental principle of the path-dependent formalism was to avoid the introduction of non-gauge-invariant quantities. Thus the electromagnetic potentials were not introduced, but were replaced by the electromagnetic field variables $F_{\mu\nu}$. Similarly, the charged field variables $\phi(x)$ were replaced by the path-dependent but gauge-invariant variables $\Phi(x,P)$. For practical purposes one would like to introduce the potentials as auxiliary variables, as one does in classical field theory. By doing so one would be able to calculate in terms of path-independent variables; one would transfer to the path-dependent variables at the end of the calculation. It is well known, however, that one cannot introduce covariant potentials without enlarging the Hilbert space and employing an indefinite metric. For electromagnetism one can use noncovariant potentials such as those of the Coulomb gauge. One can then derive the Feynman rules after a certain amount of algebraic calculation. It is possible to formulate the Yang-Mills theory in terms of noncovariant gauges, such as Schwinger's modification of the Coulomb gauge, or the Arnowitt-Fickler gauge.[6] However, the method which was used in electromagnetism for deriving the covariant Feynman rules from such gauges is not applicable here, at any rate without essential modification. To our knowledge no such consistent formalism has been given for the gravitational field.

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In the present treatment we shall avoid noncovariant quantities and we shall therefore not introduce potentials as quantum-mechanical operators. Instead, we shall introduce auxiliary Green's functions. In formalisms of quantum electrodynamics which employ potentials, whether in the Coulomb or Lorentz gauges, one can define Green's functions

$$G_\mu(x_1,\cdots;y_1,\cdots;z_1,\cdots)=\langle 0|T\{\phi(x_1)\cdots\phi^*(y_1)\cdots A_\mu(z_1)\cdots|0\rangle.$$

One can also define path-dependent but gauge-invariant Green's functions

$$G_{\mu\nu}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,\cdots)=\langle 0|T\{\Phi(x_1,P_1)\cdots\Phi^*(y_1,P_1')\cdots F_{\mu\nu}(z_1)\cdots\}|0\rangle.$$

The latter Green's functions can be expressed in terms of the former. In our present approach, the path-independent Green's functions will be introduced, not as vacuum-expectation values of time-ordered products, but as auxiliary functions in their own right. The physical, path-dependent Green's functions of our theory will then be expressed in terms of the auxiliary Green's functions by using the same formulas as in theories with potentials. The connection between the path-dependent and path-independent Green's functions will guarantee that the path-dependence equations are satisfied, as we shall verify explicitly. We then have to find the equations which the auxiliary Green's functions must satisfy in order that the path-dependent Green's functions satisfy the correct equations.

For electrodynamics, such an approach has already been carried out by Sarker.[7] He found that the equations satisfied by the auxiliary Green's functions are similar, but not identical, to the equations satisfied by the Green's functions of the Lorentz-gauge theory. The difference is due to the fact that he started with the Maxwell equations $\partial F_{\mu\nu}(x)/(\partial x_\mu)+j_\nu=0$, whereas the Lorentz-gauge theory starts with the equations $\Box^2 A_\nu(x)+j_\nu=0$. Nevertheless, he showed that the Green's functions calculated by the usual Feynman rules do satisfy the correct equations. The Feynman rules were thus derived from a procedure which was covariant throughout and which did not make use of an enlarged Hilbert space.

When we carry out a similar treatment for the Yang-Mills field, we shall again find that the equations satisfied by our auxiliary Green's functions are slightly different from the corresponding equations in the (incorrect) Lorentz-gauge theory. As with electromagnetism, the difference is due to the dropping of a term $-\partial^2 A_\nu/\partial x_\mu \partial x_\nu$ in the Lorentz gauge. In this case, however, we shall find that the difference is important, and that the solution to our equations contains terms besides those given by the Lorentz-gauge Feynman rules.

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Our results will be the same as those found by Feynman, DeWitt, and Faddeev and Popov. They showed that the correct prescription was to take all Feynman diagrams of the Lorentz-gauge theory, together with Feynman diagrams containing closed loops of fictitious scalar particles. In our treatment we shall find that integrals corresponding to closed loops of scalar particles appear directly in the solution of the Green's-function equations. We may associate such integrals with closed loops of scalar particles if we wish, but this is purely a mnemonic device. The fictitious particles never occur in external lines, nor do they appear in the intermediate states of the unitarity condition.

In our present formulation of the theory, the Feynman rules are thus rules for calculating auxiliary Green's functions. We can then proceed to calculate the gauge-invariant, path-dependent Green's functions, since we shall already have expressed them in terms of the auxiliary Green's functions. By using the reduction formulas we can then calculate the S matrix. The fundamental reduction formulas of the theory involve the path-dependent Green's functions. However, one can use these reduction formulas to derive further reduction formulas involving the auxiliary Green's functions. Thus, from the Feynman rules for the auxiliary Green's functions, one can derive Feynman rules for the S matrix by the usual reinterpretation of the external lines.

The equations for the Green's functions are coupled integral equations between an infinite number of such functions. Moreover, when expressing path-dependent Green's functions in terms of auxiliary Green's functions, one finds that a single path-dependent Green's function is equal to the sum of an infinite number of auxiliary Green's functions. It would be clumsy, if in principle possible, to carry out manipulations with such infinite systems of equations. We require a shorthand for expressing the infinite sets of equations as single equations. The Schwinger functional notation provides us with such a shorthand; Schwinger's functional differential equation is equivalent to the complete set of equations for the Green's functions. Unfortunately it does not appear to be an easy matter to express the equations for path-dependent Green's functions in Schwinger's notation. We shall therefore use another notation in which our fundamental quantity corresponds to Schwinger's $\delta/\delta\eta$ rather than to η. We shall indicate the connection between our notation and Schwinger's but we shall not assume knowledge of his notation.

In the following section we shall illustrate some of our methods by using the $\lambda\phi^3$ theory. We shall find the differential equations for the Green's functions and shall use them to construct the perturbation expansion. We shall then develop our notation for simplifying the writing of the differential equations. Essentially what we shall do is to form a linear space of all Green's functions and to write the differential equations as equations for vectors in this space. In Sec. 3 we shall treat the

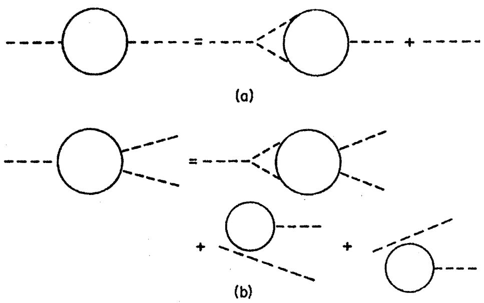

FIG. 1. Diagrammatic representation of Eqs. (2.5).

electromagnetic field. We shall write down the equations for the path-dependent Green's functions and shall reexpress them in our shorthand notation. Working within this notation, we shall then express our path-dependent Green's functions in terms of new, path-independent, auxiliary Green's functions. We shall determine the equations which the auxiliary Green's functions should satisfy in order that the path-dependent Green's functions satisfy the required equations. On solving them, we shall find that they lead to the ordinary Feynman rules. In Sec. 4 we shall treat the Yang-Mills field in a similar way. Here, however, we shall find that the perturbation expansion contains terms besides those given by naive Feynman rules.

2. DIFFERENTIAL EQUATIONS FOR GREEN'S FUNCTIONS

In this section we shall summarize the method of determining Green's functions by solving differential equations, and shall also develop our shorthand notation. The method is certainly not new but, as far as we are aware, there is no easily available reference in which it is described, and we therefore felt it worthwhile to describe its application to non-gauge fields before passing on to the gauge fields in which we are interested.

We shall treat the simple case of a neutral scalar field with $\lambda\phi^3$ coupling. The field equations will be

$$(\Box^2-\mu^2)\phi(x)-\tfrac{1}{2}\lambda\{\phi(x)\}^2=0, \tag{2.1}$$

and the ϕ's will satisfy the commutation relations

$$[\phi(\mathbf{x},t)\phi(\mathbf{y},t)]=[\dot\phi(\mathbf{x},t)\dot\phi(\mathbf{y},t)]=0, \tag{2.2a}$$

$$[\dot\phi(x,t),\phi(y,t)]=-i\delta^3(\mathbf{x}-\mathbf{y}). \tag{2.2b}$$

We can now define Green's functions

$$G_2(x_1,x_2)=\langle 0|T\{\phi(x_1),\phi(x_2)\}|0\rangle, \tag{2.3a}$$

$$G_3(x_1,x_2,x_3)=\langle 0|T\{\phi(x_1),\phi(x_2),\phi(x_3)\}|0\rangle, \tag{2.3b}$$

$$G_4(x_1,x_2,x_3,x_4)=\langle 0|T\{\phi(x_1),\phi(x_2),\phi(x_3),\phi(x_4)\}|0\rangle,\quad \text{etc.} \tag{2.3c}$$

For formal purposes we may also define the Green's functions

$$G_0=\langle 0|0\rangle=1, \tag{2.3d}$$

$$G_1(x)=\langle 0|\phi(x)|0\rangle. \tag{2.3e}$$

One method of obtaining the perturbation series for the Green's functions is to use the differential equations satisfied by them. This is the method we shall use in the following sections when treating gauge fields. Thus, G_2 will satisfy the equation

$$(\Box_1{}^2-\mu^2)G_2(x_1,x_2)=\tfrac{1}{2}\lambda G_3(x_1,x_1,x_2)+i\delta^4(x_1-x_2). \tag{2.4a}$$

Equation (2.4a) is obtained by applying the differential equation (2.1) to the factor $\phi(x_1)$ of (2.3a). The first term on the right of (2.4a) arises from the interaction term in (2.1), while the second term is obtained by applying the differential operator $-\partial^2/\partial x_0{}^2$ to the time ordering itself. In deriving this term it is of course necessary to use the commutation relation (2.2b).

The higher Green's functions will satisfy similar equations. Thus G_3 will satisfy the equation

$$(\Box_1{}^2-\mu^2)G_3(x_1,x_2,x_3)=\tfrac{1}{2}\lambda G_4(x_1,x_1,x_2,x_3)+i\delta^4(x_1-x_2)G_1(x_3)+i\delta^4(x_1-x_3)G_1(x_2). \tag{2.4b}$$

Equations (2.4a) and (2.4b) can be integrated to yield the formulas

$$G_2(x_1,x_2)=-\tfrac{1}{2}i\lambda\int dx_4\tfrac{1}{2}\Delta_F(x_1-x_4)G_3(x_4,x_4,x_2)+\tfrac{1}{2}\Delta_F(x_1-x_2), \tag{2.5a}$$

$$G_3(x_1,x_2,x_3)=-\tfrac{1}{2}i\lambda\int dx_4\tfrac{1}{2}\Delta_F(x_1-x_4)G_4(x_4,x_4,x_2,x_3)+\tfrac{1}{2}\Delta_F(x_1-x_2)G_1(x_3)+\tfrac{1}{2}\Delta_F(x_1-x_3)G_1(x_2). \tag{2.5b}$$

Equations (2.5) are illustrated diagramatically in Fig. 1.

If we are working in perturbation theory, the first Green's function on the right of (2.5a) or (2.5b) will be required to one order lower than that on the left, since it contains an explicit factor λ. The second term on the right of (2.5a) is known explicitly, while that on the right of (2.5b) only involves G_1. Hence, if we construct the perturbation series order by order and, within each order, construct the functions G_1, G_2, $\cdots$ successively, the right-hand side of (2.5) will be known in terms of previously calculated functions. We can therefore construct the entire perturbation series in this manner, and it is not difficult to see that we obtain the usual prescription for Feynman diagrams.

In a field theory with a simple Lagrangian, such as the $\lambda\phi^3$ theory, it is sufficient to write down the first few equations (2.4) and (2.5); the form of the subsequent equations is then fairly obvious. When writing down equations for gauge fields and performing manipulations with them, however, it would be somewhat cumbersome to proceed in this manner. We require a notation in which the whole series of equations (2.4) can be simply displayed. In the remainder of the section we shall develop such a notation. We emphasize that we are doing nothing more than constructing a shorthand for expressing the equations satisfied by the Green's functions.

We shall work with the linear space of the totality of functions $C_0, C_1(x_1), C_2(x_1,x_2), \cdots$. A typical vector in the linear space may be written

$$\begin{matrix} C_0 \\ C_1(x_1) \\ C_2(x_1x_2) \\ \cdots \end{matrix} \quad (2.6)$$

The linear space is thus the sum of a series of subspaces

$$\begin{matrix} C_0 & 0 & 0 & \\ 0 & C_1(x_1) & 0 & \cdots \\ 0 & 0 & C_2(x_1,x_2) & \\ \cdots & \cdots & \cdots & \end{matrix}$$

The first subspace consists of a single vector, but each of the other subspaces is itself an infinite-dimensional subspace. Thus, the second subspace is the space of all functions of a single variable x_1, the third subspace is the space of all symmetric functions of two variables x_1 and x_2, and so on. We shall denote the vector (2.6) by the symbol $|C)$. We are interested in the particular vector

$$\begin{matrix} G_0 \\ G_1(x_1) \\ G_2(x_1,x_2) \\ \cdots \end{matrix} \quad (2.7)$$

where $G_0=\langle 0|0\rangle=1$, and the remaining G's are the Green's functions of our theory. We shall denote this vector by the symbol $|G)$.

We now define an operator $\tilde{\phi}(x)$ acting in this space as follows:

$$\tilde{\phi}(x)\begin{Bmatrix} C_0 \\ C_1(x_1) \\ C_2(x_1,x_2) \\ \cdots \end{Bmatrix}=\begin{Bmatrix} C_1(x) \\ C_2(xx_1) \\ C_3(xx_1x_2) \\ \cdots \end{Bmatrix}. \quad (2.8)$$

The variable x in the vector on the right is regarded as a fixed parameter, while $x_1, x_2, \cdots$ are the variables corresponding to our linear space.

To avoid misunderstanding we must emphasize that the operator $\tilde{\phi}(x)$ *corresponds* to the quantum-mechanical operator $\phi(x)$, but that it is a totally different type of operator acting on a totally different type of space. Once we have expressed the field theory in terms of Green's functions, we need no longer consider the quantum-mechanical Hilbert space; the whole theory has been expressed in terms of the c-number equations (2.4). In order to express these c-number equations in a simple way, we have defined a new linear space on which the operators $\tilde{\phi}(x)$ act. The $\tilde{\phi}$'s do not satisfy the quantum-mechanical commutation rules. In fact, all the $\tilde{\phi}$'s and their space and time derivatives commute with one another. The linear space of vectors $|\quad)$ is thus a totally different space from the quantum-mechanical Hilbert space of vectors $|\quad\rangle$.

It would be inconvenient if we had to display equations such as (2.8) whenever we used them, and, in fact, there is a standard notation for expressing (2.8) in a more compact form. This notation is expressed in terms of vectors in the *dual space*. A vector in the dual space is written in the form $(H|$, and is *defined* by means of its scalar products $(H|C)$ with all vectors in our space $|C)$. The scalar products must depend linearly on $|C)$. We define the vectors $(H_0|$, $(H_1(x_1)|$, $(H_2(x_1,x_2)|$, $\cdots$ in the dual space by the equation.

$$\begin{aligned} (H_0|C)&=C_0, \\ (H_1(x_1)|C)&=C_1(x_1), \\ (H_2(x_1,x_2)|C)&=C_2(x_1,x_2), \end{aligned} \quad (2.9)$$

C being the general vector (2.6). There is a single vector $(H_0|$, a vector $(H_1(x_1)|$ for each value of x_1, a vector $(H_2(x_1,x_2)|$ for each combination of variables x_1, x_2, and so on. A vector in the original space is uniquely defined by its scalar products with all vectors $(H_0|$, $(H_1(x_1)|$, $(H_2(x_1,x_2)|$, $\cdots$ in the dual space.

We next construct the dual space of the space of vectors $(H|$. If $|C')$ is a vector in this space, it is defined by its scalar product will all vectors $(H|$:

$$(H_n(x_1,\cdots x_n)|C')=C_n'(x_1,\cdots x_n).$$

However, the totality of functions C_0', $C_1'(x)$, $C_2'(x_1,x_2)$, $\cdots$ define a vector in the original space with which we started. There is thus a one-one correspondence between our original space $|C)$ and our new space $|C')$, which is such that corresponding vectors have identical scalar products with all vectors $(H|$. The space of vectors $|C')$, which is the dual of the space of vectors $(H|$, may therefore be regarded as identical to our original space of vectors $|C)$.

Given any operator O acting in the dual space, we can define an operator O acting in the original space as follows:

$$(H|\{O|C)\}=\{(H|O\}|C\} \quad (2.10)$$

for all vectors H in the dual space. We can therefore use the notation $(H|O|C)$ to express the scalar product. Equation (2.10) can also be used to define an operator in the dual space once the operator in the original space is known.

It is now easy to express $\tilde{\phi}$ as an operator in the dual space. By (2.9) and (2.8)

$$\begin{aligned} (H_0|\{\tilde{\phi}(x)|C)\}&=C_1(x), \\ (H_1(x_1)|\{\tilde{\phi}(x)|C)\}&=C_2(x,x_1), \\ (H_2(x_1,x_2)|\{\tilde{\phi}(x)|C)\}&=C_3(x,x_1,x_2). \end{aligned} \quad (2.11)$$

Now, by (2.9),

$$(H_1(x)|C)=C_1(x),$$
$$(H_2(x,x_1)|C)=C_2(x,x_1), \quad (2.12)$$
$$(H_3(x,x_1,x_2)|C)=C_3(x,x_1,x_2).$$

From (2.11) and (2.12), we conclude that

$$(H_0|\tilde{\phi}(x)=(H_1(x)|,$$
$$(H_1(x_1)|\tilde{\phi}(x)=(H_2(x,x_1)|, \quad (2.13)$$
$$(\rho_2(x_1,x_2)|\tilde{\phi}(x)=(H_3(x,x_1,x_2)|,$$

since $|C)$ is an arbitrary vector. For the vector $(H_n(x_1\cdots x_n)|$, Eq. (2.13) states that

$$(H_n(x_1\cdots x_n)|\tilde{\phi}(x)=(H_{n+1}(x,x_1\cdots x_n)|. \quad (2.14)$$

The notation of the dual space therefore allows us to express (2.8) in the compact form (2.14), and we shall use this notation in the remainder of the paper. We should emphasize that the use of the dual space does not involve any deep mathematics, but is purely a concession to the printer. It would be perfectly possible to rewrite every subsequent equation in this paper without using the dual space. Every equation of the form (2.14) would then be replaced by an equation of the form (2.8).

