Shell Model in the Continuum—Application to the Four-Nucleon System*

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(Received 9 May 1968)

The continuum shell model is applied to the study of the elastic scattering of protons from He^3 as well as to the photodisintegration of He^4. A semiquantitative fit is obtained for the scattering phases and the photoabsorption cross section using a model Hamiltonian consisting of a real, local potential and a finite-range effective interaction. Two approximations for the total wave function of the system are studied, and it is found that the photoabsorption cross section is sensitive to the approximation used, while the calculated scattering phases are fairly independent of the details of the wave function. A method for improving the convergence of the Born series is proposed and is found to be useful in this calculation.

I. INTRODUCTION

OVER the past several years, a large number of papers have appeared dealing with the theory of nuclear reactions. These formulations have in part been the result of an increased understanding of nuclear structure accompanying the successful application of various nuclear models such as the shell model and the optical model. Of particular importance is the work of Feshbach,[1] who showed how projection operators may be used to elucidate the resonance structure of the S matrix. Feshbach's theory combined with the concepts of the shell model has been applied to the study of elastic scattering of neutrons from N^{15} by Lemmer and Shakin[2] and to the scattering of neutrons from C^{12} by Lovas.[3] The possibility of extending the shell model or independent-particle model to the treatment of continuum problems has stimulated many authors, and several other formal theories have been put forth.

A central problem in these studies has been the diagonalization of the continuum-continuum interaction for which various approximations have been proposed. Much of this work is described in a forthcoming monograph of Mahaux and Weidenmüller.[4] Using the shell model in the continuum a large number of topics can be studied qualitatively such as analog states, neutron strength functions, (γ,p), (γ,n) reactions and their inverse, doorway states, direct processes, (p,n) reactions, etc. The number of works attempting detailed quantitative comparison of this model with experimental data has been somewhat limited. This is in part due to the complexity of the calculations required.

Excitations of the particle-hole type have been studied fairly extensively in the O^{16} compound system[5]; however, it is rather difficult to obtain a satisfactory understanding of both the magnitude of the photoabsorption cross section and its complex structure observed experimentally.

Also, the number of resonances observed in the elastic scattering, e.g., $N^{15}(n,n)N^{15}$, is much greater than the number obtained from the particle-hole calculation. Therefore, a detailed description of the various experimental cross sections may necessitate the introduction of states of more complex structure than those of the particle-hole type.

It was thought highly desirable to study a simple nuclear system whose observed excitations are not so complex as to obscure the comparison between theoretical and experimental cross sections. Recently, Hüfner and Lemmer have introduced a formulation for nuclear reaction calculations[6] which leads to calculational prescriptions of minimum complexity.

In this work, the formalism of Hüfner and Lemmer (referred to as I) is used for a study of the nuclear reactions $He^3(p,p)He^3$ and $He^4(\gamma,p)H^3$. For these reactions there exists sufficient experimental data so that a detailed comparison between theory and experiment may be made.

In the case of the nucleon scattering experiments a fairly complete phase-shift analysis has been made by Tombrello[7] and Haeberli and Morrow.[8] Tombrello has also extracted resonance parameters using R-matrix theory. One may note that the resonances found in this manner are quite broad (of the order of 2–7 MeV) so that one may have some doubt as to the physical significance of the parameters extracted.

In Tombrello's analysis several negative-parity resonances were found and these were interpreted by de Shalit and Walecka as particle-hole excitations of the

* This work is supported in part through funds provided by the Atomic Energy Commission under Contract No. At (30-1)2098.

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four-nucleon system.[9] These authors used fairly standard techniques of the harmonic-oscillator shell model and studied the spectra obtained with various residual interactions. Their results are in reasonable agreement with the experimental spectrum as inferred from the R-matrix analysis mentioned above. However, the experimental photoabsorption cross section[10] for He^4 exhibits a single strong peak followed by a long tail while in the shell model one has two $J=1^-$ states which must in some way account for the distribution of electric-dipole transition strength. Most of the theoretical strength was found in the upper $T=1$, $J=1^-$ "state" while the peak in the experimental photoabsorption cross section is at the position of the lower state. It was suggested that more complicated residual interactions might be necessary to remove this difficulty. Barrett[11] studied the excited states of He^4 using realistic forces such as those of Tabakin, Brueckner-Gammel-Thaler, and Hamada, and found that the problem of obtaining satisfactory $E1$-transition rates was not resolved. He found the ratio of the $E1$ transition rate of the upper $J=1^-$, $T=1$ state to the ground state with respect to the lower $J=1^-$, $T=1$ state was 1.6 compared with an experimental ratio $\sim\frac{1}{2}$. An alternate approach to this problem used by Barrett, Walecka, and Meyerhof[12] is based on the observation that there are only four quantities appearing in the calculations which determine six energy splittings and the mixing of the two $T=1$, $J=1^-$ states. Using experimental data for some of the energy splittings it is possible to predict two unobserved energy splittings and the ratio of the $E1$ transition probabilities. The predictions of this approach appear to be in agreement with experiments but one may doubt whether a really satisfactory understanding of the phenomena has been achieved. In particular, one would like to take into account the continuum aspects of this problem and see whether the experimental data may be explained without resorting to the introduction of a complete parameterization of the effective interaction matrix elements as in the work of Barrett, Walecka, and Meyerhof. Also, in the continuum approach one is able to calculate the shape of the photoabsorption cross section as well as the differential scattering cross sections, quantities which are inaccessible in the ordinary shell-model treatment. As we shall see, the use of continuum wave functions turns out to be quite important in explaining the distribution of transition strength.

So far the continuum structure of the $A=4$ system has been treated[13] by the resonating-group method, which requires the solution of (coupled) integro-differential equations. The results in Ref. 13 are in fair agreement with experiment. While the essentially algebraic methods of the continuum shell model are probably more appropriate to larger systems, we have been motivated to carry through the calculations reported here to test our theory for a comparatively simple system, where much experimental information has become available.

