High-Energy Nucleus-Nucleus Collisions. I. General Theory and Applications to Deuteron-Deuteron Scattering*

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A theoretical analysis of the collisions of high-energy nuclei with nuclei is carried out by means of a simple extension of the Glauber approximation. Effects of multiple collisions are taken into account. The general formalism is applied to deuteron-deuteron collisions. Expressions are derived for single-, double-, triple-, and quadruple-scattering amplitude operators for deuteron-deuteron collisions in terms of nucleon-nucleon scattering amplitude operators. A new type of double-scattering effect, qualitatively quite different from the Glauber "shadow" effect which was discovered for particle-deuteron collisions, is described. For the case of nucleon-nucleon interactions described by purely absorbing (black) spheres, it corresponds qualitatively to a "double-counting" correction in the deuteron-deuteron absorption cross section. This effect corresponds to collisions in which one nucleon in the incident deuteron interacts with only one nucleon in the target, and the other nucleon in the incident deuteron interacts with only the other nucleon in the target. The formalism is applied to a calculation of the deuteron-deuteron total cross section σ_{dd}. It is shown that the contribution to σ_{dd} arising from the new type of double-scattering correction is approximately 50% of that arising from the usual (i.e., shadow-type) double-scattering correction. Numerical results are compared with measurements. A simple analysis of the deuteron-deuteron elastic scattering angular distribution is presented. It is shown that for a rather large range of scattering angles away from the forward direction, double scattering is the dominant process in elastic scattering, and that in this region the new type of double scattering is quantitatively much more important than the usual double-scattering process which also appears in nucleon-deuteron collisions.

I. INTRODUCTION

THE simplest nucleus-nucleus collision involving more than a total of two nucleons is the nucleon-deuteron collison. In recent years a large number of experiments involving interactions of high-energy particle beams with deuterium targets have been performed. Since the deuteron is a rather weakly bound system, the incident-particle wavelengths for such collisions may be considerably smaller than the average neutron-proton separation in the deuteron. In such cases one might be tempted to approximate particle-deuteron cross sections by sums of the corresponding free-particle–neutron and free-particle–proton cross sections, and to employ the usual impulse approximation. Various processes which occur in particle-deuteron collisions have been analyzed by Franco and Glauber.[1–3] It was shown that even at the highest available energies effects of double interactions, such as double scattering or interferences between single- and double-scattering amplitudes, are quite appreciable. The deviation from simple additivity of the nucleon cross sections in the deuteron was striking for antiproton-deuteron total cross sections, amounting at some antiproton energies to as much as 20 to 40% of the free antiproton-nucleon cross sections.[1,3] For high-energy proton-deuteron collisions, the integrated elastic scattering cross sections calculated with double interactions neglected differed from those calculated with double interactions included by approximately 20%.[2] For particle-deuteron elastic collisions double scattering was *predicted* to be the *dominant* mechanism at angles which are not too close to the forward direction.[1,3] Subsequent analyses by Franco and Coleman[4] and by Franco[5] of recent proton-deuteron elastic scattering data at 2 and 1 BeV strongly suggest that this indeed is the case. In addition, Bertocchi and Capella[6] have argued that the large backward peaks in proton-deuteron elastic scattering between 1.0 and 1.5 BeV appear to result from double collisions. A recent analysis[2] of the angular distribution of the sum of elastic plus inelastic proton-deuteron scattering in terms of single and double scattering is in good agreement with the measurements. Furthermore, it has been shown[7] that the treatment of double-scattering effects in K^+d charge-exchange collisions is necessary for the proper extraction of the K^+n charge-exchange cross section near the forward direction from the corresponding measurements of the K^+d charge-exchange cross section. Within the deuteron, therefore, double-scattering effects are seen to be rather important.

The methods employed in the above analyses of particle-deuteron collisions have also been widely utilized for extracting neutron-proton total cross sections[8] and the ratios of the real part to the imaginary part of the neutron-proton forward elastic scattering amplitudes[9] from proton-proton and proton-deuteron

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[1] V. Franco and R. J. Glauber, Phys. Rev. 142, 1195 (1965).
[2] V. Franco, Phys. Rev. Letters 16, 944 (1966).
[3] V. Franco, Ph.D. thesis, Harvard University, 1963 (unpublished).
[4] V. Franco and E. Coleman, Phys. Rev. Letters 17, 827 (1966).
[5] V. Franco, Los Alamos Scientific Laboratory Report No. LA-DC-9964 (to be published).
[6] L. Bertocchi and A. Capella, Nuovo Cimento 51A, 369 (1967).
[7] R. J. Glauber and V. Franco, Phys. Rev. 156, 1685 (1967).
[8] See, for example, D. V. Bugg, D. C. Salter, G. H. Stafford, R. F. George, K. F. Riley, and R. J. Tapper, Phys. Rev. 146, 980 (1966).
[9] See, for example, G. Bellettini, G. Cocconi, A. N. Diddens, E. Lillethun, G. Matthiae, J. P. Scanlon, and A. M. Wetherell, Phys. Letters 19, 341 (1966).

measurements. In this manner information regarding neutron-proton collisions has been obtained without having to employ neutron beams.

Experiments have shown that high-energy scattering by nucleons occurs predominantly near the forward direction. Since triple and higher-order multiple interactions in particle-deuteron collisions must take place via at least one backward scattering process, they have exceedingly small amplitudes. (We assume negligible overlap of the target nucleons.) Furthermore, since at least one backward scattering process *plus* one large-angle (i.e., $\geq 90°$ in the laboratory system) scattering process is necessary for triple or higher-order multiple collisions to result in a net scattering in the *forward* direction, such collisions would yield negligibly small contributions to the deuteron total cross section, i.e., to the imaginary part of the deuteron forward elastic scattering amplitude. On the other hand, if the incident beam consists not of single particles but rather of *composite* particles such as deuterons or more complex nuclei, we should certainly expect at least double-interaction effects to be quite significant, and it is likely that for many reactions (e.g., elastic scattering) even higher-order multiple scattering effects would be important for some range of scattering angle. Their importance would of course also be considerable if the target contained more than two nucleons. A recent experiment at Brookhaven[10] has shown that the proton-^{4}He elastic scattering angular distribution exhibits at least one and perhaps two secondary maxima. These maxima may be explained by the importance of double and triple collisions in the two angular regions in which the maxima occur, and calculations similar to those presented in Refs. 1–4 were first applied to this case by Czyż and Leśniak.[11]

In the present work we derive an expression for the scattering amplitude operator for high-energy nucleus-nucleus collisions by means of an extension of the Glauber approximation.[12] The simplest nucleus-nucleus collision in which the incident nucleus and the target each contains more than one particle is the deuteron-deuteron collision. Although the number of completed high-energy deuteron-deuteron scattering experiments is quite small, there have been several rather large-scale experiments at Berkeley and at the Princeton-Penn Accelerator which are presently being analyzed, and and another which will soon be analyzed.[13] In anticipation of data from these experiments we shall in the present work specialize the results we obtain for general nucleus-nucleus collisions to high-energy deuteron-deuteron collisions and obtain a scattering amplitude operator from which cross sections for a number of elastic and inelastic reactions may be secured. The theory leads in a natural manner to a consideration of single, double, triple, and quadruple interactions.[14] Quintuple and higher-order multiple interactions in deuteron-deuteron collisions may occur only by means of at least one backward scattering and therefore have negligibly small amplitudes, particularly for small-angle scattering (where in addition to at least one backward scattering collision a second large-angle nucleon-nucleon collision is required). The general results for the deuteron-deuteron scattering amplitude operator are then applied to a detailed investigation of the deuteron-deuteron total cross section, and comparisons are made with measurements. A brief analysis of elastic scattering is also given.

There have been few theoretical analyses of deuteron-deuteron collisions at high energies. Brander[15] developed a formalism for treating deuteron-deuteron elastic scattering in the impulse approximation. Tubis and Chern[16] used the impulse approximation to calculate both the differential cross section for elastic scattering and the vector polarization. In these analyses the effects of multiple interactions were neglected and the discussions were restricted to elastic processes. Franco[17] gave an expression for the total cross section which included double, triple, and quadruple interactions and which was applied to deuteron-deuteron collisions at an incident momentum of 2.8 BeV/c. Queen[18] treated multiple collisions in elastic and total cross sections but neglected all unbound intermediate states, an approximation which is not reliable for high-energy collisions with such a weakly bound target nucleus as the deuteron. However, an application to low-energy (i.e., 64 MeV) scattering was given.

For the present analysis we adopt the Glauber approximation.[12] This is a diffraction approximation which is asymptotically correct for high-energy scattering at small momentum transfers. It is similar in many respects to the approximations used in diffraction theory in physical optics. Each portion of the incident plane wave is assumed to traverse the region of interaction along a straight line path and to suffer a shift of phase and change of amplitude which depend only upon the path

[10] H. Palevsky, J. L. Friedes, R. J. Sutter, G. W. Bennett, G. J. Igo, W. D. Simpson, G. C. Phillips, D. M. Corley, N. S. Wall, R. L. Stearns, and B. Gottschalk, Phys. Rev. Letters **18**, 1200 (1967).

[11] W. Czyż and L. Leśniak, Phys. Letters **24B**, 227 (1967); for more recent calculations see, for example, R. H. Bassel and C. Wilkin, Phys. Rev. Letters **18**, 871 (1967).

[12] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by Wesley E. Brittin *et al.* (Interscience Publishers, Inc., New York, 1959), Vol. I, p. 315.

[13] M. Pripstein (private communication); M. Brazin (private communication).

[14] A quadruple interaction is one in which the proton and neutron in the incident deuteron each interacts with both the proton and neutron in the target deuteron. An example of a quadruple interaction is furnished by the process in which the proton in the incident deuteron is scattered by both nucleons in the target deuteron, and the neutron in the incident deuteron is scattered by the proton in the target and then interacts inelastically with the neutron in the target, producing additional particles.

[15] O. Brander, Nucl. Phys. **36**, 82 (1962).

[16] A. Tubis and B. Chern, Phys. Rev. **128**, 1352 (1962).

[17] V. Franco, University of California Lawrence Radiation Laboratory Report No. UCRL-16694, 1966 (unpublished).

[18] N. M. Queen, Phys. Letters **13**, 236 (1964).

traversed. In the analysis we do not attempt to describe the interactions themselves in any direct manner. Instead we express the various contributions to the deuteron-deuteron scattering amplitude and cross sections in terms of the nucleon-nucleon elastic scattering amplitudes and certain integrals of products of these amplitudes, and in terms of the deuteron ground-state wave function.

The analysis is begun by presenting some of the necessary results of the Glauber approximation. In Sec. II we give the expression for the elastic scattering amplitude for a simple two-particle collision. In Sec. III we obtain the scattering amplitude for a collision between a particle and a complex nucleus. In Sec. IV we calculate the scattering amplitude for collisions between two complex nuclei. We consider in Sec. V deuteron-deuteron collisions and express the corresponding scattering amplitude in terms of the free nucleon-nucleon elastic scattering amplitudes and the deuteron ground-state wave function. This amplitude may be used to calculate cross sections for both elastic and inelastic processes. An expression for the deuteron-deuteron total cross section is derived in Sec. VI and is written in terms of the proton-proton and neutron-proton elastic scattering amplitudes in Sec. VII. Several asymptotic expressions and simple approximate formulas for various special cases are obtained for the total cross section in Sec. VIII. A new type of double-collision correction to the total cross section, different from the well-known Glauber shadow effect, is derived and discussed in that section. The evaluation of the deuteron-deuteron total cross section is presented and compared with existing data in Sec. IX. In the final section we present a simple analysis of *d-d* elastic scattering and show a sample calculation at 4.42 BeV/*c*.

II. PARTICLE-PARTICLE COLLISIONS

We begin by summarizing some pertinent results of the Glauber approximation.[1,12] For collisions between two particles the scattering amplitude operator obtained in the Glauber approximation is simple in form. If $\hbar\mathbf{q}$ is the momentum transferred from the incident particle or projectile (labelled by the index p) to the target particle (labelled by the index t) and k is the wave number of the incident particle, the corresponding scattering amplitude operator $\mathcal{A}_{pt}(\mathbf{q},k)$ is given by[1,12]

$$\mathcal{A}_{pt}(\mathbf{q},k)=\frac{ik}{2\pi}\int e^{i\mathbf{q}\cdot\mathbf{b}}[1-e^{i\chi_{pt}(\mathbf{b})}]d^2b\,, \qquad (2.1)$$

which we shall find convenient to abbreviate as

$$\mathcal{A}_{pt}(\mathbf{q},k)=\frac{ik}{2\pi}\int e^{i\mathbf{q}\cdot\mathbf{b}}\Gamma_{pt}(\mathbf{b})d^2b\,, \qquad (2.2)$$

where

$$\Gamma_{pt}(\mathbf{b})=1-e^{i\chi_{pt}(\mathbf{b})}. \qquad (2.3)$$

In these expressions $\mathbf{b}$ is the impact parameter vector and is perpendicular to the direction of the incident beam, and the two-dimensional integration is over the plane of impact parameter vectors. The operator $\chi_{pt}(\mathbf{b})$ is a complex phase-shift function which depends upon the interaction between the two particles. It should be noted that the form (2.1) differs from the usual one for the high-energy approximation to the elastic scattering amplitude, which is given by a one-dimensional integral. Equation (2.1) is valid for an interaction of arbitrary shape,[1] and the phase-shift operator χ_{pt} depends in general on both the magnitude and orientation of the impact parameter vector. Although it may be valid to assume that the two-particle interaction possesses azimuthal symmetry about the direction of propagation of the incident beam, in which case the usual form for the scattering amplitude would suffice, it is generally not valid to make this assumption when the projectile or target is a bound system of nucleons. Therefore in preparation for our treatment of collisions between more complex systems we shall use the more general form for the scattering amplitude operator given by Eq. (2.1).

If we denote the initial internal state (i.e., spin, isotopic spin, etc.) of the incident and target particles by $|i\rangle$ and the final internal state by $|f\rangle$, we may write the amplitude $A_{fi,pt}(\mathbf{q},k)$ corresponding to a transition from $|i\rangle$ to $|f\rangle$, with a transfer of momentum $\hbar\mathbf{q}$ from the projectile p to the target t, as

$$A_{fi,pt}(\mathbf{q},k)=\langle f|\,\mathcal{A}_{pt}(\mathbf{q},k)\,|i\rangle. \qquad (2.4)$$

Hereafter the labels p and t shall be suppressed in the amplitudes $A_{fi,pt}(\mathbf{q},k)$. In particular, the elastic scattering amplitude for simple two-particle collisions is given by

$$A_{ii}(\mathbf{q},k)=\langle i|\,\mathcal{A}_{pt}(\mathbf{q},k)\,|i\rangle. \qquad (2.5)$$

The expressions we have written for the scattering amplitudes are of the correct form for describing the collision of the incident and target particles in their center-of-mass system. On the other hand, it has been shown[1] that these expressions undergo very little change of form when they are transformed to the laboratory system. In fact, the scattering amplitudes in the laboratory system may be found from the expressions (2.1) and (2.2) simply by substituting in them the laboratory values of the incident momentum and momentum transfers.

