Investigation of Nuclear Three- and Four-Body Systems with Soft-Core Nucleon-Nucleon Potentials*

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The nuclear three- and four-body systems are investigated with a variational method and various soft-core nucleon-nucleon potentials. A Monte Carlo method is used to compute the multidimensional integrals involved. The results show that (i) a soft-core potential and a hard-core potential which give the same effective-range parameters and S-wave phase shifts yield very nearly the same values for the binding energies and rms radii of H^3 and He^4, and (ii) as far as the ground-state properties of H^3 and He^4 are concerned, the fitting of the high-energy ($\gtrsim$100-MeV) S-wave phase shifts need not be seriously considered in choosing a nucleon-nucleon potential. Also, it is shown that by paying less attention to the higher-energy ($\gtrsim$50-MeV) nucleon-nucleon scattering data, a soft-core potential can be found which yields nearly correct values for the binding energies, rms radii, and form factors up to about 10 F^{-2} for both H^3 and He^4. The soft-core potentials proposed by Eikemeier and Hackenbroich and by Volkov have also been briefly considered. Here the results indicate that the mathematical approximations used by Hackenbroich *et al.* to investigate the properties of light nuclei are inaccurate, while the binding energy of He^4 calculated by Volkov is too small by about 2 MeV.

I. INTRODUCTION

RECENTLY, there has been a growing interest in the use of soft-core potentials for evaluating the binding energy of nuclear matter,[1] and calculations in light nuclei using shell-model[2] and resonating-group method.[3] The soft-core potentials used in these calculations usually fit the nucleon-nucleon phase shifts, scattering length, and effective range. One immediately raises the question of the equivalence of such potentials to the hard-core potentials so far used, i.e., given two potentials, one with a soft-core and the other with a hard-core, both fitting the nucleon-nucleon data, will one get the same binding energy for nuclei using these two potentials?

The aim of the present investigation is, first, to show the equivalence of hard-core and soft-core potentials in giving the same binding energies and form factors for light nuclei such as H^3 and He^4. The two potentials used are chosen to give the same nucleon-nucleon S-wave phase shifts, scattering length, and effective range. The results show that they do yield approximately the same values for these quantities of interest. However, it is also found that the use of a soft-core potential has a great advantage in a variational calculation, since it gives a lower bound that is much better than the one obtained with an equivalent hard-core potential. Second, we will examine, with another soft-core potential, to see the sensitivity of the binding energies and rms radii to changes in the high-energy phase shifts given by these potentials. Third, an attempt will be

made to construct a soft-core potential that fits the low-energy (up to $\approx$100-MeV) nucleon-nucleon data, but also yields nearly the correct binding energies and rms radii for H^3 and He^4. Such a potential will be useful for shell-model and resonating-group calculations, since especially in the latter type of calculations, it is important that one has the correct sizes for the nucleon clusters. As will be seen, it is indeed possible to find such a potential which, at the same time, fits the form factors of the three- and four-body systems up to rather large values of momentum transfer. Finally, we will discuss a soft-core potential used by other authors[4] for resonating-group calculations.

The method used for evaluating the binding energies of H^3 and He^4 is a typical variational one. In Sec. II we will first discuss the method of evaluating the upper and lower bounds on the energy and a method of estimating the eigenvalue. Then, we proceed to a discussion of the form of the Hamiltonian and trial wave function. In all our calculations the multidimensional integrals will be evaluated using a Monte Carlo method discussed previously.[5] Section III is devoted to a description of the various hard-core and soft-core potentials used in this investigation. In Sec. IV, the numerical results will be presented and in Sec. V, a discussion of the results will be made.

In the Appendix, a soft-core potential proposed by Volkov[2] for shell-model calculations on the p-shell nuclei will be considered.

II. FORMULATION AND METHOD OF CALCULATION

A variational method will be used for calculating the upper and lower bounds on the ground-state energy E_0.

¹ H. A. Bethe, J. Phys. Soc. Japan Suppl. **24**, 56 (1968).
² A. B. Volkov, Nucl. Phys. **74**, 33 (1965).
³ D. R. Thompson and Y. C. Tang, Phys. Rev. **159**, 806 (1967); Phys. Rev. Letters **19**, 87 (1967); Nucl. Phys. **A106**, 591 (1968); Phys. Letters **26B**, 194 (1968).

⁴ H. Eikemeier and H. H. Hackenbroich, Z. Physik **195**, 412 (1966).
⁵ R. C. Herndon and Y. C. Tang, Methods Computational Phys. **6**, 153 (1966).

The upper bound E_U is evaluated using the Rayleigh-Ritz method, which gives

$$E_0 \leq E_U = \langle \mathcal{3C} \rangle , \qquad (1)$$

where $\mathcal{3C}$ is the Hamiltonian of the system. The lower bound E_L is evaluated using a method of Temple,[6] i.e.,

$$E_0 \geqslant \langle \mathcal{3C} \rangle - (\langle \mathcal{3C}^2 \rangle - \langle \mathcal{3C} \rangle^2)/(E_1 - \langle \mathcal{3C} \rangle) = E_L , \qquad (2)$$

where E_1 is the energy of the first excited state having the same symmetry as the ground state, and $\langle \mathcal{3C}^2 \rangle$ is an abbreviation for $\langle \mathcal{3C}\Psi, \mathcal{3C}\Psi \rangle$. From the expression for E_L it is obvious that if $\langle \mathcal{3C} \rangle \ll E_1$, E_L is not very sensitive to the value of E_1 which, in some cases, may not be too well known. In these latter cases, it is then necessary to use estimated values of E_1. For example, in the case of H^3, E_1 is the energy of the deuteron, which can be easily calculated by numerical integration of a two-body equation. On the other hand, for He^4, E_1 is the ground-state energy of the triton, a value not known exactly. Here, however, one can simply use either the value of E_U or the estimated eigenvalue for H^3. Since the condition of $\langle \mathcal{3C} \rangle \ll E_1$ is quite well satisfied, an estimated value for E_1 should suffice.