II. THEORETICAL CONSIDERATIONS

It is worth pointing out the difference between the ordinary shell theory and the shell model in the continuum. Both theories have in common the independent-particle model of the nucleus supplemented by residual interactions. The nucleons are considered to be moving in some effective field, possibly that derived from the Hartree-Fock theory. In the usual application of the shell theory the particles are constrained to move in an infinitely deep potential and the diagonalization of the effective interaction yields a discrete set of bound states. Thus one is able to calculate energy levels, transition properties, moments, etc. On the other hand, in the continuum shell theory, one describes the effective field by a potential well of finite depth. The bound and scattering states of this well provide a basis for the solution of the problem although one is usually limited to basis functions which contain no more than one particle in a continuum orbit.

Some comments concerning the separation of the Hamiltonian into an independent-particle part H_0 and a residual interaction V may be in order. One may assume that the nucleon-nucleon interaction in the nucleus is nonsingular such that Hartree-Fock methods are applicable. In this case, H_0 will contain the kinetic energy of all the particles plus the Hartree-Fock potential. The *residual* interaction is then the actual nucleon-nucleon interaction minus the Hartree-Fock potential. One may then introduce an *effective* interaction by the following observation. The Hilbert space $\mathfrak{U}$ of the problem may be separated into two parts, one part consisting of those states which are used to form a limited basis for the solution of the problem at hand, $\mathfrak{U}_P$, and a second part consisting of the remaining states, $\mathfrak{U}_Q$. A simple application of projection operator techniques allows the construction of an effective interaction in the space $\mathfrak{U}_P$; this effective interaction will be energy-dependent and will depend on the choice of $\mathfrak{U}_P$ and $\mathfrak{U}_Q$. For example, it is now well known that core-polarization effects are particularly important in any attempt to construct an effective interaction starting from realistic residual interactions.[14]

In the case that the nucleon-nucleon interaction contains a hard core or is so strong that Hartree-Fock methods are not applicable, a more elaborate formula-

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tion is necessary. With an appropriate summation of diagrams it is possible to introduce G-matrix elements which play the role of a residual interaction. The concept of an effective interaction may again be introduced via the separation of the Hilbert space into two parts.

In this work we will not attempt to utilize the Hartree-Fock approach or to use the reaction matrix approach but will treat H_0 and the effective interaction phenomenologically. As usual in calculations of this type the average potential is determined from experimental data for the single-particle energies and nuclear radius. The residual interaction used has been determined in shell-model calculations for p-shell nuclei. Given this phenomenological viewpoint there is therefore no simple relation between H_0 and the effective interaction used in the calculations.

The application of the continuum shell model to scattering from the three-nucleon system is made with some reservations as it is not clear that the model is useful for a system of only a few particles. (We consider the relative success of the calculations reported in this paper as an *a posteriori* partial justification of the model.)

In describing the interaction of the incident particle with the target we choose a potential of the Woods-Saxon form, as has become conventional in problems of this type. The potential chosen has a bound S state and there are two single-particle resonances in the $P_{3/2}$ and $P_{1/2}$ scattering states. The latter resonance is quite broad and rather ill-defined. One notes that since there are no bound states other than the S state, the expansion of the total wave function of the system will contain only scattering states. In contrast, the usual shell-model procedure replaces the $P_{3/2}$ and $P_{1/2}$ single-particle resonances by the bound states of a harmonic oscillator.[9] The latter approximation greatly simplifies the calculations but leads to various difficulties, as mentioned in the previous section.

In order to clearly define the nature of the approximations used in this work, it is useful to review the results of I. The Hamiltonian is written as

$$H=\sum_{i=1}^{A}[T(i)+U(i)]+\tfrac{1}{2}\sum_{i\neq j}v_{ij}\equiv H_0+V, \quad (2.1)$$

with V as the residual interaction. The following functions are useful in the discussion:

$$\begin{aligned}|i\rangle&=\mathcal{A}[\varphi_{i'}(A)\Phi_{i''}(1,2,\cdots,(A-1))],\\ |\psi_{E\lambda}{}^{(0)}\rangle&=\mathcal{A}[\varphi_{E\lambda'}{}^{(0)}(A)\Phi_{\lambda''}(1,\cdots,(A-1))].\end{aligned} \quad (2.2)$$

Here $\Phi_{i''}$ and $\Phi_{\lambda''}$ are eigenfunctions of H, in the subspace spanned by bound shell-model states and $\varphi_{i'}(A)$ and $\varphi_{E\lambda'}{}^{(0)}(A)$ are bound and continuum eigenstates of $T(A)+U(A)$. If the binding energy of the target state, $\Phi_{\lambda''}$, is set equal to zero, the energy of the scattering state, E, is the total energy of the system under consideration. The operator $\mathcal{A}$ serves to antisymmetrize the basis sets whose normalization is chosen to be

$$\begin{aligned}&\langle i|j\rangle=\delta_{ij};\quad \langle i|\psi_{E\lambda}{}^{(0)}\rangle=0;\\ &\langle\psi_{E\lambda}{}^{(0)}|\psi_{E'\lambda'}{}^{(0)}\rangle=\delta_{\lambda\lambda'}\delta(E-E').\end{aligned} \quad (2.3)$$

Finally, the asymptotic behavior of the scattering states is, in the absence of the Coulomb potential,

$$\varphi_{E\lambda''}{}^{(0)}(\mathbf{r})\rightarrow\left(\frac{2}{\pi}\frac{dk}{dE}\right)^{1/2}\sin(kr+\delta_{\lambda''}-\tfrac{1}{2}\pi l)\mathcal{Y}_{lj}{}^{m}(\theta). \quad (2.4)$$

It is also useful to introduce states

$$\varphi_{E\lambda}{}^{(\pm)}(r)=e^{\pm i\delta_\lambda}\varphi_{E\lambda}{}^{(0)}(r) \quad (2.5)$$

and the corresponding many-particle states

$$|\psi_{E\lambda}{}^{(\pm)}\rangle=e^{\pm i\delta_\lambda}|\psi_{E\lambda}{}^{(0)}\rangle. \quad (2.6)$$

The S-matrix elements are related to those of the T matrix by

$$\begin{aligned}&S_{\lambda\lambda'}=\delta_{\lambda\lambda'}-2\pi iT_{\lambda\lambda'},\\ &T_{\lambda\lambda'}=T_{\lambda\lambda'}{}^{(0)}+T_{\lambda\lambda'}{}^{(1)}\equiv\langle\phi_\lambda|U|\psi_{E\lambda'}{}^{(+)}\rangle\\ &\qquad\qquad+\langle\psi_{E\lambda}{}^{(-)}|V|\Psi_{E\lambda'}{}^{(+)}\rangle,\end{aligned} \quad (2.7)$$

the first term representing the potential scattering and the second the effects of the residual interaction. Thus

$$S_{\lambda\lambda'}=e^{2i\delta_\lambda}\delta_{\lambda\lambda'}-2\pi iT_{\lambda\lambda'}{}^{(1)}. \quad (2.8)$$

The essential approximation of I is based on the observation that in the evaluation of $T_{\lambda\lambda'}{}^{(1)}$ the total wave function of the system, $|\Psi_{E\lambda}{}^{(+)}\rangle$, need only be known in the "interior" of the nucleus.