In comparing cross sections for collisions between single particles and deuterons with those for collisions between the same single particles and nucleons, use must be made of at least two center-of-mass systems. The reason for this is that in general the velocity of a particle in the center-of-mass system for particle-deuteron collisions is different from its velocity in the center-of-mass system for particle-nucleon collisions. For particle-deuteron collisions, therefore, it is convenient to refer all calculations of the scattering amplitudes to the laboratory system.

On the other hand, in comparing high-energy deuteron-deuteron cross sections with high-energy nucleon-nucleon cross sections, only one center-of-mass system need be used. That is, at high energies if we neglect the internal motion of the nucleons in the deuteron compared with the incident velocity of the deuteron, the velocity of the center of mass of a deuteron in the deuteron-deuteron center-of-mass system is very nearly the same as that which one of the nucleons in the deuteron would have in a nucleon-nucleon center-of-mass system. Nevertheless, we shall still refer all amplitudes to the laboratory system since that system is most convenient for discussing the general nucleon-nucleus or nucleus-nucleus collision, and for comparing deuteron-deuteron collisions with both nucleon-deuteron and nucleon-nucleon collisions.

It is important to note that the validity of the expression (2.2) for the scattering amplitude does not depend in any manner upon the existence of a potential function to describe the interaction. However, a complex potential may always be found to describe high-energy collisions. An application to scattering by complex potentials has recently been made in an investigation of neutron-nucleus interactions and the optical model.[19]

III. PARTICLE-NUCLEUS COLLISIONS

In this section we describe the way in which the amplitude for scattering by a bound system of nucleons may be obtained.[1,12] For scattering of a high-energy particle by a system such as a nucleus with internal degrees of freedom, we note that the individual nucleon velocities are generally small compared to the velocity of the incident projectile. Provided the relative velocities between the incident particle and the nucleons in the target do not correspond to energies of strong resonances of the particle-nucleon system, it is asymptotically correct to consider the nucleons frozen in their instantaneous positions during the passage of the incident particle through the system. For a fixed configuration of A bound nucleons $\mathbf{r}_1, \cdots, \mathbf{r}_A$ the scattering amplitude operator $F(\mathbf{q},k,\mathbf{r}_1,\cdots,\mathbf{r}_A)$ would be

$$F(\mathbf{q},k,\mathbf{r}_1,\cdots,\mathbf{r}_A)=\frac{ik}{2\pi}\int e^{i\mathbf{q}\cdot\mathbf{b}}\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)d^2b\,, \quad (3.1)$$

where

$$\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)=1-e^{i\chi_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)}\,, \quad (3.2)$$

and where $\chi_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)$ represents the accumulated effect of the passage of the wave representing the incident particle through nuclear system.

Since the nucleons are in fact not rigidly fixed, $\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)$ is to be regarded as an operator which induces appropriate changes of the internal states of the nucleus and the incident particle as well as changes of the momentum state of the incident particle. The scattering amplitude $F_{fi}(\mathbf{q},k)$ for the collision in which the particle-nucleus system makes a transition from an initial state $|i\rangle$ to a final state $|f\rangle$ and momentum $\hbar\mathbf{q}$ is transferred from the incident particle may be written as the appropriate matrix element of the operator $F(\mathbf{q},k,\mathbf{r}_1,\cdots,\mathbf{r}_A)$, so that

$$F_{fi}(\mathbf{q},k)=\frac{ik}{2\pi}\int e^{i\mathbf{q}\cdot\mathbf{b}}\langle f|\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)|i\rangle d^2b\,. \quad (3.3)$$

In this expression $\langle f|\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)|i\rangle$ denotes the matrix element of $\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)$ between initial and final states of the particle-nucleus system. Applications of this result to particle-deuteron collisions are extensive and have been referred to earlier. This expression has also been used to describe scattering of charged particles by hydrogen atoms by means of potential interactions.[20]

IV. NUCLEUS-NUCLEUS COLLISIONS

In this section we consider collisions in which the incident beam, as well as the target, consists of extended systems with internal structure, and generalize the results of Sec. III to obtain scattering amplitudes for such collisions. If the incident beam contains systems with internal degrees of freedom, the initial and final states of these systems, as well as those of the target, must of course be taken into account. Furthermore, the operator Γ_{tot} will depend not only upon the nucleon coordinates of the target, but also upon the nucleon coordinates of the incident nucleus which we shall denote by $\mathbf{r}_{A+1},\cdots,\mathbf{r}_N$. (The incident nucleus therefore contains N-A nucleons.) The scattering amplitude $F_{\varphi f i\iota}(\mathbf{q},k)$ for collisions in which a nucleus incident with momentum $\hbar k$ transfers momentum $\hbar\mathbf{q}$ to the target and makes a transition from an initial state $|\iota\rangle$ to a final state $|\varphi\rangle$ and the target makes a transition from an initial state $|i\rangle$ to a final state $|f\rangle$ is given by a generalization of Eq. (3.3) and may be written in the form

$$F_{\varphi f i\iota}(\mathbf{q},k)=\langle\varphi f|F(\mathbf{q},k,\mathbf{r}_1,\cdots,\mathbf{r}_N)|i\iota\rangle\,, \quad (4.1)$$

where the operator $F(\mathbf{q},k,\mathbf{r}_1,\cdots,\mathbf{r}_N)$ is given by

$$F(\mathbf{q},k,\mathbf{r}_1,\cdots,\mathbf{r}_N)=\frac{ik}{2\pi}\int e^{i\mathbf{q}\cdot\mathbf{b}}\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_N)d^2b\,. \quad (4.2)$$

In this expression Γ_{tot} is given by

$$\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_N)=1-e^{i\chi_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_N)}\,, \quad (4.3)$$

where χ_{tot} represents the resultant phase shift accumulated by the wave representing the incident nucleus as it passes through the target nucleus. The quantity $\langle\varphi f|\Gamma_{\text{tot}}|i\iota\rangle$ denotes the matrix element of Γ_{tot} between the initial and final states of both the incident and target nuclei. The vector $\mathbf{b}$ is the impact parameter vector of the center of mass of the incident nucleus relative to the center of mass of the target.

We shall assume that the nucleons in the incident nucleus interact with those in the target nucleus by

[19] V. Franco, Phys. Rev. **140**, B1501 (1965).

[20] V. Franco, Phys. Rev. Letters **20**, 709 (1968).

means of two-body forces. The total phase-shift function χ_{tot} may then be written as the sum of the phase-shift functions obtained by considering separately collisions between all different combinations of two nucleons subject to the restriction that one nucleon belongs to the target and the other to the incident nucleus. If we denote the components of the coordinates $\mathbf{r}_1, \cdots, \mathbf{r}_N$ perpendicular to the direction of the incident beam (i.e., parallel to the plane containing the impact parameter vector $\mathbf{b}$) by $\mathbf{s}_1, \cdots, \mathbf{s}_N$, we may express the total phase-shift function in the form

$$\chi_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_N)=\sum_{p=A+1}^{N}\sum_{t=1}^{A}\chi_{pt}(\mathbf{b}-\mathbf{s}_t+\mathbf{s}_p), \quad (4.4)$$

where $\chi_{pt}(\mathbf{b}-\mathbf{s}_t+\mathbf{s}_p)$ is the phase-shift function that would result from the interaction between a nucleon with a laboratory momentum $\hbar k/(N\text{-}A)$ at an internal coordinate $\mathbf{r}_p$ and a nucleon at rest at an internal coordinate $\mathbf{r}_t$.

Since χ_{pt} and χ_{pu}, where t and u label two target nucleons, do not commute in general, the order in which they occur in Γ_{tot} is an important feature of the terms in the scattering amplitude operator which describe multiple scattering. In order to account for this noncommutativity, the operator Γ_{tot} should be written as

$$\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_N) =\{1-\exp[i\sum_{p=A+1}^{N}\sum_{t=1}^{A}\chi_{pt}(\mathbf{b}-\mathbf{s}_t+\mathbf{s}_p)]\}_+, \quad (4.5)$$

where the symbol $\{\ \}_+$ denotes the time-ordered product. It is taken to mean that in the power-series expansion of the exponential, whenever operators χ_{pt} and χ_{pu} do not commute they appear in the order (reading from *right* to *left*) corresponding to that in which nucleons t and u interact with nucleon p.[21]

The resulting scattering amplitudes are obtained by means of Eqs. (4.1), (4.2), and (4.5). Multiple scattering effects are contained implicitly in the resulting scattering amplitude because we have summed phase shifts rather than amplitudes. A total of $(N\text{-}A)A$ orders of multiple scattering are treated. This may be explicitly demonstrated by resolving Eq. (4.2) in terms of the individual nucleon-nucleon scattering amplitudes. We shall do this for deuteron-deuteron collisions and show how single-, double-, triple-, and quadruple-scattering processes are explicitly taken into account. Higher orders of multiple scattering require at least one backward collision between nucleons and are consequently neglected.

[21] The use of completely antisymmetric wave functions to describe the N-particle system obviates the necessity of the time-ordered product. However, it is often more convenient in dealing with high-energy collision processes to antisymmetrize the wave function describing the target, and to separately antisymmetrize the wave function describing the incident nucleus. In this case some effects of the time-ordering still remain.

V. DEUTERON-DEUTERON COLLISIONS

We shall apply the results of the preceding section to collisions in which an incident beam of high-energy deuterons interacts with a deuterium target, and obtain the scattering amplitude operator for such collisions. We shall assume that the nucleon-nucleon interactions are precisely charge-independent and omit the effects of their spin dependence, some of which have been discussed for particle-deuteron collisions in Ref. 1.

We let $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ be the internal coordinate of the target deuteron, where the indices 1 and 2 label the two nucleons of that deuteron. Similarly, we let $\boldsymbol{\varrho}=\mathbf{r}_3-\mathbf{r}_4$ be the internal coordinate of the incident deuteron, where the indices 3 and 4 label the two nucleons of that deuteron. We let z and ζ be the projections of $\mathbf{r}$ and $\boldsymbol{\varrho}$ along the momentum $\hbar\mathbf{k}$ of the incident deuteron center of mass and let $\mathbf{s}$ and $\boldsymbol{\sigma}$ be the corresponding projections in the plane perpendicular to $\mathbf{k}$. Before the collision we consider both the incident and target deuterons to be in their ground states. Therefore since $|\iota\rangle$ and $|i\rangle$ will represent the same state, we set $\iota=i$. The states $|f\rangle$ and $|\varphi\rangle$ may represent excited states (i.e., unbound two-particle states) or the deuteron ground state once again. The scattering amplitude for collisions in which a deuteron with momentum $\hbar\mathbf{k}$ transfers momentum $\hbar\mathbf{q}$ to the target and emerges in a final state $|\varphi\rangle$ and the target deuteron is left in a final state $|f\rangle$ may be written by means of Eqs. (4.1), (4.2), and (4.5) as

$$F_{\varphi f i i}(\mathbf{q},k)=\langle\varphi f|F(\mathbf{q},k,\mathbf{r},\boldsymbol{\varrho})|ii\rangle, \quad (5.1)$$

where the operator F takes the form

$$F(\mathbf{q},k,\mathbf{r},\boldsymbol{\varrho})=\frac{ik}{2\pi}\int e^{i\mathbf{q}\cdot\mathbf{b}}\{1-\exp(i[\chi_{31}(\mathbf{b}-\tfrac{1}{2}\mathbf{s}+\tfrac{1}{2}\boldsymbol{\sigma}) +\chi_{41}(\mathbf{b}-\tfrac{1}{2}\mathbf{s}-\tfrac{1}{2}\boldsymbol{\sigma})+\chi_{32}(\mathbf{b}+\tfrac{1}{2}\mathbf{s}+\tfrac{1}{2}\boldsymbol{\sigma}) +\chi_{42}(\mathbf{b}+\tfrac{1}{2}\mathbf{s}-\tfrac{1}{2}\boldsymbol{\sigma})])\}_+. \quad (5.2)$$

As a first step in separating the contributions of the individual nucleons to the scattering processes, we introduce the functions Γ_{31}, Γ_{41}, Γ_{32}, and Γ_{42} defined in terms of χ_{31}, χ_{41}, χ_{32}, and χ_{42} by means of Eq. (2.3). Let the notation $1\leftrightarrow 2$ denote the interchange of indices 1 and 2 *together with the inversion* $\mathbf{r}\to-\mathbf{r}$, and let $3\leftrightarrow 4$ denote the interchange of indices 3 and 4 *together with the inversion* $\boldsymbol{\varrho}\to-\boldsymbol{\varrho}$. We may then write the identity

$$\{1-\exp[i(\chi_{31}+\chi_{41}+\chi_{32}+\chi_{42})]\}_+=\sum_{j=1}^{4}\Gamma_j, \quad (5.3)$$

where the Γ_j are given by

$$\Gamma_1=\sum_{p=3}^{4}\sum_{t=1}^{2}\Gamma_{pt}, \quad (5.4)$$

$$
\begin{aligned}
-\Gamma_2=&\{[\Gamma_{41}\Gamma_{31}\theta(\zeta)+3\leftrightarrow 4]+[1\leftrightarrow 2]\}\\
&+\{[\Gamma_{41}\Gamma_{42}\theta(z)+1\leftrightarrow 2]+[3\leftrightarrow 4]\}\\
&+\{\Gamma_{31}\Gamma_{42}+3\leftrightarrow 4\}, \qquad (5.5)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_3=&[\Gamma_{41}\Gamma_{31}\Gamma_{42}\theta(z)\theta(\zeta)+\Gamma_{42}\Gamma_{41}\Gamma_{31}\theta(-z)\theta(\zeta)\\
&+\Gamma_{31}\Gamma_{41}\Gamma_{42}\theta(z)\theta(-\zeta)+\Gamma_{42}\Gamma_{31}\Gamma_{41}\theta(-z)\theta(-\zeta)]\\
&+[1\leftrightarrow 2]+[3\leftrightarrow 4]+[1\leftrightarrow 2, 3\leftrightarrow 4], \qquad (5.6)
\end{aligned}
$$

and

$$
\begin{aligned}
-\Gamma_4=&[\Gamma_{41}\Gamma_{31}\Gamma_{42}\Gamma_{32}\theta(z)\theta(\zeta)+\Gamma_{31}\Gamma_{41}\Gamma_{32}\Gamma_{42}\theta(z)\theta(-\zeta)]\\
&+[1\leftrightarrow 2, 3\leftrightarrow 4]. \qquad (5.7)
\end{aligned}
$$

The function $\theta(z)$ is defined by

$$
\begin{aligned}
\theta(z)&=1, \quad \text{for} \quad z>0\\
&=0, \quad \text{for} \quad z<0.
\end{aligned} \qquad (5.8)
$$

We see that in Γ_1, Γ_2, Γ_3, and Γ_4 there are 4, 10, 16, and 4 terms, respectively. The arguments of the various operators on the right-hand sides of Eqs. (5.3)–(5.7) have been suppressed, but are determined simply by their indices which label the incident and target nucleons.