We now define a new operator η, which we shall require when writing the right-hand side of (2.4). It is defined as follows:

$$(H_n(x_1,x_2,\cdots,x_n)|\eta(x) = \sum_{r=1}^{n} (H_{n-1}(x_1,x_2,\cdots,[x_r],\cdots x_n)|\delta(x-x_r). \quad (2.15)$$

The vector $(H_{n-1}(x_1,x_2,\cdots,[x_r],\cdots,x_n)|$ denotes the vector which is obtained from $(H_n(x_1,x_2,\cdots,x_r,\cdots,x_n)|$ by removing the variable x_r. From (2.14) and (2.15) it is easily seen that η and ϕ obey the commutation relations:

$$[\eta(x_1),\tilde{\phi}(x_2)]=-\delta^4(x_1-x_2). \quad (2.16)$$

Furthermore, from (2.15),

$$(H_0|\eta(x)=0. \quad (2.17)$$

Equations (2.16) and (2.17) are sufficient to determine η, since $(H_{n+1}|\eta$ can be found from (2.14) and (2.16) once $(H_n|\eta|$ is known. We may therefore regard (2.16) and (2.17) as the definitions of η; we can then easily obtain (2.15).

Having defined the operators $\tilde{\phi}$ and η, we can express the Green's functions equations in a compact form. Our Eq. (2.9), which defines the vectors $(H_n|$, shows that

$$(H_n(x_1\cdots x_n)|G)=G(x_1\cdots x_n). \quad (2.18)$$

Hence Eq. (2.4a) may be written as follows:

$$(\Box_1{}^2-\mu^2)(H_2(x_1,x_2)|G)=\tfrac{1}{2}\lambda(H_3(x_1,x_1,x_2)|G) + i\delta_4(x_1-x_2)(H_0|G). \quad (2.19)$$

We now use (2.14) and (2.15) to express the vectors $(H_2(x_1,x_2)|$, $(H_3(x_1,x_2,x_2)|$, and $(H_0|\delta(x_1-x_2)$ in terms of the single vector $(H_1(x_2)|$:

$$(H_2(x_1,x_2)|=(H_1(x_2)|\tilde{\phi}(x_1), \quad (2.20a)$$
$$(H_3(x_1,x_1,x_2)|=(H_1(x_2)|\tilde{\phi}^2(x_1), \quad (2.20b)$$
$$(H_0|\delta(x_1-x_2)=(H_1(x_2)|\eta(x_1). \quad (2.20c)$$

Equation (2.19) therefore becomes

$$(\Box_1{}^2-\mu^2)(H_1(x_2)|\tilde{\phi}(x_1)|G)=\tfrac{1}{2}\lambda(H_1(x_2)|\tilde{\phi}^2(x_1)|G) + i(H_1(x_2)|\eta(x_1)|G). \quad (2.21)$$

Since the operator $(\Box_1{}^2-\mu^2)$ in the first term of (2.21) acts only on the variable x_1, it may be taken inside the scalar product. Thus

$$(H_1(x_2)|(\Box_1{}^2-\mu^2)\tilde{\phi}(x_1)|G)=\tfrac{1}{2}\lambda(H_1(x_2)|\tilde{\phi}^2(x_1)|G) + i(H_1(x_2)|\eta(x_1)|G). \quad (2.22a)$$

Similarly, Eq. (2.4b) may be written in the form

$$(H_2(x_2x_3)|(\Box_1{}^2-\mu^2)\tilde{\phi}(x_1)|G)=\tfrac{1}{2}\lambda(H_2(x_2,x_3)|\tilde{\phi}^2(x_1)|G) + i(H_2(x_2,x_3)|\eta(x_1)|G). \quad (2.22b)$$

The general Eq. (2.4) can be obtained by replacing the vectors $(H_1|$ and $(H_2|$ in (2.23) by the general vector $(H_n|$:

$$(H_n(x_2\cdots x_{n+1})|(\Box_1{}^2-\mu^2)\tilde{\phi}(x_1)|G) = \tfrac{1}{2}\lambda(H_n(x_2\cdots x_{n+1})|\tilde{\phi}^2(x_1)|G) + i(H_n(x_2\cdots x_{n+1})|\eta(x_1)|G). \quad (2.22c)$$

Since the vector $(H_n(x_2\cdots x_{n+1})|$ in (2.22c) is arbitrary, we may rewrite the equation as an equation for the vector G:

$$(\Box_1{}^2-\mu^2)\tilde{\phi}(x_1)|G)=\tfrac{1}{2}\lambda\tilde{\phi}^2(x_1)|G)+i\eta(x_1)|G). \quad (2.23)$$

We have thus replaced the infinite set of equations for the Green's functions by a single equation for the vector (G).

We can easily integrate (2.23) to give the result:

$$\left(\tilde{\phi}(x)+\tfrac{1}{2}i\lambda\int dx'\tfrac{1}{2}\Delta_F(x-x')\tilde{\phi}^2(x') - \int dx'\tfrac{1}{2}\Delta_F(x-x')\eta(x')\right)|G)=0. \quad (2.24)$$

Equation (2.24) is of course equivalent to the series of Eqs. (2.5) in our new notation.

It is worthwhile noticing that Eq. (2.23) has the same form as the field Eq. (2.1), except for the term $i\eta(x)$. This last term corresponds to the δ-function terms in the equations for the Green's functions.

We can relate our notation to Schwinger's functional notation by making the correspondence

$$\tilde{\phi}(x)\rightarrow\delta/\delta\eta(x), \quad \eta(x)\rightarrow\eta(x),$$

and regarding the operators as acting on the Schwinger functional. We have chosen to define $\tilde{\phi}$ as our fundamental quantity and to define η in terms of it, whereas Schwinger proceeds in the reverse direction. The reason why we have set up a new notation is that quantities similar to $\tilde{\phi}$ may easily be defined in the path-dependent formalism, whereas the quantities analogous to η are not quite so simple. Our notation is therefore more easily applicable to the theories treated in this paper.

3. ELECTROMAGNETIC FIELD

Fundamental Equations

In this section we shall derive the Feynman rules for the electromagnetic field from the path-dependent formalism. Such a derivation has already been given by Sarker,[7] who used a method somewhat different from that to be followed here. Nevertheless, it is worthwhile to rederive the results by the methods which we shall use for the Yang-Mills field, since the methods will be most easily understood by applying them first to the simpler case of the electromagnetic field.

We shall treat an electromagnetic field in interaction with a charged scalar field. The equations of motion in the path-dependent formalism are as follows:

$$(\Box^2-\mu^2)\Phi(x,P)=0\,, \qquad (3.1a)$$

$$(\Box^2-\mu^2)\Phi^*(x,P)=0\,, \qquad (3.1b)$$

$$\frac{\partial F_{\mu\nu}(x)}{\partial x_\mu}+j_\nu(x)=0\,, \qquad (3.1c)$$

where Φ and F are the scalar and electromagnetic field variables, and the current density j is given by the equation

$$j_\nu(x)=-ie[\Phi^*(x,P)\partial_\nu\Phi(x,P)-\Phi(x,P)\partial_\nu\Phi^*(x,P)]. \qquad (3.2)$$

The Φ's and Φ^*'s satisfy the usual commutation relations, which we shall not write down. The commutation relations between the Φ's and the F's are

$$[\Phi(\mathbf{x},t,P),F_{ij}(\mathbf{y},t)]=[\Phi^*(\mathbf{x},t,P),F_{ij}(\mathbf{y},t)]=0\,, \qquad (3.3a)$$

$$[\Phi(\mathbf{x},t,P),F_{0i}(\mathbf{y},t)]=-e\int^x d\xi_i\delta^3(\mathbf{y}-\boldsymbol{\xi})\Phi(\mathbf{x},P)\,, \qquad (3.3b)$$

$$[\Phi^*(\mathbf{x},t,P),F_{0i}(\mathbf{y},t)]=e\int^x d\xi_i\delta^3(\mathbf{y}-\boldsymbol{\xi})\Phi(\mathbf{x},P)\,. \qquad (3.3c)$$

We employ the usual convention that Latin indices range from 1 to 3, Greek indices from 1 to 4. The integrals in (3.3) are to be taken along the path P. There will be similar commutation relations between the $\tilde{\Phi}$'s and the F's. The electromagnetic field variables themselves obey the commutation relations

$$[F_{ij}(\mathbf{x},t),F_{i'j'}(\mathbf{y},t)]=[F_{0i}(\mathbf{x},t),F_{0i'}(\mathbf{y},t)]=0\,, \qquad (3.4a)$$

$$[F_{0i}(\mathbf{x},t),F_{jk}(\mathbf{y},t)]=-i\left(\delta_{ik}\frac{\partial}{\partial y_j}-\delta_{ij}\frac{\partial}{\partial y_k}\right)\times\delta^3(\mathbf{x}-\mathbf{y})\,. \qquad (3.4b)$$

The dependence of the operators on the path P is given by the equations[8]

$$\boldsymbol{\delta}_z\Phi(x,P)=-ie\Phi(x,P)\times F_{\mu\nu}(z)\sigma_{\mu\nu}\,, \qquad (3.5a)$$

$$\boldsymbol{\delta}_z\Phi^*(x,P)=ie\Phi^*(x,P)\times F_{\mu\nu}(z)\sigma_{\mu\nu}\,, \qquad (3.5b)$$

where $\boldsymbol{\delta}_z\Phi$ is the change of Φ caused by a change in the path by an infinitesimal area $\sigma_{\mu\nu}$ at the point z. If the variation of F over the area is non-negligible; for instance, if F contains δ functions, we must rewrite (3.5a) in the form

$$\boldsymbol{\delta}_z\Phi(x,P)=-ie\Phi(x,P)\int d\sigma_{\mu\nu}(z)F_{\mu\nu}(z)\,. \qquad (3.5c)$$

For consistency of the path-dependence equations, we require the homogeneous Maxwell equations

$$\epsilon_{\mu\nu\rho\sigma}\frac{\partial F_{\mu\nu}(x)}{\partial x_\rho}=0\,. \qquad (3.1d)$$

We could replace (3.1d) by the more general condition that the integral of the left-hand side over any volume be a multiple of 2π; we would then obtain the Cabibbo-Ferrari-Coleman theory of magnetic monopoles. We shall assume that there are no monopoles present.

Path-Dependent Green's Functions

One defines Green's functions of the path-dependent variables in the usual way:

$$\hat{G}(x,P;y,P';)=\langle 0|T\{\Phi(x,P),\Phi^*(y,P')\}|0\rangle\,, \qquad (3.6a)$$

$$\hat{G}_{\mu\nu}(x,P;y,P';z)=\langle 0|T\{\Phi(x,P),\Phi^*(y,P'),F_{\mu\nu}(z)\}|0\rangle\,, \qquad (3.6b)$$

and, in general,

$$\hat{G}_{\mu\nu,\rho\sigma,\ldots}(x_1,P_1,x_2,P_2,\cdots;y_1,P_1',y_2,P_2',\cdots;z_1,z_2,\cdots)=\langle 0|T\{\Phi(x_1,P_1)\Phi(x_2,P_2)\cdots\Phi^*(y_1,P_1)\Phi^*(y_2,P_2)\cdots F_{\mu\nu}(z_1),F_{\rho\sigma}(z_2)\cdots\}|0\rangle+\delta\text{-function terms}. \qquad (3.6c)$$

We employ the circumflex to distinguish the path-dependent Green's functions from the auxiliary Green's functions which we shall subsequently define and which we shall denote by the symbol G.

When defining Green's functions of the F's, one has to add an extra term in order to make them covariant.

[8] In order to avoid any possible confusion with the δ function, we shall use a boldface $\boldsymbol{\delta}$ to denote a change in a quantity due to a change in the path.

Such a term is always necessary when the commutators contain derivatives of δ functions. The definition is as follows:

$$\hat{G}_{\mu\nu,ij}(;\,;z,w)=\langle 0|T\{F_{\mu\nu}(z),F_{ij}(w)\}|0\rangle, \quad (3.7a)$$

$$\hat{G}_{0i,0j}(;\,;z,w)=\langle 0|T\{F_{0i}(z),F_{0j}(w)\}|0\rangle+i\delta_{ij}\delta^4(z-w). \quad (3.7b)$$

It is not difficult to check that the Green's function defined by (3.7) with the extra term in (3.7b) are covariant. Green's functions involving more than two F's are similarly defined.

When writing down the field equations and path-dependence equations for time-ordered products, we shall as usual obtain terms from the change of the time ordering. The information contained in the field equations refers to differentiation with respect to the end point of the path, while the path-dependence equation gives the change of a variable resulting from deformation of the path. In the process of such differentiation or deformation, we may reach a situation where one path P, together with its end point x, has some points which are earlier and others which are later than another point y. We then have to define what we mean by time ordering. The commutation relations between $\Phi(x,P)$ and $\Phi(y,P')$ contain contributions from the two end points x and y, while the commutation relations (3.3b) contain contributions from the point y and an element $d\xi_1$ of the path P. We shall adopt the convention that, whenever an end point or an element of a path has its time ordering relative to an end point or element of another path changed in the process of differentiation or of deformation, we add the corresponding contribution from the commutator to the time-ordered product. This convention obviously fulfils the requirement that the time-ordered product of two operators $A(x,P)$ and $B(y,P')$ be changed by their commutator when the time ordering of x,P with y,P' is changed completely.

We can now rewrite the field equations (3.1) as equations for Green's functions. We quote three such equations for purposes of illustration. The Klein-Gordon equation (3.1) gives us the following equation for the two-point Green's function of the scalar particles:

$$(\Box_x{}^2-\mu^2)\hat{G}(x,P;y,P')=i\delta^4(x-y). \quad (3.8a)$$

The Maxwell equation (3.1c) gives us a similar equation for the two-point photon Green's function:

$$\frac{\partial}{\partial z_\mu}\hat{G}_{\mu\nu,\rho\sigma}(;;z,w)=ie\left(\frac{\partial}{\partial z_{1\nu}}-\frac{\partial}{\partial z_{2\nu}}\right)\hat{G}_{\rho\sigma}(z_1,P;z_2,P;w)_{z_1=z_2=z}$$
$$+i\left(\frac{\partial}{\partial w_\rho}\delta_{\sigma\nu}-\frac{\partial}{\partial w_\sigma}\delta_{\rho\nu}\right)\delta^4(z-w). \quad (3.8b)$$

When applied to the Green's function

$$\hat{G}_{\mu\nu,\rho\sigma}(x,P;y,P';z,w),$$

the Maxwell equation becomes

$$\frac{\partial}{\partial z_\mu}\hat{G}_{\mu\nu,\rho\sigma}(x,P;y,P';z,w)$$
$$=ie\left(\frac{\partial}{\partial z_{1\nu}}-\frac{\partial}{\partial z_{2\nu}}\right)\hat{G}_{\rho\sigma}(x,P,z_1,P'';y,P',z_2,P'';w)_{z_1=z_2=z}$$
$$+i\left(\frac{\partial}{\partial w_\rho}\delta_{\sigma\nu}-\frac{\partial}{\partial w_\sigma}\delta_{\rho\nu}\right)\delta^4(z-w)\hat{G}(x,P;y,P')$$
$$+e\left(\int_P^x-\int_{P'}^y\right)d\xi_\nu\delta^4(z-\xi)\hat{G}_{\rho\sigma}(x,P;y,P';w). \quad (3.8c)$$

The second and third terms on the right of (3.8c) result from differentiating the time ordering; the second term comes from the commutator (3.4) between $F_{\mu\nu}(z)$ and $F_{\rho\sigma}(w)$, and the third from the commutators (3.3b) and (3.3c) between $F_{\mu\nu}(z)$ and $\Phi(x,P)$ or $\Phi^*(y,P')$. Higher Green's functions will satisfy equations similar to (3.8c), with a sum of δ-function terms on the right.

The homogeneous Maxwell equation (3.1d) gives simple equations when applied to Green's function. For instance,

$$\epsilon_{\mu\nu\rho\sigma}\frac{\partial}{\partial z_\rho}\hat{G}_{\mu\nu}(x,P;y,P';z)=0. \quad (3.8d)$$

One can also obtain path-dependence equations for the Green's functions from the path-dependence equations (3.5) for the field variables. We shall treat the Green's function

$$\hat{G}_{\mu\nu}(x,P;y,P';z)=\langle 0|T\{\Phi(x,P),\Phi^*(y,P'),F_{\mu\nu}(z)\}|0\rangle, \quad (3.9)$$

which is the simplest example where all the general features occur. We are interested in the change of $\hat{G}$ due to a change of the path P by an infinitesimal area $\sigma_{\mu\nu}$ at the point w. The change will consist of two parts. The first is obtained simply by applying (3.5c) to the factor Φ in (3.9):

$$\boldsymbol{\delta}_w{}^{(1)}\hat{G}_{\mu\nu}(x,P;y,P';z)=-ie\int d\sigma_{\rho\sigma}(w)$$
$$\times\langle 0|T\{\Phi(x,P),\Phi^*(y,P),F_{\mu\nu}(z),F_{\rho\sigma}(w)\}|0\rangle. \quad (3.10a)$$

When expressing the time-ordered product on the right of (3.10) as a Green's function, one has to subtract terms similar to the second term of (3.7b). Thus,

$$\boldsymbol{\delta}_w{}^{(1)}\hat{G}_{\mu\nu}(x,P;y,P';z)=-ie\int d\sigma_{\rho\sigma}(x,P;y,P';z,w)$$
$$+e\int\{d\sigma_{0\nu}(w)\delta_{\mu0}-d\sigma_{0\mu}(w)\delta_{\nu0}\}$$
$$\times\hat{G}(x,P;y,P')\delta^4(z-w). \quad (3.11)$$

The second possible effect of the deformation of the path is the change of the time ordering itself. If part of the path P was originally timelike earlier than the point z, but became timelike later after the change, we obtain a contribution from the commutator. On applying the commutation relation (3.3b) we obtain the following additional change in the Green's function:

$$\delta^{(2)}\hat{G}_{\mu\nu}(x,P,y,P',z)=-e\int^{x}(d\xi_\nu\delta_{\mu0}-d\xi_\mu\delta_{\nu0})\delta^3(\mathbf{z}-\boldsymbol{\xi})\times\hat{G}(x,P;y,P')\tfrac{1}{2}\delta\{\epsilon(\xi_0-z_0)\}. \quad (3.12)$$

The function $\epsilon(\xi_0-z_0)$ is the function which is ±1 according to the sign of the argument. The factor $\frac{1}{2}\delta\{\epsilon(\xi_0-z_0)\}$ is thus equal to ±1 if the time ordering of the path element $d\xi$ and the point z is changed by the change of the path, otherwise it is zero. Equation (3.12) may be rewritten in the form

$$\delta_w{}^{(2)}G_{\mu\nu}(x,P;y,P';z)=-e\int\{d\sigma_{0\nu}(\xi)\delta_{\mu0}-d\sigma_{0\mu}(\xi)\delta_{\nu0}\}\times\delta^4(z-\xi)\hat{G}(x,P;y,P'). \quad (3.10b)$$

The integral in (3.10b) is to be taken over the area between the old and new paths.