Since it is desirable to avoid the solution of integral equations, the expansion of $|\Psi_{E\lambda}{}^{(+)}\rangle$ is made over a *discrete* set of functions $|\chi_k\rangle$. If the $\{|\chi_k\rangle\}$ form a complete set, at least in the interior, the solution for the problem is exact.

The interior may be defined by choosing a radius which is somewhat larger than the nuclear radius plus the range of the effective interaction. A complete set of functions in this region of configuration space can either be generated by specifying a boundary condition at this radius (in analogy to R-matrix concepts), or in a more natural way by using Weinberg functions. Harmonic-oscillator functions provide an alternate choice. If one uses only a limited number of these functions, the choice of these expansion functions matters and one should be careful in selecting functions $|\chi_k\rangle$ which can approximate the final solution as well as possible. Here one has to be guided by more or less intuitive ideas concerning the reaction mechanism.

It is not necessary to introduce a variational principle for the K matrix (or T matrix) as in I, but one may proceed directly from the Lippman-Schwinger equation for $V|\Psi_{E\lambda}{}^{(+)}\rangle$, which is the quantity required for the calculation of the T matrix. If the state $V|\Psi_{E\lambda}{}^{(+)}\rangle$ is expanded as

$$V|\Psi_{E\lambda}{}^{(+)}\rangle\equiv\sum_k C_k{}^{(E\lambda)}V|\chi_k\rangle, \quad (2.9)$$

the equation for the $C_k^{(E\lambda)}$ is readily found:

$$\sum_k \langle\chi_i|V-VG_0^{(+)}(E)V|\chi_k\rangle C_k^{(E\lambda)} = \langle\chi_i|V|\psi_{E\lambda}^{(+)}\rangle, \quad (2.10)$$

with

$$G_0^{(+)}(E)=(E-H_0+i\epsilon)^{-1}. \quad (2.11)$$

Because of the finite number of expansion functions, we need only deal with the algebraic problem of Eq. (2.10). One finds

$$C_k^{(E\lambda)}=\sum_k A_{ik}^{-1}\langle\chi_k|V|\psi_{E\lambda}^{(+)}\rangle, \quad (2.12)$$

where

$$A_{ik}=\langle\chi_i|V-VG_0^{(+)}V|\chi_k\rangle. \quad (2.13)$$

Further,

$$T_{\lambda'\lambda}^{(1)}=\sum_{i,k}\langle\psi_{E\lambda'}^{(-)}|V|\chi_i\rangle A_{ik}^{-1}\langle\chi_k|V|\psi_{E\lambda}^{(+)}\rangle = e^{i(\delta_\lambda+\delta_{\lambda'})}\sum_{ik}\langle\psi_{E\lambda'}^{(0)}|V|\chi_i\rangle A_{ik}^{-1} \times\langle\chi_k|V|\psi_{E\lambda}^{(0)}\rangle. \quad (2.14)$$

As in I, the S matrix is given by

$$S_{\lambda\lambda'}=e^{i(\delta_\lambda+\delta_{\lambda'})}\left\{\delta_{\lambda\lambda'}+2\pi i \times\det\begin{pmatrix} 0 & \langle\psi_{E\lambda}^{(0)}|V|\chi_k\rangle \\ \langle\chi_i|V|\psi_{E\lambda'}^{(0)}\rangle & A_{ik}\end{pmatrix}\Big/\det(A_{ik})\right\}. \quad (2.15)$$

We stress that the expression for the S matrix is invariant to changing the normalization of any of the expansion functions $|\chi_k\rangle$.

The specific choice of the $|\chi_k\rangle$ and the Green's function $G_0^{(+)}(E)$ will be discussed later. However, it is worth pointing out that the construction of the Green's function is independent of the choice of the set $\{|\chi_k\rangle\}$ and is only dependent on the specification of H_0.

In concluding this chapter, we describe a method which we found useful for improving the rate of convergence of the Born series. In the framework of the shell model in the continuum, the Hamiltonian

$$H=H_0+V \quad (2.16)$$

is diagonalized in a subspace of the eigenfunctions of H_0. Or, in another language, the T matrix is calculated by the "two-potential formula"

$$T_{\lambda\lambda'}=\langle\phi_\lambda|U|\psi_{E\lambda'}^{(+)}\rangle+\langle\psi_{E\lambda}^{(-)}|V+VG_0^{(+)}V +VG_0^{(+)}VG_0^{(+)}V+\cdots|\psi_{E\lambda'}^{(+)}\rangle. \quad (2.17)$$

Only by using very complicated methods (coupled integro-differential equations) is it possible to sum up the Born series in the second expression of Eq. (2.17). Many authors have therefore proposed using perturbation methods for evaluating the second expression. These methods depend critically on whether the operator $G_0^{(+)}V$ is "small." Fortunately, it has been found that the effective interaction to be used in the shell model is such that $G_0^{(+)}V$ is indeed small, provided there are no narrow single-particle resonances in $G_0^{(+)}(E)$. The main effort in improving the convergence of the Born series has been concentrated upon the problem of removing the single-particle resonances from $G_0^{(+)}(E)$. We are not aware of any attempt to make $G_0^{(+)}V$ small by taking out the most important parts of V and introducing them into the operator H_0. Indeed, there exists some arbitrariness in splitting H into two parts. One may write

$$H=\{\sum_{i=1}^{A}[T(i)+U(i)+\Delta U(i)]\} +\{\tfrac{1}{2}\sum_{i\neq j}v_{ij}-\sum_i\Delta U(i)\} =\hat{H}_0+\hat{V}, \quad (2.18)$$

where $\Delta U(i)$ may be chosen at one's convenience. In complete formal analogy, $T_{\lambda\lambda'}$ can be expressed in terms of the quantities referring to the separation [Eq. (2.18)]

$$T_{\lambda\lambda'}=\langle\phi_\lambda|U+\Delta U|\hat{\psi}_{E\lambda'}^{(+)}\rangle\times\langle\hat{\psi}_{E\lambda}^{(-)}|\hat{V}+\hat{V}\hat{G}_0^{(+)}\hat{V} +\hat{V}\hat{G}_0^{(+)}\hat{V}\hat{G}_0^{(+)}\hat{V}+\cdots|\hat{\psi}_{E\lambda'}^{(+)}\rangle. \quad (2.19)$$