An alternative and useful way of writing the ordered products occurring in Eq. (5.2) is based on the identity

$$\theta(z)=\tfrac{1}{2}[1+\epsilon(z)], \qquad (5.9)$$

where

$$\epsilon(z)=z/|z|. \qquad (5.10)$$

By substituting this identity and the corresponding one for $\theta(\zeta)$ into Eqs. (5.5)–(5.7) we obtain

$$
\begin{aligned}
-\Gamma_2=&\tfrac{1}{2}\{[\{\Gamma_{41},\Gamma_{31}\}+[\Gamma_{41},\Gamma_{31}]\epsilon(\zeta)]+[1\leftrightarrow 2]\}\\
&+\tfrac{1}{2}\{[\{\Gamma_{41},\Gamma_{42}\}+[\Gamma_{41},\Gamma_{42}]\epsilon(z)]+[3\leftrightarrow 4]\}\\
&+\{[\Gamma_{31}\Gamma_{42}]+[3\leftrightarrow 4]\}, \qquad (5.11)
\end{aligned}
$$

$$
\begin{aligned}
6\Gamma_3=&[\{\{\Gamma_{41},\Gamma_{31}\},\Gamma_{42}\}+\{\Gamma_{41},\Gamma_{31}\Gamma_{42}\}[1+\epsilon(z)\epsilon(\zeta)]\\
&+[\{\Gamma_{41},\Gamma_{31}\},\Gamma_{42}]\epsilon(z)+[\Gamma_{41},\Gamma_{31}\Gamma_{42}][\epsilon(z)+\epsilon(\zeta)]\\
&+[\{\Gamma_{41},\Gamma_{42}\}\Gamma_{31}]\epsilon(\zeta)+[[\Gamma_{41},\Gamma_{31}],\Gamma_{42}]\epsilon(z)\epsilon(\zeta)]\\
&+[1\leftrightarrow 2]+[3\leftrightarrow 4]+[1\leftrightarrow 2, 3\leftrightarrow 4], \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
4\Gamma_4=&\{[\{\Gamma_{41}\Gamma_{42},\Gamma_{31}\Gamma_{32}\}+[\Gamma_{41}\Gamma_{42},\Gamma_{31}\Gamma_{32}]\epsilon(\zeta)]\\
&+[1\leftrightarrow 2]\}+\{[\Gamma_{41}\Gamma_{31},\Gamma_{42}\Gamma_{32}]\epsilon(z)+3\leftrightarrow 4\}\\
&+\{[\Gamma_{41}\Gamma_{42},\Gamma_{31}\Gamma_{32}]\epsilon(z)\epsilon(\zeta)+(1\leftrightarrow 2, 3\leftrightarrow 4)\}, \qquad (5.13)
\end{aligned}
$$

where the brackets $[\ ,\ ]$ designate the commutator and $\{\ ,\ \}$ the anticommutator.

If we substitute the expressions given by Eqs. (5.3), (5.4), and (5.11)–(5.13) into the integral (5.2) and shift the origin in the **b** plane to carry out the first four integrals, we obtain

$$F(\mathbf{q},k,\mathbf{r},\boldsymbol{\rho})=\sum_{j=1}^{4} F_j(\mathbf{q},k,\mathbf{r},\boldsymbol{\rho}), \qquad (5.14)$$

where the contribution $F_1(\mathbf{q},k,\mathbf{r},\boldsymbol{\rho})$ arises from single-scattering processes and is given by

$$
\begin{aligned}
F_1(\mathbf{q},k,\mathbf{r},\boldsymbol{\rho})=&2[e^{\frac{1}{2}i\mathbf{q}\cdot(\mathbf{s}-\boldsymbol{\sigma})}a_{31}(\mathbf{q},\tfrac{1}{2}k)+e^{\frac{1}{2}i\mathbf{q}\cdot(\mathbf{s}+\boldsymbol{\sigma})}a_{41}(\mathbf{q},\tfrac{1}{2}k)\\
&+e^{-\frac{1}{2}i\mathbf{q}\cdot(\mathbf{s}+\boldsymbol{\sigma})}a_{32}(\mathbf{q},\tfrac{1}{2}k)+e^{-\frac{1}{2}i\mathbf{q}\cdot(\mathbf{s}-\boldsymbol{\sigma})}a_{42}(\mathbf{q},\tfrac{1}{2}k)]. \qquad (5.15)
\end{aligned}
$$

In this expression $a_{pt}(\mathbf{q},\frac{1}{2}k)$ is the scattering amplitude operator for collisions of incident nucleon p having momentum $\frac{1}{2}\hbar k$ with target nucleon t, in which a momentum $\hbar\mathbf{q}$ is transferred to the target nucleon.[22]

In order to express the contributions F_2, F_3, and F_4 to the deuteron-deuteron scattering amplitude operator which arise from multiple interactions in terms of the basic free nucleon-nucleon scattering amplitude operators a_{pt}, we note from Eq. (2.2) that a_{pt} is a Fourier transform of the function Γ_{pt}. An approximate inversion of the transform is obtained by multiplying Eq. (2.2) by $\exp(-i\mathbf{q}\cdot\mathbf{b})$ and integrating with respect to $\mathbf{q}$ over a plane perpendicular to the direction of the incident beam. We then secure

$$\Gamma_{pt}(\mathbf{b})=(2\pi ik)^{-1}\int e^{-i\mathbf{q}\cdot\mathbf{b}}a_{pt}(\mathbf{q})d^2q, \qquad (5.16)$$

$$t=1, 2;\ p=3, 4.$$

If we utilize this expression and the Fourier integral representation of the two-dimensional δ function, we obtain for the multiple scattering contributions to the deuteron-deuteron scattering amplitude operator the results

$$
\begin{aligned}
F_2(\mathbf{q},k,\mathbf{r},\boldsymbol{\rho})=&\frac{i}{\pi k}\int(\{[e^{i(\frac{1}{2}\mathbf{q}\cdot\mathbf{s}+\mathbf{q}'\cdot\boldsymbol{\sigma})}(\{a_{41}(\mathbf{u}),a_{31}(\mathbf{v})\}+[a_{41}(\mathbf{u}),a_{31}(\mathbf{v})]\epsilon(\zeta))]+[1\leftrightarrow 2]\}\\
&+\{[e^{i(\frac{1}{2}\mathbf{q}\cdot\boldsymbol{\sigma}+\mathbf{q}'\cdot\mathbf{s})}(\{a_{41}(\mathbf{u}),a_{42}(\mathbf{v})\}+[a_{41}(\mathbf{u}),a_{42}(\mathbf{v})]\epsilon(z))]+[3\leftrightarrow 4]\}\\
&+\{[2e^{i\mathbf{q}'\cdot(\mathbf{s}-\boldsymbol{\sigma})}a_{31}(\mathbf{u})a_{42}(\mathbf{v})]+[3\leftrightarrow 4]\})d^2q', \qquad (5.17)
\end{aligned}
$$

$$
\begin{aligned}
F_3(\mathbf{q},k,\mathbf{r},\boldsymbol{\rho})=&-(3\pi^2k^2)^{-1}\int([e^{i(\mathbf{q}'\cdot\mathbf{s}+\mathbf{q}''\cdot\boldsymbol{\sigma})}(\{\{a_{41}(\mathbf{w}),a_{31}(\mathbf{u}-\mathbf{w})\},a_{42}(\mathbf{v})\}+\{a_{41}(\mathbf{w}),a_{31}(\mathbf{u}-\mathbf{w})a_{42}(\mathbf{v})\}[1+\epsilon(z)\epsilon(\zeta)]\\
&+[\{a_{41}(\mathbf{w}),a_{31}(\mathbf{u}-\mathbf{w})\},a_{42}(\mathbf{v})]\epsilon(z)+[a_{41}(\mathbf{w}),a_{31}(\mathbf{u}-\mathbf{w})a_{42}(\mathbf{v})][\epsilon(z)+\epsilon(\zeta)]\\
&+[\{a_{41}(\mathbf{w}),a_{42}(\mathbf{v})\},a_{31}(\mathbf{u}-\mathbf{w})]\epsilon(\zeta)+[[a_{41}(\mathbf{w}),a_{31}(\mathbf{u}-\mathbf{w})],a_{42}(\mathbf{v})]\epsilon(z)\epsilon(\zeta))]\\
&+[1\leftrightarrow 2]+[3\leftrightarrow 4]+[1\leftrightarrow 2, 3\leftrightarrow 4])d^2q'd^2q'', \qquad (5.18)
\end{aligned}
$$

[22] From this point forward we shall often suppress the arguments k and $\frac{1}{2}k$ since, for a given experiment, they remain constant.

and

$$F_4(\mathbf{q},k,\mathbf{r},\boldsymbol{\varrho})=-\frac{i}{2\pi^3k^3}\int\Big(\{[e^{-i(\mathbf{q}'\cdot\mathbf{s}+\mathbf{q}''\cdot\boldsymbol{\sigma})}(\{a_{41}(\mathbf{x}-\mathbf{w})a_{42}(\mathbf{u}-\mathbf{x}),a_{31}(\mathbf{v}+\mathbf{w}-\mathbf{x})a_{32}(\mathbf{x})\}$$
$$+[a_{41}(\mathbf{x}-\mathbf{w})a_{42}(\mathbf{u}-\mathbf{x}),a_{31}(\mathbf{v}+\mathbf{w}-\mathbf{x})a_{32}(\mathbf{x})]\epsilon(\zeta))]+[1\leftrightarrow 2]\}$$
$$+\{e^{-i(\mathbf{q}'\cdot\mathbf{s}+\mathbf{q}''\cdot\boldsymbol{\sigma})}[a_{41}(\mathbf{x}-\mathbf{w})a_{31}(\mathbf{v}+\mathbf{w}-\mathbf{x}),a_{42}(\mathbf{u}-\mathbf{x})a_{32}(\mathbf{x})]\epsilon(z)+3\leftrightarrow 4\}$$
$$+\{e^{-i(\mathbf{q}'\cdot\mathbf{s}+\mathbf{q}''\cdot\boldsymbol{\sigma})}[a_{41}(\mathbf{x}-\mathbf{w})a_{42}(\mathbf{u}-\mathbf{x}),a_{31}(\mathbf{v}+\mathbf{w}-\mathbf{x})a_{32}(\mathbf{x})]\epsilon(z)\epsilon(\zeta)+(1\leftrightarrow 2,\,3\leftrightarrow 4)\}\Big)d^2q'd^2q''d^2q'''. \quad (5.19)$$

In these expressions **u**, **v**, **w**, and **x** are defined by

$$\mathbf{u}=\tfrac{1}{2}\mathbf{q}+\mathbf{q}', \quad (5.20)$$

$$\mathbf{v}=\tfrac{1}{2}\mathbf{q}-\mathbf{q}', \quad (5.21)$$

$$\mathbf{w}=\mathbf{q}'+\mathbf{q}'', \quad (5.22)$$

and

$$\mathbf{x}=\mathbf{q}'+\mathbf{q}''+\mathbf{q}'''. \quad (5.23)$$

Equations (5.1), (5.14), (5.15), and (5.17)–(5.19) are the general expressions we obtain for the deuteron-deuteron scattering amplitude operator in terms of the basic nucleon-nucleon amplitudes and the initial and final states of the deuterons.

The effects of single, double, triple, and quadruple interactions have been separated by means of Eqs. (5.15) and (5.17)–(5.19). Each of these equations gives the contribution arising from a particular degree of multiple collision. Single collisions are described by F_1, double collisions by F_2, etc. Each of these expressions may be analyzed further according to *which* nucleons are involved in the collision and the *order* in which the interactions take place. The resulting terms may be characterized in part by the different combinations of two-body interactions between incident and target nucleons which can be formed under the high-energy small-angle scattering assumptions that all collisions involve small momentum transfers and that the relative instantaneous configurations of the nucleons within each deuteron do not change appreciably during the collision.

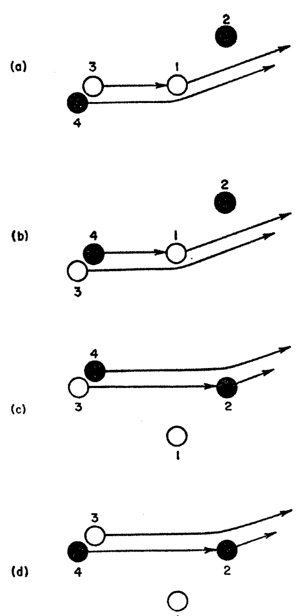

FIG. 1. Schematic representation of single-collision processes which contribute to deuteron-deuteron scattering. The particles are labelled in their initial configurations, and the positions of the circles indicate instantaneous positions of these particles. Particles 1 and 2 belong to the target deuteron, and particles 3 and 4 to the incident deuteron.

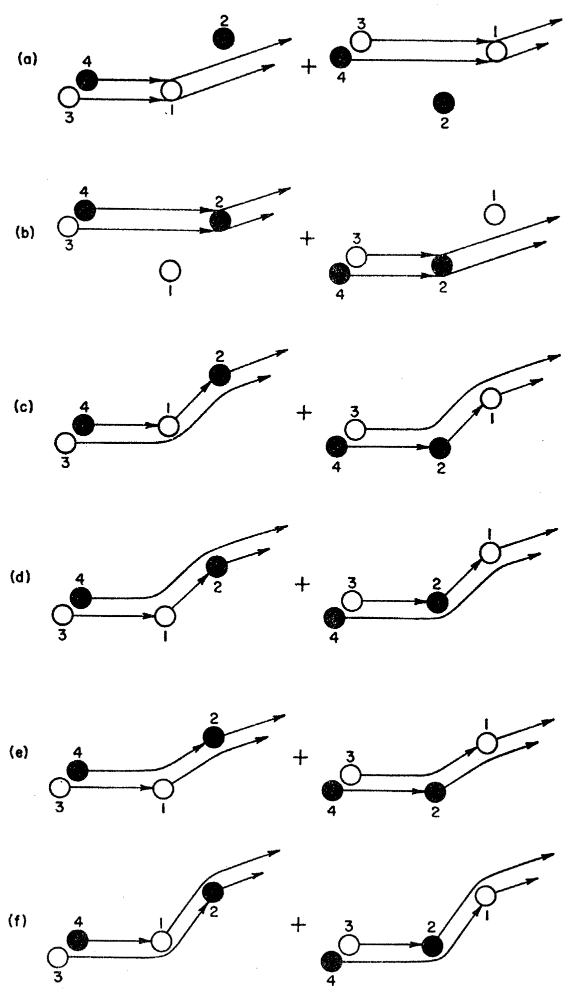

FIG. 2. Schematic representation of double-collision processes which contribute to deuteron-deuteron scattering. The particles are again labelled in their initial configurations, and the positions of the circles indicate instantaneous positions of these particles. Particles 1 and 2 belong to the target deuteron, and particles 3 and 4 to the incident deuteron.