A knowledge of E_U and E_L allows us to estimate the eigenvalue for the ground state of the system. From calculations on the ground state of the helium atom,[7] and from problems where one can evaluate the eigenvalue (e.g., deuteron),[8] it was found that the eigenvalue is much closer to E_U than to E_L. In particular, if one introduces a quantity η defined as

$$\eta = (E_0 - E_L)/(E_U - E_0) , \qquad (3)$$

then one finds that in these particular cases η is of the order of 50. The value of zero for η is certainly allowed; however, as has been indicated previously,[9] this possibility is very unlikely to occur. To have a good estimate of E_0, it is therefore necessary to have a good estimate for η. This can be done[9] by examining the behavior of the wave function in the configuration space. In the cases that we are mainly concerned with, i.e., the calculation of the binding energies of H^3 and He^4 with soft-core potentials, this rather tedious procedure of estimating η is, however, not necessary since it will be shown that the upper and lower bounds are so close to each other that even an order-of-magnitude estimate for η will yield an estimate for the eigenvalue to within 0.1 MeV.

Using the values of E_U and E_L, we can also estimate how good the trial wave function is by calculating the overlap $\langle \Phi_0 | \Psi \rangle$, where Φ_0 is the ground-state eigen-function. For this purpose let us write

$$\Psi = [1 - \sum_{n=1}^{\infty} |a_n|^2]^{1/2} \Phi_0 + \sum_{n=1}^{\infty} a_n \Phi_n , \qquad (4)$$

where Φ_n's are the eigenfunctions of the Hamiltonian. From this equation we obtain

$$E_U = E_0 + \sum_{n=1}^{\infty} |a_n|^2 (E_n - E_0) \qquad (5)$$

and

$$\langle \Phi_0 | \Psi \rangle = [1 - \sum_{n=1}^{\infty} |a_n|^2]^{1/2} . \qquad (6)$$

Using Eqs. (5) and (6) it is easy to show that

$$\langle \Phi_0 | \Psi \rangle \geqslant \left(\frac{E_1 - E_U}{E_1 - E_0} \right)^{1/2} \geqslant \left(\frac{E_1 - E_U}{E_1 - E_L} \right)^{1/2} . \qquad (7)$$

Thus, given E_1, E_U, and E_L a lower bound on the overlap can be evaluated. We can go even further in studying the wave function to see in what region of the configuration space the wave function is poor. This is done by studying $\mathcal{3C}\Psi/\Psi$ as a function of the sum of the interparticle distances. If Ψ were the eigenfunction, then $\mathcal{3C}\Psi/\Psi$ would take on a constant value, equal to the eigenvalue everywhere in this space. Thus by studying the deviation of $\mathcal{3C}\Psi/\Psi$ from the eigenvalue, information about deficiencies in the trial function Ψ in different parts of this space can be obtained.

Having discussed the variational method for evaluating the upper and lower bounds which are given in terms of $\langle \mathcal{3C} \rangle$ and $\langle \mathcal{3C}^2 \rangle$, we turn our attention to the evaluation of these matrix elements. The Hamiltonian for a system of N interacting nucleons is written as

$$\mathcal{3C} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \nabla_i^2 + \sum_{i<j=1}^{N} V_{ij} + \sum_{i<j=1}^{N} V_c(r_{ij}) \epsilon_{ij} , \qquad (8)$$

where V_{ij} is the nucleon-nucleon potential and is taken to have the form

$$V_{ij} = \frac{1}{2}(1 + P_{ij}^\sigma) V_t(r_{ij}) + \frac{1}{2}(1 - P_{ij}^\sigma) V_s(r_{ij}) . \qquad (9)$$

Here P_{ij}^σ is the spin exchange operator, and $V_t(r)$ and $V_s(r)$ are the triplet and singlet nucleon-nucleon potentials in the even states,[10] respectively. The last term in the Hamiltonian is the Coulomb interaction, with ϵ_{ij} equal to one if i and j label protons, and zero otherwise.

The trial wave function will be chosen to have the form

$$\Psi = \chi \psi , \qquad (10)$$

where χ is the spin-isospin part of the wave function and ψ is the space part of the wave function. For the sake of

[6] G. Temple, Proc. Roy. Soc. (London) A119, 276 (1928).
[7] T. Kinoshita, Phys. Rev. 105, 1490 (1957); 115, 366 (1959).
[8] E. W. Schmid, Y. C. Tang, and R. C. Herndon, Nucl. Phys. 42, 95 (1963).
[9] Y. C. Tang, E. W. Schmid, and R. C. Herndon, Nucl. Phys. 65, 203 (1965).

[10] The potential in the odd states need not be specified, since a fully space symmetric trial function will be used in our calculation.

simplicity we are going to choose the space part to be totally symmetric and of the form

$$\psi = \prod_{i<j=1}^{N} f(r_{ij}).$$ (11)

It is quite clear that this is not the most general trial function, since the presence of spin dependence in the two-body potential requires that the eigenfunction has less symmetric parts. What we shall do below is to replace the actual Hamiltonian $\mathcal{H}$ of Eq. (8) by an effective Hamiltonian H, which contains an effective nucleon-nucleon potential of the form

$$V_e = \tfrac{1}{2}[V_t(r) + V_s(r)],$$ (12)

which has no spin dependence. With this latter simplification, the choice of ψ in Eq. (11) would then be quite appropriate. Of course, the quantity of most interest is the eigenvalue of the Hamiltonian $\mathcal{H}$. But, as has been indicated by a number of calculations for the three-body system,[11] the difference in the eigenvalues of $\mathcal{H}$ and H is only about a few tenths of a MeV.[12] Thus, for the purposes of this investigation, a detailed examination of this simpler problem should be quite sufficient.