In certain cases, the introduction of the additional diagonal potential ΔU improves the convergence considerably. [After having introduced ΔU one might still remove dangerous single-particle resonances from $\hat{G}_0^{(+)}(E)$.] The additional potential ΔU may depend on the total spin or the total isospin of the system. When diagonalizing H exactly, the solution should not depend on ΔU, but any method based on perturbation theory will. The proposed method is not generally applicable, but is limited to those cases where the main strength of the residual interaction is concentrated in the diagonal matrix elements and where the diagonal matrix elements all have about the same magnitude and sign. For example, this holds for all matrix elements between $T=1$ states where the residual interaction is always strongly repulsive.

We are only aware of the work of MacDonald,[5] who used the freedom of introducing a term ΔU in order to remove dangerous single-particle resonances from $G_0^{(+)}$ and to make them bound states. He did not introduce ΔU in order to increase the rate of convergence of the Born series. On the contrary, in the application to $O^{16}(\gamma,p)N^{15}$ the ΔU, which was chosen to convert the $d_{3/2}$ single-particle resonance into a bound state, *in-*

creased the diagonal matrix elements, thus making the Born series converge more slowly.

III. APPLICATION TO THE FOUR-NUCLEON SYSTEM

In the application of this formalism[15] to the reactions $He^3(p,p)He^3$ or $He^4(\gamma,p)H^3$ the question arises as to the appropriate treatment of the $P_{1/2}$ or $P_{3/2}$ "single-particle resonances." In particular, the $P_{1/2}$ resonance is too broad and ill-defined to apply the prescriptions of I for the treatment of sharp single-particle resonances. It is felt, however, that these resonances play some role in the reaction process and their presence should be taken into account. As the resonances are too broad to make a pole approximation for the Green's function, this function is treated exactly. For the $|\chi_k\rangle$, however, two approximations are considered which correspond to two extreme points of view concerning the reaction mechanism. In the first case, it is assumed that the $P_{3/2}$ and $P_{1/2}$ waves exhibit sufficiently sharp resonances such that these resonances are important in the formation of a compound nucleus resonance during the scattering process. Thus, the $P_{3/2}$ and $P_{1/2}$ waves *at their resonant energies, E_0 and E_1* play an important role, and it is these wave functions which are used in the expansion of $V|\Psi_{E\lambda}^{(+)}\rangle$ [Eq. (2.9)],

$$V|\Psi_{E\lambda}^{(+)}\rangle=\alpha V|\varphi_{E_0,P_{3/2}}^{(0)},S_{1/2}^{-1}\rangle+\beta V|\varphi_{E_1,P_{1/2}}^{(0)},S_{1/2}^{-1}\rangle \quad (3.1)$$

(resonance approximation).

In the second approximation it is assumed that these resonances are rather ill-defined and no significant development of a compound state takes place. Therefore, we have essentially a direct interaction approximation [or distorted-wave Born approximation (DWBA)], in which case the scattered particle may be said to spend only a short time in the vicinity of the target. In this case it becomes more natural to expand $V|\Psi_{E\lambda}^{(+)}\rangle$ using the $P_{3/2}$ and $P_{1/2}$ scattering states *at the energy of the scattering process, E.*

$$V|\Psi_{E\lambda}^{(+)}\rangle=\alpha V|\varphi_{E,P_{3/2}}^{(0)},S_{1/2}^{-1}\rangle+\beta V|\varphi_{E,P_{1/2}}^{(0)},S_{1/2}^{-1}\rangle \quad (3.2)$$

(direct-interaction approximation).

It is difficult to estimate the accuracy of the first approximation [Eq. (3.1)]. However, the study of the convergence of the Born series gives a reliable estimate for the accuracy of the approximation [Eq. (3.2)]. Inserting this approximation for $V|\Psi_{E\lambda}^{(+)}\rangle$ into the general expression for the S matrix [Eq. (2.15)] we find for the T matrix (for simplicity we restrict the argument to one channel only)

$$T_{\lambda\lambda}^{(1)}=\frac{\langle\psi_{E\lambda}^{(-)}|V|\psi_{E\lambda}^{(0)}\rangle\langle\psi_{E\lambda}^{(0)}|V|\psi_{E\lambda}^{(+)}\rangle}{\langle\psi_{E\lambda}^{(0)}|V-VG_0^{(+)}V|\psi_{E\lambda}^{(0)}\rangle}=\langle\psi_{E\lambda}^{(-)}|V|\psi_{E\lambda}^{(+)}\rangle+\langle\psi_{E\lambda}^{(-)}|VG_0^{(+)}V|\psi_{E\lambda}^{(+)}\rangle+\frac{\langle\psi_{E\lambda}^{(-)}|VG_0^{(+)}V|\psi_{E\lambda}^{(0)}\rangle\langle\psi_{E\lambda}^{(0)}|VG_0^{(+)}V|\psi_{E\lambda}^{(+)}\rangle}{\langle\psi_{E\lambda}^{(0)}|V|\psi_{E\lambda}^{(0)}\rangle}+\cdots. \quad (3.3)$$

Comparing to the exact expression for the Born series [Eq. (2.17)], one observes that the expression (3.3) is exact up to first order in $G_0^{(+)}V$ while the term of second order in $G_0^{(+)}V$ differs. However, provided that the approximation

$$V|\varphi_{E\lambda}^{(0)}\rangle=f(E,\lambda)V|\psi_{E_0,\lambda}^{(0)}\rangle, \quad (3.4)$$

with $f(E,\lambda)$ being a scalar function, is a good one, the above expression (3.3) for $T^{(1)}$ coincides in all orders with the exact one. (The approximation defined by Eq. (3.4) is not used in this paper.)