The contributions to the scattering amplitude operator which arise when the individual nucleon-nucleon *single* interactions are considered separately are given by the four terms comprising Eq. (5.15). Each term represents a possible two-particle collision which may be obtained by considering a nucleon in the incident deuteron as the incident particle and a nucleon in the target deuteron as the target particle. These kinds of collisions are illustrated schematically in Fig. 1 (a)–(d). In this figure, and in the three to follow, we have labelled the particles in their initial configurations, and the positions of the circles representing them are meant to indicate instantaneous positions of these particles. Particles 1 and 2 belong to the target deuteron, and particles 3 and 4 to the incident deuteron. The positions of the two arrowheads at the ends of the paths of particles 3 and 4 represent instantaneous positions of these particles after the collision. The matrix elements of the four terms in Eq. (5.15), taken between appropriate initial and final states of the incident and target deuterons, add coherently to form the amplitudes obtained in the usual single-scattering impulse approximation.

Although we do not explicitly show in Figs. 1–4 the states of the nucleons after the collisions, they may of course differ from their initial states. For example, charge-exchange reactions may take place, so that the charge states of various nucleons may change.

Equation (5.17) represents an approximate expression for the double-scattering amplitude operator. Each term in this equation results from a possible type of double interaction. These types of collision processes are illustrated schematically in Fig. 2. In each of the parts (a)–(f) in this figure two diagrams are shown since the term they represent includes double collisions in which the order of the collisions may be reversed. Thus in Fig. 2(a), for example, the first diagram represents collisions in which first particle 4 interacts with particle 1, and then particle 3 interacts with particle 1. The second diagram in Fig. 2(a) represents collisions in which first particle 3 interacts with particle 1, and then particle 4 interacts with particle 1. We should note that only one diagram is actually necessary for each of Figs. 2(e) and 2(f) since Γ_{42} and Γ_{31} generally commute, as do Γ_{41} and Γ_{32}. We have included a redundant diagram in each of these figures mainly for purposes of symmetry. Note that parts (e) and (f) are qualitatively different from parts (a)–(d).

Equation (5.18) is an approximate expression for the triple-scattering amplitude operator. Each of the terms in this expression results from a possible type of triple interaction. These types are illustrated in Fig. 3. In Fig. 3(a) four diagrams are shown since the term represented includes triple collisions in which the order of the individual collisions may be different. Two additional diagrams may be drawn in Fig. 3(a). These would represent the sequence of collisions 41, 42, 31 and 31, 42, 41. However, since Γ_{31} and Γ_{42} generally commute, these diagrams are equivalent to the first and fourth diagrams, respectively, in Fig. 3(a). The latter two diagrams represent the sequence of collisions 41, 31, 42 and 42, 31, 41. The terms in Eq. (5.19) which involve different pairs of particles in the triple collision than those represented by Fig. 3(a) may be illustrated schematically by interchanging the nucleons in one or both of the deuterons, as indicated in parts (b)–(d) of Fig. 3.

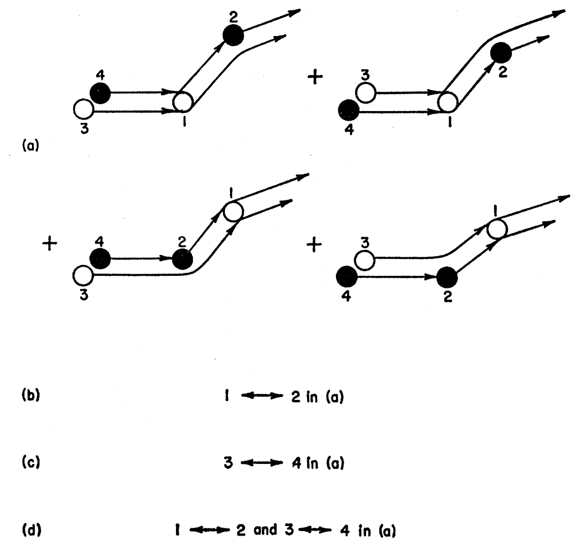

FIG. 3. Schematic representation of triple-collision processes which contribute to deuteron-deuteron scattering. The particles are labelled in their initial configurations, and the positions of the circles indicate instantaneous positions of these particles. Particles 1 and 2 belong to the target deuteron, and particles 3 and 4 to the incident deuteron. The notation "1 ↔ 2 in (a)" denotes the diagram which is obtained upon interchange of particles 1 and 2 in part (a), with corresponding meanings for the notation given in parts (c) and (d).

Equation (5.20) represents an approximate expression for the quadruple-scattering amplitude. It arises from collisions in which each nucleon in the incident deuteron interacts with both nucleons in the target deuteron. These interactions are illustrated schematically in Fig. 4. Additional diagrams may be drawn but, as we have

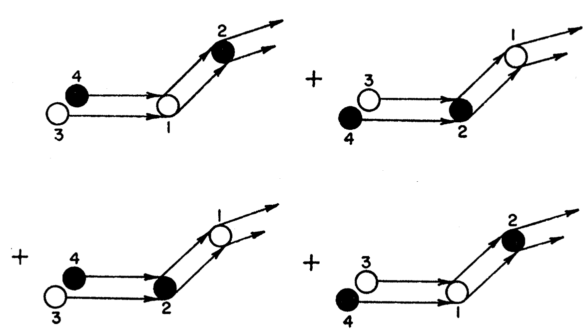

FIG. 4. Schematic representation of quadruple-collision processes which contribute to deuteron-deuteron scattering. The particles are labelled in their initial configurations, and the positions of the circles indicate instantaneous positions of these particles. Particles 1 and 2 belong to the target deuteron, and particles 3 and 4 to the incident deuteron.

pointed out in our discussion of Fig. 3, they would be equivalent to those already shown. The number of diagrams needed to represent possible quadruple interactions is greatly restricted by the high-energy small-angle approximations mentioned earlier. Thus, for example, the sequence of collisions 32, 31, 41, 42 is omitted since the sequence 32, 31 requires the target deuteron to be in an instantaneous configuration such that $z>0$ whereas the sequence 41, 42 requires the instantaneous configuration to be such that $z<0$.

Thus we see how, in this high-energy small-angle approximation, the effects of multiple interactions are explicitly included in the formalism. Quintuple and higher-order scattering effects are absent from the approximation since they cannot occur when all collisions are confined to small scattering angles.

VI. DEUTERON-DEUTERON TOTAL CROSS SECTION

To consider the effects of multiple interactions on the total cross section in a quantitative manner, we define a total cross-section defect $\delta\sigma$ to be the difference between the sum of the four free nucleon-nucleon total cross sections at momentum $\frac{1}{2}\hbar k$ and the deuteron-deuteron cross section $\sigma_{dd}(k)$ at momentum $\hbar k$, and write

$$\sigma_{dd}(k)=\sigma_{nn}(\tfrac{1}{2}k)+\sigma_{pn}(\tfrac{1}{2}k)+\sigma_{pp}(\tfrac{1}{2}k)+\sigma_{np}(\tfrac{1}{2}k)-\delta\sigma\,, \quad (6.1)$$

where the subscripts n and p in this expression refer to neutron and proton, respectively. (If we assume charge symmetry of nuclear forces, then $\sigma_{nn}=\sigma_{pp}$ and $\sigma_{pn}=\sigma_{np}$.) If the effects of multiple interactions were neglected $\delta\sigma$ would be zero and the deuteron-deuteron total cross section would be simply given by the sum of the four individual free nucleon-nucleon total cross sections, a result given in a simple impulse approximation.

It will be instructive to analyze $\delta\sigma$ in terms of contributions $\delta\sigma_2$, $\delta\sigma_3$, and $\delta\sigma_4$ which arise from double, triple, and quadruple interactions, respectively. We therefore shall write

$$\delta\sigma=\delta\sigma_2+\delta\sigma_3+\delta\sigma_4. \quad (6.2)$$

The deuteron-deuteron total cross section may be calculated by applying to Eqs. (5.1), (5.14), (5.15), and (5.17)–(5.19) the optical theorem, which relates the total cross section to the imaginary part of the forward elastic scattering amplitude. The total cross section $\sigma_{dd}(k)$ is given by

$$\sigma_{dd}(k)=(4\pi/k)\,\mathrm{Im}F_{iiii}(0,k,\mathbf{s},\boldsymbol{\sigma}). \quad (6.3)$$

The free nucleon-nucleon total cross sections $\sigma_{pt}(\frac{1}{2}k)$, where p and t represent a nucleon in the incident deuteron and a nucleon in the target deuteron, respectively, are given similarly by

$$\sigma_{pt}(\tfrac{1}{2}k)=(8\pi/k)\,\mathrm{Im}\langle ii|a_{pt}(0,\tfrac{1}{2}k)|ii\rangle. \quad (6.4)$$

To evaluate the cross-section defect we must make use of integrals of the form

$$\int e^{i\mathbf{q}\cdot\mathbf{s}}|\psi_i(\mathbf{r})|^2d\mathbf{r}\,,$$

where $\psi_i(\mathbf{r})$ is the configuration-space wave function of the deuteron ground state. Since $\mathbf{s}$ is the component of the coordinate $\mathbf{r}$ lying parallel to the plane which contains the momentum transfers $\mathbf{q}$, this integral is equivalent to the expression

$$S(\mathbf{q})=\int e^{i\mathbf{q}\cdot\mathbf{r}}|\psi_i(\mathbf{r})|^2d\mathbf{r}=S(-\mathbf{q})\,, \quad (6.5)$$

which we recognize to be the form factor of the deuteron ground state.

We notice immediately that the diagonal matrix element in the deuteron ground state of all terms in Eqs. (5.17)–(5.19) which are linear in $\epsilon(z)$ or $\epsilon(\zeta)$ vanish. If we set q equal to zero in these expressions in Eq. (5.15) and use the optical theorem (6.3) and (6.4) we obtain

$$\delta\sigma_2=-\frac{32}{k^2}\mathrm{Re}\langle ii|\int[S(\mathbf{q})a_{41}(\mathbf{q})a_{31}(-\mathbf{q})+\tfrac{1}{2}S^2(\mathbf{q})a_{31}(\mathbf{q})a_{42}(-\mathbf{q})]d^2q|ii\rangle\,, \quad (6.6)$$

$$\delta\sigma_3=\frac{16}{3\pi k^3}\mathrm{Im}\langle ii|\int S(\mathbf{q})S(\mathbf{q}')\times[\{\{a_{41}(\mathbf{q}+\mathbf{q}'),a_{31}(-\mathbf{q}')\},a_{42}(-\mathbf{q})\}+\{a_{41}(\mathbf{q}+\mathbf{q}'),a_{31}(-\mathbf{q}')a_{42}(-\mathbf{q})\}]d^2qd^2q'|ii\rangle\,, \quad (6.7)$$

and

$$\delta\sigma_4=\frac{8}{\pi^2k^4}\mathrm{Re}\langle ii|\int S(\mathbf{q})S(\mathbf{q}')a_{41}(\mathbf{q}'')a_{42}(-\mathbf{q}'-\mathbf{q}'')\times a_{31}(-\mathbf{q}-\mathbf{q}'')a_{32}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')d^2qd^2q'd^2q''|ii\rangle. \quad (6.8)$$

In obtaining these expressions we have made use of the antisymmetry of the total deuteron ground-state wave functions with respect to interchange of particles 1 and 2 and with respect to interchange of particles 3 and 4.

It will be convenient and useful to further analyze the double-scattering correction $\delta\sigma_2$ into two distinct contributions. The first, denoted by $\delta\sigma_{21}$, is given by that part of $\delta\sigma_2$ which contains the factor $S(q)$ in the integrand. The second, denoted by $\delta\sigma_{22}$, is given by that part of $\delta\sigma_2$ which contains the factor $S^2(q)$ in the integrand. We may therefore write

$$\delta\sigma_2=\delta\sigma_{21}+\delta\sigma_{22}. \quad (6.9)$$

The correction $\delta\sigma_{21}$ corresponds to double collisions in which *both* nucleons in one of the deuterons interact with *only one* of the nucleons in the remaining deuteron, while the other nucleon acts as a "spectator" particle. The correction $\delta\sigma_{22}$ corresponds to double collisions in which one nucleon in the incident deuteron interacts with *only one* nucleon in the target, and the other nucleon in the incident deuteron interacts with *only the other* nucleon in the target, so that all four nucleons take part in the collision, and there are no "spectators." The significance of this latter double collision correction to the total cross section will be discussed more fully in Sec. VIII.