Having simplified the problem to one with an effective interaction that is spin-independent, let us turn our attention to the choice of $f(r)$. In choosing $f(r)$ we have to keep in mind two important factors. First, the function ψ should have the right asymptotic behavior when one of the particles is far away from the rest. Second, when two nucleons are very close to each other, the function $f(r)$ should depend primarily on the interaction between these two nucleons, and comparatively little on the presence of other nucleons in the system. Such a wave function has in fact been proposed,[9,13] and is of the form

$$f(r) = u(r)/r, \qquad\qquad r < r_d$$
$$= Ar^{-1/(N-1)}[\exp(-\alpha r) + B\exp(-\beta r)], \quad r > r_d \quad (13)$$

where $u(r)$ is a solution to the equation

$$-(\hbar^2/m)d^2u/dr^2 + [V_e(r) - \epsilon]u(r) = 0.$$ (14)

In Eq. (13), the constants A and B are chosen such that $f(r)$ and its first derivative are continuous at r_d. The quantities α, β, ϵ, and r_d are variational parameters. The factor $r^{-1/(N-1)}$ is chosen such that $f(r)$ and thus ψ has the right asymptotic behavior. For example, if we take $\alpha < \beta$, we have for the case when the Nth particle is far away from the rest,

$$\psi \sim \left(\prod_{i<j=1}^{N-1} f(r_{ij}) \right) \frac{1}{R} e^{-(N-1)\alpha R},$$ (15)

where R is the distance of the Nth nucleon from the center of mass of the other $(N-1)$ nucleons. The parameter α is therefore related to the separation energy of the Nth nucleon. This enables us to make an intelligent guess for the starting value of α in the variational procedure.

It is appropriate to mention that the type of trial wave function adopted here is capable of yielding upper bounds very close to the eigenvalues. For a three-body problem, the results of a large number of calculations,[14] all using the same two-body potential, indicate that the upper bound obtained with our trial function is only a few tenths of a percent away from the eigenvalue. In the case of a four-body problem,[15] similar calculations have not been made. However, the results for E_U and E_L obtained with a purely attractive potential also show that our trial function is certainly a very adequate one. Also, it should be mentioned that our trial function is quite suitable from the viewpoint of numerical computation. Because of the physical significance which some of the parameters possess, it is usually only necessary to vary the parameters once to obtain their optimum values. Thus, the amount of computer time needed is not at all large. For example, in the case of a three-body problem, it takes only about 8 min on the CDC 6600 computer to obtain the optimum value of the upper bound.[16]

Using the optimum wave functions, we can calculate the body form factors F_b of the three- and four-body systems. In the first Born approximation, F_b is given by

$$F_b(q^2) = \int e^{i\mathbf{q}\cdot\mathbf{r}}\rho(\mathbf{r})d\mathbf{r},$$ (16)

with $\rho(\mathbf{r})$ being the nucleon density function, which is normalized according to the condition

$$\int \rho(\mathbf{r})d\mathbf{r} = 1.$$ (17)

The density function is related to the function ψ by the expression

$$\rho(\mathbf{r}) = \int \psi^* \frac{1}{N} \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i)\psi d\tau,$$ (18)

where $\mathbf{r}_i$ is the distance of the ith nucleon from the center of mass of the system. Since the trial wave function used in this investigation is totally space sym-

[11] Y. C. Tang, R. C. Herndon, and E. W. Schmid, Phys. Rev. 134, B743 (1964); S. Rosati and M. Barbi, *ibid.* 147, 730 (1966).

[12] One can also obtain some idea about the difference in the eigenvalues for these two Hamiltonians by computing the respective lower bounds. For example, see Ref. 9.

[13] N. Austern and P. Iano, Nucl. Phys. 18, 672 (1960).

[14] G. A. Baker, Jr., J. L. Gammel, B. J. Hill, and J. G. Wills, Phys. Rev. 125, 1754 (1962); M. H. Kalos, *ibid.* 128, 1791 (1962); S. Aranoff and J. K. Percus, *ibid.* 162, 878 (1967); Y. C. Tang, R. C. Herndon, and E. W. Schmid, *ibid.* 134, B743 (1964); Y. C. Tang and R. C. Herndon, *ibid.* 138, B637 (1965); R. T. Folk, Nucl. Phys. 85, 449 (1966); S. Rosati and M. Barbi, Phys. Rev. 147, 730 (1966); M. Banville and P. D. Kunz (to be published); R. Van Wageningen and L. P. Kok, Nucl. Phys. A98, 365 (1967).

[15] Y. C. Tang and R. C. Herndon, Nucl. Phys. A93, 692 (1967).

[16] For He4, the computing time needed is about 40% longer.

TABLE I. Parameters of soft-core potentials.

Potential	V_{t1} (MeV)	β_{t1} (F^{-2})	V_{t2} (MeV)	β_{t2} (F^{-2})	V_{t3} (MeV)	β_{t3} (F^{-2})	V_{s1} (MeV)	β_{s1} (F^{-2})	V_{s2} (MeV)	β_{s2} (F^{-2})	V_{s3} (MeV)	β_{s3} (F^{-2})
S1	1000.0	5.4	−143.4	0.82	−43.0	0.60	880.0	5.2	− 67.1	0.62	−21.0	0.38
S2	1100.0	4.8	−162.6	0.82	−43.0	0.60	880.0	5.2	− 67.1	0.62	−21.0	0.38
S3	1000.0	3.0	−326.7	1.05	−43.0	0.60	1000.0	3.0	−166.0	0.80	−23.0	0.40
S4	600.0	5.5	− 70.0	0.50	−27.6	0.38	880.0	5.4	− 70.0	0.64	−21.0	0.48

metric, $\rho(\mathbf{r})$ can be written as

$$\rho(\mathbf{r}) = \frac{1}{4\pi r^2} \int \psi^* \frac{1}{N} \sum_{i=1}^{N} \delta(r-r_i)\psi d\tau. \qquad (19)$$