We now observe the crucial feature that *the right-hand side of* (3.10b) *cancels against the second term of the right-hand side of* (3.11). The final result is thus

$$\delta_w\hat{G}_{\mu\nu}(x,P;y,P';z)=-ie\int d\sigma_{\rho\sigma}(w)\hat{G}_{\mu\nu,\rho\sigma}(x,P;y,P';z,w). \quad (3.10c)$$

The higher Green's functions obey similar path-dependence equations.

Equation (3.10c) shows that the path-dependence equation of the covariant time-ordered products is similar to that of the field variables themselves. The ordinary time-ordered products satisfy somewhat more complicated path-dependence equations, which contain terms such as (3.10b) that arise from the change of the time ordering. Such terms are exactly cancelled by the delta functions in the definition of the covariant time-ordered product.

Condensed Notation for Path-Dependent Quantities

We now wish to write the field equations and path-dependence equations for the Green's functions in our condensed notation. The method of doing so is a perfectly straightforward generalization of the procedure followed in the previous section. We consider the totality of functions

$$\hat{C}_{\mu\nu,\rho\sigma,\cdots}(x_1,P_1,x_2,P_2,\cdots;y_1,P_1',y_2,P_2',\cdots;z_1,z_2,\cdots)$$

as vectors in a linear space. We denote a typical vector by the symbol $|C)$. The path-dependence equations of the form (3.10c) are assumed, so that a vector is fully specified once the functions are given for one choice of path. Furthermore, we only consider functions which satisfy the homogeneous Maxwell equation (3.8d); indeed, we are forced to impose this limitation in order that the path-dependence equations be consistent.

We now define the following vectors in the dual space:

$$(\hat{H}_{\mu\nu,\cdots}(x_1,P_1,x_2,P_2,\cdots;y_1,P_1',\cdots;z_1,\cdots|\hat{C})=\hat{C}_{\mu\nu,\cdots}(x_1,P_1,x_2,P_2,\cdots;y_1,P_1',\cdots;z_1,\cdots). \quad (3.13a)$$

Applying the path-dependence equations (3.10c) to (3.13), we notice that the vector $(\hat{H}_{\mu\nu}(x,P;y,P';z)|$ satisfies the path-dependence equation

$$\delta_w(\hat{H}_{\mu\nu}(x,P;y,P';z)|=-ie\int d\sigma_{\rho\sigma}(w)(\hat{H}_{\mu\nu,\rho\sigma}(x,P;y,P';z,w)|. \quad (3.14a)$$

Similarly, the general vector in the dual space will satisfy the path-dependence equation

$$\delta_w(\hat{H}_{\mu\nu,\cdots}(x_1,P_1,x_2,P_2,\cdots;y_1,P_1',\cdots;z_1,\cdots)|=-ie\int d\sigma_{\rho\sigma}(w)(\hat{H}_{\mu\nu,\rho\sigma}(x_1,P_1,x_2P_2,\cdots;y_1,P_1',\cdots;z_1,w,\cdots)|., \quad (3.14b)$$

where δ_w represents the change in $(\hat{H}|$ due to a small change in the path P_1 by an amount $\int d\sigma_{\rho\sigma}(w)$.

We are particularly interested in the vector obtained by setting the functions $\hat{C}$ equal to the Green's functions $\hat{G}$, and we denote this vector by the symbol $|\hat{G})$. Thus, from (3.13a),

$$(\hat{H}_{\mu\nu,\cdots}(x_1,P_1,x_2,P_2,\cdots;y_1,P_1',\cdots;z_1,\cdots|\hat{G})=\hat{G}_{\mu\nu,\cdots}(x_1,P_1,x_2,P_2,\cdots;y_1,P_1',\cdots;z_1,\cdots). \quad (3.13b)$$

Following the procedure of the $\lambda\phi^3$ theory, we next define operators $\Phi(x,P)$, $\Phi^*(x,P)$, and $F_{\mu\nu}(x)$ as follows:

$$(\hat{H}_{\mu\nu}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,\cdots)|\tilde{\Phi}(x,P)=(\hat{H}_{\mu\nu}(x_1,P_1,x,P,\cdots;y_1,P_1',\cdots;z_1,\cdots)|, \quad (3.15a)$$

$$(\hat{H}_{\mu\nu\cdots}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,\cdots)|\tilde{\Phi}^*(x,P)=(\hat{H}_{\mu\nu}(x_1,P_1,\cdots;y_1,P_1',x,P,\cdots;z_1,\cdots)|, \quad (3.15b)$$

$$(\hat{H}_{\rho\sigma\cdots}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,\cdots)|\tilde{F}_{\mu\nu}(x)=(\hat{H}_{\rho\sigma,\mu\nu,\cdots}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,x,\cdots)|. \quad (3.15c)$$

We also introduce quantities $U(x,P)$, $\bar{U}(x,P)$, and $X_\nu(z)$, analogous to the η's of the previous section. They are defined to correspond to the δ-function terms on the right-hand sides of (3.8). Thus

$$(\hat{H}_{\mu\nu\ldots}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,\cdots)|U(x,P)$$
$$=\sum_r(\hat{H}_{\mu\nu\ldots}(x_1,P_1,\cdots,[x_r,P_r],\cdots;y_1,P_1',\cdots;z_1\cdots)|\delta^4(x-x_r),\quad(3.16\text{a})$$

$$(\hat{H}_{\mu\nu\ldots}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,\cdots)|\bar{U}(x,P)$$
$$=\sum_r(\hat{H}_{\mu\nu}(x_1,P_1,\cdots;y_1,P_1,\cdots,[y_r,P_r'],\cdots;z_1,\cdots|\delta^4(x-y_r),\quad(3.16\text{b})$$

$$\hat{H}_{\kappa\lambda,\ldots}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,\cdots)|X_\nu(z)$$
$$=\sum_r\left(\frac{\partial}{\partial z_{r\rho}}\delta_{\sigma\nu}-\frac{\partial}{\partial z_{r\sigma}}\delta_{\rho\nu}\right)(\hat{H}_{\kappa\lambda,\ldots,[\rho\sigma],\ldots}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,\cdots[z_r]\cdots)|\delta^4(z-z_r)$$
$$-ie\left(\sum_r\int_{P_r}^{x_r}-\sum_r\int_{P_r'}^{y_r}\right)d\xi_\nu(\hat{H}_{\kappa\lambda,\ldots}(x_1,P_1,\cdots;y_1,P_1',\cdots;z_1,\cdots)|\delta^4(z-\xi),\quad(3.16\text{c})$$

where $\rho\sigma$ are the subscripts corresponding to the coordinate z_r. The operators U, $\bar{U}$, and X_ν can alternatively be defined from their commutation relations with the $\tilde{\Phi}$'s, $\tilde{\Phi}^*$'s, and F's. It follows from (3.15) and (3.16) that

$$[U(x_1,P_1),\tilde{\Phi}(x_2,P_2)]=-\delta^4(x_1-x_2),\quad(3.17\text{a})$$

$$[\bar{U}(x_1,P_1),\tilde{\Phi}^*(x_2,P_2)]=[U(x_1,P_1),\tilde{F}_{\mu\nu}(x_2)]=0,\quad(3.17\text{b})$$

$$[\bar{U}(x_1,P_1),\tilde{\Phi}^*(x_2,P_2)]=-\delta^4(x_1-x_2),\quad(3.18\text{a})$$

$$[\bar{U}(x_1,P_1),\tilde{\Phi}(x_2,P_2)]=[\bar{U}(x_1,P_1),F_{\mu\nu}(x_2)]=0,\quad(3.18\text{b})$$

$$[X_\nu(x_1),\tilde{F}_{\rho\sigma}(x_2)]=-\left(\frac{\partial}{\partial x_{2\rho}}\delta_{\nu\sigma}-\frac{\partial}{\partial x_{2\sigma}}\delta_{\nu\rho}\right)\times\delta^4(x_1-x_2),\quad(3.19\text{a})$$

$$[X_\nu(x_1),\tilde{\Phi}(x_2,P_2)]=ie\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)\times\tilde{\Phi}(x_2,P_2),\quad(3.19\text{b})$$

$$[X_\nu(x_1),\tilde{\Phi}^*(x_2,P_2)]=-ie\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)\times\tilde{\Phi}^*(x_2,P_2).\quad(3.19\text{c})$$

Furthermore,

$$(H_0|U(x,P)=(H_0|\bar{U}(x,P)=(H_0|X_\nu(x)=0.\quad(3.20)$$

As in the case of the scalar field, Eqs. (3.17)-(3.20) define U, $\bar{U}$, and X_ν completely. Perhaps it is worthwhile stressing again that (3.17)-(3.19) are in no sense quantum-mechanical commutation relations.

The equations for the Green's functions such as (3.8) can easily be expressed in terms of the operators we have just defined. Thus

$$[(\Box^2-\mu^2)\tilde{\Phi}(x,P)-i\bar{U}(x,P)]|\hat{G})=0,\quad(3.21\text{a})$$

$$[(\Box^2-\mu^2)\tilde{\Phi}^*(x,P)-iU(x,P)]|\hat{G})=0,\quad(3.21\text{b})$$

$$\left[\frac{\partial}{\partial x_\mu}\tilde{F}_{\mu\nu}(x)-ie\left(\frac{\partial}{\partial x_{1\nu}}-\frac{\partial}{\partial x_{2\nu}}\right)\tilde{\Phi}^*(x_2,P_2)\tilde{\Phi}(x_1,P_1)|_{x_1=x_2=x}-iX_\nu(x)\right]|\hat{G})=0.\quad(3.21\text{c})$$

The derivation of (3.21) is exactly the same as the derivation of (2.25). Apart from the last terms in the square brackets, Eqs. (3.21) have the same form as the field equations.

The path-dependence equation (3.14b) may be written as follows:

$$(\hat{H}_{\mu\nu}(x_2,P_2,\cdots;y_1,P_1',\cdots;z_1,\cdots)|\boldsymbol{\delta}_w\tilde{\Phi}(x,P)$$
$$=(\hat{H}_{\mu\nu}(x_2,P_2,\cdots;y_1,P_1',\cdots;z_1,\cdots|\times[-ie\tilde{\Phi}(x,P)\tilde{F}_{\rho\sigma}(z)].\quad(3.22)$$

Since the vector $(H|$ on the left is a general vector in the dual space, we may write this equation as an operator equation

$$\boldsymbol{\delta}_w\tilde{\Phi}(x,P)=-ie\tilde{\Phi}(x,P)\tilde{F}_{\rho\sigma}(z)\sigma_{\rho\sigma}.\quad(3.23\text{a})$$

In the same way we can write the following path-dependence equation for $\tilde{\Phi}^*$:

$$\boldsymbol{\delta}_w\tilde{\Phi}^*(x,P)=ie\tilde{\Phi}^*(x,P)\tilde{F}_{\rho\sigma}(z)\sigma_{\rho\sigma}.\quad(3.23\text{b})$$

Thus the path-dependence equations, like the field equations, can be expressed in a compact form in our linear space. Again, the form of the equations (3.22) is identical to the form of the path-dependence equations (3.5) for the field variables.

It should be noted that the path-dependence equations and the field equations are treated differently in our formalism. When defining our linear space, we exclude all functions $\hat{C}$ which do not satisfy the path-dependence equations. The path-dependence equations thus appear as equations on the vectors $(H|$ of the dual space [Eqs. (3.14)], and finally as the operator equations (3.23). On the other hand, the field equations select out a particular vector $|G)$ from the linear space of vectors $|C)$. They are not operator equations, but equations on the vector $|G)$.

Auxiliary Variables

We now wish to obtain a perturbation solution for the system of field equations (3.19) and path-dependence equations (3.23). We shall do so by introducing auxiliary path-independent quantities which we shall call $\tilde{\phi}$, $\tilde{\phi}^*$, and $\tilde{A}_\mu$. The connection between $\tilde{\Phi}$, $\tilde{\Phi}^*$, and $\tilde{F}_{\mu\nu}$ on the one hand, and $\tilde{\phi}$, $\tilde{\phi}^*$, and $\tilde{A}_\mu$ on the other, will be defined to be of the same form as the connection between the corresponding field variables. Thus

$$\tilde{\Phi}(x,P)=\tilde{\phi}(x)\bigg(1-ie\int_P^x d\xi_\mu\tilde{A}_\mu(\xi)-\tfrac{1}{2}e^2\int_P^x d\xi_\mu \times\int_P^x d\xi_\nu'\tilde{A}_\mu(\xi)\tilde{A}_\nu(\xi')+\cdots\bigg), \quad (3.24a)$$

$$\tilde{\Phi}^*(x,P)=\tilde{\phi}^*(x)\bigg(1+ie\int_P^x d\xi_\mu\tilde{A}_\mu(\xi)-\tfrac{1}{2}e^2\int_P^x d\xi_\mu \times\int_P^x d\xi_\nu'\tilde{A}_\mu(\xi)\tilde{A}_\nu(\xi')+\cdots\bigg), \quad (3.24b)$$

$$\tilde{F}_{\mu\nu}(x)=\frac{\partial\tilde{A}_\nu(x)}{\partial x_\mu}-\frac{\partial\tilde{A}_\mu(x)}{\partial x_\nu}. \quad (3.24c)$$

Equations (3.24) leave the variables $\tilde{\phi}$, $\tilde{\phi}^*$, and $\tilde{A}_\mu$ arbitrary to within a gauge transformation, and we shall not define them further.

The definitions (3.24) require an enlargement of our linear space. Let us define the new vectors in the dual space:

$$(H_0|=(\hat{H}_0|, \quad (3.25a)$$

$$(H(x)|=(\hat{H}^0|\tilde{\phi}(x), \quad (3.25b)$$

$$(H(x;y)|=(\hat{H}_0|\tilde{\phi}(x)\tilde{\phi}^*(y), \quad (3.25c)$$

$$(H_\mu(x;y;z)|=(\hat{H}_0|\tilde{\phi}(x)\tilde{\phi}^*(y)\tilde{A}_\mu(z), \quad \text{etc.} \quad (3.25d)$$

The path-dependent vectors $(\hat{H}|$ of our original dual space may then be expressed in terms of the vectors $(H|$ of our auxiliary dual space. To take an example,

$$(\hat{H}(x,P;y,P')|=(\hat{H}_0|\tilde{\Phi}(x,P)\Phi^*(y,P') \quad \text{[from (3.15)]}$$

$$=(H_0|\tilde{\phi}(x)\tilde{\phi}^*(y)\bigg[1-ie\bigg(\int_P^x-\int_{P'}^y\bigg)d\xi_\mu\tilde{A}_\mu(\xi) -\tfrac{1}{2}e^2\bigg(\int_P^x-\int_{P'}^y\bigg)d\xi_\mu\bigg(\int_P^x-\int_{P'}^y\bigg)d\xi_\nu' \times\tilde{A}_\mu(\xi)\tilde{A}_\nu(\xi')+\cdots\bigg] \quad \text{[from (3.24)]}$$

$$=(H(x;y)|-ie\bigg(\int_P^x-\int_{P'}^y\bigg)d\xi_\mu(H_\mu(x' y;\xi)| -\tfrac{1}{2}e^2\bigg(\int_P^x-\int_{P'}^y\bigg)d\xi_\mu\bigg(\int^x-\int_{P'}^y\bigg)d\xi_\nu' \times(H_{\mu\nu}(x;y;\xi,\xi')|+\cdots \quad \text{[from (3.25)]}. \quad (3.26a)$$

Similarly we may show that

$$(\hat{H}_{\mu\nu}(x,P;y,P';z)|=\bigg(\frac{\partial}{\partial z_\mu}\delta_{\nu\lambda}-\frac{\partial}{\partial z_\nu}\delta_{\mu\lambda}\bigg) \times\bigg\{(H_\lambda(x;y;z)|-ie\bigg(\int_P^x-\int_{P'}^y\bigg)d\xi_\rho(H_{\lambda\rho}(x;y;z,\xi)| -\tfrac{1}{2}e^2\bigg(\int_P^x-\int_{P'}^y\bigg)d\xi_\rho\bigg(\int_P^x-\int_{P'}^y\bigg)d\xi_\sigma' \times(H_{\lambda\rho\sigma}(x;y;z,\xi,\xi')|+\cdots\bigg\}. \quad (3.26b)$$

We can similarly obtain formulas for any dual-space vector $(\hat{H}|$ as a linear combination of auxiliary dual-space vectors $(H|$.

It is not true, however, that the vectors $(H|$ in our auxiliary dual space can be expressed as linear combinations of the vectors $(\hat{H}|$ in our path-dependent dual space. For instance, in the limit $e\to 0$, Eq. (3.26) gives us an equation for the curl of $(H_\lambda(x;y;z)|$ with respect to z, but we have no equation for the divergence of $(H_\lambda(x;y;z)|$. Thus the space of vectors $(H|$ is larger than the space of vectors $(\hat{H}|$.

Our next step is to introduce the space of vectors $|C)$ dual to the space $(H|$. A vector $|C)$ is defined by its scalar products $(H|C)$ with all vectors in the space $|H)$. For any vector $|\hat{C})$ in our original space, the scalar products $(\hat{H}|\hat{C})$ with any vector in the space $(\hat{H}|$ is defined. We can therefore regard our vector $|C)$ as corresponding to a vector $|\hat{C})$ in the original space by equating the scalar products

$$(\hat{H}|C)=(\hat{H}|\hat{C}). \quad (3.27)$$

Since the space of vectors $(\hat{H}|$ is only a subspace of the space of vectors $(H|$, Eq. (3.27) does not define a unique vector $|C)$ if $|\hat{C})$ is known. By saying that a vector $|C)$ corresponds to a vector $|\hat{C})$ of our original space we therefore fix certain of its components, but we do not define it completely.