The Green's function $G_0^{(+)}(E)$ is treated exactly,

$$G_0^{(+)}=\sum_\lambda\int dE'\psi^*_{E'\lambda}{}^{(0)}(\mathbf{r})\psi_{E'\lambda}{}^{(0)}(\mathbf{r}')/(E-E'+i\epsilon), \quad (3.5)$$

except that the integral over the continuum states is cut off at the energy $E=30$ MeV. This cutoff is in keeping with the solution of the problem in a restricted basis and is in analogy with the neglect of $2P_{3/2}$ and $2P_{1/2}$ states, for example, in a corresponding bound-state shell-model treatment. The cutoff also provides a method for eliminating particle-hole–like components of high-lying $T=1$ spurious states. The numerical results are found to be quite insensitive to an increase of the cutoff energy.

In the calculations reported here a j-j coupling scheme has been used with the coupling order $(sl)j$. Since the S matrix is given in the literature[7,8] in the L-S scheme, it is necessary to discuss the relation of these matrices for $J=1^-$, where the S matrix is of dimension 2. The parameterization of the experiment is done in terms of the eigenvalues of the S matrix and the mixing parameter ϵ. The eigenvalues are denoted by $\exp(2i\delta_{0,1}{}^1)$ and $\exp(2i\delta_{1,1}{}^1)$, which correspond to the singlet and triplet phase shifts $\delta_{S,L}{}^J$ in the limit $\epsilon\to 0$.

The elements of the S matrix, $S_{LS,L'S'}{}^J$, are given by

$$\begin{aligned}S_{10;10}{}^1&=\cos^2\epsilon\exp(2i\delta_{0,1}{}^1)+\sin^2\epsilon\exp(2i\delta_{1,1}{}^1),\\ S_{11;11}{}^1&=\sin^2\epsilon\exp(2i\delta_{0,1}{}^1)+\cos^2\epsilon\exp(2i\delta_{1,1}{}^1),\end{aligned} \quad (3.6)$$

and

$$S_{10;11}{}^1=S_{11;10}{}^1=\tfrac{1}{2}\sin 2\epsilon[\exp(2i\delta_{0,1}{}^1)-\exp(2i\delta_{1,1}{}^1)].$$

[15] We are completely neglecting the presence of the breakup channels, $He^3(p,d)2p$ ($Q=-5.5$ MeV) and $He^3(p,n)3p$ ($Q=-7.7$ MeV).

The elements $S_{LS;L'S'}{}^J$ may be obtained from the S matrix in j-j coupling, $S_{lj,l'j'}{}^J$, using the following relations:

$$\begin{aligned} S_{10;10}{}^1&=\tfrac{1}{3}S_{1\frac{1}{2};1\frac{1}{2}}{}^1+\tfrac{2}{3}S_{1\frac{3}{2};1\frac{3}{2}}{}^1+\tfrac{1}{3}2\sqrt{2}S_{1\frac{1}{2};1\frac{3}{2}}{}^1,\\ S_{10;11}{}^1&=-\tfrac{1}{3}\sqrt{2}S_{1\frac{1}{2};1\frac{1}{2}}{}^1+\tfrac{1}{3}\sqrt{2}S_{1\frac{3}{2};1\frac{3}{2}}{}^1-\tfrac{1}{3}S_{1\frac{1}{2};1\frac{3}{2}}{}^1,\\ S_{11;11}{}^1&=\tfrac{2}{3}S_{1\frac{1}{2};1\frac{1}{2}}{}^1+\tfrac{1}{3}S_{1\frac{3}{2};1\frac{3}{2}}{}^1-\tfrac{1}{3}2\sqrt{2}S_{1\frac{1}{2};1\frac{3}{2}}{}^1. \end{aligned} \tag{3.7}$$

Since $L=1$ and $l=1$ in the above formulas, we may use the notation $S_{SS'}$ and $S_{j,j'}$ for the following discussion.

It is easy to solve for the eigenvalues of the S matrix and the mixing parameter:

$$\exp(2i\delta_{0,1}{}^1)=\tfrac{1}{2}[S_{0,0}+S_{1,1}+2S_{1,0}/\sin2\epsilon],$$

$$\exp(2i\delta_{1,1}{}^1)=\tfrac{1}{2}[S_{0,0}+S_{1,1}-2S_{1,0}/\sin2\epsilon],$$

$$\begin{aligned} \tan(2\epsilon)&=\frac{2S_{1,0}}{S_{0,0}-S_{1,1}}\\ &=2\sqrt{2}\left(\frac{-S_{1/2,1/2}+S_{3/2,3/2}-S_{1/2,3/2}/\sqrt{2}}{-S_{1/2,1/2}+S_{3/2,3/2}+4\sqrt{2}S_{1/2,3/2}}\right). \end{aligned} \tag{3.8}$$

Note that this solution is invariant to the replacement of ϵ by $\epsilon'=\epsilon\pm\frac{1}{2}\pi$ accompanied by the interchange of $\delta_{0,1}{}^1$ and $\delta_{1,1}{}^1$. Further, if there is no mixing in the j-j coupling scheme (i.e., $S_{1/2,3/2}=0$) the mixing parameter ϵ_0 is given by $\tan(2\epsilon_0)=2\sqrt{2}$. One may write

$$\tan(2\epsilon)=\tan2(\epsilon_0+\delta), \tag{3.9}$$

where

$$\tan2\delta=2S_{1/2,3/2}/(S_{1/2,1/2}-S_{3/2,3/2}). \tag{3.10}$$

Since the sign of $S_{1/2,3/2}$ may be changed without changing any observables there are two possible solutions for ϵ other than the $\frac{1}{2}\pi$ ambiguity,