The body form factor calculated using Eqs. (16) and (19) can be compared with that determined experimentally. This latter quantity is obtained from the charge form factor which has been experimentally measured. In the case of the three-body system, the body form factor is expressed,[17] under reasonable assumptions, in terms of the charge form factor for $H^3[F_{ch}(H^3)]$ and the charge form factor for He^3 $[F_{ch}(He^3)]$ as

$$F_b(q^2) = \frac{2F_{ch}(He^3) + F_{ch}(H^3)}{3[F_{ch}{}^p(q^2) + F_{ch}{}^n(q^2)]}, \qquad (20)$$

where $F_{ch}{}^p(q^2)$ and $F_{ch}{}^n(q^2)$ are the proton and neutron charge form factors, respectively. For He^4, an analogous expression can be derived; this is

$$F_b(q^2) = \frac{F_{ch}(He^4)}{F_{ch}{}^p(q^2) + F_{ch}{}^n(q^2)}. \qquad (21)$$

In both Eqs. (19) and (20) a knowledge of

$$[F_{ch}{}^p(q^2) + F_{ch}{}^n(q^2)]$$

is required; in this work, we use

$$F_{ch}{}^p(q^2) + F_{ch}{}^n(q^2) = \frac{2.50}{1+q^2/15.7} - \frac{1.60}{1+q^2/26.7} + 0.1, \qquad (22)$$

which was obtained by Janssens et al.[18]

TABLE II. Fit to nucleon-nucleon effective-range parameters.

Potential	n-p triplet Scattering length (F)	n-p triplet Effective range (F)	n-p singlet Scattering length (F)	n-p singlet Effective range (F)	p-p singlet Scattering length (F)	p-p singlet Effective range (F)
A	5.38	1.70	−16.69	2.73	−7.70	2.61
S1	5.38	1.70	−16.78	2.67	−7.66	2.53
S2	5.40	1.74	−16.78	2.67	−7.66	2.53
S3	5.39	1.74	−16.32	2.76	−7.69	2.60
S4	5.60	1.94	− 7.94	2.67	−5.14	2.53
Experiment	5.39	1.704	−23.74	2.67	−7.68	2.65

[17] L. I. Schiff, Phys. Rev. **133**, B802 (1964).
[18] T. Janssens, R. Hofstadter, E. B. Hughes, and M. R. Yearian, Phys. Rev. Letters. **16**, 169 (1965).

Most of the formulas presented above involve integrals that are multidimensional. To evaluate these integrals, we shall make use of a Monte Carlo method. Since this method has been discussed in detail recently,[5] we shall not further describe it here, except to say that this method has been well tested in many cases and shown to yield accurate results even with a moderate number of estimates which take rather short computing periods on modern high-speed computers.

III. TWO-BODY POTENTIALS

The soft-core potentials used in this investigation have the form of Eq. (9) with

$$V_t(r) = \sum_{i=1}^{3} V_{ti} \exp(-\beta_{ti} r^2) \qquad (23)$$

and

$$V_s(r) = \sum_{i=1}^{3} V_{si} \exp(-\beta_{si} r^2). \qquad (24)$$

Four different potentials will be considered, each for a different purpose. These potentials will be denoted as potentials S1, S2, S3, and S4, with the values of the parameters presented in Table I.

The potential S1 is chosen to show the equivalence of hard-core and soft-core potentials. For the hard-core potential, we use potential A of a previous calculation,[9] which has the form

$$V_t(r) = -V_{0t} \exp[-\beta_t(r-r_c)], \quad r>r_c \qquad (25)$$
$$= +\infty, \quad r<r_c$$

and

$$V_s(r) = -V_{0s} \exp[-\beta_s(r-r_c)], \quad r>r_c \qquad (26)$$
$$= +\infty, \quad r<r_c$$

TABLE III. Comparison of calculated S-wave phase shifts with experimental values.

E_{LAB} (MeV)	n-p $^3S_1$ phase shifts (rad) A	S1	S2	YLAN $3M^3\theta_1{}^s$	p-p $^1S_0$ phase shifts (rad) A	S1 or S2	YLAM K_0
20	1.515	1.512	1.497	1.400	0.871	0.874	0.865
60	1.024	1.010	0.983	0.945	0.581	0.569	0.580
100	0.774	0.756	0.718	0.680	0.394	0.376	0.387
140	0.600	0.582	0.537	0.520	0.255	0.238	0.242
180	0.464	0.452	0.400	0.400	0.143	0.132	0.137
220	0.351	0.349	0.290	0.290	0.048	0.047	0.047
260	0.254	0.264	0.199	0.200	−0.035	−0.023	−0.037
300	0.169	0.192	0.122	0.110	−0.108	−0.083	−0.122
340	0.092	0.131	0.056	0.020	−0.175	−0.134	−0.190

TABLE IV. Variational results for potentials A and S1.

Nucleus	Potential	α (F^{-1})	β (F^{-1})	r_d (F)	ϵ (MeV)	E_U (MeV)	E_L (MeV)	rms radius (F)	Variation performed for optimum values of
H^3	A	0.275	3.0	1.2	$-$ 6.0	$-$ 7.63$\pm$0.05	-17.24 ± 1.72	1.62	E_U
	S1	0.303	1.85	1.3	0.5	$-$ 7.74$\pm$0.05	-12.38 ± 0.15	1.59	E_U
	A	0.245	2.8	1.3	$-$ 4.8	$-$ 7.36$\pm$0.05	-14.52 ± 1.20	1.77	E_L
	S1	0.336	2.15	1.16	$-$ 3.3	$-$ 7.70$\pm$0.03	$-$ 8.88$\pm$0.06	1.49	E_L
He^4	A	0.310	3.2	1.2	-16.0	-30.28 ± 0.16	-54.20 ± 1.79	1.39	E_U
	S1	0.345	2.4	1.2	$-$ 8.3	-31.09 ± 0.16	-32.43 ± 0.34	1.34	E_U
	A	0.285	2.6	1.1	-14.0	-29.73 ± 0.18	-46.62 ± 1.70	1.47	E_L
	S1	0.340	2.45	1.16	$-$ 8.8	-31.08 ± 0.15	-32.38 ± 0.15	1.35	E_L

where $r_c = 0.35$ F, $V_{0t} = 434.0$ MeV, $\beta_t = 2.4$ F^{-1}, $V_{0s} = 216.0$ MeV, and $\beta_s = 1.97$ F^{-1}. The parameters of the soft-core potential S1 are then adjusted to yield the same scattering length, effective range, and S-wave phase shifts as those obtained from potential A.