We can now define a set of functions

$$C_{\mu\ldots}(x,\cdots;y,\cdots;z,\cdots)$$

by taking the scalar product of the vector $|C)$ with all the vectors in the space $(H|$:

$$C_{\mu\ldots}(x_1,\cdots;y_1,\cdots;z_1,\cdots) =(H_{\mu\ldots}(x_1,\cdots;y_1,\cdots;z_1,\cdots)|C). \quad (3.28a)$$

The vector C is thus defined by the totality of functions $C_{\mu\ldots}(x_1,\cdots;y_1,\cdots;z_1,\cdots)$. We are particularly interested in the vectors $|G)$ corresponding to our original vector $|\hat{G})$. We therefore define

$$G_\mu(x_1,\cdots;y_1,\cdots;z_1,\cdots) =(H_{\mu\ldots}(x_1,\cdots;y_1,\cdots;z_1,\cdots)|G). \quad (3.28b)$$

By taking the scalar product of $|G)$ with the dual-space vectors on the left- and right-hand sides of (3.26a), and applying (3.13b) and (3.28b), we obtain the equation

$$\hat{G}(x,P;y,P')=G(x,y)-ie\left(\int_P^x-\int_{P'}^y\right)d\xi_\mu G_\mu(x;y;\xi)$$
$$-\tfrac{1}{2}e^2\left(\int_P^x-\int_{P'}^y\right)d\xi_\mu\left(\int_P^x-\int_{P'}^y\right)$$
$$\times d\xi_\nu' G_{\mu\nu}(x;y;\xi,\xi')+\cdots. \quad (3.29a)$$

From Eq. (3.26b), we obtain the equation

$$\hat{G}_{\mu\nu}(x,P;y,P';z)$$
$$=\left(\frac{\partial}{\partial z_\mu}\delta_{\nu\lambda}-\frac{\partial}{\partial z_\nu}\delta_{\mu\lambda}\right)\Bigg[G_\lambda(x;y;z)-ie\left(\int_P^x-\int_{P'}^y\right)d\xi_\rho$$
$$\times G_{\lambda\rho}(x;y;z,\xi)-\tfrac{1}{2}e^2\left(\int_P^x-\int_{P'}^y\right)d\xi_\rho\left(\int_P^x-\int_{P'}^y\right)$$
$$\times d\xi_\sigma' G_{\lambda\rho\sigma}(x;y;z,\xi,\xi')+\cdots\Bigg]. \quad (3.29b)$$

We can similarly obtain formulas for any Green's function $\hat{G}$ in terms of the auxiliary Green's functions G and, indeed, Eqs. (3.25) are the general expressions for such formulas in our shorthand notation. On the other hand, equations such as (3.29) do not uniquely define the G's in terms of the $\hat{G}$'s. This corresponds to the fact that the vector $|G)$ is not uniquely defined by the vector $|\hat{G})$.

The relations between the $\hat{G}$'s and the G's are the same as we would have obtained if we had defined the G's by the equation

$$G_{\mu\cdots}(x_1,\cdots;y_1,\cdots;z_1,\cdots)$$
$$=\langle 0|T\{\phi(x_1)\cdots\phi^*(y_1)\cdots A_\nu(z_1)\cdots\}|0\rangle$$

and expressed the Φ's, Φ^*'s, and F's in terms of the ϕ's, ϕ^*'s, and A's by the usual formulas. The ambiguity in the definition of the G's corresponds to the gauge ambiguity in the definition of the ϕ's, ϕ^*'s, and A's. Needless to say, our present approach makes no use of the ϕ's, ϕ^*'s, and A's considered as quantum-mechanical operators. The auxiliary Green's functions G or, equivalently, the auxiliary quantities $\tilde{\phi}$, $\tilde{\phi}^*$, and $\tilde{A}_\mu$ are introduced directly, and the path-dependent Green's functions are related to them.

Our enlargement of the dual space $(\hat{H}|$ is thus equivalent to introducing path-independent, auxiliary Green's functions G, and expressing the path-dependent Green's functions in terms of them by (3.29). The reader may feel that we have adopted a rather long-winded procedure for introducing these auxiliary Green's functions. However, it would be very cumbersome to deal with the infinite set of equations (3.29) in practice, whereas the compact equations (3.24) are easily handled. The necessity for expressing the Green's-function equations in a compact form will be even greater when we treat the Yang-Mills and gravitational fields.

Once we have defined the auxiliary quantities, our first requirement is to verify that the definitions do imply the correct path-dependence equations for the $\tilde{\Phi}$'s. It is not difficult to see that (3.24) does indeed lead to (3.23).

We next define quantities η, $\bar{\eta}$, and ζ_μ, analogous to the η of the previous section. We shall define them directly by their commutation relations with the $\tilde{\phi}$, $\tilde{\phi}^*$, and A_μ. Thus,

$$[\eta(x_1),\tilde{\phi}(x_2)]=-\delta^4(x_1-x_2)\,, \quad (3.30a)$$
$$[\eta(x_1),\tilde{\phi}^*(x_2)]=[\eta(x_1),A_\mu(x_2)]=0\,, \quad (3.30b)$$
$$[\bar{\eta}(x_1),\tilde{\phi}^*(x_2)]=-\delta^4(x_1-x_2)\,, \quad (3.30c)$$
$$[\bar{\eta}(x_1),\tilde{\phi}(x_2)]=[\bar{\eta}(x_1),A_\mu(x_2)]=0\,, \quad (3.30d)$$
$$[\zeta_\mu(x_1),\tilde{A}_\nu(x_2)]=-\delta_{\mu\nu}\delta^4(x_1-x_2)\,, \quad (3.30e)$$
$$[\zeta_\mu(x_1),\tilde{\phi}(x_2)]=\zeta_\mu(x_1),\tilde{\phi}^*(x_2)]=0. \quad (3.30f)$$

The definitions are completed by the equations

$$(H_0|\eta(x)=(H_0|\bar{\eta}(x)=(H_0|\zeta_\nu(x)=0. \quad (3.31)$$

Equations (3.30) are equivalent to the definitions

$$(H_{\mu\cdots}(x_1,\cdots;y_1,\cdots;z_1,\cdots)|\eta(x)$$
$$=\sum_r(H_{\mu\cdots}(x_1,\cdots[x_r],\cdots;y_1,\cdots;z_1,\cdots)|$$
$$\times\delta^4(x-x_r), \quad (3.32a)$$
$$(H_{\mu\cdots}(x_1,\cdots;y_1,\cdots;z_1,\cdots)|\bar{\eta}(x)$$
$$=\sum_r(H_{\mu\cdots}(x_1,\cdots;y_1,\cdots[y_r],\cdots;z_1,\cdots)|$$
$$\times\delta^4(x-y_r), \quad (3.32b)$$
$$(H_{\mu\cdots}(x_1,\cdots;y_1,\cdots;z_1,\cdots)|\zeta_\nu(x)$$
$$=\sum_r(H_{\mu\cdots[\nu]\cdots}(x_1,\cdots;y_1,\cdots;z_1,\cdots[z_r]\cdots)|$$
$$\times\delta^4(x-z_r). \quad (3.32c)$$

In the following development it will be useful to have a special notation for the functions appearing in (3.24). We thus define

$$V(P,x)=1-ie\int_P^x d\xi_\mu\tilde{A}_\mu(\xi)$$
$$-\tfrac{1}{2}e^2\int_P^x d\xi_\mu\int_P^x d\xi_\nu'\tilde{A}_\mu(\xi)\tilde{A}_\nu(\xi')+\cdots, \quad (3.33a)$$

$$\bar{V}(P,x)=1+ie\int_P^x d\xi_\mu \tilde{A}_\mu(\xi)$$
$$-\tfrac{1}{2}e^2\int_P^x d\xi_\mu \int_P^x d\xi_\nu' \tilde{A}_\mu(\xi)\tilde{A}_\nu(\xi')+\cdots \quad (3.33b)$$
$$=V^{-1}(P,x). \quad (3.33c)$$

Equations (3.24a) and (3.24b) may then be written

$$\tilde{\Phi}(x,P)=V(P,x)\tilde{\phi}(x)\,, \quad (3.34a)$$
$$\tilde{\Phi}^*(x,P)=\bar{V}(P,x)\tilde{\phi}^*(x)=V^{-1}(P,x)\tilde{\phi}^*(x)\,. \quad (3.34b)$$

The functions V and $\bar{V}$ satisfy the equations

$$\frac{\partial}{\partial x_\mu}V(x,P)=-ieA_\mu(x)V(x,P)\,, \quad (3.35a)$$
$$\frac{\partial}{\partial x_\mu}\bar{V}(x,P)=ieA_\mu(x)\bar{V}(x,P)\,. \quad (3.35b)$$

Gauge Transformations

Equations (3.24) do not define the auxiliary quantities $\tilde{\phi}$, $\tilde{\phi}^*$, and A uniquely. If one makes the infinitesimal transformation

$$\tilde{\phi}(x)\to\tilde{\phi}(x)+ie\lambda\chi(x)\tilde{\phi}(x)\,, \quad (3.36a)$$
$$\tilde{\phi}^*(x)\to\tilde{\phi}^*(x)-ie\lambda\chi(x)\tilde{\phi}^*(x)\,, \quad (3.36b)$$
$$A_\mu(x)\to A_\mu(x)+\lambda\frac{\partial\chi(x)}{\partial x_\mu}\,, \quad (3.36c)$$

the path-dependent quantities defined by (3.24) remain unaltered. The transformation (3.36) is precisely analogous to a gauge transformation in the usual formalism.

We shall have to apply the transformation (3.36) in writing down the equations of motion, and it will be useful to construct the generator of such a transformation. Let us define

$$Y(y)=-ie\eta(y)\tilde{\phi}(y)+ie\bar{\eta}(y)\tilde{\phi}^*(y)+\frac{\partial\zeta_\mu(y)}{\partial y_\mu}\,. \quad (3.37)$$

From the commutation relations (3.30) one obtains the equations

$$\left[\int dy\, Y(y)\chi(y),\tilde{\phi}(x)\right]=ie\chi(x)\tilde{\phi}(x)\,, \quad (3.38a)$$
$$\left[\int dy\, Y(y)\chi(y),\tilde{\phi}^*(x)\right]=-ie\chi(x)\tilde{\phi}^*(x)\,, \quad (3.38b)$$
$$\left[\int dy\, Y(y)\chi(y),\tilde{A}_\mu(x)\right]=\frac{\partial\chi(x)}{\partial x_\mu}\,. \quad (3.38c)$$

Thus, comparing (3.36) with (3.38), we conclude that the integral $\int dy\, Y(y)\chi(y)$ does generate a gauge transformation when it is commuted with any of our auxiliary quantities. In particular, the integral $\int dy\, Y(y)\chi(y)$ commutes with any of our path-dependent quantities $\tilde{\Phi}$, $\tilde{\Phi}^*$, and F, since such quantities remain invariant under a gauge transformation.

Field Equations

We now reexpress the field equations (3.21) for the path-dependent quantities as field equations for the auxiliary quantities. Our aim is to show that the auxiliary Green's functions can be expressed as a sum of Feynman diagrams in the familiar manner.

The first term in the curly bracket of (3.21a) is easily rewritten in terms of the auxiliary quantities. From (3.34a) and (3.35a) it follows that

$$\frac{\partial}{\partial x_\mu}\tilde{\Phi}(x,P)=V(x,P)\left(\frac{\partial}{\partial x_\mu}-ie\tilde{A}_\mu(x)\right)\tilde{\phi}(x)\,, \quad (3.39)$$

and therefore

$$(\Box^2-\mu^2)\tilde{\Phi}(x,P)=V(x,P)\times\left[\left(\frac{\partial}{\partial x_\mu}-ie\tilde{A}_\mu(x)\right)^2-\mu^2\right]\tilde{\phi}(x)\,. \quad (3.40)$$

It is not much more difficult to express the second term in (3.21a), namely the function $\bar{U}$, in terms of auxiliary quantities. The function $\bar{U}$ is defined by (3.17) and (3.20). One can show immediately from (3.30c), (3.30d), and (3.34) that the quantity $V(x,P)\bar{\eta}(x)$ has precisely the commutation relations (3.17) with $\tilde{\Phi}(x,P)$, $\tilde{\Phi}^*(x,P)$, and $\tilde{F}_{\mu\nu}(z)$. Further, from (3.31) and the fact that $\bar{\eta}$ commutes with V, we deduce that

$$(H_0|\,V(x,P)\bar{\eta}(x)=0\,.$$

Hence the quantity $V(x,P)\bar{\eta}(x)$ satisfies all the requirements of (3.17) and (3.18), and we may write

$$\bar{U}(x,P)=V(x,P)\bar{\eta}(x)\,. \quad (3.41)$$

From (3.21a), (3.39), and (3.41), we may therefore write the equation

$$V(x,P)\left[\left(\frac{\partial}{\partial x_\mu}-ie\tilde{A}_\mu\right)^2\times\tilde{\phi}(x)-\mu^2\tilde{\phi}(x)-i\bar{\eta}(x)\right]|G)=0\,. \quad (3.42)$$

Equation (3.42) implies (3.21a). The converse is not true, since (3.21a) is an equation for the vector $|\hat{G})$, which does not define the vector $|G)$ uniquely. Since those components of $|G)$ which are not determined by $|\hat{G})$ have no physical significance, we can define them in any convenient manner, and we shall require that

(3.42) be true. On multiplying by $\bar{V}(x,P)$ and applying (3.33c), we obtain the field equation in its final form:

$$\left[\left(\frac{\partial}{\partial x_\mu}-ie\tilde{A}_\mu\right)^2\tilde{\phi}(x)-\mu^2\tilde{\phi}(x)-i\bar{\eta}(x)\right]|G)=0. \quad (3.43a)$$

The path dependence has been removed from Eq. (3.43a), which is identical to the equation satisfied by the Green's functions of the Lorentz-gauge theory. In our present derivation we have made no reference to quantum-mechanical operators ϕ, ϕ^*, or A, however.

Equation (3.21b) may similarly be expressed as an equation involving auxiliary variables:

$$\left[\left(\frac{\partial}{\partial x_\mu}+ie\tilde{A}_\mu\right)\tilde{\phi}^*(x)-\mu^2\tilde{\phi}^*(x)-i\eta(x)\right]|G)=0. \quad (3.43b)$$

Before going on to express (3.21c) as an equation for our auxiliary variables, we shall introduce an expression for the current and shall find the equations for its divergence. We thus define

$$\tilde{j}_\mu(x)=-ie\left(\tilde{\phi}^*(x)\frac{\partial}{\partial x_\mu}\tilde{\phi}(x)-\tilde{\phi}(x)\frac{\partial}{\partial x_\mu}\tilde{\phi}^*(x)\right) -e^2[\tilde{A}_\lambda(x)]^2\tilde{\phi}^*(x)\tilde{\phi}(x). \quad (3.44)$$

Again, the definition of $\tilde{j}_\mu$ in terms of $\tilde{\phi}$, $\tilde{\phi}^*$, and $\tilde{A}$ corresponds to the definition of the quantum-mechanical operator j_μ in terms of ϕ, ϕ^*, and A. From (3.43) we can easily show that

$$\left(\frac{\partial\tilde{j}_\mu(x)}{\partial x_\mu}+e\eta(x)\tilde{\phi}(x)-e\bar{\eta}(x)\tilde{\phi}^*(x)\right)|G)=0. \quad (3.45)$$

Equation (3.45) is the Takahashi-Ward identity in our notation.

We now examine (3.21c). From (3.24c), we may write

$$\frac{\partial}{\partial x_\mu}\tilde{\tilde{F}}_{\mu\nu}=\frac{\partial}{\partial x_\mu}\left(\frac{\partial\tilde{A}_\nu}{\partial x_\mu}-\frac{\partial\tilde{A}_\mu}{\partial x_\nu}\right) \quad (3.46a)$$

and, from (3.34) and (3.35),

$$\begin{aligned}-ie\left(\frac{\partial}{\partial x_{1\nu}}-\frac{\partial}{\partial x_{2\nu}}\right)&\tilde{\Phi}^*(x_2,P_2)\tilde{\Phi}(x_1,P_1)|_{x_1=x_2}\\ &=-ie\left[\tilde{\phi}^*(x)\left(\frac{\partial}{\partial x_\mu}-ie\tilde{A}_\mu\right)\tilde{\phi}(x) -\tilde{\phi}(x)\left(\frac{\partial}{\partial x_\mu}+ie\tilde{A}_\mu\right)\tilde{\phi}^*(x)\right]\\ &=\tilde{j}_\nu(x). \end{aligned} \quad (3.46b)$$

The first two terms in the square brackets of (3.21c) are thus easily expressed. Furthermore, if we use (3.30e) and (3.30f) to calculate the commutation relations of the operator $\zeta_\nu(x)$ with the path-dependent quantities defined in (3.24), we find that $\zeta_\nu(x)$ fulfills all the requirements of the function $X_\nu(x)$, defined in (3.19) and (3.20). We might therefore be tempted to write

$$X_\nu(x)=\zeta_\nu(x) \quad (3.47)$$

and to write (3.21c) in the form

$$\left[\frac{\partial}{\partial x_\mu}\left(\frac{\partial\tilde{A}_\nu(x)}{\partial x_\mu}-\frac{\partial\tilde{A}_\mu(x)}{\partial x_\nu}\right)+\tilde{j}_\nu(x)-i\zeta_\nu(x)\right]|G)=0. \quad (3.48)$$

Equation (3.48) is not consistent, however. By differentiating (3.48) with respect to x_ν, we obtain the consistency condition

$$\left(\frac{\partial\tilde{j}_\nu(x)}{\partial x_\nu}-i\frac{\partial\zeta_\nu(x)}{\partial x_\nu}\right)|G)=0.$$

The variable $\tilde{j}$ does not satisfy this divergence condition. Instead, it satisfies the divergence condition (3.45).

We therefore require a generalization of (3.48), which we shall achieve by generalizing (3.47). The commutation relations (3.19) define X_ν within the original linear space of our path-dependent Green's functions, but they do not fully define it within the enlarged linear space. We are free to extend the definition into the enlarged space in any convenient manner, but we must do so consistently. The definition (3.47), as we have just seen, is not consistent with the field equations.

We shall generalize the definition of X_ν by writing

$$X_\nu(x)=\zeta_\nu(x)+\int dy\,\chi_\nu(x,y)Y(y), \quad (3.49)$$

where the function χ has still to be specified. We have shown that the second term of (3.49) generates a gauge transformation and that it therefore commutes with the path-dependent variables Φ, Φ^*, and F. Furthermore, $(H_0|Y(y)=0$, by (3.31) and (3.37). The right-hand side of (3.49) therefore satisfies the conditions (3.19) and (3.20) which define the operator $X_\nu(x)$.