The scattering amplitude operators $a_{pt}(\mathbf{q})$ which describe collisions between nucleons may be dealt with most compactly by writing them in the form

$$a_{pt}(\mathbf{q})=f(\mathbf{q})+\boldsymbol{\tau}_p\cdot\boldsymbol{\tau}_t g(\mathbf{q}), \tag{6.10}$$

where $\boldsymbol{\tau}_p$ and $\boldsymbol{\tau}_t$ are the isotopic spin operators for a nucleon in the projectile and a nucleon in the target, respectively. If we substitute this form in Eqs. (6.6)–(6.8) and evaluate the expectation values of the isotopic spin operators which occur in them, we find

$$\delta\sigma_{21}=-\frac{32}{k^2}\,\mathrm{Re}\int S(q)[f(\mathbf{q})f(-\mathbf{q})-3g(\mathbf{q})g(-\mathbf{q})]d^2q, \tag{6.11}$$

$$\delta\sigma_{22}=-\frac{16}{k^2}\,\mathrm{Re}\int S^2(q)[f(\mathbf{q})f(-\mathbf{q})+3g(\mathbf{q})g(-\mathbf{q})]d^2q, \tag{6.12}$$

$$\delta\sigma_3=\frac{32}{\pi k^3}\int S(\mathbf{q})S(\mathbf{q}')\,\mathrm{Im}[f(\mathbf{q}+\mathbf{q}')f(-\mathbf{q}')\times f(-\mathbf{q})-5g(\mathbf{q}+\mathbf{q}')f(-\mathbf{q}')g(-\mathbf{q})+2f(\mathbf{q}+\mathbf{q}')g(-\mathbf{q}')g(-\mathbf{q})]d^2qd^2q', \tag{6.13}$$

and

$$\delta\sigma_4=\frac{8}{\pi^2k^4}\int S(\mathbf{q})S(\mathbf{q}')\,\mathrm{Re}[f(\mathbf{q}'')f(-\mathbf{q}'-\mathbf{q}'')f(-\mathbf{q}-\mathbf{q}'')\times f(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')+21g(\mathbf{q}'')g(-\mathbf{q}'-\mathbf{q}'')g(-\mathbf{q}-\mathbf{q}'')\times g(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')+6f(\mathbf{q}'')g(-\mathbf{q}'-\mathbf{q}'')g(-\mathbf{q}-\mathbf{q}'')\times f(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')-12f(\mathbf{q}'')f(-\mathbf{q}'-\mathbf{q}'')g(-\mathbf{q}-\mathbf{q}'')\times g(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')]d^2qd^2q'd^2q''. \tag{6.14}$$

We wish to note that an alternative way of writing the deuteron-deuteron cross section which will be particularly useful for comparing the theory with experiment is to express $\sigma_{dd}(k)$ in terms of the nucleon-*deuteron* total cross section $\sigma_{Nd}(\frac{1}{2}k)$ plus correction terms. (We assume charge symmetry of nuclear forces so that $\sigma_{pd}=\sigma_{nd}=\sigma_{Nd}$.) The expression for σ_{Nd} is[7]

$$\sigma_{Nd}(\tfrac{1}{2}k)=\sigma_{pp}(\tfrac{1}{2}k)+\sigma_{np}(\tfrac{1}{2}k)-\tfrac{1}{4}\delta\sigma_{21}. \tag{6.15}$$

The deuteron-deuteron total cross section may then be written as

$$\sigma_{dd}(k)=2\sigma_{Nd}(\tfrac{1}{2}k)-\delta\sigma', \tag{6.16}$$

which may be regarded as a defining equation for the cross-section defect $\delta\sigma'$. In analogy with Eqs. (6.2) and (6.9), we may write

$$\delta\sigma'=\delta\sigma_2'+\delta\sigma_3+\delta\sigma_4 \tag{6.17}$$

and

$$\delta\sigma_2'=\delta\sigma_{21}'+\delta\sigma_{22}, \tag{6.18}$$

where

$$\delta\sigma_{21}'=\tfrac{1}{2}\delta\sigma_{21}. \tag{6.19}$$

Equations (6.16)–(6.19) have an advantage over Eqs. (6.1) and (6.2) since the main contribution to σ_{dd} is written as $2\sigma_{Nd}$, which is *directly* measurable. Furthermore, the magnitude of the double-scattering correction $\delta\sigma_2'$ is generally substantially smaller than the magnitude of the correction $\delta\sigma_2$. Consequently any uncertainty in the evaluation of $\delta\sigma_2'$ leads to a smaller uncertainty in the calculated value of σ_{dd} than does the corresponding uncertainty in the evaluation of $\delta\sigma_2$.

The contributions to the deuteron total cross section which arise from double, triple, and quadruple interactions are represented by negatives of $\delta\sigma_2'$ (or $\delta\sigma_2$), $\delta\sigma_3$, and $\delta\sigma_4$. The signs of these contributions depend mainly upon the phases of the free nucleon-nucleon elastic scattering amplitudes. In the limiting case of purely imaginary nucleon-nucleon elastic amplitudes and no charge exchange, for example, the contributions to the deuteron cross section which result from double and quadruple interactions would both be negative, whereas the contribution from triple interactions would be positive.

VII. CROSS SECTION IN TERMS OF NEUTRON AND PROTON ELASTIC SCATTERING AMPLITUDES

In order to compare the measured deuteron-deuteron cross section with the expression which we have derived for it, it is convenient to rewrite the theoretical result in terms of the experimentally measured amplitudes for nucleon-nucleon scattering. If we assume charge symmetry of nuclear forces, so that $f_{pp}=f_{nn}$ and $f_{pn}=f_{np}$, then we may express all nucleon-nucleon elastic amplitudes in terms of two independent ones, say f_{pp} and f_{np}. The amplitudes f and g which we have used in constructing the expression for the deuteron-deuteron total cross section are related to the directly observable amplitudes f_{pp} and f_{np} via the equations

$$f(\mathbf{q})=\tfrac{1}{2}[f_{pp}(\mathbf{q})+f_{np}(\mathbf{q})] \tag{7.1}$$

and

$$g(\mathbf{q})=\tfrac{1}{2}[f_{pp}(\mathbf{q})-f_{np}(\mathbf{q})]. \tag{7.2}$$

Furthermore, $g(\mathbf{q})$ is simply related to the charge-exchange amplitude $f_c(\mathbf{q})$ by

$$g(\mathbf{q})=\tfrac{1}{2}f_c(\mathbf{q}). \tag{7.3}$$

The possibility of finding the amplitude g in two ways, either through direct measurement of the charge-exchange amplitude or by taking the difference of the pp and np elastic amplitudes, leads to a variety of useful ways of expressing our results.

We may, for example, write the cross-section defects for the deuteron-deuteron total cross section, given by Eqs. (6.11)–(6.14), in terms of the observable amplitudes in the form

$$\delta\sigma_{21}=-\frac{32}{k^2}\,\mathrm{Re}\int S(\mathbf{q})[f_{np}(\mathbf{q})f_{pp}(-\mathbf{q})-\tfrac{1}{2}f_c(\mathbf{q})f_c(-\mathbf{q})]d^2q\,, \tag{7.4}$$

$$\delta\sigma_{22}=-\frac{8}{k^2}\,\mathrm{Re}\int S^2(\mathbf{q})[f_{pp}(\mathbf{q})f_{pp}(-\mathbf{q})+f_{np}(\mathbf{q})f_{np}(-\mathbf{q})+f_c(\mathbf{q})f_c(-\mathbf{q})]d^2q\,, \tag{7.5}$$

$$\begin{aligned}\delta\sigma_3=\frac{2}{\pi k^3}\,\mathrm{Im}\int S(\mathbf{q})S(\mathbf{q}')\{&6[f_{pp}(\mathbf{q}+\mathbf{q}')f_{pp}(-\mathbf{q}')f_{np}(-\mathbf{q})+f_{np}(\mathbf{q}+\mathbf{q}')f_{np}(-\mathbf{q}')f_{pp}(-\mathbf{q})]\\&+2[f_{pp}(\mathbf{q}+\mathbf{q}')f_{np}(-\mathbf{q}')f_{np}(-\mathbf{q})+f_{np}(\mathbf{q}+\mathbf{q}')f_{pp}(-\mathbf{q}')f_{pp}(-\mathbf{q})]+5[f_{pp}(\mathbf{q}+\mathbf{q}')+f_{np}(\mathbf{q}+\mathbf{q}')]\\&\times f_c(-\mathbf{q}')f_c(-\mathbf{q})-9f_c(\mathbf{q}+\mathbf{q}')f_c(-\mathbf{q}')[f_{pp}(-\mathbf{q})+f_{np}(-\mathbf{q})]\}d^2qd^2q'\,,\end{aligned} \tag{7.6}$$

and

$$\begin{aligned}\delta\sigma_4=(\pi^2k^4)^{-1}\,\mathrm{Re}\int S(\mathbf{q})S(\mathbf{q}')[&4f_{pp}(\mathbf{q}'')f_{pp}(-\mathbf{q}'-\mathbf{q}'')f_{np}(-\mathbf{q}-\mathbf{q}'')f_{np}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')\\&+4f_{pp}(\mathbf{q}'')f_{np}(-\mathbf{q}'-\mathbf{q}'')f_{np}(-\mathbf{q}-\mathbf{q}'')f_{pp}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')+f_{pp}(\mathbf{q}'')f_c(-\mathbf{q}'-\mathbf{q}'')f_c(-\mathbf{q}-\mathbf{q}'')f_{pp}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')\\&+f_{np}(\mathbf{q}'')f_c(-\mathbf{q}'-\mathbf{q}'')f_c(-\mathbf{q}-\mathbf{q}'')f_{np}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')-10f_c(\mathbf{q}'')f_c(-\mathbf{q}'-\mathbf{q}'')f_{np}(-\mathbf{q}-\mathbf{q}'')f_{pp}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')\\&+7f_c(\mathbf{q}'')f_c(-\mathbf{q}'-\mathbf{q}'')f_c(-\mathbf{q}-\mathbf{q}'')f_c(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')]d^2qd^2q'd^2q''.\end{aligned} \tag{7.7}$$

In Eq. (7.4) a contribution $f_{np}f_{pp}-\frac{1}{2}f_cf_c$ arises from each of the four diagrams in Figs. 2(a)–2(d). The term $f_{pp}f_{pp}$ in Eq. (7.5) corresponds to Fig. 2(e), and the term $f_{np}f_{np}+f_cf_c$ corresponds to Fig. 2(f). In Eq. (7.6) the contributions which are bilinear in f_c could have been written somewhat more symmetrically by using a term of the form $f_cf_{pp}f_c$. However, such a sequence of collisions in deuteron-deuteron elastic scattering in fact is not possible since as a result of the first charge-exchange process the charge states of the nucleons of the incident two-particle system are different from those of the target nucleons. Consequently only a *pn* or *np* collision (not a *pp* or *nn* collision) may occur as the second scattering process.

Similarly, Eq. (7.7) could have been written more symmetrically by using terms such as $f_{pp}f_{pp}f_{pp}f_{pp}$, for example. However, such a sequence of collisions in *d-d* elastic scattering is not possible within the framework of the approximation that each incident nucleon interacts no more than once with a given target nucleon.

We may alternatively write the cross-section defect completely in terms of scattering amplitudes for charge-preserving processes. If we note that these amplitudes can only depend on the magnitude, and not on the direction, of the momentum transfers, we may write

$$\delta\sigma_{21}=-\frac{32}{k^2}\,\mathrm{Re}\int S(q)[2f_{np}(q)f_{pp}(q)-\tfrac{1}{2}f_{np}{}^2(q)-\tfrac{1}{2}f_{pp}{}^2(q)]d^2q\,, \tag{7.8}$$

$$\delta\sigma_{22}=-\frac{16}{k^2}\,\mathrm{Re}\int S^2(q)[f_{pp}{}^2(q)+f_{np}{}^2(q)-f_{np}(q)f_{pp}(q)]d^2q\,, \tag{7.9}$$

$$\begin{aligned}\delta\sigma_3=\frac{8}{\pi k^3}\,\mathrm{Im}\int S(q)S(q')[&5f_{np}(\mathbf{q}+\mathbf{q}')f_{pp}(q')f_{pp}(q)+5f_{pp}(\mathbf{q}+\mathbf{q}')f_{np}(q')f_{np}(q)\\&-3f_{pp}(\mathbf{q}+\mathbf{q}')f_{pp}(q')f_{np}(q)-3f_{np}(\mathbf{q}+\mathbf{q}')f_{np}(q')f_{pp}(q)]d^2qd^2q'\,,\end{aligned} \tag{7.10}$$

and

$$\begin{aligned}\delta\sigma_4=\frac{8}{\pi^2k^4}\,\mathrm{Re}\int S(q)S(q')[&5f_{np}(q'')f_{pp}(\mathbf{q}'+\mathbf{q}'')f_{pp}(\mathbf{q}+\mathbf{q}'')f_{np}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')\\&+4f_{pp}(q'')f_{pp}(\mathbf{q}'+\mathbf{q}'')f_{np}(\mathbf{q}+\mathbf{q}'')f_{np}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')-5f_{pp}(q'')f_{pp}(\mathbf{q}'+\mathbf{q}'')f_{pp}(\mathbf{q}+\mathbf{q}'')f_{np}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')\\&-5f_{np}(q'')f_{np}(\mathbf{q}'+\mathbf{q}'')f_{np}(\mathbf{q}+\mathbf{q}'')f_{pp}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')+f_{pp}(q'')f_{pp}(\mathbf{q}'+\mathbf{q}'')f_{pp}(\mathbf{q}+\mathbf{q}'')f_{pp}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')\\&+f_{np}(q'')f_{np}(\mathbf{q}'+\mathbf{q}'')f_{np}(\mathbf{q}+\mathbf{q}'')f_{np}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')]d^2qd^2q'd^2q''.\end{aligned} \tag{7.11}$$

VIII. ASYMPTOTIC APPROXIMATIONS FOR THE CROSS-SECTION DEFECTS

We have derived the expressions for the deuteron-deuteron cross section in Secs. VI and VII by means of the Glauber approximation. To obtain quantitative estimates of the cross-section defect $\delta\sigma'$ or $\delta\sigma$, a number of integrals must be carried out, some of which involve multiple integrations. In order to allow estimates of the cross-section defect to be easily made, we shall reduce the general results for $\delta\sigma'$ and $\delta\sigma$ to a number of simpler forms by means of the various approximations regarding the deuteron ground state and the nucleon-nucleon elastic scattering amplitudes. We wish to emphasize, however, that *although the resulting expressions will be quite easy to evaluate, greater accuracy would be obtained by explicitly performing the necessary integrals which appear in the general expressions.* We shall do these integrals numerically in the next section.