The values of the low-energy effective-range parameters for potential A and the various soft-core potentials are given in Table II. As is seen, the requirement stated above for potential S1 is very well satisfied. The S-wave phase shifts for potentials A and S1 are listed in Table III. Here again we see that both potentials give nearly the same values for the S-wave phase shifts in both triplet and singlet states. In fact, even for the D-wave phase shifts in the energy range up to 340 MeV, the difference is less than one degree.

From Table III, it is seen that the p-p $^1S_0$ phases of potential S1 agree quite well with the values of the YLAM set of Breit et al.[19] On the other hand, then n-p $^3S_1$ phases seem to lie above the YLAN3M values of Hull et al.[20] at energies greater than about 100 MeV. To see the sensitivity of our results for the four-body system to the high-energy phase shifts, we shall also calculate with a potential (S2) which yields the same effective-range parameters (see Table II), but gives better fit to the n-p $^3S_1$ phases at high energies (Table III). As will be seen below, the binding energy and rms radius obtained with S2 are very nearly the same as those obtained with S1. This gives a strong indication that, as far as the low-energy properties of light nuclear systems are concerned, the fitting of the S-wave phases above 100 MeV is not an essential condition.

In nuclear-physics calculations, such as the resonating-group calculation, it is important that the nucleon-nucleon potential used gives about the right sizes and binding energies for the nucleon clusters. For this reason, we shall see if it is possible to find a potential which fits the low-energy nucleon-nucleon scattering data and at the same time, yields nearly correct values for the binding energies and rms radii of H^3 and He^4. The results show that this potential can indeed be found and will be called potential S3 in this investigation.

Finally, we have also computed with a potential (S4) proposed by Eikemeier and Hackenbroich.[4] Even though this potential does not fit the low-energy experimental data (Table II), it has been employed by Hackenbroich et al.[21] to compute the binding energies of H^3, He^4, Li^6, and Li^7, and the scattering phases of He^4 by He^4 and He^3. To avoid the use of a Monte Carlo technique as has been done previously,[22] mathematical approximations were introduced in their calculations. As these approximations seem quite drastic to us, we feel that it is important to check their results on H^3 and He^4 with our variational method. The results of our investigation show that the use of such approximations does introduce rather serious errors and should, therefore, be avoided if accurate results are desired.

IV. RESULTS OF CALCULATIONS

A. Potential S1

In the previous section we presented the hard-core potential A and the soft-core potential S1 which give nearly the same S-wave phase shifts in both singlet and triplet states. The question is, will these two potentials give the same binding energies, rms radii, and form factors for H^3 and He^4. The reasons for choosing these two nuclei are as follows. First, it is relatively easy to perform variational calculations on the upper and lower bounds of the energies and calculate the rms radii and form factors without any approximations. This eliminates the question of inaccuracy with which one has to deal if mathematical approximations are employed, or the question of convergence in those cases when only the first few terms of an expansion are taken into consideration. Second, the calculations for both H^3 and He^4 have already been performed for potential A[9] and an equivalent velocity-dependent potential,[23] so that a comparison with the soft-core-potential result can readily be made.

In Table IV the results of the variation on the upper

[19] G. Breit, M. H. Hull, Jr., K. E. Lassila, K. D. Pyatt, Jr., and H. M. Ruppel, Phys. Rev. 128, 826 (1962).
[20] M. H. Hull, Jr., K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. 128, 830 (1962).

[21] H. H. Hackenbroich, K. Wildermuth, and H. W. Wittern, J. Phys. Soc. Japan Suppl. 24, 627 (1968); H. H. Hackenbroich, Habilitationsschrift, University of Tübingen, 1967 (unpublished).
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[23] R. C. Herndon, E. W. Schmid, and Y. C. Tang, Nucl. Phys. 42, 113 (1963).

and lower bounds of the energies using potentials A and S1 are presented for both H^3 and He^4. Here, we notice that not only the optimum values of the upper bounds on the energies for both nuclei are very close, but also the rms radii are about the same. This indicates strongly that the binding energy and rms radius of a few-nucleon system are not sensitive to the form of the potential used. In fact, it has been shown by Herndon et al.[28] that even with an equivalent velocity-dependent potential (an equivalent velocity-dependent potential is one which gives the same S-wave scattering phases as potential A or S1) very similar values for the upper bounds and rms radii were obtained.

If we compare the lower bounds on the energies using the hard-core potential and the equivalent soft-core potential in the cases of H^3 and He^4, we find that the lower bounds obtained using potential S1 (Table IV) are much closer to the upper bounds than in the case of potential A. As a matter of fact, the upper and lower bounds on the energy using S1 are close enough so that the eigenvalue for the potential can be estimated fairly accurately without any reliance on how well one can estimate η. This is one of the advantages in using a soft-core potential, since a common argument against a variational calculation is that it is difficult to know how close the upper bound is to the eigenvalue. Since our experience[9] with the hard-core-potential calculations showed that the quantity η is unlikely to have a value less than 10, the closeness of the bounds in the case of potential S1 allows us to predict with confidence that the eigenvalue is within 0.1 MeV of the upper bound.

Let us now turn our attention to the optimum wave functions obtained using potentials A and S1. In Fig. 1 we have plotted $f(r)$, normalized to unity at the peaks, for both H^3 and He^4. In this figure the dots represent the values of $f(r)$ obtained with potential A, while the solid line is that obtained with potential S1. Here, one immediately notices that the two wave functions agree very well for $r \gtrsim 0.8$ F. This is the reason why the values of the upper bounds and the rms radii for the two
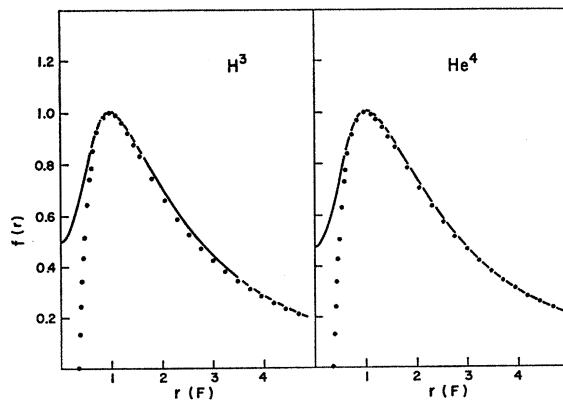

FIG. 1. The functions $f(r)$ for potentials S1 and A. The solid line is for potential S1, while the dots represent the values for potential A. The peaks are normalized to unity.

potentials agree, since both of these quantities depend to a large extent on the tail of the wave function and the position of the peak, rather than on the short-range part of the wave function.