Equation (3.48) is therefore generalized to read:

$$\left[\frac{\partial}{\partial x_\mu}\left(\frac{\partial\tilde{A}_\nu(x)}{\partial x_\mu}-\frac{\partial\tilde{A}_\mu(x)}{\partial x_\nu}\right)+\tilde{j}_\nu(x)-iX_\nu(x)\right]|G)=0, \quad (3.50)$$

with the term $X_\nu(x)$ given by (3.49), and the operator Y given in turn by (3.37). Equation (3.50), like (3.48), does imply the truth of Eq. (3.21c).

We now have to choose the function χ_ν in such a way that the consistency condition

$$\left(\frac{\partial\tilde{j}_\nu(x)}{\partial x_\nu}-i\frac{\partial X_\nu(x)}{\partial x_\nu}\right)|G)=0 \quad (3.51)$$

is satisfied. From (3.45) we observe that we can achieve this by writing[9]

$$X_\nu(x)=\zeta_\nu(x)-\frac{\partial}{\partial x_\nu}\Box^{-2}\times\left(\frac{\partial\zeta_\mu(x)}{\partial x_\mu}-ie\eta(x)\tilde{\phi}(x)+ie\bar{\eta}(x)\tilde{\phi}^*(x)\right). \quad (3.52)$$

Furthermore, (3.52) does have the required form (3.49), with

$$\chi_\nu(x,y)=-\frac{\partial}{\partial x_\nu}\Box^{-2}\delta^4(x-y). \quad (3.53)$$

We therefore write Eq. (3.50) as follows:

$$\left[\frac{\partial}{\partial x_\mu}\left(\frac{\partial\tilde{A}_\nu(x)}{\partial x_\mu}-\frac{\partial\tilde{A}_\mu(x)}{\partial x_\nu}\right)+\tilde{j}_\nu(x)-i\zeta_\nu(x)+i\frac{\partial}{\partial x_\nu}\Box^{-2}\times\left(\frac{\partial\zeta_\mu(x)}{\partial x_\mu}-ie\eta(x)\tilde{\phi}(x)+ie\bar{\eta}(x)\tilde{\phi}^*(x)\right)\right]|G)=0. \quad (3.54)$$

Equation (3.54) is consistent with the divergence condition and, since it is of the form (3.50), it implies Eq. (3.21c) as required.

Our final field equations are (3.43) and (3.54). Equation (3.43a) can easily be expressed in integral form:

$$\left(\tilde{\phi}(x)-e\int dx'\tfrac{1}{2}\Delta_F(x-x')\tilde{A}_\mu(x')\frac{\partial\tilde{\phi}(x')}{\partial x_\mu}+i\int dx'\tfrac{1}{2}\Delta_F(x-x')\tilde{A}_\mu(x')\tilde{A}_\mu(x')\tilde{\phi}(x')+\int dx'\tfrac{1}{2}\Delta_F(x-x')\eta(x')\right)|G)=0. \quad (3.55)$$

Equation (3.43b) can be expressed in integral form in a similar way. Turning to (3.54), we notice that it has the general form

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial f_\nu}{\partial x_\mu}-\frac{\partial f_\mu}{\partial x_\nu}\right)+g_\nu+\frac{\partial h}{\partial x_\nu}=0, \quad (3.56a)$$

where the consistency condition

$$\frac{\partial}{\partial x_\nu}\left(g_\nu+\frac{\partial h}{\partial x_\nu}\right)=0 \quad (3.56b)$$

is satisfied. Equation (3.56) (with Feynman boundary conditions) has the solution

$$f_\nu(x)=\tfrac{1}{2}i\int dx'\Delta_F(x-x')\times\left(g_\nu(x')+\frac{\partial h(x')}{\partial x_\nu'}\right)+\frac{\partial a(x)}{\partial x_\nu}, \quad (3.57a)$$

where the function $a(x)$ is arbitrary. We may rewrite the integral

$$\int dx'\Delta_F(x-x')\frac{\partial h(x')}{\partial x_\nu'} \quad \text{as} \quad \frac{\partial}{\partial x_\nu}\int dx'\Delta_F(x-x')h(x'),$$

and we may then absorb it in the function a. Thus

$$f_\nu(x)=\tfrac{1}{2}i\int dx'\Delta_F(x-x')g_\nu(x')+\frac{\partial a'(x)}{\partial x_\nu}. \quad (3.57b)$$

Applying the solution (3.57b) to (3.54) and putting $a'=0$, we find that

$$\left(\tilde{A}_\nu(x)-i\int dx'\tfrac{1}{2}\Delta_F(x-x')\tilde{j}_\nu(x')-\int dx'\tfrac{1}{2}\Delta_F(x-x')\zeta_\nu(x')\right)|G)=0. \quad (3.58)$$

We can obtain more general solutions of (3.54) by adding a pure divergence to the terms within the curly bracket. Such solutions correspond to taking nonzero values for the function a' in (3.57b). In particular, another solution of (3.54) is the following:

$$\left[\tilde{A}_\nu(x)-i\int dx'\left(\delta_{\nu\rho}-c\frac{\partial^2}{\partial x_\nu\partial x_\rho}\Box^{-2}\right)\times\tfrac{1}{2}\Delta_F(x-x')\tilde{j}_\rho(x')-\int dx'\left(\delta_{\nu\rho}-c\frac{\partial^2}{\partial x_\nu\partial x_\rho}\Box^{-2}\right)\times\tfrac{1}{2}\Delta_F(x-x')\zeta_\rho(x')\right]|G)=0, \quad (3.59)$$

where c is any constant.

Equations (3.55) and (3.58) are exactly the equations we would have obtained if we had started with the Lagrangian

$$\mathfrak{L}=-\frac{1}{2}\left[\left(\frac{\partial}{\partial x_\mu}+ieA_\mu\right)\phi^*\right]\left(\frac{\partial}{\partial x_\mu}-ieA_\mu\right)\phi-\frac{1}{2}\left(\frac{\partial A_\nu}{\partial x_\mu}\right)^2$$

and quantized in the usual way. Equations (3.55) and (3.58) therefore lead to the usual Feynman rules, as may also be seen by rewriting them as equations for Green's functions. Equation (3.59) leads to the Feynman rules with more general covariant gauges; the choice $c=1$ corresponds to the Landau gauge.

If we wish we may reexpress the equations for our auxiliary quantities in the Schwinger functional nota-

[9] One can obtain a more general form for $X_\nu(x)$ by replacing the factor $(\partial/\partial x_\nu)\Box^{-2}$ outside the parentheses of (3.52) by a general vector a_ν satisfying $\partial_\nu a_\nu=1$. One can then obtain Green's functions in noncovariant gauges such as the Coulomb gauge. See Ref. 10.

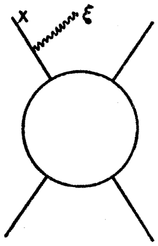

FIG. 2. Diagrams which correspond to renormalization of the path-dependent electron wave function.

tion by making the correspondence $\tilde{\phi}\rightarrow\delta/\delta\eta$, $\tilde{\phi}^*\rightarrow\delta/\delta\bar{\eta}$, $\tilde{A}_\mu\rightarrow\delta/\delta\zeta_\mu$. We then obtain equations which are essentially equivalent to those of Zumino.[10] It is not so straightforward, however, to express the equations for our path-dependent quantities in the Schwinger functional notation, and we have therefore introduced our present notation.

The auxiliary Green's functions may be used to calculate the S matrix in the usual way. The reduction formulas of the path-dependent formalism relate the S matrix elements to the mass-shell singularities of the path-dependent Green's functions $\hat{G}$ in p space. It can be shown that the terms on the right of (3.29) which involve integrations over ξ do not lead to such singularities, so that we may use the auxiliary Green's functions G instead of the path-dependent Green's functions $\hat{G}$. Strictly speaking it is not quite correct that the terms on the right of (3.29a) which involve integrals over ξ do not contribute to the singularities of G. In an integral such as $\int_P{}^x d\xi_\lambda G_\lambda(x,\cdots;y,\cdots;\xi,\cdots)$, we obtain a singularity from the diagram in which the external photon line leading to the point ξ is attached to the external (scalar) electron line leading to the point x (Fig. 2). Such a diagram is associated with the renormalization of the path-dependent electron wave-function and makes no contribution to the S matrix.

4. YANG-MILLS FIELD

The massless Yang-Mills field appears to possess all the essential complications of the gravitational field while lacking some of the algebraic complications. It is therefore instructive to consider this field before going on to the gravitational field. We shall treat a self-interacting Yang-Mills field, since interaction with other fields does not introduce any new features.

The path-dependent formalism for the Yang-Mills field has been examined by Bialynicki-Birula.[5] The procedure followed is analogous to that used for the electromagnetic field, with the difference that the Yang-Mills field plays the dual role of the gauge field and the charged field. The field equations are simpler *in appearance* than the Maxwell equations of electrodynamics, since there is no additional current term. They take the form

$$\frac{\partial F_{\mu\nu}{}^{\alpha}(x,P)}{\partial x_\mu}=0. \qquad (4.1a)$$

We shall always use indices from the beginning of the Greek alphabet to denote components in isotopic space, indices from the middle of the Greek alphabet to denote components in ordinary space. Note that F, being a charged field, is path-dependent.

The path-dependence equation is a straightforward generalization of (3.5):

$$\delta_z F_{\mu\nu}{}^{\alpha}(x,P)=g\epsilon_{\alpha\beta\gamma}F_{\mu\nu}{}^{\gamma}(x,P)F_{\rho\sigma}{}^{\beta}(z,P')\sigma_{\rho\sigma}. \qquad (4.2)$$

As usual, $\delta_z F_{\mu\nu}{}^{\alpha}$ is the change of the variable $F_{\mu\nu}{}^{\alpha}$ caused by a change in the path by an infinitesimal area $\sigma_{\rho\sigma}$ at the point z. The path P' is that portion of P leading to z. Equation (4.2) requires the following consistency condition, which is analogous to the homogeneous Maxwell equations:

$$\epsilon_{\mu\nu\rho\sigma}\frac{\partial F_{\mu\nu}{}^{\alpha}(x,P)}{\partial x_\rho}=0. \qquad (4.1b)$$

The equal-time commutators between the F's will contain terms analogous to (3.3) and (3.4). Thus

$$[F_{ij}{}^{\alpha}(\mathbf{x},t,P),F_{i'j'}{}^{\beta}(\mathbf{y},t,P')]=0, \qquad (4.3a)$$

$$\begin{aligned}[F_{0i}{}^{\alpha}(\mathbf{x},t,P),F_{jk}{}^{\beta}(\mathbf{y},t,P')]&=-i\delta_{\alpha\beta}\left(\delta_{ik}\frac{\partial}{\partial y_j}-\delta_{ij}\frac{\partial}{\partial y_k}\right)\delta^3(\mathbf{x}-\mathbf{y})\\&\quad+i\epsilon_{\alpha\beta\gamma}\int^{y}d\xi_i\delta^3(\mathbf{x}-\boldsymbol{\xi})F_{jk}{}^{\gamma}(\mathbf{y},t),\end{aligned} \qquad (4.3b)$$

$$\begin{aligned}[F_{0i}{}^{\alpha}(\mathbf{x},t,P),F_{0j}{}^{\beta}(\mathbf{y},t,P')]&=i\epsilon_{\alpha\beta\gamma}\int_{P'}^{y}d\xi_i\delta^3(\mathbf{x}-\boldsymbol{\xi})F_{0j}{}^{\gamma}(\mathbf{y},t)\\&\quad+i\epsilon_{\alpha\beta\gamma}\int_{P}^{x}d\xi_j\delta^3(\mathbf{y}-\boldsymbol{\xi})F_{0i}{}^{\gamma}(\mathbf{x},t).\end{aligned} \qquad (4.3c)$$

It is not difficult to check that Eqs. (4.1)–(4.3) are consistent with one another and with Lorentz transformations. In fact, the equations of motion and commutation relations may be derived from the Lagrangian

$$\mathfrak{L}=-\tfrac{1}{4}[F_{\mu\nu}{}^{\alpha}(x,P)]^2.$$

One may define path-dependent Green's functions in the usual way. As in the electromagnetic case, it is necessary to include δ-function terms if the Green's functions are to be covariant. The definitions are therefore as follows:

$$\hat{G}^{\alpha\beta}{}_{\mu\nu,ij}(x_1,P_1,x_2,P_2)=\langle 0|T\{F_{\mu\nu}{}^{\alpha}(x_1,P_1),F_{ij}{}^{\beta}(x_2,P_2)\}|0\rangle, \qquad (4.4a)$$

$$\hat{G}^{\alpha\beta}{}_{0i,0j}(x_1,P_1,x_2,P_2)=\langle 0|T\{F_{0i}{}^{\alpha}(x_1,P_1),F_{0j}{}^{\beta}(x_2,P_2)\}|0\rangle+i\delta_{\alpha\beta}\delta_{ij}\delta^4(x_1-x_2). \qquad (4.4b)$$

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[10] B. Zumino, J. Math. Phys. 1, 1 (1960).

Higher Green's functions may be defined in a similar way.

The field equations (4.1a) may be rewritten as equations for the Green's function, analogous to Eqs. (3.8) for the electromagnetic Green's functions. Thus the two-point Green's function satisfies the equation

$$\frac{\partial}{\partial x_{1\mu}}\hat{G}^{\alpha\beta}{}_{\mu\nu,\rho\sigma}(x_1,P_1,x_2,P_2) = i\left(\frac{\partial}{\partial x_{2\rho}}\delta_{\sigma\nu}-\frac{\partial}{\partial x_{2\sigma}}\delta_{\rho\nu}\right)\delta_{\alpha\beta}\delta^4(x_1-x_2). \quad (4.5a)$$

The four-point Green's function satisfies the equation

$$\frac{\partial}{\partial x_{1\mu}}\hat{G}^{\alpha\beta\gamma\delta}{}_{\mu\nu,\rho\sigma,\kappa\lambda,\tau\omega}(x_1P_1,x_2P_2,x_3P_3,x_4P_4) = i\left(\frac{\partial}{\partial x_{2\rho}}\delta_{\sigma\nu}-\frac{\partial}{\partial x_{2\sigma}}\delta_{\rho\nu}\right)\delta_{\alpha\beta}\delta^4(x_1-x_2)\hat{G}^{\gamma\delta}{}_{\kappa\lambda,\tau\omega}(x_3P_3,x_4P_4)$$

$$+\text{two similar terms with } 2\leftrightarrow 3,\ 2\leftrightarrow 4$$

$$-ig\epsilon_{\alpha\beta\epsilon}\int_{P_2}^{x_2} d\xi_\nu\delta^4(x_1-\xi)G^{\epsilon\gamma\delta}{}_{\rho\sigma,\kappa\lambda,\tau\omega}(x_2P_2,x_3P_3,x_4P_4)$$

$$+\text{two similar terms with } 2\leftrightarrow 3,\ 2\leftrightarrow 4. \quad (4.5b)$$

The right-hand side of (4.5b) is obtained by differentiating the time ordering and applying the commutation relations (4.3). Higher Green's functions will satisfy equations similar to (4.5b).

Equation (4.1b) implies that the Green's functions satisfy equations such as

$$\epsilon_{\mu\nu\rho\sigma}\frac{\partial}{\partial x_{1\rho}}\hat{G}^{\alpha\beta}{}_{\mu\nu,\kappa\lambda}(x_1,P_1,x_2,P_2)=0. \quad (4.5c)$$

The path-dependence equations satisfied by the Green's function can be obtained from (4.2), by following reasoning identical to that used for the electromagnetic field. Again the term arising from the variation of the time-ordering cancels against the term obtained from the δ function in the Green's function [the second term on the right of (4.4b)], and the final result is

$$\mathbf{\delta}_{x_3}\hat{G}^{\alpha\beta}{}_{\mu\nu,\rho\sigma}(x_1,P_1,x_2,P_2)=g\epsilon_{\alpha\gamma\epsilon}\int d\sigma_{\kappa\lambda}(x_3)\hat{G}^{\epsilon\beta\gamma}{}_{\mu\nu,\rho\sigma,\kappa\lambda}\times(x_1,P_1,x_2,P_2,x_3,P_1'). \quad (4.6)$$

In this equation, $\mathbf{\delta}_{x_3}$ is the variation of G caused by a variation of the path P_1 at the point x_3, and P_1' is the portion of P_1 leading to x_3. The higher Green's functions satisfy similar path-dependence equations.

Condensed Notation

The condensed notation which we shall use is very similar to that used in the two previous sections, and we need not explain it in detail again. We construct the linear space of the totality of all functions $\hat{C}_{\mu\nu\cdots}{}^{\alpha\cdots}(x_1,P_1,\cdots)$. We then construct the dual space and define the vector $(\hat{H}_{\mu\nu\cdots}{}^{\alpha\cdots}(x_1,P_1,\cdots)|$ in the usual way. We next define the operators $\tilde{F}^{\alpha}{}_{\mu\nu}(x,P)$ by the equations

$$(\hat{H}_{\rho\sigma\cdots}{}^{\beta\cdots}(x_1,P_1,\cdots)|\tilde{F}^{\alpha}{}_{\mu\nu}(x,P) =(\hat{H}_{\rho\sigma,\mu\nu\cdots}{}^{\beta\alpha\cdots}(x_1,P_1,x,P,\cdots)|. \quad (4.7)$$

We also define an operator $U(x,P)$ which corresponds to the right-hand side of (4.5):

$$(\hat{H}_{\kappa\lambda,\cdots}{}^{\beta\cdots}(x_1,P_1),\cdots|U_\nu{}^{\alpha}(x,P) =\sum_r\left(\frac{\partial}{\partial x_{r\rho}}\delta_{\sigma\nu}-\frac{\partial}{\partial x_{r\sigma}}\delta_{\rho\nu}\right)\times(\hat{H}_{\kappa\lambda,\cdots[\rho\sigma]\cdots}{}^{\beta\cdots[\gamma]\cdots}(x_1,P_1,\cdots[x_r,P_r]\cdots)|$$

$$\times\delta_{\alpha\gamma}\delta^4(x-x_r)-g\sum_r\epsilon_{\alpha\gamma\delta}\int_{P_r}^{x_r}d\xi_\nu \times(\hat{H}_{\kappa\lambda,\cdots}{}^{\beta\cdots[\gamma]\delta\cdots}(x_1,P_1,\cdots)|\delta^4(x_r-\xi), \quad (4.8)$$

where γ and $\rho\sigma$ are the indices corresponding to the coordinates x_r, and the superscript $[\gamma]\delta$ in (4.8) indicates that γ is to be replaced by δ. The operator U may be defined by its commutation relations

$$[U_\nu{}^{\alpha}(x_1,P_1),\tilde{F}_{\rho\sigma}{}^{\beta}(x_2,P_2)] =-\left(\frac{\partial}{\partial x_{2\rho}}\delta_{\nu\sigma}-\frac{\partial}{\partial x_{2\sigma}}\delta_{\nu\rho}\right)\delta_{\alpha\beta}\delta^4(x_1-x_2) +g\epsilon_{\alpha\beta\gamma}\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)F_{\rho\sigma}{}^{\gamma}(x_2,P_2). \quad (4.9)$$

Note that the right-hand side of (4.9) has terms corresponding to the right-hand sides of both (3.19a) and (3.19b), this is because the variable F in the Yang-Mills field plays the roles of the gauge field and the charged field. Equation (4.9) must be supplemented by the equation

$$(H_0|U_\nu{}^{\alpha}(x_1,P_1)=0, \quad (4.10)$$

to complete the definition of U.