We begin by considering the form which the cross-section defect takes in the asymptotic limit of the average deuteron radius, i.e., neutron-proton separation, being much larger than the ranges of the high-energy nucleon-nucleon interactions. In this limit the deuteron form factor $S(q)$ decreases from its value of unity at $q=0$ much more rapidly than do the nucleon-nucleon elastic scattering amplitudes. Consequently we may approximate the contributions to the cross-section defect which arise from double and triple interactions directly in terms of the forward elastic scattering amplitudes $a_{pt}(0)$ and integrals of the deuteron ground-state form factor and of the square of the form factor. The contribution from quadruple scattering is reduced to an integral over a single momentum transfer variable q. The integrals involving the form factors may be written in terms of expectation values, in the deuteron ground state, of functions of the neutron-proton separation r. For spherically symmetric wave functions these relations, which we derive in the Appendix, are

$$\int S(\mathbf{q})d^2q=2\pi\int \psi_i^*(\mathbf{r})\frac{1}{r^2}\psi_i(\mathbf{r})d\mathbf{r} \tag{8.1}$$

$$=2\pi\langle r^{-2}\rangle_d \tag{8.2}$$

and

$$\int S^2(\mathbf{q})d^2q=2\pi\int \psi_i^*(\boldsymbol{\rho})\psi_i^*(\mathbf{r})\frac{1}{2r\rho}\times\ln\left(\frac{r+\rho}{|r-\rho|}\right)\psi_i(\mathbf{r})\psi_i(\boldsymbol{\rho})d\mathbf{r}d\boldsymbol{\rho} \tag{8.3}$$

$$=2\pi\left\langle\left\langle\frac{1}{2r\rho}\ln\left(\frac{r+\rho}{|r-\rho|}\right)\right\rangle_d\right\rangle_d. \tag{8.4}$$

In the asymptotic limit of large deuteron radius, therefore, the contributions $\delta\sigma_{21}^{(A)}$, $\delta\sigma_{22}^{(A)}$, $\delta\sigma_3^{(A)}$, and $\delta\sigma_4^{(A)}$ to the cross-section defect may be written by means of Eqs. (7.4)–(7.7) and (8.1)–(8.4) as

$$\delta\sigma_{21}^{(A)}\approx-(64\pi/k^2)\times\mathrm{Re}[f_{np}(0)f_{pp}(0)-\tfrac{1}{2}f_c^2(0)]\langle r^{-2}\rangle_d, \tag{8.5}$$

$$\delta\sigma_{22}^{(A)}\approx-(16\pi/k^2)\,\mathrm{Re}[f_{pp}^2(0)+f_{np}^2(0)+f_c^2(0)]\times\left\langle\left\langle\frac{1}{2r\rho}\ln\left(\frac{r+\rho}{|r-\rho|}\right)\right\rangle_d\right\rangle_d, \tag{8.6}$$

$$\delta\sigma_3^{(A)}\approx(32/k^3)\,\mathrm{Im}\{2f_{pp}^2(0)f_{np}(0)+2f_{np}^2(0)f_{pp}(0)-f_c^2(0)[f_{pp}(0)+f_{np}(0)]\}(\langle r^{-2}\rangle_d)^2, \tag{8.7}$$

and

$$\delta\sigma_4^{(A)}\approx\frac{4}{k^4}\mathrm{Re}\int\{8f_{pp}^2(q)f_{np}^2(q)+f_c^2(q)\times[f_{pp}^2(q)+f_{np}^2(q)-10f_{pp}(q)f_{np}(q)]+7f_c^4(q)\}d^2q(\langle r^{-2}\rangle_d)^2. \tag{8.8}$$

The various real and imaginary parts of the different products of nucleon-nucleon amplitudes may be expressed in terms of the real and imaginary parts of the individual amplitudes. The imaginary parts of the forward elastic scattering amplitudes may, in turn, be expressed in terms of the nucleon-nucleon total cross sections by means of the optical theorem. If we define α_{lj} to be the ratio of the real to the imaginary parts of the nucleon-nucleon forward elastic scattering amplitudes,

$$\alpha_{lj}=\mathrm{Re}f_{lj}(0)/\mathrm{Im}f_{lj}(0),\quad l,j=n,p \tag{8.9}$$

we obtain for $\delta\sigma_{21}^{(A)}$, $\delta\sigma_{22}^{(A)}$, and $\delta\sigma_3^{(A)}$ the expressions

$$\delta\sigma_{21}^{(A)}\approx\pi^{-1}\{(1-\alpha_{np}\alpha_{pp})\sigma_{np}\sigma_{pp}-\tfrac{1}{2}[(\sigma_{np}-\sigma_{pp})^2-(\alpha_{np}\sigma_{np}-\alpha_{pp}\sigma_{pp})^2]\}\langle r^{-2}\rangle_d, \tag{8.10}$$

$$\delta\sigma_{22}^{(A)}\approx(1/4\pi)\{(1-\alpha_{pp}^2)\sigma_{pp}^2+(1-\alpha_{np}^2)\sigma_{np}^2+[(\sigma_{np}-\sigma_{pp})^2-(\alpha_{np}\sigma_{np}-\alpha_{pp}\sigma_{pp})^2]\}\times\left\langle\left\langle\frac{1}{2r\rho}\ln\left(\frac{r+\rho}{|r-\rho|}\right)\right\rangle_d\right\rangle_d, \tag{8.11}$$

and

$$\begin{aligned}\delta\sigma_3^{(A)}\approx&-(1/16\pi^2)\{2\sigma_{np}\sigma_{pp}^2(1-2\alpha_{np}\alpha_{pp}-\alpha_{pp}^2)\\&+2\sigma_{pp}\sigma_{np}^2(1-2\alpha_{pp}\alpha_{np}-\alpha_{np}^2)-(\sigma_{np}+\sigma_{pp})\\&\times[(\sigma_{np}-\sigma_{pp})^2-(\alpha_{np}\sigma_{np}-\alpha_{pp}\sigma_{pp})^2]\\&+2(\alpha_{np}\sigma_{np}+\alpha_{pp}\sigma_{pp})(\sigma_{np}-\sigma_{pp})\\&\times(\alpha_{np}\sigma_{np}-\alpha_{pp}\sigma_{pp})\}(\langle r^{-2}\rangle_d)^2.\end{aligned} \tag{8.12}$$

The expression (8.8) for $\delta\sigma_4^{(A)}$ may also be simplified if we make the additional assumption that the nucleon-nucleon elastic scattering amplitudes $f_{lj}(q)$ in the integral may be represented by $f_{lj}(0)\exp(-\tfrac{1}{2}A_{lj}q^2)$, where

A_{lj} is a real constant. In that case we obtain the result

$$\delta\sigma_4{}^{(A)} \approx (1/128\pi^3)\{9\sigma_{np}{}^2\sigma_{pp}{}^2[(1-\alpha_{np}\alpha_{pp})^2 - (\alpha_{np}+\alpha_{pp})^2]/(A_{np}+A_{pp}) + \sigma_{pp}{}^4(1-6\alpha_{pp}{}^2+\alpha_{pp}{}^4)/2A_{pp} + \sigma_{np}{}^4(1-6\alpha_{np}{}^2+\alpha_{np}{}^4)/2A_{np} - 5\sigma_{pp}{}^3\sigma_{np}[1-3\alpha_{pp}(\alpha_{pp}+\alpha_{np})+\alpha_{np}\alpha_{pp}{}^3]/(\tfrac{1}{2}A_{np}+\tfrac{3}{2}A_{pp}) - 5\sigma_{np}{}^3\sigma_{pp}[1-3\alpha_{np}(\alpha_{np}+\alpha_{pp}) + \alpha_{pp}\alpha_{np}{}^3]/(\tfrac{1}{2}A_{pp}+\tfrac{3}{2}A_{np})\}(\langle r^{-2}\rangle_d)^2. \quad (8.13)$$

Equations (8.10)–(8.12) require only np and pp total cross sections, the ratio of the real to the imaginary part of the np and pp forward scattering amplitudes, and a deuteron ground-state wave function so that the various expectation values may be evaluated. We shall give estimates for these expectation values in the next section. For $\delta\sigma_4$ the slopes of the pp and np forward elastic diffraction peaks are also needed.

In the high-energy limit we may expect the nucleon-nucleon elastic scattering amplitude to be purely imaginary (or nearly so) in the forward direction. If $\alpha_{np}{}^2$ and $\alpha_{pp}{}^2$ are negligibly small compared to unity, Eqs. (8.10)–(8.13) reduce to

$$\delta\sigma_{21}{}^{(A)} \approx \pi^{-1}[\sigma_{np}\sigma_{pp} - \tfrac{1}{2}(\sigma_{np}-\sigma_{pp})^2]\langle r^{-2}\rangle_d, \quad (8.14)$$

$$\delta\sigma_{22}{}^{(A)} \approx (1/4\pi)[\sigma_{pp}{}^2+\sigma_{np}{}^2+(\sigma_{np}-\sigma_{pp})^2] \times\left\langle\left\langle\frac{1}{2r\rho}\ln\left(\frac{r+\rho}{|r-\rho|}\right)\right\rangle_d\right\rangle_d, \quad (8.15)$$

$$\delta\sigma_3{}^{(A)} \approx -(1/16\pi^2)(\sigma_{np}+\sigma_{pp}) \times[2\sigma_{np}\sigma_{pp} - (\sigma_{np}-\sigma_{pp})^2](\langle r^{-2}\rangle_d)^2, \quad (8.16)$$

and

$$\delta\sigma_4{}^{(A)} \approx \frac{1}{128\pi^3}\left(\frac{9\sigma_{np}{}^2\sigma_{pp}{}^2}{A_{np}+A_{pp}} + \frac{\sigma_{pp}{}^4}{2A_{pp}} + \frac{\sigma_{np}{}^4}{2A_{np}} - \frac{5\sigma_{pp}{}^3\sigma_{np}}{\tfrac{1}{2}A_{np}+\tfrac{3}{2}A_{pp}} - \frac{5\sigma_{np}{}^3\sigma_{pp}}{\tfrac{1}{2}A_{pp}+\tfrac{3}{2}A_{np}}\right)(\langle r^{-2}\rangle_d)^2. \quad (8.17)$$

Of course other combinations of simplifying assumptions may be made, and we shall not list here all the possibilities.

The contributions to the asymptotic cross-section defect which arise from triple and quadruple interactions contain as a factor the *square* of the expectation value, in the deuteron ground state, of the inverse-square neutron-proton separation. Therefore in the asymptotic limit of very large deuteron radius $\delta\sigma_3{}^{(A)}$ and $\delta\sigma_4{}^{(A)}$ are negligibly small in magnitude and the asymptotic cross-section defect $\delta\sigma^{(A)}$ is approximately given by $\delta\sigma^{(A)} \approx \delta\sigma_{21}{}^{(A)} + \delta\sigma_{22}{}^{(A)}$ and $\delta\sigma'^{(A)}$ is approximately given by $\delta\sigma'^{(A)} \approx \frac{1}{2}\delta\sigma_{21}{}^{(A)} + \delta\sigma_{22}{}^{(A)}$.

The double-collision correction $\delta\sigma_{21}{}^{(A)}$ given by Eq. (8.10) is precisely four times the asymptotic result obtained for the *single-particle*–deuteron cross-section defect.[7] This is not unexpected since it corrects for double collisions of the type for which one nucleon in one of the deuterons interacts with *each* nucleon in the deuteron, i.e., for double collisions of the same type as occur in particle-deuteron collisions. In *d-d* collisions there are four times as many ways in which these collisions may occur. On the other hand the terms in Eq. (8.10) represent processes in which one nucleon in the incident deuteron interacts with one nucleon in the target and the other nucleon in the incident deuteron interacts with the other nucleon in the target. Since the relative neutron-proton separation of *each* deuteron is important in such processes, the wave function for each deuteron appears in these terms. The origin of the two terms may be illustrated by considering the deuteron absorption cross section for purely absorptive nucleon-nucleon interactions, so that $\alpha_{np}=\alpha_{pp}=0$, and neglecting charge-exchange effects. If we merely equate the deuteron-deuteron absorption cross section to the sum of the four free nucleon-nucleon absorption cross sections we must of course correct for the shadow effect described by Glauber[23] for the single-particle–deuteron case. For the absorption cross section, twice this correction is given by setting $\alpha_{np}=\alpha_{pp}=0$ in Eq. (8.10) with the nucleon-nucleon total cross sections σ_{lj} replaced by the corresponding absorption cross sections $(\sigma_{lj})_{\text{abs}}$. But in addition, we must now correct for counting certain double-absorption processes *twice.* For example, $(\sigma_{nn})_{\text{abs}}$ corresponds to processes in which the incident neutron is absorbed by the target neutron while the incident proton may or may not be absorbed by the target proton. Similarly $(\sigma_{pp})_{\text{abs}}$ corresponds to processes in which the incident proton is absorbed by the target neutron. These processes are not mutually exclusive. Each of these two cross sections contain contributions from those processes in which the incident neutron is absorbed by the target neutron *and* the incident proton is absorbed by the target proton. To correct for counting these processes twice we must subtract a term of the form

$$\frac{1}{2\pi}\left\langle\frac{1}{2r\rho}\ln\frac{r+\rho}{|r-\rho|}\right\rangle(\sigma_{nn})_{\text{abs}}(\sigma_{pp})_{\text{abs}}. \quad (8.18)$$

A similar term with $(\sigma_{nn})_{\text{abs}}(\sigma_{pp})_{\text{abs}}$ replaced by $(\sigma_{np})_{\text{abs}}(\sigma_{pn})_{\text{abs}}$ occurs for processes in which the incident neutron is absorbed by the target proton and the incident proton is absorbed by the target neutron.

The two correction terms for the absorption cross section may also be obtained from a geometrical calculation. We again assume purely absorptive nucleon-nucleon interactions and omit the effects of charge exchange. What is determined in such a calculation is the probability that given one nucleon in the incident deuteron is absorbed by one nucleon in the target, the other nucleon in the incident deuteron is absorbed by the other nucleon in the target. When this probability is multiplied by the cross section for the absorption of

[23] R. J. Glauber, Phys. Rev. **100**, 242 (1955).

one of the incident nucleons by one of the target nucleons (which is the condition under which the probability is being calculated), the result which is obtained is

$$(\sigma_{mk})_{\rm abs}\int|\psi_i(\mathbf{s},z)|^2d\mathbf{s}dz\int_{\mathcal{S}}d\boldsymbol{\sigma}\int_{-\infty}^{\infty}|\psi_i(\boldsymbol{\sigma},\zeta)|^2d\zeta\,, \quad (8.19)$$

in which $\mathcal{S}$ is the region $|\mathbf{s}-\boldsymbol{\sigma}|\leq[(\sigma_{lj})_{\rm abs}/\pi]^{1/2}$, and l and m correspond to the incident nucleons and j and k to the target nucleons. In these expressions we have used $\mathbf{r}=\mathbf{s}+\mathbf{z}$ and $\boldsymbol{\varrho}=\boldsymbol{\sigma}+\boldsymbol{\zeta}$. For average neutron-proton separations much greater than the ranges of nucleon-nucleon forces we may write

$$\int_{\mathcal{S}}d\boldsymbol{\sigma}\int_{-\infty}^{\infty}|\psi_i(\boldsymbol{\sigma},\zeta)|^2d\zeta\approx\int_{-\infty}^{\infty}|\psi_i(\mathbf{s},\zeta)|^2d\zeta\int_{\mathcal{S}}d\boldsymbol{\sigma} \quad (8.20)$$

$$\approx(\sigma_{lj})_{\rm abs}\int_{-\infty}^{\infty}|\psi_i(\mathbf{s},\zeta)|^2d\zeta\,. \quad (8.21)$$

Therefore the resulting double-collision correction is given by

$$(\sigma_{mk})_{\rm abs}(\sigma_{lj})_{\rm abs}\int|\psi_i(\mathbf{s},\zeta)|^2|\psi_i(\mathbf{s},z)|^2d\mathbf{s}d\zeta dz\,. \quad (8.22)$$

It is shown in the Appendix that the expression (8.22), with the indices mk and lj replaced by nn and pp, is equal to the expression (8.18). The corresponding correction to the total cross section is very easily obtained for the case in which the regions surrounding the nucleons are black spheres by noting that in such a case the total cross sections are equal to twice the absorption cross sections. In this manner we obtain the first two terms of Eq. (8.15). The last term is a charge-exchange correction which we have omitted in the geometrical derivation. We should point out, however, that we have previously derived Eq. (8.15) from more general grounds and its validity does not require the black sphere model.