From Fig. 1, one sees that one way to define the degree of softness of a potential is by using the value of $f(r)$ at $r=0$. For potential S1, this value is about equal to 0.5, which is half of the value at the peak.

Using the optimum wave function, the overlapping integrals of Eq. (7) can also be calculated. With the parameters of the lower bounds, we obtain

$$\langle \Phi_0 | \Psi \rangle \geqslant 0.94 \quad \text{for } H^3 \tag{27}$$

and

$$\langle \Phi_0 | \Psi \rangle \geqslant 0.98 \quad \text{for } He^4 \tag{28}$$

with potential S1. Since it is clear from Eq. (7) that these are very conservative lower limits, the fact that they have such large values indicates that our optimum wave function represents the eigenfunction quite closely.
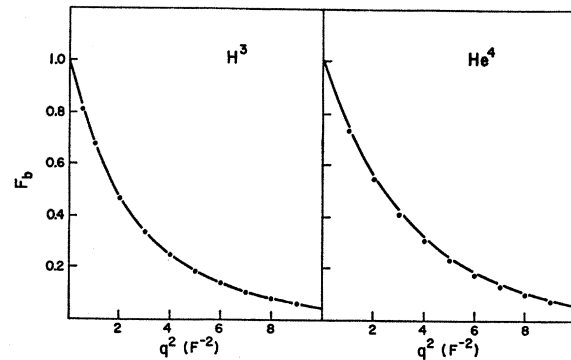

FIG. 2. Comparison of body form factors obtained with potentials S1 and A. The solid line is for potential S1, while the dots represent the values for potential A.

In Fig. 2 we show the body form factor F_b for both H^3 and He^4, where again, the dots represent the form factor for potential A, and the solid line is that for potential S1. Here also, we find that the agreement is good, a reflection of the fact that the long-range parts of the optimum wave functions for these potentials are almost indistinguishable.

The above results indicate strongly that it is not possible to distinguish between a hard-core and a soft-core potential, or even a velocity-dependent potential, on the basis of a calculation of the binding energy, rms radius, or form factor of a light nuclear system, provided these potentials give the same scattering length, effective range, and S-wave phase shifts. However, it is important to keep in mind the advantage of the soft-core potential in giving a much better lower bound on the energy. This is a very useful feature in a variational calculation, since a good lower bound can clearly enable us to make a reliable estimate of the eigenvalue.

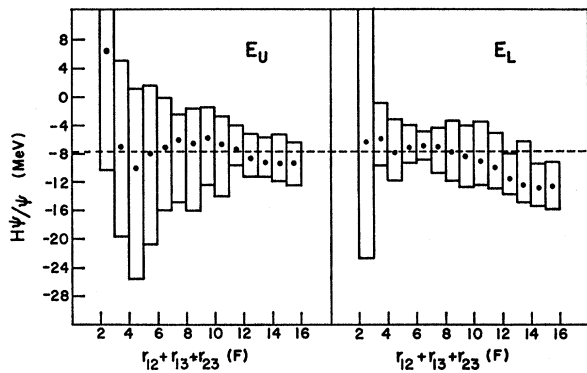To see why we get a better lower bound on the energy

FIG. 3. $H\psi/\psi$ as a function of $r_{12}+r_{13}+r_{23}$ obtained with potential S1.

using potential S1, and in what region of the configuration space the optimum wave function is poor as compared to the eigenfunction, we will study the behavior of the quantity $H\psi/\psi$. In Fig. 3 we have plotted $H\psi/\psi$ against $(r_{12}+r_{13}+r_{23})$ for H^3, using potential S1. The plots are made with the parameters corresponding to the optimum values of the upper and lower bounds. In this figure, the rectangles include 95% of the points, while the dots represent the weighted average of all the points in the rectangle. The dashed line is the upper bound, which in this case is almost equal to the eigenvalue. Now, if we compare the results of $H\psi/\psi$ for S1 with that for potential A,[9] we find that in the case of potential A, the points for $r_{12}+r_{13}+r_{23}<4$ F are so widely distributed that they were not even included in the plot, while this is not the case for potential S1. This indicates that when three particles are close to each other the wave function for potential S1 is closer to the eigenfunction than that for potential A. Clearly, then, the reason for a better lower bound with potential S1 is that the quantity $\langle H^2\rangle$ is very sensitive to the behavior of the trial function in this region of the configuration space.

The plots of $H\psi/\psi$ in Fig. 3 also show that the parameters of the lower bound try to adjust themselves such that the trial wave function resembles the eigenfunction closely in the region of the configuration space where $r_{12}+r_{13}+r_{23}$ is small. By so doing, however, the fit to the eigenfunction in the asymptotic region is somewhat sacrificed. These are demonstrated by the fact that the dots are close to the dashed line when $r_{12}+r_{13}+r_{23}$ is less than about 8 F, but begin to deviate from the dashed line quite severely for $r_{12}+r_{13}+r_{23}$ greater than 8 F. For the upper bound, on the other hand, the trial function seems to aim for a good over-all representation of the eigenfunction; thus, the dots oscillate about the dashed line, but without large deviations.

It should also be noted that in the case of He^4 the optimum parameters for the upper and lower bounds are almost the same. This is another indication that the optimum wave functions for either the upper bound or

the lower bound must be a very good representation of the eigenfunction.