In our present notation, the field equations (4.5) become

$$\left(\frac{\partial}{\partial x_\mu}\tilde{F}_{\mu\nu}{}^{\alpha}(x,P)-iU_\nu{}^{\alpha}(x,P)\right)|\hat{G})=0. \quad (4.11)$$

The path-dependence equation (4.6) is

$$\mathbf{\delta}_z\tilde{F}_{\mu\nu}{}^{\alpha}(x,P)=g\epsilon_{\alpha\beta\gamma}\tilde{F}_{\kappa\lambda}{}^{\beta}(x,P')\tilde{F}_{\mu\nu}{}^{\gamma}(x,P)\sigma_{\kappa\lambda}, \quad (4.12)$$

where P' represents the portion of P leading to the point z.

Auxiliary Variables

Following the procedure used in electrodynamics, we shall attempt to express our path-dependent quantities $\tilde{F}$ in terms of auxiliary path-independent quantities $\tilde{A}$. The formulas relating the $\tilde{F}$'s to the $\tilde{A}$'s will be the same as the formulas relating the field variables F to the potentials A. Thus, following the results of Ref. 5, we make the connection as follows:

$$\tilde{F}_{\mu\nu}{}^{\alpha}(x,P)=V_{\alpha\gamma}(x,P)\tilde{f}_{\mu\nu}{}^{\gamma}(x)\,, \qquad (4.13a)$$

where

$$\tilde{f}_{\mu\nu}{}^{\gamma}(x)=\frac{\partial\tilde{A}_{\nu}{}^{\gamma}(x)}{\partial x_{\mu}}-\frac{\partial\tilde{A}_{\mu}{}^{\gamma}(x)}{\partial x_{\nu}}+g\epsilon_{\gamma\delta\epsilon}\tilde{A}_{\mu}{}^{\delta}(x)\tilde{A}_{\nu}{}^{\epsilon}(x)\,, \qquad (4.13b)$$

$$V_{\alpha\gamma}(x,P)=\delta_{\alpha\gamma}+g\epsilon_{\alpha\beta\gamma}\int_P^x d\xi_\rho\tilde{A}_\rho{}^\beta(\xi)+g^2\epsilon_{\alpha\beta\epsilon}\epsilon_{\epsilon\delta\gamma}\int_P^x d\xi_\rho\int_P^\xi d\xi_\sigma{}'\tilde{A}_\sigma{}^\beta(\xi')\tilde{A}_\rho{}^\delta(\xi)+g^3\epsilon_{\alpha\beta\epsilon}\epsilon_{\epsilon\delta\eta}\epsilon_{\eta\zeta\gamma}\int_P^x d\xi_\rho\int_P^\xi d\xi_\sigma\int_P^{\xi'} d\xi_\tau{}''\tilde{A}_\tau{}^\beta(\xi'')\times\tilde{A}_\sigma{}^\zeta(\xi')\tilde{A}_\rho{}^\zeta(\xi)+\cdots. \qquad (4.13c)$$

If we compare (4.13) with (3.24) and (3.34), we observe that we have to take the curl of A *and* multiply it by the function V in order to obtain the path-dependent function $\tilde{F}$. Again, this is because the Yang-Mills field is both a gauge field and a charged field.

Before going further it will be useful to obtain two identities satisfied by the function V. For this purpose it is convenient to obtain an alternative definition of V, due to Bialynicki-Birula.[5] We introduce the following unitary matrix in the Pauli spin space:

$$W(x,P)=L\exp\left\{-ig\int_P^x d\xi_\rho A_\rho{}^\alpha(\xi)\tau_\alpha\right\}, \qquad (4.14)$$

where the matrices τ_α are the Pauli matrices. The symbol L indicates that the τ's are to be ordered from the beginning to the end of the path when expanding the exponential. It is then not difficult to see that

$$W^\dagger(x,P)\tau_\gamma W(x,P)=V_{\alpha\gamma}(x,P)\tau_\alpha. \qquad (4.15)$$

From (4.15) we can derive the identities

$$V_{\alpha\gamma}V_{\alpha\gamma'}=V_{\gamma\alpha}V_{\gamma'\alpha}=\delta_{\gamma\gamma'}\,, \qquad (4.16a)$$

$$\epsilon_{\zeta\delta\epsilon}V_{\delta\alpha}V_{\epsilon\beta}=\epsilon_{\alpha\beta\gamma}V_{\zeta\gamma}\,, \qquad (4.16b)$$

$$\epsilon_{\zeta\delta\epsilon}V_{\alpha\delta}V_{\beta\epsilon}=\epsilon_{\alpha\beta\gamma}V_{\gamma\zeta}. \qquad (4.16c)$$

For future reference we shall add the following trivial identity involving the ϵ's;

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\delta\epsilon}+\epsilon_{\alpha\beta\delta}\epsilon_{\alpha\epsilon\gamma}+\epsilon_{\alpha\beta\epsilon}\epsilon_{\alpha\gamma\delta}=0. \qquad (4.16d)$$

Another symbol which it will be convenient to introduce is the following:

$$V_{\alpha\gamma}(x',x,P)=\delta_{\alpha\gamma}+g\epsilon_{\alpha\beta\gamma}\int_{x',P}^x d\xi_\rho\tilde{A}_\rho{}^\beta(\xi)+g^2\epsilon_{\alpha\beta\epsilon}\epsilon_{\epsilon\delta\gamma}\times\int_{x',P}^x d\xi_\rho\int_{x',P}^x d\xi_\sigma{}'\tilde{A}_\sigma{}^\beta(\xi')A_\rho{}^\delta(\xi)+\cdots, \qquad (4.17)$$

where x' is a point on the path P. In other words, $V_{\alpha\gamma}(x',x,P)$ is defined in a similar way to $V_{\alpha\gamma}(x,P)$ except that all integrals are taken from x' to x instead of from $-\infty$ to x. The following relation may be verified directly:

$$V_{\alpha\gamma}(x,P)=V_{\alpha\delta}(x',P')V_{\delta\gamma}(x',x,P)\,, \qquad (4.18)$$

where as usual P' is the portion of P leading to x'.

If we differentiate (4.13c) with respect to the end-point of the path of integration, we obtain the equation

$$\frac{\partial}{\partial x_\mu}V_{\alpha\gamma}(x,P)=g\epsilon_{\gamma\delta\epsilon}\tilde{A}_\mu{}^\epsilon(x)V_{\epsilon\delta}(x,P). \qquad (4.19)$$

As we shall see below, this equation will enable us to express derivatives of path-dependent functions in terms of derivatives of auxiliary functions. Equation (4.19) can be written in integral form,

$$V_{\alpha\gamma}(x,P)=\delta_{\alpha\gamma}+g\epsilon_{\gamma\delta\epsilon}\int_P^x d\xi_\mu\tilde{A}_\mu{}^\epsilon(\xi)V_{\alpha\delta}(\xi,P)\,, \qquad (4.20)$$

where we have used the boundary condition $V_{\alpha\gamma}=\delta_{\alpha\gamma}$ when $g=0$ or when $x\to\infty$. By expanding (4.20) in a power series in g, we recover the definition (4.13c). Equation (4.19) or (4.20) may therefore be taken as a definition of V in place of (4.13c).

We can now show that the definitions (4.13) do lead to the correct path-dependence equations (4.12) for F, and we have carried out the algebra in Appendix A. The formulas (4.13) can therefore be used to define path-dependent quantities in terms of auxiliary quantities.

The operator η will be defined in exactly the same way as the analogous operators were defined in the two preceding sections:

$$[\eta_\mu{}^\alpha(x_1),\tilde{A}_\nu{}^\beta(x_2)]=-\delta_{\alpha\beta}\delta_{\mu\nu}\delta^4(x_1-x_2)\,, \qquad (4.21)$$

$$(H_0|\eta_\mu{}^\alpha(x)=0. \qquad (4.22)$$

We require an expression for the corresponding path-dependent quantity $U_\nu{}^\alpha(x,P)$, defined by (4.9) and (4.10), in terms of the auxiliary variables. In electrodynamics the quantity $U(x,P)$ was equal to $V(x,P)\times\eta(x)$, and this suggests that the Yang-Mills operator $U_\nu{}^\alpha(x,P)$ might be given by a similar equation:

$$U_\nu{}^\alpha(x,P)\approx V_{\alpha\gamma}(x,P)\eta_\nu{}^\gamma(x). \qquad (4.23)$$

We shall verify (4.23) in Appendix A. The equality between the two sides of (4.23) is to be interpreted in the sense that they both have the same commutation relations (4.9) with the operators $F_{\rho\sigma}{}^{\beta}(x',P')$.

Once we have defined the operators $\tilde{A}_\mu{}^\alpha(x)$, we can construct our enlarged dual space of vectors $(H_{\mu\cdots}{}^{\alpha\cdots}(x_1,\cdots)|$. We can then construct the linear space of vectors $|G)$ and can define auxiliary, path-independent Green's functions G. The path-dependent Green's functions $\hat{G}$ can be expressed in terms of the G's by formulas analogous to (3.29). We shall not give the details, which are the exact analog of the corresponding details in electrodynamics. We can also write equations similar to (3.32) for the operators η.

Gauge Transformations

The gauge transformations are given by the equation

$$\tilde{A}_\mu{}^\alpha(x) \to \tilde{A}_\mu{}^\alpha(x) +\lambda\left(\frac{\partial\chi_\alpha(x)}{\partial x_\mu}+g\epsilon_{\alpha\beta\gamma}\chi_\gamma(x)\tilde{A}_\mu{}^\beta(x)\right). \quad (4.24)$$

If we compare (4.24) with (3.36) we notice that terms corresponding to the right-hand side of (3.36a) and (3.36c) appear on the right of (4.24). This is once more due to the fact that the Yang-Mills field plays the dual role of gauge field and charged field.

Let us first investigate how the operators defined in (4.13) transform when $\tilde{A}$ undergoes the transformation (4.24). From (4.13b) one can verify directly that

$$\tilde{f}_{\mu\nu}{}^\alpha(x) \to \tilde{f}_{\mu\nu}{}^\alpha(x)+\lambda g\epsilon_{\alpha\beta\gamma}\chi_\gamma(x)\tilde{A}_\mu{}^\beta(x). \quad (4.25)$$

The function V transforms in a similar way:

$$V_{\alpha\gamma}(x,P) \to V_{\alpha\gamma}(x,P)+\lambda g\epsilon_{\gamma\delta\epsilon}\chi_\epsilon(x)V_{\alpha\delta}(x,P). \quad (4.26)$$

The easiest way of verifying (4.26) is to show that (4.19), which may be taken as the defining equation for V, remains invariant when $\tilde{A}$ undergoes the transformation (4.24) and V the transformation (4.26). Under such a transformation, the two sides of (4.19) transform as follows:

$$\frac{\partial}{\partial x_\mu}V_{\alpha\gamma}(x,P) \to \frac{\partial}{\partial x_\mu}V_{\alpha\gamma}(x,P)+\lambda g\epsilon_{\gamma\delta\epsilon}\left(\frac{\partial\chi_\epsilon(x,P)}{\partial x_\mu}V_{\alpha\delta}(x,P)+\chi_\epsilon\frac{\partial}{\partial x_\mu}V_{\alpha\delta}(x,P)\right)=\frac{\partial}{\partial x_\mu}V_{\alpha\gamma}(x,P)+\lambda g\epsilon_{\gamma\delta\epsilon}$$
$$\times\frac{\partial\chi_\epsilon(x,P)}{\partial x_\mu}V_{\alpha\delta}(x,P)+\lambda g^2\epsilon_{\gamma\delta\epsilon}\epsilon_{\delta\zeta\eta}\chi_\epsilon(x)\tilde{A}_\mu{}^\eta(x)V_{\alpha\zeta}(x,P), \quad (4.27)$$

from (4.19),

$$g\epsilon_{\gamma\delta\epsilon}\tilde{A}_\mu{}^\epsilon(x)V_{\alpha\delta}(x,P) \to g\epsilon_{\gamma\delta\epsilon}\tilde{A}_\mu{}^\epsilon(x)V_{\alpha\delta}(x,P)+\lambda g\epsilon_{\gamma\delta\epsilon}\frac{\partial\chi_\epsilon(x)}{\partial x_\mu}V_{\alpha\delta}(x,P)$$
$$+\lambda g^2\epsilon_{\gamma\delta\epsilon}\epsilon_{\epsilon\zeta\eta}\chi_\eta(x)\tilde{A}_\mu{}^\zeta(x)V_{\alpha\delta}(x,P)+\lambda g^2\epsilon_{\gamma\delta\epsilon}\epsilon_{\delta\zeta\eta}\tilde{A}_\mu{}^\epsilon(x)\chi_\eta(x)V_{\alpha\zeta}(x,P)$$
$$=g\epsilon_{\gamma\delta\epsilon}\tilde{A}_\mu{}^\epsilon(x)V_{\alpha\delta}(x,P)+\lambda g\epsilon_{\gamma\delta\epsilon}\frac{\partial\chi_\epsilon(x)}{\partial x_\mu}V_{\alpha\delta}(x,P)+\lambda g^2\epsilon_{\gamma\epsilon\eta}\epsilon_{\epsilon\delta\zeta}\chi_\eta(x)\tilde{A}_\mu{}^\zeta(x)V_{\alpha\delta}(x,P), \quad (4.28)$$

from (4.16d). Comparing (4.27) and (4.28), we observe that the changes in the two sides of Eq. (4.19) are the same, so that (4.19) is invariant under the transformation (4.24), (4.26). Thus the change in V is given by (4.26).

We can now find the change in $\tilde{F}$ as defined by (4.13):

$$\tilde{F}_{\mu\nu}{}^\alpha(x,P) \to \tilde{F}_{\mu\nu}{}^\alpha(x,P)+\lambda g\epsilon_{\gamma\delta\epsilon}\chi_\epsilon(x)V_{\alpha\delta}(x,P)\tilde{f}_{\mu\nu}{}^\gamma(x) +\lambda g\epsilon_{\gamma\delta\epsilon}V_{\alpha\gamma}(x,P)\chi_\epsilon(x)\tilde{f}_{\mu\nu}{}^\delta(x) \quad \text{[from (4.25) and (4.26)]}$$
$$=\tilde{F}_{\mu\nu}{}^\alpha(x,P).$$

The variable $\tilde{F}_{\mu\nu}{}^\alpha(x,P)$ is therefore invariant under the transformation (4.24), and we are justified in calling it a gauge transformation.

To define a generator of the gauge transformation, we construct the operator

$$Y_\beta(y)=\frac{\partial\eta_\mu{}^\beta(y)}{\partial y_\mu}+\epsilon_{\beta\epsilon\delta}\eta_\mu{}^\delta(y)\tilde{A}_\mu{}^\epsilon(y) =\left(\delta_{\beta\delta}\frac{\partial}{\partial y_\mu}+\epsilon_{\beta\epsilon\delta}\tilde{A}_\mu{}^\epsilon(y)\right)\eta_\mu{}^\delta(y). \quad (4.29)$$

The integral $\lambda\int dy Y_\beta(y)\chi_\beta(y)$, when commuted with $\tilde{A}_\mu{}^\alpha(x)$, does give (4.24). We thus conclude that the integral $\int dy Y_\beta(y)\chi_\beta(y)$ commutes with all our path-dependent variables $\tilde{F}_{\mu\nu}{}^\alpha(x,P)$. Furthermore, since V undergoes the transformation (4.26), we conclude that

$$\left[\int dy\, Y_\beta(y)\chi_\beta(y), V_{\alpha\gamma}(x,P)\right] =-g\epsilon_{\gamma\delta\epsilon}\chi_\delta(x)V_{\alpha\epsilon}(x,P). \quad (4.30)$$

Field Equations

Our aim is now to express the field equations (4.11) as equations for the auxiliary variables. The first term of (4.11) is easily transformed:

$$\frac{\partial}{\partial x_\mu}\tilde{F}_{\mu\nu}{}^{\alpha}(x,P)$$

$$=\frac{\partial}{\partial x_\mu}V_{\alpha\gamma}(x,P)\tilde{f}_{\mu\nu}{}^{\gamma}(x)$$

$$=V_{\alpha\gamma}(x,P)\frac{\partial}{\partial x_\mu}\tilde{f}_{\mu\nu}{}^{\gamma}(x)+\epsilon_{\gamma\delta\epsilon}V_{\alpha\delta}(x)\tilde{A}_\mu{}^{\epsilon}(x)\tilde{f}_{\mu\nu}{}^{\gamma}(x)$$

[from (4.19)]

$$=V_{\alpha\gamma}(x,P)\left(\frac{\partial}{\partial x_\mu}\tilde{f}_{\mu\nu}{}^{\gamma}(x)+\epsilon_{\gamma\delta\epsilon}\tilde{A}_\mu{}^{\delta}(x)\tilde{f}_{\mu\nu}{}^{\epsilon}(x)\right). \quad (4.31)$$

We have seen that the second term $-iU_\nu{}^{\alpha}(x,P)$ of (4.11) is equivalent to the operator $-iV_{\alpha\gamma}(x,P)\eta_\nu{}^{\gamma}(x)$ in the sense that they both have the same commutation relations with the operator $F_{\rho\sigma}{}^{\beta}(x',P')$. We may therefore be tempted to rewrite the field equations (4.11) as follows:

$$V_{\alpha\gamma}(x,P)\left(\frac{\partial}{\partial x_\mu}\tilde{f}_{\mu\nu}{}^{\gamma}(x)+\epsilon_{\gamma\delta\epsilon}\tilde{A}_\mu{}^{\delta}(x)\times\tilde{f}_{\mu\nu}{}^{\epsilon}(x)-i\eta_\nu{}^{\gamma}(x)\right)|G)=0. \quad (4.32)$$

However, we shall show below that the consistency of Eq. (4.32) requires that its last term satisfy a divergence condition similar to the corresponding condition in the Maxwell equation of electrodynamics, and we shall have to generalize it if the condition is to be satisfied.