$$\tan(2\epsilon_\pm)=\tan2(\epsilon_0\pm\delta). \tag{3.11}$$

Also it is easy to see that the change in sign of $S_{1/2,3/2}$ does not change the eigenvalues of the S matrix nor their order. However, this modification changes the sign of δ. Similarly, $S_{1,0}$ may be replaced by its negative with a corresponding sign change of ϵ. These changes in $S_{1,0}$ and $S_{1/2,3/2}$ may not be made independently because of the linear relationship [Eq. (3.7)] between the S-matrix elements in the two representations. The rotation parameters ϵ, δ, and ϵ_0 have a simple interpretation. If one defines the matrix

$$U(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}, \tag{3.12}$$

then the S matrix in diagonal form, S^D, is taken to the S matrix in L-S coupling by

$$S_{[L,S]}=U^\dagger(\epsilon)S^DU(\epsilon). \tag{3.13}$$

The matrix taking one from the L-S scheme to the j-j scheme is $U(-\epsilon_0)$, i.e.,

$$S_{[j,j]}=U^\dagger(-\epsilon_0)S_{[L,S]}U(-\epsilon_0). \tag{3.14}$$

Thus,

$$S_{[j,j]}=U^\dagger(\delta)S^DU(\delta), \tag{3.15}$$

with $\delta=\epsilon-\epsilon_0$.

IV. RESULTS

In the spirit of what has been said in Sec. III, a semiphenomenological shell-model Hamiltonian

$$H=\sum_i[T(i)+U(i)]+\tfrac{1}{2}\sum_{i\neq j}v(i,j) \tag{4.1}$$

has been set up. The potential U has a Woods-Saxon form plus a Coulomb potential,

$$U(r)=-U_0f(r)-U_{so}(m_\pi)^{-2}\mathbf{l}\cdot\mathbf{s}\frac{d}{dr}V(r)+V^{\text{Coul}}(r), \tag{4.2}$$

with

$$\begin{aligned} f(r)&=[1+\exp((r-R)/a)]^{-1},\\ R&=1.74\text{ fm},\quad a=0.4\text{ fm},\\ U_0&=71.4\text{ MeV},\quad U_{so}=4.5\text{ MeV}. \end{aligned} \tag{4.3}$$

After choosing the listed values for R and a, the depth of the central part is fixed to provide a bound proton S state at 19.8 MeV corresponding to the photodisintegration threshold of the reaction $He^4(\gamma,p)He^3$. Then, the neutron S state in this potential is found at 20.7 MeV [compared to an energy of 20.6 MeV for the threshold of $He^4(\gamma,n)He^3$]. The root-mean-square radius of the proton orbit is 1.75 fm (experimental value for He^4 is 1.63 ± 0.04 fm). The spin-orbit splitting is essentially a free parameter chosen to be of the order of magnitude appropriate to nucleon scattering from He^4. For the calculation of the scattering phase shifts a Coulomb potential with $Z=2$ was used, while in the calculation of the photodisintegration the continuum waves were constructed with $Z=1$. The Kurath force[16] is used for the residual interaction. This force is an effective interaction which is determined by fitting the low-energy properties of the nuclei of the p shell. The force is given by

$$\begin{aligned} V(r)&=-V_0[P_M+\tfrac{1}{4}P_\sigma](e^{-\mu r}/\mu r),\\ V_0&=36\text{ MeV},\quad \mu=0.714\text{ fm}^{-1}, \end{aligned} \tag{4.4}$$

and P_M and P_σ are the space and spin-exchange operators. The diagonal matrix elements of this force in the $T=1$ states, to which we restrict ourselves in this calculation, are rather large. They are positive and all of the same order of magnitude (about a factor 2 to 3 larger than the nondiagonal matrix elements.) Therefore, the proposed method for improving the convergence of the Born series can be applied. And additional central potential ΔU is chosen with a strength of 8.5 MeV. With this modification, diagonal elements of

[16] D. Kurath, Phys. Rev. 101, 216 (1956).

$\hat{V}\equiv V-\Delta U$ become of the same order of magnitude as the (unchanged) nondiagonal ones of V. Moreover, the introduction of ΔU shifts the single-particle resonances, which occur in $G_0^{(+)}$ even further off the real axis. This combined effect of ΔU causes the ratio

$$R=\left(\frac{\langle\hat{\psi}_{E\lambda}|\,\hat{V}\hat{G}_0^{(+)}\hat{V}\,|\hat{\psi}_{E\lambda}\rangle}{\langle\hat{\psi}_{E\lambda}|\,\hat{V}\,|\hat{\psi}_{E\lambda}\rangle}\right) \tag{4.5}$$

to be smaller than $\frac{1}{3}$ at the energy of the $P_{3/2}$ resonance ($E_{\rm res}\sim 6$ MeV), while it is of the order of $\frac{1}{10}$ for the other energies. This is to be contrasted to the case $\Delta U=0$, where the ratio R takes values as large as 2, indicating the divergence of the Born series. Therefore, we conclude that after having introduced ΔU of the given strength, the direct-interaction approximation for our wave function $\Psi_{E\lambda}^{(+)}$ provides a T matrix which is accurate to at least 10%. We cannot estimate the accuracy of the resonance approximation for $\Psi_{E\lambda}^{(+)}$. However, we find that the calculated scattering phase shifts are very insensitive as to which approximation is used (differences of at most one degree occur in the phase shifts even at 20 MeV, i.e., far from the resonances). The reason for this lies in the fact that the matrix elements of the S matrix for elastic scattering test the wave function $\Psi_{E\lambda}^{(+)}$ only in the interior region. Indeed, we find that 85% of the value of the matrix elements of V originate from regions of space with $r<2.5$ fm and we infer that the approximation

$$V\,|\,\Psi_{E\lambda}\rangle=f(E,\lambda)V\,|\,\Psi_{E_0\lambda}\rangle \tag{4.6}$$

is a good one for the calculation of the S matrix. Recall that the solution with the approximation [Eq. (3.1)] is even exact to the degree relation (4.6) holds. Therefore, in the present calculation of the elastic scattering the accuracy might be much better than the value of 10% given above.