B. Potential S2

Next, we turn to potential S2 which gives a better fit to the high-energy triplet S-wave phase shifts than does potential S1. The purpose of this calculation is to see the sensitivity of the binding energy and rms radius on the nucleon-nucleon phase shifts at high energies. The results for the variation on the upper bound for He^4 are

$$\alpha=0.342\ \mathrm{F}^{-1},\quad \beta=2.6\ \mathrm{F}^{-1},\quad r_d=1.18\ \mathrm{F},$$
$$\epsilon=-11.0\ \mathrm{MeV},\quad (29)$$
$$E_U=-30.25\pm0.15\ \mathrm{MeV},\quad E_L=-32.23\pm0.35\ \mathrm{MeV},$$
$$\mathrm{rms\ radius}=1.35\ \mathrm{F}.$$

A variation to obtain the optimum value of the lower bound has not been performed, since the lower bound obtained with the optimum parameters of the upper bound [see Eq. (29)] is already very close to the upper bound. If we compare the above results for potential S2 with those of potential S1, we notice that there is little change in the optimum parameters of the trial function and in the values of the rms radius and the upper and lower bounds on the energy. This indicates that a fit to nucleon-nucleon phase shifts up to E_{lab} $\simeq$100 MeV is sufficient for the evaluation of the binding energy and rms radius. We realize, of course, that this conclusion is reached only on the basis of a four-body calculation. However, we feel that, since the average nucleon-nucleon distance in heavier nuclei is about the same as that in an α particle, it is very likely that this conclusion is also valid when there are more than four nucleons in a system.

C. Potential S3

As was mentioned in Sec. III, it is important in low-energy nuclear physics to use a nucleon-nucleon potential that gives the correct sizes for the nucleon clusters. For example, in the calculation on He^3+He^3 elastic scattering by resonating-group method,[24] it was found that agreement with experimental data can be achieved only when the rms radius of the He^3 cluster is chosen to have a value close to that determined experimentally. Since in many resonating-group calculations the clusters involved are deuteron, He^3, and He^4 clusters, we shall try to find a soft-core potential which will yield nearly the right sizes and binding energies for these clusters, and, at the same time, fit the low-energy nucleon-nucleon scattering data.

In Table V, the results of the variational calculation for H^3 and He^4 using potential S3 are presented. From this table, it is observed that the rms radii obtained for H^3 and He^4 are in good agreement with experimental values of 1.70 ± 0.10 F and 1.44 ± 0.07 F. Also, we note that, here again, the lower bounds on the energies are close enough to the upper bounds so that the eigen-

[24] D. R. Thompson and Y. C. Tang, Phys. Rev. 159, 806 (1967).

TABLE V. Variational results for potential S3.

Nucleus	α (F⁻¹)	β (F⁻¹)	r_d (F)	ϵ (MeV)	E_U (MeV)	E_L (MeV)	rms radius (F)	Variation performed for optimum values of
H³	0.285	2.2	1.18	3.5	-6.56 ± 0.05	-16.59 ± 0.23	1.69	E_U
	0.348	2.29	1.22	-5.7	-6.32 ± 0.03	-9.79 ± 0.09	1.51	E_L
He⁴	0.317	3.08	1.10	-18.0	-26.47 ± 0.13	-35.81 ± 0.37	1.44	E_U
	0.324	2.68	1.22	-11.0	-26.46 ± 0.11	-29.34 ± 0.29	1.45	E_L

values for H³ and He⁴ can be fairly well estimated; these turn out to be around -6.7 and -26.7 MeV, respectively, which are in fair agreement with the values found experimentally.

With the peak of $f(r)$ normalized to unity, the values of $f(0)$ are equal to 0.40 and 0.37 for H³ and He⁴, respectively; these values are smaller than the corresponding values of 0.50 and 0.48 obtained with potential S1, indicating that potential S3 has a stronger repulsive component. This means that, to obtain an agreement with the experimental values of the rms radii, it is necessary to use a core which is more repulsive than that obtained by fitting the nucleon-nucleon scattering data at high energies. The reason for this is that we have used a purely central force with no tensor component in our calculation, and it is well known that a tensor force is not as effective in binding three- and four-body systems as in binding a two-body system.

There have been extensive measurements on the charge form factors for both H³ and He⁴. In Fig. 4 we have compared the calculated body form factors for H³ and He⁴ with experimental values.[25] As was explained, the experimental body form factors are obtained from the measured charge form factors by using Eqs. (20)–(22). From this figure, it is seen that the agreement between the experimental and calculated form factors is quite good even at relatively large values of momentum transfer. This indicates that potential S3 does
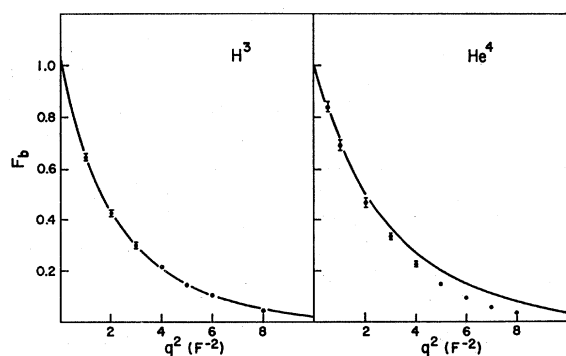
satisfy our purposes stated above and can be used as an effective potential in resonating-group and shell-model calculations.

D. Potential S4

A soft-core potential of the same form as we have used has been reported by Eikemeier et al.[4] The parameters of this potential (potential S4) are given in Table I. With this potential we have also performed variational calculations on H³ and He⁴. The results are presented in Table VI, where the results obtained by Hackenbroich et al.[21] have also been included. Here we notice that the value of E_U of He⁴ given by Hackenbroich et al. is even lower than the lower bound (-28.74 ± 0.22 MeV) of our calculation. This indicates clearly that the results of Hackenbroich et al. are inaccurate. The reason for this is probably because they have used some mathematical approximations to simplify their numerical computation, which are too drastic.