We shall follow the procedure used in electrodynamics and shall make use of the fact that the commutation relations (4.9), which define the operator $U_\nu{}^{\alpha}(x,P)$ uniquely in the original linear space, do not define it uniquely in the enlarged linear space. We employ this freedom to find a definition of $U_\nu{}^{\alpha}(x,P)$ which gives consistent field equations. We begin by writing

$$U_\nu{}^{\alpha}(x,P)=V_{\alpha\gamma}(x,P)\eta_\nu{}^{\gamma}(x)+\int dy\,Y_\beta(y)\chi_{\alpha\beta}'(x,y) \quad (4.33)$$

where χ' is arbitrary. Since the second term commutes with every gauge-invariant operator, the right-hand side of (4.33) maintains the correct commutation relations (4.9). All the terms in the equation of motion (4.32) have a factor $V_{\alpha\gamma}(x,P)$ in front of them, and it will be convenient for us if the last term in (4.33) also such a factor. We therefore define

$$\chi_{\alpha\beta}'(x,y)=V_{\alpha\gamma}(x,P)\chi_{\gamma\beta}(x).$$

Thus,

$$U_\nu{}^{\alpha}(x,P)=V_{\alpha\gamma}(x,P)\eta_\nu{}^{\gamma}(x)+\int dy\,Y_\beta(y)V_{\alpha\gamma}(x,P)\chi_{\gamma\beta}(x,y). \quad (4.34)$$

The factor $V_{\alpha\gamma}$ in the second term of (4.34) is still not in front of the other factors, but we can move it into this position by using the commutation relation (4.30) The equation then becomes

$$U_\nu{}^{\alpha}(x,P)=V_{\alpha\gamma}(x,P)\eta_\nu{}^{\gamma}(x)+\int dy V_{\alpha\gamma}(x,P)Y_\beta(y)\chi_{\gamma\beta}(x,y)-g\epsilon_{\gamma\delta\epsilon}V_{\alpha\epsilon}(x,x)\chi_{\gamma\delta}(x,x)=V_{\alpha\gamma}(x,P)\theta_\nu{}^{\gamma}(x), \quad (4.35)$$

where

$$\theta_\nu{}^{\gamma}(x)=\eta_\nu{}^{\gamma}(x)+\int dy\,Y_\beta(y)\chi_{\gamma\beta}(x,y)+g\epsilon_{\gamma\delta\epsilon}\chi_{\epsilon\delta}(x,x)$$

$$=\eta_\nu{}^{\gamma}(x)+\int dy\left(\frac{\partial\eta_\mu{}^{\beta}(y)}{\partial y_\mu}+g\epsilon_{\beta\epsilon\delta}\eta_\mu{}^{\delta}(y)\tilde{A}_\mu{}^{\epsilon}(y)\right)\chi_{\gamma\beta}(x,y)+g\epsilon_{\gamma\delta\epsilon}\chi_{\epsilon\delta}(x,x), \quad (4.36)$$

from (4.29).

We can thus generalize the field equations (4.32) to read

$$V_{\alpha\gamma}(x,P)\left(\frac{\partial}{\partial x_\mu}\tilde{f}_{\mu\nu}{}^{\gamma}(x)+\epsilon_{\gamma\delta\epsilon}\tilde{A}_\mu{}^{\delta}(x)\tilde{f}_{\mu\nu}{}^{\epsilon}(x)-i\theta_\nu{}^{\gamma}(x)\right)|G)=0, \quad (4.37)$$

with θ given by (4.36). If we multiply (4.37) by $V_{\alpha\gamma'}$, sum over α, and apply (4.16a), we obtain the equation

$$\left(\frac{\partial}{\partial x_\mu}\tilde{f}_{\mu\nu}{}^{\gamma}(x)+\epsilon_{\gamma\delta\epsilon}\tilde{A}_\mu{}^{\delta}(x)\tilde{f}_{\mu\nu}{}^{\epsilon}(x)-i\theta_\nu{}^{\gamma}(x)\right)|G)=0. \quad (4.38)$$

The path dependence has been removed from (4.38), and we shall adopt it as our field equation. By taking the gauge-invariant derivative

$$\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\nu{}^{\beta}$$

of the factor within the parentheses we can easily show that the last term must satisfy the consistency condition

$$\left(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\nu{}^{\beta}(x)\right)\theta_\nu{}^{\gamma}(x)=0. \quad (4.39)$$

We have to choose the function χ in the definition (4.36) of θ so that (4.39) is satisfied.

In order to orient ourselves we shall first find a function $\theta_\nu{}^{\alpha}$, of the form $\eta_\nu{}^{\alpha}+\theta_\nu'{}^{\alpha}$, which satisfies (4.39). The term $\theta_\nu'^\alpha$ will not have precisely the form of the second term of (4.36), but we shall then be able to modify it so that all conditions are satisfied. The following function clearly satisfies (4.39):

$$\theta_\nu^{(1)\alpha}(x)=\eta_\nu^\alpha(x)-\frac{\partial}{\partial x_\nu}\left[\left(I\frac{\partial}{\partial x_\lambda}+gE_\beta\tilde{A}_\lambda^\beta(x)\right)\frac{\partial}{\partial x_\lambda}\right]^{-1}_{\alpha\gamma}\times\left[\delta_{\gamma\delta}\frac{\partial}{\partial x_\mu}+g\epsilon_{\gamma\epsilon\delta}\tilde{A}_\mu^\epsilon(x)\right]\eta_\mu^\delta(x),\quad(4.40)$$

where the matrices I and E_β represent the symbols $\delta_{\alpha\gamma}$ and $\epsilon_{\alpha\beta\gamma}$, considered as matrices in α and γ. The superscript (1) on θ indicates that it is not our final definition of this operator. Equation (4.40) can be rewritten without using reciprocals of operators as follows:

$$\theta_\nu^{(1)\alpha}(x)=\eta_\nu^\alpha(x)-\frac{\partial}{\partial x_\nu}\int dy\, O_{\alpha\gamma}(x,y)\times\left(\delta_{\gamma\delta}\frac{\partial}{\partial y_\mu}+g\epsilon_{\gamma\epsilon\delta}\tilde{A}_\mu^\epsilon(y)\right)\eta_\mu^\delta(y),\quad(4.41)$$

where

$$\left(\frac{\partial}{\partial x_\lambda}\delta_{\alpha\gamma}+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\lambda^\beta(x)\right)\frac{\partial}{\partial x_\lambda}O_{\gamma\delta}(x,y)=\delta_{\alpha\delta}\delta^4(x-y).\quad(4.42)$$

The right-hand side of (4.41) resembles that of (4.36). The differences are, first that the operator η in the second term of (4.41) is ordered to the right of the other operators whereas it should be ordered to the left, and second that the last term of (4.36) is missing. Let us therefore change (4.42) to bring it into the correct form (4.36):

$$\theta_\nu^\alpha(x)=\eta_\nu^\alpha(x)-\int dy\left(\frac{\partial\eta_\mu^\gamma(y)}{\partial y_\mu}+g\epsilon_{\gamma\epsilon\delta}\eta_\mu^\delta(y)\tilde{A}_\mu^\epsilon(y)\right)\times\frac{\partial}{\partial x_\nu}O_{\alpha\gamma}(x,y)-g\epsilon_{\alpha\beta\gamma}\frac{\partial}{\partial x_\nu}O_{\beta\gamma}(x,y)\big|_{x=y},\quad(4.43)$$

with the function O still defined by (4.42). In Appendix B we shall calculate the value of the expression

$$\left(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}+g_{\alpha\beta\gamma}\tilde{A}_\nu^\beta(x)\right)\theta_\nu^\gamma(x)\quad(4.44)$$

and shall show that it is zero, so that (4.43) is a permissible, consistent choice for the function θ.

Rules for Feynman Diagrams

We can rewrite Eq. (4.38) in the form

$$\left[\frac{\partial}{\partial x_\mu}\left(\frac{\partial\tilde{A}_\nu^\alpha(x)}{\partial x_\mu}-\frac{\partial\tilde{A}_\mu^\alpha(x)}{\partial x_\nu}\right)+g\tilde{j}_\nu^\alpha(x)-i\theta_\nu(x)\right]|G)=0,\quad(4.45)$$

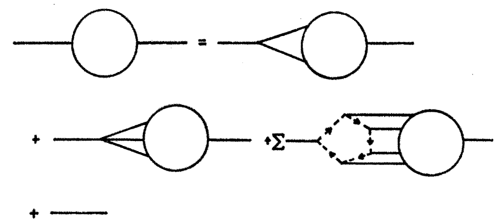

FIG. 3. Diagrammatic representation of the equation for the two-point Green's function in the Yang-Mills theory.

where

$$\tilde{j}_\nu^\alpha(x)=\epsilon_{\alpha\beta\gamma}\left(-\tilde{A}_\nu^\beta(x)\frac{\partial A_\mu^\gamma(x)}{\partial x_\mu}+2\tilde{A}_\mu^\beta(x)\frac{\partial A_\nu^\gamma(x)}{\partial x_\mu}-A_\mu^\beta(x)\frac{\partial A_\mu^\gamma(x)}{\partial x_\nu}\right)+g\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\delta\epsilon}A_\mu^\beta(x)A_\mu^\delta(x)A_\nu^\epsilon(x).\quad(4.46)$$

By integrating (4.45) in the usual way and using (4.43) for θ, we obtain the result

$$\left(A_\nu^\alpha(x)-ig\int dx'\tfrac{1}{2}\Delta_F(x-x')\tilde{j}_\nu^\alpha(x')-\int dx'\tfrac{1}{2}\Delta_F(x-x')\eta_\nu^\alpha(x')-g\epsilon_{\alpha\beta\gamma}\int dx'\tfrac{1}{2}\Delta_F(x-x')\times\frac{\partial}{\partial x_\nu'}O_{\beta\gamma}(x',y)\big|_{x'=y}\right)|G)=0,\quad(4.47)$$

where O is defined by (4.42). We have omitted the middle term of (4.43), as it is a pure divergence. If we wish we may generalize (4.47) by replacing the propagator $\delta_{\nu\rho}\frac{1}{2}\Delta(x-x')$ by $[\delta_{\nu\rho}-c(\partial^2/\partial x_\nu\partial x_\rho)\Box^{-2}]\frac{1}{2}\Delta_F\times(x-x')$; we then obtain other gauges such as the Landau gauge.

If the last term of (4.47) had been absent, we would have obtained Feynman rules similar to those for electrodynamics. The equation for the Green's function could then be represented graphically as in Fig. 3, without the second-last diagram. For simplicity we have exhibited the equation for the two-point Green's function; equations for higher Green's functions can be similarly represented. The three- and four-point vertices have the following factors associated with them:

$$v_3(p_1,\alpha,\mu;\,p_2,\beta,\nu;\,p_3,\gamma,\rho)=i(2\pi)^4\epsilon_{\alpha\beta\gamma}[(p_2-p_3)_\mu\delta_{\nu\rho}+(p_3-p_1)_\nu\delta_{\mu\rho}+(p_1-p_2)_\rho\delta_{\mu\nu}]\quad(4.48)$$

$$v_4(p_1,\alpha,\mu;\,p_2,\beta,\nu;\,p_3,\gamma,\rho;\,p_4,\delta,\sigma)=-(2\pi)^4\epsilon_{\alpha\beta\epsilon}\epsilon_{\epsilon\gamma\delta}(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})-(2\pi)^4\epsilon_{\alpha\gamma\epsilon}\epsilon_{\epsilon\beta\delta}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})-(2\pi)^4\epsilon_{\alpha\delta\epsilon}\epsilon_{\epsilon\beta\gamma}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\rho}\delta_{\nu\sigma}).\quad(4.49)$$

One could then construct Feynman diagrams by arranging the vertices (4.48) and (4.49) in all possible ways. In fact, Eq. (4.47) without the last term is identical to the equation we would have obtained by starting from the Lagrangian

$$\mathcal{L}=\frac{1}{2}\left(\frac{\partial A_\nu{}^\alpha}{\partial x_\mu}\right)^2+\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\left(\frac{\partial A_\nu{}^\alpha}{\partial x_\mu}-\frac{\partial A_\mu{}^\alpha}{\partial x_\nu}\right)A_\mu{}^\beta A_\nu{}^\gamma$$
$$+\tfrac{1}{4}\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\delta\epsilon}A_\mu{}^\beta A_\nu{}^\gamma A_\mu{}^\delta A_\nu{}^\epsilon \quad (4.50)$$

and writing down "naive" Feynman rules in the usual way.

The presence of the last term in (4.47) shows that the naive Feynman rules are not correct and that there are additional terms in the perturbation expansion. From (4.42), we can expand O as a perturbation series in g as follows:

$$O_{\beta\gamma}(x,y)=-i\sum_{n=0}^{\infty}\int dx_1\cdots dx_r\tfrac{1}{2}\Delta_F(x-x_1)ig\epsilon_{\beta\delta\epsilon}A_\lambda{}^\delta(x_1)$$
$$\times\frac{\partial}{\partial x_{1\lambda}}\tfrac{1}{2}\Delta_F(x_1-x_2)ig\epsilon_{\epsilon\zeta\eta}A_\mu{}^\zeta(x_2)\cdots\epsilon_{\theta\iota\gamma}A_\sigma{}^\iota(x_r)$$
$$\times\frac{\partial}{\partial x_{r\sigma}}\frac{1}{2}\Delta_F(x_n-y). \quad (4.51)$$

When (4.51) is substituted in the last term of (4.47), we obtain the result

$$\sum_{n=0}^{\infty}\int dx'dx_1\cdots dx_r\tfrac{1}{2}\Delta_F(x-x')ig\epsilon_{\gamma\alpha\beta}\frac{\partial}{\partial x_\nu'}\frac{1}{2}\Delta_F(x'-x_1)$$
$$\times ig\epsilon_{\beta\delta\epsilon}A_\lambda{}^\delta(x_1)\frac{\partial}{\partial x_{1\lambda}}\frac{1}{2}\Delta_F(x_1-x_2)$$
$$\times ig\epsilon_{\epsilon\zeta\eta}A_\mu{}^\zeta(x_2)\cdots\epsilon_{\theta\iota\gamma}A_\sigma{}^\iota(x_r)\frac{\partial}{\partial x_{r\sigma}}\frac{1}{2}\Delta_F(x_n-x'). \quad (4.52)$$

The expression (4.52) has the form of an integral which occurs in Feynman diagrams, and the contribution (4.52) to (4.47) has been represented by the second-last diagram of Fig. 3. The summation sign represents the sum over polygons with any number of dashed lines, and corresponds to the summation over n in (4.52). The dashed lines and vertices are associated respectively with the factors $\frac{1}{2}\Delta_F(x_r-x_{r+1})$ and $ig\epsilon(\partial/\partial x_r)$ in (4.52). Thus, in momentum space, the following factors are associated with the dashed lines and the vertices at which they end:

$$\text{dashed lines:}\quad \frac{i}{(2\pi)^4}\frac{\delta_{\alpha\beta}}{-p^2+i\epsilon}; \quad (4.53a)$$

$$\text{vertices:}\quad w(p_1\alpha,p_2\beta\sigma,p_3\gamma)=-(2\pi)^4g\epsilon_{\alpha\beta\gamma}p_{3\sigma}; \quad (4.53b)$$

an over-all factor -1. (4.53c)

In (4.53c), the quantities p_1, α and p_3, γ refer to the dashed lines, the quantities p_2, $\beta\sigma$ to the solid lines representing the Yang-Mills quanta. We notice that the vertex factor is not symmetric in the two dashed lines; it involves a factor $p_{3\sigma}$ but no factor $p_{1\sigma}$. It is for this reason that we have drawn arrows on the dashed lines in Fig. 3. The factor $p_{3\sigma}$ in (4.53b) is associated with the line directed away from the vertex.

The presecription for constructing Feynman diagrams is therefore to draw three-particle and four-particle vertices with factors (4.48) and (4.49), and also polygons with any number of dashed lines and with factors (4.53) associated with them. The three- and four-point vertices, as well as the vertices of the polygons, are then to be joined by solid Yang-Mills lines in all possible ways.

The Feynman rules for our theory are the same as those for a theory with ficititious scalar particles as well as the Yang-Mills particles. The Feynman diagrams contain three- and four-point vertices involving the Yang-Mills lines alone. The factors (4.48) and (4.49) are associated with these vertices. In addition, the diagrams contain vertices involving two scalar lines and one Yang-Mills line. Associated with such vertices are the factors (4.53b). There is a further factor -1 associated with each closed loop of scalar particles. The scalar lines only occur as internal lines and only in closed loops.

Note added in manuscript. Faddeev and Popov (unpublished) have shown that their functional-integration prescription[3] can be related to Schwinger's formulation of the Yang-Mills theory.[6] This therefore provides an alternative derivation of the Feynman rules from a quantized field theory. Faddeev and Popov have restricted themselves to Landau gauge.

