Gell-Mann–Goldberger Relation for Reactions of the Form (d,pn)*

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For the breakup of a particle d into its constituents n and p in the field of a nucleus A, a Gell-Mann–Goldberger relation is derived for which the first potential excludes the np interaction. Effects of the recoil and structure of A are neglected. The limit problems that arise in the use of Lippmann-Schwinger equations with three-particle final states are treated for the case of short-range forces, and the discussion is extended to include Coulomb forces. The final exact "post" form of the transition matrix element resembles the distorted-wave Born approximation result given by Huby and Mines, and may be considered a justification of it.

1. INTRODUCTION AND PRELIMINARY DEFINITIONS

FOR the sake of definiteness, consider a deuteron d disintegrating into a neutron n and a proton p in the field of a nucleus A, i.e.,

$$A+d\rightarrow A+n+p+Q, \tag{1}$$

where $-Q$ is the binding energy of d. For simplicity, take A to be structureless and infinitely massive, so that it can be treated in terms of fixed potentials acting on n and p. The problem then corresponds to an assumption of pure direct reaction, and has been treated in the distorted-wave approximation by Huby and Mines.[1] The aim of the present paper is to examine the limiting processes involved in greater detail, starting from a more fundamental expression for the transition matrix element. We shall consider the specific effects of the three-particle final state on the derivation of the Gell-Mann–Goldberger relation, and the validity of the Lippmann-Schwinger equation generating the complete scattering state (including a deuteron channel) from an "individual particle" distorted wave without a deuteron channel.

Considerable use will be made of the concepts of weak and strong limits. A family of vectors (wave functions) $y(\lambda)$ is said to tend *weakly* to y as $\lambda\rightarrow\lambda_0$ if for every fixed *normalizable* vector n

$$\langle n|y(\lambda)\rangle\rightarrow\langle n|y\rangle \text{ as } \lambda\rightarrow\lambda_0.$$

A family of vectors can have a weak limit even though the point-by-point limit of the corresponding coordinate space representations does not exist. Thus the family $y(\lambda)$ represented by $\exp(i\lambda x)$ has the weak limit 0 as $\lambda\rightarrow\infty$, since if $n(x)$ is normalizable then

$$\int n(x)\exp(i\lambda x)dx\rightarrow 0 \text{ as } \lambda\rightarrow\infty$$

by the Riemann-Lebegue lemma. In contrast, a family $y(\lambda)$ tends *strongly* to y if

$$\langle y(\lambda)-y|y(\lambda)-y\rangle\rightarrow 0 \text{ as } \lambda\rightarrow\lambda_0.$$

The family represented by $\exp(i\lambda x)$ has no strong limit.

Weak and strong convergence can also be defined for families of operators. The family $O(\lambda)$ is said to tend weakly (strongly) to O as $\lambda\rightarrow\lambda_0$ if $(O(\lambda)-O)n$ tends weakly (strongly) to O for every fixed normalizable n.

Weak and strong limits will be denoted by wlim and slim.

Let subscripts a and b label the initial and final channels, and $\mathbf{r}_n$ and $\mathbf{r}_p$ be the respective position vectors of n and p, relative to the (fixed) center of mass of A. We disregard spin coordinates, and suppose that n and p have equal mass m. The complete Hamiltonian is

$$H=K+V_n(r_n)+V_p(r_p)+V_{np}(r_r),\quad K=(p_n^2+p_p^2)/2m, \tag{2}$$

where $r_r=|\mathbf{r}_n-\mathbf{r}_p|$. In channels a and b, H reduces to the corresponding free Hamiltonians

$$H_a=K+V_{np},\quad H_b=K. \tag{3}$$

The initial- and final-channel interactions are

$$V_a\equiv H-H_a=V_n+V_p,\quad V_b\equiv H-H_b\equiv V_n+V_p+V_{np}. \tag{4}$$

The free resolvents

$$G_a(E)=(E-H_a)^{-1},\quad G_b(E)=(E-H_b)^{-1} \tag{5}$$

and the complete resolvent

$$\mathcal{G}(E)=(E-H)^{-1} \tag{6}$$

are also needed for complex E. We assume for all these resolvents that $\mathcal{G}(E)$ and $\mathcal{G}(E^*)$ are Hermitian conjugates in the sense that $\langle\mathcal{G}(E)f|n\rangle=\langle f|\mathcal{G}(E^*)n\rangle$, where n is normalizable but f need not be.

If relative and center-of-mass coordinates for n and p are defined by

$$\mathbf{r}_r=\mathbf{r}_n-\mathbf{r}_p,\quad \mathbf{r}_d=\tfrac{1}{2}(\mathbf{r}_n+\mathbf{r}_p), \tag{7}$$

a free wave incident in the direction of k_d is

$$\phi_a=\beta(r_r)\exp(i\mathbf{k}_d\cdot\mathbf{r}_d). \tag{8a}$$

Here $\beta(r_r)$ is the bound-state wave function of the deuteron, and

$$\hbar^2k_d^2/4m=E-Q, \tag{9a}$$

where E is the total kinetic energy when all particles

* Work performed under the auspices of the U. S. Atomic Energy Commission.
[1] R. Huby and J. R. Mines, Rev. Mod. Phys. 37, 406 (1965).