V. CONCLUSION

The results of our calculation indicate that, on the basis of the binding energy, the rms radius, and the form factor up to q^2 of about 10 F⁻², it is not possible to differentiate between a soft-core potential and a hard-core potential, which yield the same effective-range parameters and high-energy S-wave phase shifts. We realize, of course, that this conclusion is reached only from a calculation on the light systems H³ and He⁴, but we do expect it to be also valid for heavier nuclei. The reason for this is that the average separation distance in a heavy nucleus is nearly the same as that in He⁴. Thus, it seems reasonable to assume that the binding energy of a heavier nucleus will not be sensitive to the shape of the repulsive core just as in the case of He⁴. In fact, the close resemblance of the functions $f(r)$, as exhibited in Fig. 1, leads us to speculate that even the properties of low-lying states in nuclei may not be too sensitive to the form of the repulsive part of the nucleon-nucleon potential. It seems that, in order to determine the characteristics of the repulsive part, one may have to study high-energy phenomena, such as high-energy nucleon-nucleus scattering.

Although the hard-core and equivalent soft-core potentials give the same results for the binding energies and rms radii of H³ and He⁴, it is found that a soft-core potential is more suitable for a variational calculation.

FIG. 4. Comparison of body form factors obtained for H³ and He⁴ with experiment. The potential used is potential S3.

[25] H. Collard, R. Hofstadter, E. B. Hughes, A. Johanson, M. R. Yearian, R. B. Day, and R. T. Wagner, Phys. Rev. **138**, B57 (1965); R. F. Frosh, J. S. McCarthy, R. E. Rand, and M. R. Yearian, Bull. Am. Phys. Soc. **12**, 16 (1967); Phys. Rev. **160**, 874 (1967).

TABLE VI. Comparison of Hackenbroich's and our results.

Nucleus	E_U (MeV)	E_L (MeV)	rms radius (F)	Remarks
H³	− 7.30		1.72	Hackenbroich et al.[a]
	− 7.00±0.03	− 8.04±0.05	1.71	Present calculations, E_U varied
	− 6.97±0.05	− 7.57±0.10	1.60	Present calculations, E_L varied
He⁴	−31.91		1.45	Hackenbroich et al.[a]
	−27.74±0.11	−33.80±0.31	1.44	Present calculations, E_U varied
	−27.61±0.09	−28.74±0.22	1.52	Present calculations, E_L varied

[a] Reference 21.

The reason is that a soft-core potential gives a much better lower bound than an equivalent hard-core potential. This is an important factor in favor of a soft-core potential, since a good lower bound enables one to estimate the eigenvalues of H³ and He⁴ quite well, even without an accurate estimate of the quantity η, which requires a detailed study of the behavior of the function $H\psi/\psi$ in the configuration space. This is unfortunately not the case when a hard-core potential is used. Thus, in the case of H³ with the Hamada-Johnston potential, Davies[26] has obtained upper and lower bounds of −5.735 and −135.8 MeV, respectively, which are very much separated. It is our belief that, with an equivalent soft-core potential, the lower bound may be greatly improved.

Also, it is found that the binding energy and rms radius are not sensitive to the nucleon-nucleon phase shifts at high energies. Our calculations with two potentials which differ only in their fits to the high-energy ($\gtrsim$100-MeV) phase shifts show that the binding energies of He⁴ are different by only about 0.8 MeV.

Our investigation further shows that it is possible to construct a soft-core effective nucleon-nucleon potential which yields values for the energies, radii, and form factors for H³ and He⁴ that are in good agreement with the corresponding experimental values. Such a potential will be very useful for resonating-group calculations in light nuclei, where one would like to use a potential that has these properties.

Finally, a brief study of the soft-core potential proposed by Eikemeier and Hackenbroich has also been undertaken. Here, we find that the approximations used by Hackenbroich et al. to study the properties of light nuclei lead to rather poor results. This is rather unfortunate, since it means that for a careful investigation of the light nuclear system with a realistic potential, one may have to adopt a Monte Carlo technique such as the one being used here.

ACKNOWLEDGMENT

[26] B. Davies, Nucl. Phys. A103, 165 (1967).

APPENDIX: He⁴ BINDING ENERGY WITH VOLKOV POTENTIAL

In this Appendix, we use Volkov's potential[2] to calculate the rms radius and the upper and lower bounds on the energy of He⁴. The reason for this calculation is to compare our results for He⁴ with those obtained by Volkov,[2] where a simpler trial function of a product of 1s harmonic-oscillator wave functions was used.

The potential used by Volkov is

$$V(r) = -83.34 \, e^{-[r/(1.6\,\text{F})]^2} + 144.86 \, e^{-[r/(0.82\,\text{F})]^2} \quad \text{MeV} \tag{A1}$$

in the even states and has no Coulomb component. With this potential, the results of our calculation on the upper and lower bounds are

$$E_U = -29.95 \pm 0.13 \text{ MeV}, \quad E_L = -33.86 \pm 0.31 \text{ MeV},$$
$$\text{rms radius} = 1.55 \text{ F}. \quad \text{(A2)}$$

The rms radius reported above is that calculated with the optimum parameters of the upper bound, since, as has been mentioned, the upper bound is almost always much closer to the eigenvalue than the lower bound is. These results should be compared with those obtained by Volkov, which are

$$E_U = -28.01 \text{ MeV}, \quad \text{rms radius} = 1.67 \text{ F}. \quad \text{(A3)}$$

A comparison of these results with those in Eq. (A2) shows that the upper bound we obtain is about 2 MeV lower than that in Eq. (A3). For the rms radius, both our value and his value are larger than the experimental value. The reason for this is that the Volkov potential has a tail that is too long-ranged.

The Volkov potential is a rather soft potential; this is reflected in the value of $f(0)$, which is equal to 86% of the value at the peak of $f(r)$.

Even though the Volkov potential does not fit the low-energy nucleon-nucleon scattering data too well, this investigation does indicate that it is possible to obtain another potential, which gives a better agreement with the nucleon-nucleon scattering data and, at the same time, yields the ground-state properties of H³ and He⁴. Such a potential, being soft enough, may be very useful for Hartree-Fock calculations in nuclei heavier than He⁴.