APPENDIX A

We first show that the definitions (4.13) do lead to the correct path-dependence equation for the Yang-Mills $\tilde{F}$'s. Our results will be almost trivial once we have found the path-dependence equation for V. We shall begin by finding the change of V due to a small change of the path P *near its end point* by an amount $\sigma_{\mu\nu}$. It follows at once from (4.20) that

$$\boldsymbol{\delta}_xV_{\alpha\gamma}(x,P)=g\epsilon_{\gamma\delta\epsilon}\left(\frac{\partial}{\partial x_\mu}\delta_{\alpha\rho}-\frac{\partial}{\partial x_\nu}\delta_{\mu\rho}\right)$$
$$\times\{\tilde{A}_\rho{}^\epsilon(x)V_{\alpha\delta}(x,P)\}\sigma_{\mu\nu}, \quad (A1)$$

where $\boldsymbol{\delta}_x$ is the change of $V_{\alpha\gamma}(x,P)$ caused by a change of the path P near the point x itself. By using (4.19) to differentiate the factor $V_{\alpha\delta}(x,P)$ on the right of

(A.1), we can transform the equation to read

$$\begin{aligned}\delta_z V_{\alpha\gamma}(x,P) &= g\epsilon_{\gamma\delta\epsilon}V_{\alpha\delta}(x,P)\left(\frac{\partial\tilde{A}_\nu(x)}{\partial x_\mu}-\frac{\partial\tilde{A}_\mu(x)}{\partial x_\nu}\right)\sigma_{\mu\nu}+g^2\epsilon_{\gamma\delta\epsilon}\epsilon_{\delta\zeta\eta}V_{\alpha\zeta}(x,P)\{\tilde{A}_\nu{}^\epsilon(x)\tilde{A}_\mu{}^\eta(x)-\tilde{A}_\mu{}^\epsilon(x)\tilde{A}_\nu{}^\eta(x)\}\sigma_{\mu\nu}\\
&= g\epsilon_{\gamma\delta\epsilon}V_{\alpha\delta}(x,P)\left(\frac{\partial\tilde{A}_\nu{}^\epsilon(x)}{\partial x_\mu}-\frac{\partial\tilde{A}_\mu{}^\epsilon(x)}{\partial x_\nu}\right)\sigma_{\mu\nu}+g^2\epsilon_{\gamma\zeta\delta}V_{\alpha\zeta}(x,P)\epsilon_{\delta\eta\epsilon}\tilde{A}_\nu{}^\epsilon(x)\tilde{A}_\mu{}^\eta(x)\sigma_{\mu\nu}\quad[\text{from (4.16d)}]\\
&= g\epsilon_{\gamma\delta\epsilon}V_{\alpha\delta}(x,P)\left(\frac{\partial\tilde{A}_\nu{}^\epsilon(x)}{\partial x_\mu}-\frac{\partial\tilde{A}_\mu{}^\epsilon(x)}{\partial x_\nu}+g\epsilon_{\epsilon\eta\zeta}\tilde{A}_\mu{}^\eta(x)\tilde{A}_\nu{}^\zeta(x)\right)\sigma_{\mu\nu}=g\epsilon_{\gamma\delta\epsilon}V_{\alpha\delta}(x,P)\tilde{f}_{\mu\nu}{}^\epsilon(x)\sigma_{\mu\nu}\quad[\text{from (4.13b)}]\\
&= g\epsilon_{\alpha\beta\zeta}V_{\beta\epsilon}(x,P)V_{\zeta\gamma}(x,P)\tilde{f}_{\mu\nu}{}^\epsilon(x)\sigma_{\mu\nu}\quad[\text{from (4.16b)}]\\
&= g\epsilon_{\alpha\beta\zeta}\tilde{F}_{\mu\nu}{}^\beta(x,P)V_{\zeta\gamma}(x,P)\sigma_{\mu\nu}\quad[\text{from (4.13a)}].\end{aligned}\qquad\text{(A2)}$$

We can now generalize (A2) to obtain the change in V caused by a change in the path P at an arbitrary point z. We write

$$V_{\alpha\gamma}(x,P)=V_{\alpha\epsilon}(z,P')V_{\epsilon\gamma}(z,x,P)\,,$$

the point z being chosen just beyond the region where the path is varied, so that the factor $V_{\epsilon\gamma}(z,x,P)$ remains unchanged. Applying (A2) to the factor $V_{\alpha\epsilon}(z,P')$, we find

$$\begin{aligned}\delta_z V_{\alpha\gamma}(x,P) &= [\delta_z V_{\alpha\epsilon}(z,P')]V_{\epsilon\gamma}(z,x,P)\\
&= g\epsilon_{\alpha\beta\zeta}\tilde{F}_{\mu\nu}{}^\beta(z,P')V_{\zeta\epsilon}(z,P')V_{\epsilon\gamma}(z,x,P)\sigma_{\mu\nu}\\
&= g\epsilon_{\alpha\beta\zeta}\tilde{F}_{\mu\nu}{}^\beta(z,P')V_{\zeta\gamma}(x,P)\sigma_{\mu\nu}\,,\end{aligned}\qquad\text{(A3)}$$

from (4.18).

Equation (A3) is the path-dependence equation for V. By substituting in (4.13a), we find the path-dependence equation for $\tilde{F}$, and it does have the required form (4.12).

We now turn to the verification of (4.23). We have to find the commutator between the operators $V_{\alpha\gamma}(x_1,P_1)\times\eta_\nu{}^\gamma(x_1)$ and $\tilde{F}_{\rho\sigma}{}^\beta(x_2,P_2)$, and to show that it is given by (4.9). We begin by finding the commutator between the operators $V_{\alpha\gamma}(x_1,P_1)\eta_\nu{}^\gamma(x_1)$ and $V_{\beta\delta}(x_2,P_2)$. From (4.21),

$$[V_{\alpha\gamma}(x_1,P_1)\eta_\nu{}^\gamma(x_1),V_{\beta\delta}(x_2,P_2)]=-V_{\alpha\gamma}(x_1,P_1)\frac{\delta}{\delta\tilde{A}_\nu{}^\gamma(x_1)}V_{\beta\delta}(x_2,P_2)\,.\qquad\text{(A4)}$$

To evaluate the right-hand side of (A4), we first show that Eq. (4.19), which defines V, is invariant under the transformation

$$V_{\beta\delta}(x_2,P_2)\to V_{\beta\delta}(x_2,P_2)+\lambda g\epsilon_{\beta\alpha\zeta}V_{\zeta\delta}(x_2,P_2)\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)\,,\qquad\text{(A5a)}$$

$$\tilde{A}_\mu{}^\gamma(x)\to\tilde{A}_\mu{}^\gamma(x)+\lambda V_{\alpha\gamma}(x_1,P_1)\delta_{\mu\nu}\delta^4(x-x_1)\,.\qquad\text{(A5b)}$$

For, under the transformation (A5), the additions to the two sides of (4.19) are as follows:

$$\begin{aligned}\delta\left(\frac{\partial}{\partial x_\mu}V_{\beta\delta}(x_2,P_2)\right) &= \lambda g\epsilon_{\beta\alpha\zeta}V_{\zeta\delta}(x_2,P_2)\delta_{\mu\nu}\delta^4(x_1-x_2)+\lambda\epsilon_{\beta\alpha\zeta}\left(\frac{\partial}{\partial x_{2,\mu}}V_{\zeta\delta}(x_2,P_2)\right)\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)\\
&= \lambda g\epsilon_{\beta\alpha\zeta}V_{\zeta\delta}(x_2,P_2)\delta_{\mu\nu}\delta^4(x_1-x_2)+\lambda\epsilon_{\beta\alpha\zeta}g\epsilon_{\delta\eta\epsilon}\tilde{A}_\mu{}^\epsilon(x_2)V_{\zeta\delta}(x_2,P_2)\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)\,,\end{aligned}\qquad\text{(A6a)}$$

from (4.19).

$$\begin{aligned}\delta[g\epsilon_{\delta\eta\epsilon}\tilde{A}_\mu{}^\epsilon(x_2)V_{\beta\eta}(x_2,P_2)] &= \lambda g\epsilon_{\delta\eta\epsilon}V_{\alpha\epsilon}(x_1,P_1)V_{\beta\eta}(x_2,P_2)\delta_{\mu\nu}\delta^4(x_1-x_2)\\
&\quad+\lambda\epsilon_{\beta\alpha\zeta}g\epsilon_{\delta\eta\epsilon}\tilde{A}_\mu{}^\epsilon(x_2)V_{\zeta\eta}(x_2,P_2)\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)\\
&= \lambda g\epsilon_{\beta\alpha\zeta}V_{\zeta\delta}(x_2,P_2)\delta_{\mu\nu}\delta^4(x_1-x_2)+\lambda\epsilon_{\beta\alpha\zeta}g\epsilon_{\delta\eta\epsilon}\tilde{A}_\mu{}^\epsilon(x_2)V_{\zeta\eta}(x_2,P_2)\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)\,,\end{aligned}\qquad\text{(A6b)}$$

from (4.16c). We observe that the right-hand sides of (A6a) and (A6b) are equal, so that Eq. (4.19) is invariant under the transformation (A5).

From the fact that the change (A5b) in $\tilde{A}_\mu{}^\nu$ results in the change (A5a) in $V_{\beta\delta}(x_2,P_2)$, it follows that

$$V_{\alpha\gamma}(x_1,P_1)\frac{\delta}{\delta A_\nu{}^\gamma(x_1)}V_{\beta\delta}(x_2,P_2)=g\epsilon_{\beta\alpha\zeta}V_{\zeta\delta}(x_2,P_2)\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)\,, \tag{A7}$$

and therefore that

$$[V_{\alpha\gamma}(x_1,P_1)\eta_\nu{}^\gamma(x_1),V_{\beta\delta}(x_2,P_2)]=-g\epsilon_{\beta\alpha\zeta}V_{\zeta\delta}(x_2,P_2)\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)\,. \tag{A8}$$

The commutator between the operators $V_{\alpha\gamma}(x_1,P_1)\eta_\nu{}^\gamma(x_1)$ and $\tilde{f}_{\rho\sigma}{}^\delta(x_2)$ is easily found from (4.13b):

$$\begin{aligned}[V_{\alpha\gamma}(x_1,P_1)\eta_\nu{}^\gamma(x_1),\tilde{f}_{\rho\sigma}{}^\delta(x_2)]&=-V_{\alpha\gamma}(x_1,P_1)\frac{\delta}{\delta\tilde{A}_\nu{}^\gamma(x_1)}\tilde{f}_{\rho\sigma}{}^\delta(x_2)=-V_{\alpha\gamma}(x_1,P_1)\bigg[\bigg(\frac{\partial}{\partial x_{2\rho}}\delta_{\nu\sigma}-\frac{\partial}{\partial x_{2\sigma}}\delta_{\nu\rho}\bigg)\delta_{\delta\gamma}\delta^4(x_1-x_2)\\&+g\epsilon_{\delta\gamma\epsilon}[\delta_{\nu\rho}A_\sigma{}^\epsilon(x_2)-\delta_{\nu\sigma}A_\rho{}^\epsilon(x_2)]\delta^4(x_1-x_2)\bigg]=-V_{\alpha\gamma}(x_2,P_2)\bigg[\bigg(\frac{\partial}{\partial x_{2\rho}}\delta_{\nu\sigma}-\frac{\partial}{\partial x_{2\sigma}}\delta_{\nu\rho}\bigg)\delta_{\delta\gamma}\delta^4(x_1-x_2)\bigg]\end{aligned} \tag{A9}$$

on using Eq. (4.19) for the derivative of the operator $V_{\alpha\gamma}$.

We can now finally find the commutator between the operators $V_{\alpha\gamma}(x_1,P_1)\eta_\nu{}^\gamma(x_1)$ and $\tilde{F}_{\rho\sigma}{}^\beta(x_2)$:

$$\begin{aligned}[V_{\alpha\gamma}(x_1,P_1)\eta_\nu{}^\gamma(x_1),\tilde{F}_{\rho\sigma}{}^\beta(x_2,P_2)]&=[V_{\alpha\gamma}(x_1,P_1)\eta_\nu{}^\gamma(x_1),V_{\beta\delta}(x_2,P_2)\tilde{f}_{\rho\sigma}{}^\delta(x_2)]\\&=-g\epsilon_{\beta\alpha\zeta}V_{\zeta\delta}(x_2,P_2)\tilde{f}_{\rho\sigma}{}^\delta(x_2)\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)-V_{\alpha\gamma}(x_2,P_2)V_{\beta\delta}(x_2,P_2)\\&\times\bigg(\frac{\partial}{\partial x_{2\rho}}\delta_{\nu\sigma}-\frac{\partial}{\partial x_{2\sigma}}\delta_{\nu\rho}\bigg)\delta_{\delta\gamma}\delta^4(x_1-x_2)\quad\text{[from (A8) and (A9)]},\\&=g\epsilon_{\alpha\beta\zeta}F_{\rho\sigma}{}^\zeta(x_2,P_2)\int_{P_2}^{x_2}d\xi_\nu\delta^4(x_1-\xi)-\bigg(\frac{\partial}{\partial x_{2\rho}}\delta_{\nu\sigma}-\frac{\partial}{\partial x_{2\sigma}}\delta_{\nu\rho}\bigg)\delta_{\alpha\beta}\delta(x_1-x_2)\,,\end{aligned} \tag{A10}$$

from (4.16b). We observe that the operators $V_{\alpha\gamma}(x_1,P_1)\eta_\nu{}^\gamma(x_1)$ and $U_\nu{}^\alpha(x_1,P_1)$ do satisfy the same commutation relations (4.9) with the operators $\tilde{F}_{\rho\sigma}{}^\beta(x_2,P_2)$.

APPENDIX B

In this Appendix we shall evaluate the expression (4.44), with θ given by (4.43), and shall show that it is zero. We shall thereby have verified the consistency condition on the Yang-Mills field equations.

We can divide (4.44) into the sum of four terms:

(i)
$$\bigg(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\gamma{}^\beta(x)\bigg)\eta_\nu{}^\gamma(x)=\frac{\partial}{\partial x_\nu}\eta_\nu{}^\alpha(x)+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\nu{}^\beta(x)\eta_\nu{}^\gamma(x)\,, \tag{B1a}$$

(ii)
$$\begin{aligned}-\int dy\bigg[\bigg(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\nu{}^\beta(x)\bigg),\bigg(\frac{\partial\eta_\mu{}^\zeta(y)}{\partial y_\mu}+g\epsilon_{\zeta\epsilon\delta}\eta_\mu{}^\delta(y)\tilde{A}_\mu{}^\epsilon(y)\bigg)\bigg]\frac{\partial}{\partial x_\nu}O_{\gamma\zeta}(x,y)\\=g\epsilon_{\alpha\beta\gamma}\bigg(\frac{\partial^2}{\partial y_\nu\partial x_\nu}O_{\gamma\beta}(x,y)\bigg|_{x=y}-g^2\epsilon_{\zeta\epsilon\beta}\tilde{A}_\nu{}^\epsilon(x)\frac{\partial}{\partial x_\nu}O_{\gamma\zeta}(x,y)\bigg|_{x=y}\bigg),\end{aligned} \tag{B1b}$$

from (4.27),

(iii)
$$\begin{aligned}-\int dy\bigg(\frac{\partial\eta_\mu{}^\zeta(y)}{\partial y_\mu}+g\epsilon_{\zeta\epsilon\delta}\eta_\mu{}^\delta(y)\tilde{A}_\mu{}^\epsilon(y)\bigg)\bigg(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}\times g\epsilon_{\alpha\beta\gamma}\tilde{A}_\nu{}^\beta(x)\bigg)\frac{\partial}{\partial x_\nu}O_{\gamma\zeta}(x,y)\\=-\frac{\partial}{\partial x_\nu}\eta_\nu{}^\alpha(x)-g\epsilon_{\alpha\beta\gamma}\eta_\nu{}^\gamma(x)\tilde{A}_\nu{}^\beta(x)\,,\end{aligned} \tag{B1c}$$

from (4.42),

(iv) $$\left(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\gamma^{\beta}(x)\right)\left(-g\epsilon_{\gamma\zeta\delta}\frac{\partial}{\partial x_\nu}O_{\delta\zeta}(x,y)\,|_{x=y}\right)$$

$$=-g\epsilon_{\alpha\gamma\beta}\frac{\partial}{\partial x_\nu}\left(\frac{\partial}{\partial x_\nu}O_{\beta\gamma}(x,y)\,|_{x=y}\right)-g^2\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\zeta\delta}\tilde{A}_\nu^{\beta}(x)\frac{\partial}{\partial x_\nu}O_{\delta\zeta}(x,y)\,|_{x=y}. \quad \text{(B1d)}$$

Expressions (B1a) and (B1c) are equal and opposite, except for the different ordering of the factors $\eta_\nu{}^\gamma(x)\tilde{A}_\nu{}^\beta(x)$. Thus the total contribution of these two terms to (4.44) is

$$g\epsilon_{\alpha\beta\gamma}[\tilde{A}_\nu{}^\beta(x)\eta_\nu{}^\gamma(x)]. \quad \text{(B2a)}$$

On adding (B1b) and (B1d) and using (4.16d), we find the result (4.16d), we find the result

$$-g\epsilon_{\alpha\gamma\beta}\left(\frac{\partial}{\partial x_\nu}\delta_{\beta\epsilon}+g\epsilon_{\beta\delta\epsilon}\tilde{A}_\nu{}^\delta(x)\right)\frac{\partial}{\partial x_\nu}O_{\epsilon\gamma}(x,y)\,|_{x=y}. \quad \text{(B2b)}$$

Finally, on adding (B2a) and (B2b), we obtain the equation

$$\left(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\nu{}^\beta(x)\right)\theta_\nu{}^\gamma(x)=g\epsilon_{\alpha\beta\gamma}[\tilde{A}_\nu{}^\beta(x)\eta_\nu{}^\gamma(x)]$$

$$-g\epsilon_{\alpha\gamma\beta}\left(\frac{\partial}{\partial x_\nu}\delta_{\beta\epsilon}+g\epsilon_{\beta\delta\epsilon}\tilde{A}_\nu{}^\delta(x)\right)\frac{\partial}{\partial x_\nu}O_{\epsilon\gamma}(x,y)\,|_{x=y}. \quad \text{(B3)}$$

With the aid of (4.21) and (4.42), we can rewrite (B3) in the form

$$\left(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\nu{}^\beta(x)\right)\theta_\nu{}^\gamma(x)=g\epsilon_{\alpha\beta\gamma}\delta_{\beta\gamma}\delta_{\nu\nu}\delta^4(x-x)$$

$$-g\epsilon_{\alpha\gamma\beta}\delta_{\beta\gamma}\delta_{\nu\nu}\delta^4(x-x). \quad \text{(B4)}$$

If we allow subtraction of infinites as one usually does in perturbation theory, we can set the right-hand side of (B4) equal to zero. If we had been working in momentum space, the δ function in (B4) would have taken the form

$$\int d^4p \quad \text{(B5)}$$

and we would have set (B4) equal to zero owing to the vanishing of the integrand. Thus, insofar as perturbation theory for local fields has any meaning at all, we may write

$$\left(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\nu}+g\epsilon_{\alpha\beta\gamma}\tilde{A}_\nu{}^\beta(x)\right)\theta_\nu{}^\gamma(x)=0. \quad \text{(B6)}$$

We conclude that (4.38), with θ given by (4.43), is consistent.