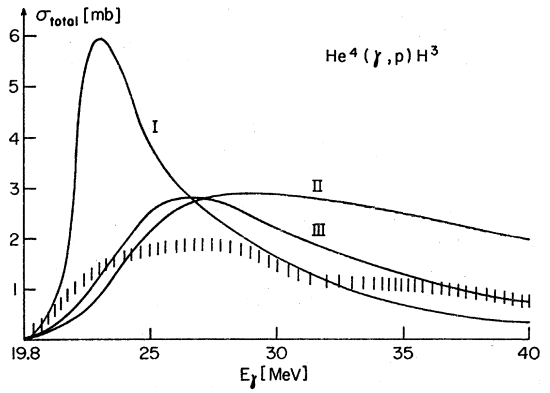

FIG. 1. The calculated (γ,p) cross section under various approximations. Curve I: direct-interaction calculation (residual interaction $V=0$). Curve II: resonance approximation for $|\Psi E\lambda^{(+)}\rangle$ [Eq. (3.1)]. Curve III: direct-interaction approximation for $|\Psi_{E\lambda}^{(+)}\rangle$ [Eq. (3.2)]. Potential parameters of [Eq. (4.3)] (non-l-dependent potential.)

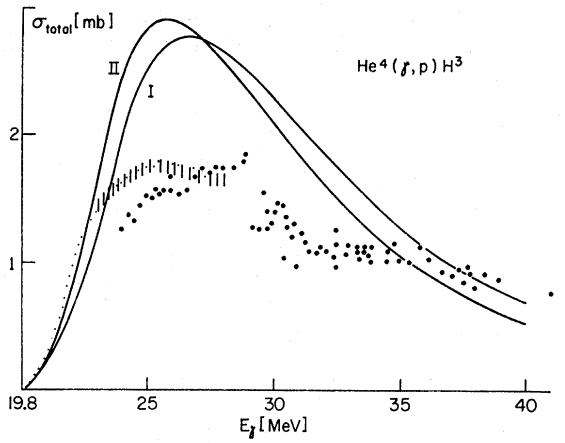

FIG. 2. The calculated (γ,p) cross section compared to various experimental data (Ref. 10) (points, vertical bars, and solid circles). Direct-interaction approximation for $|\Psi_{E\lambda}^{(+)}\rangle$ [Eq. (3.2)]. Curve I: l-independent potential. Curve II: l-dependent potential [Eq. (4.7)].

These estimates do not hold for the calculation of the (γ,p) cross section. Here, we have found that only 50% of the value of the matrix element originates from regions with $r\leq 2.5$ fm. The shape and magnitude of the (γ,p) cross section is more sensitive to the details of the wave function. Figure 1 supports this idea. The (γ,p) total cross section is shown for the two approximations for $|\Psi_{E\lambda}\rangle$ [Eqs. (3.1) and (3.2)]. The difference is remarkable and experiment clearly favors the second choice. On the basis of arguments concerning the convergence of the Born series we estimate the accuracy of the (γ,p) calculation to be about 30% around $E_\gamma=26$ MeV and about 10% elsewhere.

Calculations were performed using the potential parameters of Eq. (4.3). In addition, an l-dependent potential was used with the parameters

$$\begin{aligned} R&=1.74\text{ fm},\quad a=0.4\text{ fm},\quad U_{so}=4.5\text{ MeV},\\ U_0&=75.4\text{ MeV}\quad (l=1)\\ &=71.4\text{ MeV}\quad (l=0), \end{aligned} \tag{4.7}$$

in order to investigate the sensitivity of the results to the shell-model potential. The results of the calculation with the two choices for the potential parameters, Eqs. (4.3) and (4.7), are given in Figs. 2–5.

A word concerning the experimental phase shifts in the 1^- channel may be appropriate. Here, the two solutions of Refs. 7 and 8 which fit the data seem to differ rather drastically. However, by using the freedom which consists in adding $\pm\frac{1}{2}\pi$ to the value of ϵ, associated with an interchange of the eigenphases, one can make them fall on top of each other. We subtracted $\frac{1}{2}\pi$ from the ϵ of solution[8] and used the reflection invariance around $\epsilon_0=-55°$ [Eq. (3.10)] to bring the two derived values of ϵ as close as possible.

The two choices for the shell potential, Eqs. (4.3) and (4.7), yield similar results. The threshold behavior is well reproduced; however, both choices give a cross sec-

tion which is about 50% too large at their maxima. This discrepancy does not seem to arise from the approximations which are made in solving the model problem but probably represents a deficiency in the model itself. Similar discrepancies have also arisen in other calculations of photonuclear reactions. The source of the discrepancy in this calculation may be due to the use of a local and energy-independent shell-model potential. This suggestion is supported by the observation that at the proton energies $E_p \sim 5$ MeV ($E_\gamma \sim 25$ MeV), where the (γ,p) cross section becomes too large, the phase shifts in the 2^- and in one 1^- channel are found to exceed the experimental ones (Figs. 4 and 5). Recall that the potential U is chosen to give a bound S state about -20 MeV. We essentially have no information about the energy dependence of this potential or its strength in the P states.

V. CONCLUSIONS

The following conclusions may be drawn:

1. In this application of the shell model in the continuum, a detailed comparison between experiment and theory has been made. An over-all qualitative agreement is obtained with essentially no free parameters. The results tend to support the basic ideas of the model. The effort involved was not appreciably greater than in an ordinary shell-model calculation. A calculational accuracy of a few percent seems to be more than sufficient for a calculation of this type because the shell-model approach itself as a way to solve the many-body problem does not seem to give a more accurate description of reality. This is true at least for the system considered here, where the relative success of the model is rather surprising.

2. It has been found that the calculation of the photoabsorption cross section provides a more sensitive test of the approximations made for the wave function than

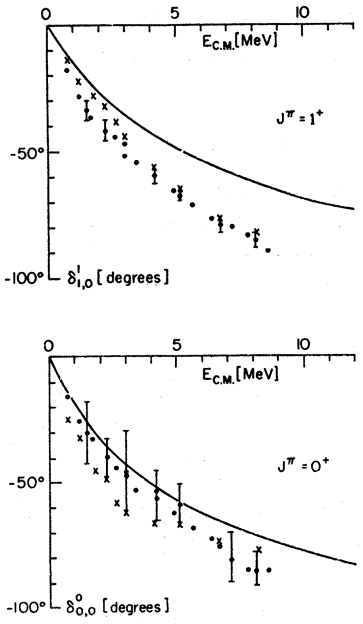

FIG. 3. The S-wave phase shifts δ_{S,L^J} for singlet ($\delta_{0,0^0}$)and triplet ($\delta_{1,0^1}$) channel spin. The solid circles (Ref. 7) and the crosses (Ref. 8) were derived from analyzing experimental data. A direct-interaction approximation analogous to Eq. (3.2) is used for $|\Psi_{E\lambda}{}^{(+)}\rangle$.