We have seen therefore that the double-scattering correction $\delta\sigma_{21}'$ corresponds to the Glauber shadow, or eclipse, effect for absorption, but that the double-scattering correction $\delta\sigma_{22}$ is of a different nature. It may be considered largely a *double counting* correction since we have seen that for purely absorptive nucleon-nucleon interactions its analog for the absorption cross section corrects for counting certain double-absorption processes (e.g., absorption of nucleon l by nucleon j and absorption of nucleon m by nucleon k) twice.[24]

We may obtain an estimate of the importance of the double-counting correction relative to the Glauber shadow correction in *d*-*d* collisions by means of Eqs. (8.1)–(8.4), (8.13), and (8.14). If we assume a Gaussian function $e^{-\beta^2q^2}$ for the deuteron form factor and assume $\sigma_{np}=\sigma_{pp}$ and $\alpha_{np}=\alpha_{pp}$ we are led to the result $\delta\sigma_{22}^{(A)}=\frac{1}{2}\delta\sigma_{21}'^{(A)}$, so that the double-counting correction is approximately 50% as large as the Glauber shadow correction.

IX. EVALUATION OF THE TOTAL CROSS SECTION

Deuteron-deuteron total cross-section measurements have been made by Debaisieux *et al.*[25] and by Pripstein and Eberhard[26] at incident deuteron momenta of 3.0 and 4.42 BeV/c, respectively. We shall therefore evaluate σ_{dd} at these momenta. A deuteron momentum of 3.0 BeV/c corresponds to a kinetic energy of an *individual nucleon* of ~832 MeV.[27] Similarly, a deuteron momentum of 4.42 BeV/c corresponds to a kinetic energy of an individual nucleon of ~1.46 BeV. Since the approximation we have used is a high-energy approximation, a more desisive comparison of the theory with experiment must await future measurements at higher momenta.[28]

We shall evaluate the deuteron-deuteron total cross sections by means of the relatively simple asymptotic formulas derived in Sec. VIII and also by means of the more general expressions obtained in Sec. VII. This approach will give us an indication of the numerical accuracy of the asymptotic approximations.

We begin by considering the asymptotic multiple scattering corrections Eqs. (8.10)–(8.13). Equations (8.10)–(8.12) were obtained from the general expressions by assuming that the average neutron-proton separation in the deuteron is much larger than the range of nucleon-nucleon interactions. Equation (8.13) was obtained by making the additional assumption that the nucleon-nucleon elastic scattering amplitudes $f_{lj}(q)$ are proportional to $\exp(-\frac{1}{2}A_{lj}q^2)$, where A_{lj} is real a constant. To explicitly calculate the dd total cross section we shall use Eqs. (6.16)–(6.19). The alternative choice of Eqs. (6.1), (6.2), and (6.9) leads to identical results for σ_{dd} provided σ_{np} is extracted from measurements of σ_{pd} and σ_{pp} consistently in both cases.

To perform the calculations, we require the expectation values $\langle r^{-2}\rangle_d$ and $\langle\langle(2\rho r)^{-1}\ln[(r+\rho)/|r-\rho|]\rangle_d\rangle_d$. These may be calculated from explicit representations of the deuteron ground-state wave function. We shall consider two such representations. The first, ψ_1, is the most accurate fit given by Moravcsik[29] to the Gartenhaus wave function and includes a contribution from the D state. The second, ψ_2, is the ground-state wave function referred to as φ_5 in Ref. 1. The expectation

[24] The double-scattering correction was first derived in Ref. 17. It has also been employed by the author in the double-scattering corrections to π-π scattering in the quark model. See V. Franco, Phys. Rev. Letters 18, 1159 (1967).

[25] J. Debaisieux, F. Grard, J. Heughebaert, R. Servranckx, and R. Windmolders, Nucl. Phys. 70, 603 (1965); F. Grard (private communication).

[26] M. Pripstein and P. Eberhard (private communication).

[27] We neglect the internal motion of the nucleons in the deuteron.

[28] Measurements of *d*-*d* cross sections at approximately 7.85 and 13.5 BeV are presently being analyzed. We thank Dr. M. Bazin for this information.

[29] M. J. Moravcsik, Nucl. Phys. 7, 113 (1958).

TABLE I. Neutron-proton total cross sections $\sigma_{np}^{(A)}$ predicted from pp and pd measurements. Values calculated using two different deuteron wave functions ψ_1 and ψ_2 are shown. The asymptotic relation (9.1) was used in the calculation. P_{lab} refers to the nucleon momentum.

P_{lab} (BeV/c)	σ_{pp}(expt) (mb)	σ_{pd}(expt) (mb)	α_{pp}	α_{np}	$\sigma_{np}^{(A)}(\psi_1)$ (mb)	$\sigma_{np}^{(A)}(\psi_2)$ (mb)
1.50	47.0	81.5	−0.01	−0.48	39.4	40.8
2.21	47.0	84.5	−0.20	−0.47	42.0	43.3

values of r^{-2} are 0.313 F^{-2} for ψ_1 and 0.384 F^{-2} for ψ_2. The expectation values of $(2r\rho)^{-1}\ln[(r+\rho)/|r-\rho|]$ are 0.162 F^{-2} for ψ_1 and 0.192 F^{-2} for ψ_2.[30]

In our calculations we also require a knowledge of the total cross sections σ_{pd}, σ_{pp}, and σ_{np}, of the ratio of the real to imaginary parts of the forward elastic NN amplitudes α_{pp} and α_{np}, and of the slopes of the NN forward diffraction peaks A_{pp} and A_{np}. In principle these quantities can be measured. In practice, for a calculation at a given momentum it may be necessary to use assumed or theoretical values for those quantities for which measurements at or near the appropriate momentum do not exist. It may also be necessary to use measurements made at slightly different momenta for those quantities which have not been measured at precisely the appropriate momentum.

For our calculations of dd cross sections at 3.0 and 4.42 BeV/c we require measurements at nucleon momenta close to 1.5 and 2.21 BeV/c. We are therefore led to the measurements of Bugg *et al.*[8] for σ_{pd} and σ_{pp} at 1.41, 1.61, and 2.21 BeV/c. For α_{pp} we use measurements[31] between 1.29 and 1.69 BeV/c and theoretical predictions[8] for 2.21 BeV/c. For α_{np} we use measurements[31,32] at 1.69 BeV/c and theoretical predictions[8] for 2.21 BeV/c.

The neutron-proton cross section σ_{np} may be obtained by means of the asymptotic relation[7]

$$\sigma_{pd}^{(A)} \approx \sigma_{pp} + \sigma_{np} - \tfrac{1}{4}\delta\sigma_{21}^{(A)}. \tag{9.1}$$

In Table I we list average values of σ_{pp}, σ_{pd}, α_{pp}, and α_{np} obtained from Refs. 8, 31, and 32 at average nucleon incident momenta of 1.50 and 2.21 BeV/c. In the last two columns we present the neutron-proton cross sections calculated by means of Eq. (9.1) with the wave functions ψ_1 and ψ_2. We note that use of the two different deuteron wave functions results in a difference of $\sim$1.4 mb in the neutron-proton total cross section as deduced from the pp and pd cross sections. Consequently, in addition to the possible inaccuracies arising from the *asymptotic* approximations, which we shall soon investigate, an additional uncertainty of the order of 1 mb may be expected from the uncertainty in the deuteron wave function. The use of the present analysis to detect effects of the order of 1 mb should therefore be made with great caution.

In the asymptotic formulas the quantities A_{pp} and A_{np} appear only in $\delta\sigma_4$. Since $\delta\sigma_4$ will turn out to be quite small, inaccuracies in measurements of A_{pp} and A_{np} will not affect the calculated values of σ_{dd} significantly. For our calculation at $\frac{1}{2}\hbar k = 1.50$ BeV/c we obtained a value of $A_{pp}=0.189$ F^2 by making a least-squares fit of pp data at 1.39 BeV/c.[33] Since np measurements in this energy region are lacking, we have assumed $A_{np}=A_{pp}$. For our calculation at $\frac{1}{2}\hbar k = 2.21$ BeV/c we have used the measured value[34] $A_{pp}=0.207$ F^2, and again we have assumed $A_{np}=A_{pp}$.

In Table II we present the values of σ_{dd} and the various cross-section defects calculated from the asymptotic relations (8.10)–(8.13). We note that use of the different wave functions results in a difference of $\sim$2.2 mb in the calculated values of σ_{dd}. The cross-section defect resulting from double-scattering corrections ($\delta\sigma_{21}'+\delta\sigma_{22}$) is an order of magnitude greater than the magnitude of the cross-section defect resulting from triple-scattering corrections, which in turn is an order of magnitude (or more) greater than the cross-section defect resulting from quadruple-scattering corrections. The new type of

TABLE II. Calculated values of the asymptotic cross-section defects and dd total cross sections. ψ_1 and ψ_2 refer to the two deuteron wave functions used in the calculations. P_{lab} refers to the deuteron momentum. The measured pd cross sections σ_{pd}(expt) refer to proton momenta of $\frac{1}{2}P_{\text{lab}}$.

P_{lab} (BeV/c)	ψ	$\delta\sigma_{21}'^{(A)}$ (mb)	$\delta\sigma_{22}^{(A)}$ (mb)	$\delta\sigma_3^{(A)}$ (mb)	$\delta\sigma_4^{(A)}$ (mb)	$\delta\sigma'^{(A)}$ (mb)	$\sigma_{dd}^{(A)}$(theor) (mb)	σ_{dd}(expt) (mb)	$2\sigma_{pd}$(expt) (mb)
3.00	ψ_1	9.9	4.0	−1.9	0.2	12.2	150.8	123±6	163.0
3.00	ψ_2	12.2	4.8	−2.8	0.3	14.5	148.5	123±6	163.0
4.42	ψ_1	9.1	4.4	−1.5	0.0	12.0	157.0	a	169.0
4.42	ψ_2	11.2	5.2	−2.3	0.0	14.1	154.9	a	169.0

a Preliminary estimates indicate that σ_{dd}(expt) is approximately >140 mb (Ref. 35). This is a lower limit since events for $0<-t\lesssim 0.01$ (BeV/c)^2 have not been included in the analysis of the measurements (Ref. 35). Our calculation of the dd elastic scattering, presented in Sec. X, indicates that the elastic scattering cross section for $0\lesssim -t\lesssim 0.01$ (BeV/c)^2 is $\sim$11 mb. When this is added to $\sim$140 mb, we arrive at a value for σ_{dd}(expt) of $\sim$151 mb, which is very close to the predicted values.

[30] The two expectation values for ψ_2 are related by a factor of exactly 2. The factor relating the two expectation values for ψ_1 happens to be nearly, but not exactly, 2.

[31] L. M. C. Dutton, R. J. W. Howells, J. D. Jafar, and H. B. Van der Raay, Phys. Letters **25B**, 245 (1967), and references cited therein.

[32] L. Kirillova, V. Nikitin, M. Shafranova, V. Sviridov, L. Zolin, Z. Korbel, L. Rob, Kh. Chernev, P. Devinski, L. Khristov, P. Markov, Z. Zlatanov, N. Dalkhazhav, D. Tuvdendorzh, Ngo quang Huy, Hquyen dinh Tu, and Truong Bien (unpublished).

[33] P. G. McManigal, R. D. Eandi, S. N. Kaplan, and B. J. Moyer, Phys. Rev. **137**, B620 (1965).

[34] A. M. Eisner, E. L. Hart, R. I. Louttit, and T. W. Morris, Phys. Rev. **138**, B670 (1965).

double-scattering correction, $\delta\sigma_{22}$, which we described in detail earlier, is approximately 43% of the Glauber "shadow correction" $\delta\sigma_{21}'$, in good agreement with our rough theoretical estimate of 50% described in the previous section.

We shall now repeat our calculations using the more general relations (7.8)–(7.11). We assume that the nucleon-nucleon amplitudes may be expressed as

$$f_{lp}(q,\tfrac{1}{2}k)=\frac{k}{8\pi}\sigma_{lp}(i+\alpha_{lp})e^{-\frac{1}{2}A_{lp}q^2}, \quad l=n,p.$$

The two-, four-, and six-dimensional integrals can then be reduced to single and double integrals which may be done analytically for ψ_2 and numerically for ψ_1.

In Table III we show the neutron-proton cross section calculated by means of Eq. (6.15). [The corresponding results obtained from the asymptotic expression (9.1) were shown in Table I.] By comparing Tables I and III we note that use of the asymptotic expression (9.1) to determine σ_{np} from measurements of σ_{pd} and σ_{pp} tends to introduce a numerical inaccuracy of $\sim$1 mb in magnitude.

In Table IV we present the values of σ_{dd} and the various cross-section defects calculated with Eqs. (7.8)–(7.11). (The corresponding results obtained from the asymptotic formulas were shown in Table II.) By comparing Tables II and IV we note that use of the asymptotic formulas tends to introduce a numerical inaccuracy in the calculated deuteron-deuteron cross section of $\sim$1 mb in magnitude. Although the cross-section defects $\delta\sigma'$ calculated by the two methods differ by $\sim$5%, the dd total cross sections differ by only $\sim\frac{1}{2}$%. At high energies, therefore, the asymptotic formulas appear reasonably adequate for calculating σ_{dd} to a numerical accuracy of a few percent.

At 3.0 BeV/c, the dd cross sections calculated with the impulse approximation (i.e., $2\sigma_{Nd}$) are $\sim$33% higher than the measured value, whereas the values calculated with the multiple scattering corrections are $\sim$22% higher. (However, see *Note added in proof* in Table IV. Taking 141 mb as the measured value, we would replace 33% and 22% by 16% and 6%, respectively.) To compare the calculations at the higher momentum of 4.42 BeV/c we must await the completion of the analysis of the data. However, preliminary estimates indicate that the measured value of the dd total cross section is approximately 140 mb.[35] This is a lower limit since events for $0\leq -t\lesssim 0.01$ (BeV/c)^2 have not been included in the analysis of the measurements.[35] Our calculation of the dd elastic scattering cross section for $0\leq -t\leq 0.01$ (BeV/c)^2 is $\sim$11 mb. When this is added to $\sim$140 mb, we arrive at a value for σ_{dd}(expt) of $\sim$151 mb which is very close to the predicted value.