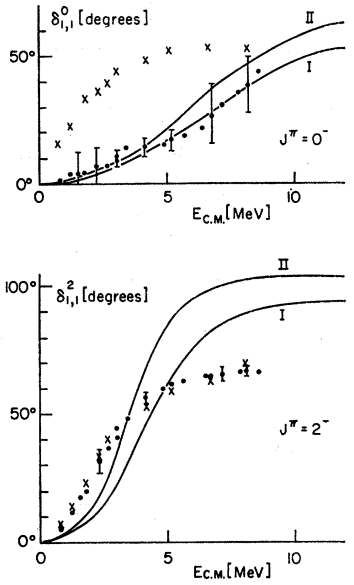

FIG. 4. The P-wave phase shifts δ_{S,L^J} for $J^\pi = 0^-$ ($\delta_{1,1^0}$) and $J^\pi = 2^-$($\delta_{1,1^2}$). The solid circles (Ref. 7) and the crosses (Ref. 8) were derived from analyzing experimental data. A direct-interaction approximation (3.2) is used for $|\Psi_{E\lambda}{}^{(+)}\rangle$. Curve I: l-independent potential [Eq. (4.3)]. Curve II: l-dependent potential [Eq. (4.7)].

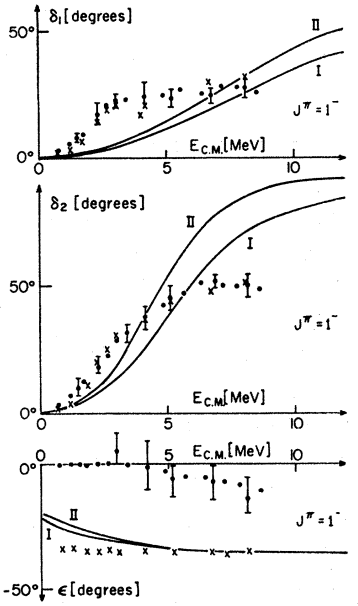

FIG. 5. The S matrix in the 1^- channel. The two eigenphases δ_1 and δ_2 are given together with the mixing parameter ϵ (in L-S representation) [Eqs. (3.5) to (3.14)]. The experimental data of Refs. 7 and 8 have been made to coincide as well as possible by a procedure described in the text. Curve I: l-independent potential [Eq. (4.3)]. Curve II: l-dependent potential [Eq. (4.7)].

the calculation of the scattering phase shifts. This observation parallels the well-known fact in bound-state shell-model calculations that energy levels are less sensitive to the model wave functions than transition rates. A fairly good description of the scattering process is probably obtained even with rather crude approximations. The energy dependence of the total wave function of the system appears to be quite important in explaining the shape of the photoabsorption cross section.

3. Not too much can be said concerning the energy dependence (or nonlocality) of the shell-model potential. However, we feel that at least part of the discrepancy may be removed by a more refined choice of the shell-model potential or the effective interaction.

Note added in proof. Recently another calculation of the photodisintegration cross section of He^4 has appeared [F. Beck and A. Müller Aruke, Phys. Letters **27B**, 343 (1968)].

ACKNOWLEDGMENT

One of the authors (J.H.) would like to thank Professor Feshbach for the hospitality extended to him at the Center for Theoretical Physics.

PHYSICAL REVIEW VOLUME 175, NUMBER 4 20 NOVEMBER 1968

π^--^{4}He Inelastic and Capture Reactions Leading to Excited and Multineutron Final States

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A beam of (140±0.5)-MeV π^- was produced at the Berkely 184-in. cyclotron and used to study the final-state interactions of three and four neutrons, and to look for excited levels of the α particle through the reactions

$$\begin{aligned}\pi^-+{}^4\mathrm{He} &\rightarrow \pi^-+{}^4\mathrm{He}^*,\\ &\rightarrow p+3n,\\ &\rightarrow d+2n,\\ &\rightarrow \pi^++4n.\end{aligned}$$

Only one such level is found, with an excitation energy of 32 MeV and an intrinsic width smaller than our 1-MeV resolution. We find that our data on the four-neutron final state, although not inconsistent with phase space, agree more closely with the assumption that there is a $^1S_0$ final-state interaction between two of the neutrons, the other two not interacting. We find, too, that deuteron production is down by a factor of $\approx 10^3$ from proton production, and that the proton spectrum indicates a stronger-than-expected interaction between the three neutrons in the final state. Lower limits for the production of a tri- or tetraneutron are set.

I. INTRODUCTION

THE n-n interaction at low energies has been extensively studied through reactions such as $D(n,p)2n$,[1] $^3H(n,d)2n$,[2] and $T(d,^3He)2n$,[3] and through a different approach by the reaction $\pi^-D \rightarrow 2n\gamma$,[4] where in the final state only the two neutrons are strongly interacting. The theory for the analysis of the data obtained in these experiments is well enough known[3,5–7] that it is not discussed here.

On the other hand, data on the three- and four-neutron systems are scarce and inadequate, and theoretical predictions are contradicting and inconclusive.

A. Three-Neutron System

The ^3n has been searched for through the reaction $^3H(n,p)3n$. In 1965 Ajdacic *et al.* reported observing a proton distribution of energy that led to a ^3n bound by about 1 MeV.[8] This experiment was repeated later at Oak Ridge National Laboratory,[9] and no evidence for the existence of the ^3n system was observed.

A paper by Mitra and Bhasin[10] predicts the existence of the ^3n. They argue that only a moderate ^3P attractive force is needed between all neutron pairs to yield a bound ^3n system, and they predict an $(LSJ)=(1,\frac{3}{2},\frac{1}{2})$ state as the most likely, with $(1,\frac{3}{2},\frac{3}{2})$ a second best. Mitra and Bhasin comment that the existence of the ^3n is independent of the ^4n, for in the latter the $^1S_0$ re-

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