Table III. Neutron-proton total cross sections σ_{np} predicted from pp and pd measurements. Values calculated using two different wave functions ψ_1 and ψ_2 are shown. The relation (6.15) was used in the calculation. P_{lab} refers to the nucleon momentum.

P_{lab} (BeV/c)	σ_{pp}(expt) (mb)	σ_{pd}(expt) (mb)	$\sigma_{np}(\psi_1)$ (mb)	$\sigma_{np}(\psi_2)$ (mb)
1.50	47.0	81.5	39.9	39.9
2.21	47.0	84.5	42.4	42.4

X. ELASTIC SCATTERING ANGULAR DISTRIBUTION

In this section we shall discuss briefly the elastic scattering of deuterons by deuterons. The differential cross section is given by

$$(d\sigma/d\Omega)_{el}=|F_{iiii}(\mathbf{q},k)|^2. \tag{10.1}$$

In order to determine some *qualitative* aspects of the elastic scattering processes, we shall make the simplifying assumptions that

$$f_{np}(q,\tfrac{1}{2}k)=f_{pp}(q,\tfrac{1}{2}k), \tag{10.2}$$

$$f_{pp}(q,\tfrac{1}{2}k)=(k\sigma/8\pi)(i+\alpha)e^{-\frac{1}{2}Aq^2}, \tag{10.3}$$

and

$$S(q)=e^{-Bq^2}. \tag{10.4}$$

The differential cross section may then be written

Table IV. Calculated values of the cross-section defects and dd total cross sections. ψ_1 and ψ_2 refer to the two deuteron wave functions used in the calculations. P_{lab} refers to the deuteron momentum. The measured pd cross sections σ_{pd}(expt) refer to proton momenta of $\frac{1}{2}P_{\text{lab}}$.

P_{lab} (BeV/c)	ψ	$\delta\sigma_{21}'$ (mb)	$\delta\sigma_{22}$ (mb)	$\delta\sigma_3$ (mb)	$\delta\sigma_4$ (mb)	$\delta\sigma'$ (mb)	σ_{dd}(theoret) (mb)	σ_{dd}(expt) (mb)	$2\sigma_{pd}$(expt) (mb)
3.00	ψ_1	10.7	3.7	−2.2	0.2	12.5	150.5	123 ± 6[b]	163.0
3.00	ψ_2	10.8	4.5	−2.2	0.3	13.3	149.7	123 ± 6[b]	163.0
4.42	ψ_1	9.7	4.0	−1.7	0.1	12.1	156.9	a	169.0
4.42	ψ_2	9.8	4.9	−1.7	0.1	13.0	156.0	a	169.0

[a] Preliminary estimates indicate that σ_{dd}(expt) is approximately >140 mb (Ref. 35). This is a lower limit since events for $0<-t\lesssim 0.01$ (BeV/c)^2 have not been included in the anslysis of the measurements (Ref. 35). Our calculation of the dd elastic scattering, presented in Sec. X, indicates that the elastic scattering cross section for $0\lesssim -t\lesssim 0.01$ (BeV/c)^2 is $\sim$11 mb. When this is added to $\sim$140 mb, we arrive at a value for σ_{dd}(expt) of $\sim$151 mb, which is very close to the predicted values.

[b] *Note added in proof.* Professor Grard has kindly informed us that the measurements at 3.00 BeV/c do not include corrections for elastic events in which $\lesssim$150 MeV/c is transferred to the target deuteron. We calculate this correction to be $\sim$18 mb, which when added to 123 mb yields $\sim$141 mb. This result is quite close to our predicted value. (This *Note* also applies to Table II.)

[35] M. Pripstein (private communication). We wish to thank Dr. Pripstein for this information.

simply as

$$\left(\frac{d\sigma}{d\Omega}\right)_{el}=\left|\frac{k\sigma(i+\alpha)}{\pi}\right|^2\Bigg|e^{-\frac{1}{2}(A+B)q^2}+i\frac{\sigma}{8\pi}(i+\alpha)\left[\frac{e^{-\frac{1}{4}Aq^2}}{4B+2A}+\frac{e^{-\frac{1}{4}(A+B)q^2}}{B+A}\right]-\left[\frac{\sigma}{8\pi}(i+\alpha)\right]^2\frac{e^{-A(B+A)q^2/(4B+6A)}}{(B+\frac{1}{2}A)(B+\frac{3}{2}A)}-i\left[\frac{\sigma}{8\pi}(i+\alpha)\right]^3\frac{e^{-Aq^2/8}}{8A(B+\frac{1}{2}A)^2}\Bigg|^2. \quad (10.5)$$

For a deuteron momentum of 4.42 BeV/c, corresponding to a nucleon momentum of 2.21 BeV/c, we have used the average values $\sigma=44.5$ mb, $\alpha=-0.335$, and $A=0.207$ F^2. The value of B is taken to be[1] 1.30 F^2. The results are shown in Fig. 5 where four intensities are plotted against $-t$, the negative of the square of the four-momentum transfer. The curve labelled 1 contains only the single-scattering contribution. The curve labelled 2 contains both single- and double-scattering contributions. The curve labelled 3 contains single-, double-, and triple-scattering contributions. The curve labelled 4 contains contributions from all four orders of multiple scattering. Near the forward direction single scattering is dominant, but it falls off extremely rapidly. For 0.1 (BeV/c)$^2\lesssim -t\lesssim 0.2$ (BeV/c)^2, the single- and double-scattering amplitudes interfere appreciably, resulting in a minimum near $-t=0.16$ (BeV/c)^2. Although in the forward direction the magnitude of the double-scattering amplitude is much smaller than that of the single-scattering amplitude, it decreases much more slowly. Consequently, for $-t\gtrsim 0.16$ (BeV/v)^2 double scattering is more important than single scattering. In addition to an interference minimum, a double-scattering maximum appears at $-t\sim 0.26$ (BeV/c)^2, beyond which the differential cross-section decreases, but at a much slower rate than it does near the forward direction.

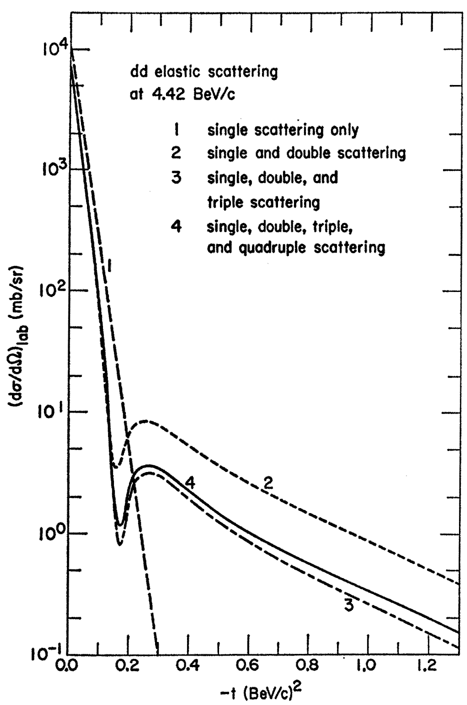

FIG. 5. Differential cross sections in the laboratory system for deuteron-deuteron elastic scattering for an incident deuteron momentum of 4.42 BeV/c plotted as a function of the negative squared four-momentum transfer $-t$. The solid curve labelled 4 is the theoretical prediction, using nucleon-nucleon data, with all single-, double-, triple, and quadruple-collision processes taken into account. Curve 1 represents the contribution of single-scattering processes alone. Curve 2 represents the contribution of single- and double-scattering processes. Curve 3 represents the contribution of single-, double-, and triple-scattering processes. The shape of the curve in the interference region, $-t\sim 0.16$ (BeV/c)^2, is quite sensitive to the phases of the nucleon-nucleon scattering amplitudes which are not yet known accurately.

It is interesting to note that in the double-scattering region, the contribution to $d\sigma/d\Omega$ arising from the new type of double-scattering correction described in Sec. VIII (i.e., the one corresponding to $\delta\sigma_{22}$ for the total cross section) is *substantially* greater than the familiar type of double-scattering correction first described by Glauber for particle-deuteron collisions. This is not surprising. This new double-scattering process takes place when one nucleon in the incident deuteron interacts with one nucleon in the target, and the other nucleon in the incident deuteron interacts with the other nucleon in the target. These two interactions can occur simultaneously. Consequently a large momentum may be transferred to a deuteron without there being a large momentum transferred to one nucleon in a deuteron *relative* to the other nucleon in that deuteron. In other words, a large momentum may be transferred to the nucleons in the deuteron without breaking it up, by transferring the momentum to the deuteron center of mass. This is easily done by having the two nucleons in the incident deuteron collide *simultaneously* with the two nucleons in the target, and this can be done quite readily by the new type of double-scattering process described in Sec. VIII. The usual familiar double-scattering process describes collisions of an incident nucleon *first* with one nucleon in the target and *later* with a second nucleon. A *simultaneous* double scattering in this type of process could only occur if the neutron and proton in the deuteron were overlapping, a condition which rarely occurs in such a weakly bound system.

A more analytic explanation for the greater importance of the new type of double-scattering process in $d\sigma/d\Omega$ is obtained by inspection of Eq. (5.17) for the double-scattering amplitude operator. The only q dependence of the new double-scattering process arises from terms like $a_{31}(\mathbf{u})a_{42}(\mathbf{v})$. However, the q dependence of the usual familiar double-scattering process arises from, in addition to bilinear forms in the a_{pt}'s, a factor such as $\exp(\frac{1}{2}\mathbf{q}\cdot\mathbf{s})$ which in turn yields an additional factor $S(\frac{1}{2}q)$ in the elastic amplitude. This is a very rapidly decreasing factor and hence away from the forward direction the resulting terms are quite small compared with those that do not contain this factor, i.e., compared with the terms representing the new type of double-scattering process.

APPENDIX

In this Appendix we derive Eqs. (8.2) and (8.4). Using the definition of the form factor of the deuteron ground state, Eq. (6.5), and assuming a spherically symmetric wave function, we may write

$$\int S(\mathbf{q})d^2q=\int e^{i\mathbf{q}\cdot\mathbf{r}}|\psi(r)|^2 d\mathbf{r}d^2q. \tag{A1}$$

Let $\mathbf{r}=\mathbf{s}+\mathbf{z}$, where $\mathbf{s}$ is the component of the coordinate $\mathbf{r}$ lying parallel to the plane which contains $\mathbf{q}$. The integral (A1) may then be expressed in the form

$$\int S(\mathbf{q})d^2q=\int e^{i\mathbf{q}\cdot\mathbf{s}}|\psi(s,z)|^2 d^2s dz d^2q \tag{A2}$$

$$=4\pi^2\int \delta^{(2)}(\mathbf{s})|\psi(s,z)|^2 d^2s dz, \tag{A3}$$

where $\delta^{(2)}(\mathbf{s})$ is a two-dimensional δ-function. Carrying out the integration over $\mathbf{s}$, we obtain

$$\int S(\mathbf{q})d^2q=4\pi^2\int_{-\infty}^{\infty}|\psi(0,z)|^2 dz \tag{A4}$$

$$=8\pi^2\int_0^{\infty}|\psi(r)|^2 dr \tag{A5}$$

$$=2\pi\int \psi^*(r)\frac{1}{r^2}\psi(r)d\mathbf{r} \tag{A6}$$

$$=2\pi\langle r^{-2}\rangle_d. \tag{A7}$$

This is the result stated in Eq. (8.2).

To derive Eq. (8.4) we write

$$\int S^2(\mathbf{q})d^2q=\int e^{i\mathbf{q}\cdot(\mathbf{r}+\mathbf{r}')}|\psi(r)|^2|\psi(r')|^2 d\mathbf{r}d\mathbf{r}'d^2q \tag{A8}$$

$$=4\pi^2\int \delta^{(2)}(\mathbf{s}+\mathbf{s}')|\psi(s,z)|^2 \times|\psi(s',z')|^2 d^2s d^2s' dz dz' \tag{A9}$$

$$=16\pi^2\int g^2(s)d^2s, \tag{A10}$$

where

$$g(s)=\frac{1}{2}\int_{-\infty}^{\infty}|\psi(s,z)|^2 dz \tag{A11}$$

$$=\int_s^{\infty}\frac{|\psi(\rho)|^2}{(\rho^2-s^2)^{1/2}}\rho d\rho. \tag{A12}$$

Therefore we may write

$$\int S^2(\mathbf{q})d^2q=32\pi^3\int_0^{\infty}s ds\, g(s)\int_s^{\infty}\frac{|\psi(r)|^2}{(r^2-s^2)^{1/2}}r dr. \tag{A13}$$

If we interchange the order of integration we obtain

$$\int S^2(\mathbf{q})d^2q=32\pi^3\int_0^{\infty}|\psi(r)|^2 r\, dr \times\int_0^r\frac{s\, ds}{(r^2-s^2)^{1/2}}g(s) \tag{A14}$$

$$=32\pi^3\int_0^{\infty}|\psi(r)|^2 r\, dr\int_0^r\frac{s\, ds}{(r^2-s^2)^{1/2}} \times\int_s^{\infty}\frac{|\psi(\rho)|^2}{(\rho^2-s^2)^{1/2}}\rho d\rho. \tag{A15}$$

If we now interchange the order of the ρ and s integrations, we find

$$\int S^2(\mathbf{q})d^2q=32\pi^3\int_0^{\infty}|\psi(r)|^2 r\, dr\left\{\int_0^r|\psi(\rho)|^2\rho d\rho \times\int_0^{\rho}\frac{s\, ds}{[(\rho^2-s^2)(r^2-s^2)]^{1/2}}+\int_r^{\infty}|\psi(\rho)|^2\rho d\rho \times\int_0^r\frac{s\, ds}{[(\rho^2-s^2)(r^2-s^2)]^{1/2}}\right\}. \tag{A16}$$

The s integration is evaluated to be

$$\int_0^{\rho}\frac{s\, ds}{[s^4-(\rho^2+r^2)s^2+\rho^2r^2]^{1/2}}=\tfrac{1}{2}\ln\left(\frac{r+\rho}{r-\rho}\right). \tag{A17}$$

Therefore Eq. (A16) becomes

$$\int S^2(q)d^2q=16\pi^3\int_0^{\infty}|\psi(r)|^2 r\, dr \times\int_0^{\infty}|\psi(\rho)|^2\rho d\rho\ln\left(\frac{r+\rho}{|r-\rho|}\right) \tag{A18}$$

$$=2\pi\left\langle\left\langle(2r\rho)^{-1}\ln\left[\frac{(r+\rho)}{|r-\rho|}\right]\right\rangle_d\right\rangle_d. \tag{A19}$$

This is the result stated in Eq. (8.2).