are separated. A free wave in channel b is

$$\phi_b = \exp[i(\mathbf{k}_n \cdot \mathbf{r}_n + \mathbf{k}_p \cdot \mathbf{r}_p)] \tag{8b}$$

with

$$\hbar^2(k_n^2 + k_p^2)/2m = E. \tag{9b}$$

The complete scattering wave functions are

$$\psi_a^\pm = \phi_a + \operatorname*{wlim}_{\epsilon\to 0+} \mathcal{G}(E \pm i\epsilon) V_a \phi_a, \tag{10a}$$

$$\psi_b^\pm = \phi_b + \operatorname*{wlim}_{\epsilon\to 0+} \mathcal{G}(E \pm i\epsilon) V_b \phi_b. \tag{10b}$$

The weak limit wlim is to be understood to mean that

$$\langle n | \operatorname*{wlim}_{\epsilon\to 0+} | y(\epsilon)\rangle = \lim_{\epsilon\to 0+} \langle n | y(\epsilon)\rangle \tag{11}$$

for every *normalizable* function n.

Following Huby and Mines, we want to use the "independent-particle" potential

$$\bar{V} = V_n + V_p \tag{12}$$

as "first potential" in a Gell-Mann–Goldberger[2] relation. Corresponding to $\bar{V}$, the Hamiltonian, interactions, and resolvent are

$$\begin{aligned} \bar{H} &= K + V_n + V_p, & \bar{\mathcal{G}}(E) &= (E - \bar{H})^{-1}, \\ \bar{V}_a &= V_n + V_p - V_{np}, & \bar{V}_b &= V_n + V_p, \end{aligned} \tag{13}$$

and the complete scattering functions are

$$\chi_a^\pm = \phi_a + \operatorname*{wlim}_{\delta\to 0+} \bar{\mathcal{G}}(E \pm i\delta) \bar{V}_a \phi_a, \tag{14a}$$

$$\chi_b^\pm = \phi_b + \operatorname*{wlim}_{\delta\to 0+} \bar{\mathcal{G}}(E \pm i\delta) \bar{V}_b \phi_b. \tag{14b}$$

Instead of (14b) we can use

$$\chi_b^\pm = \phi_b + \operatorname*{wlim}_{\delta\to 0+} G_b(E \pm i\delta) \bar{V}_b \chi_b^\pm. \tag{15}$$

Equation (14b), of course, uniquely defines $\chi_b^\pm$. Equation (15) may not *define* $\chi_b^\pm$ uniquely,[3] but it is nevertheless a correct equation since it is just the Fourier transform of the corresponding equation in the general time-dependent theory.

Since both ϕ_b and $\bar{V}_b$ have a separable form, $\chi_b^\pm$ should be easier to calculate than $\chi_a^\pm$. But (14b) does not show this separability. We therefore use the time-dependent expression for the wave operator[4] $\bar{\Omega}_b^\pm$ which satisfies

$$\bar{\Omega}_b^\pm \phi_b = \chi_b^\pm, \tag{16}$$

$$\bar{\Omega}_b^\pm = \operatorname*{slim}_{t\to\pm\infty} \exp(-i\bar{H}t)\exp(iH_b t). \tag{17}$$

Since $\bar{H}$ and H_b can be analyzed into commuting n and p parts,

$$\bar{H} = \bar{H}_n + \bar{H}_p, \quad H_b = H_{bn} + H_{bp}.$$

Then Eq. (17) can be written

$$\begin{aligned} \bar{\Omega}_b^\pm = \operatorname*{slim}_{t\to\pm\infty} \{ &[\exp(-i\bar{H}_n t)\exp(iH_{bn}t)] \\ &\times [\exp(-i\bar{H}_p t)\exp(iH_{bp}t)]\} \\ = \bar{\Omega}_{bn}^\pm \bar{\Omega}_{bp}^\pm, & \end{aligned} \tag{18}$$

where, for example,

$$\bar{\Omega}_{bn}^\pm = \operatorname*{slim}_{t\to\pm\infty} \exp(-i\bar{H}_n t)\exp(iH_{bn}t)$$

is the wave operator for n in the potential V_n. The replacement of the strong limit of the product of operators by the product of the strong limits is easily justified, since the operators act on independent spaces. From (16) and (18), it follows that

$$\begin{aligned} \chi_b^\pm &= \bar{\Omega}_{bn}^\pm \bar{\Omega}_{bp}^\pm \phi_b \\ &= \chi_{bn}^\pm \chi_{bp}^\pm, \end{aligned} \tag{19}$$

where, for example,

$$\chi_{bn}^\pm = \bar{\Omega}_{bn}^\pm \exp(i\mathbf{k}_n \cdot \mathbf{r}_n) \tag{20}$$

is the complete scattering wave function of n in the potential V_n.

It is interesting that χ_b^- contains *outgoing* as well as incoming n and p waves. This is because the resolvent in (14b) acts on the non-normalizable function $\bar{V}_b\phi_b$. The function $V_b\phi_b$ is not normalizable even if V_n and V_p are of short range; it can be rendered normalizable only if $\bar{V}_b$ is replaced by a three-body force tending strongly to zero as $r_d \to \infty$.

2. TRANSFORMATION OF THE TRANSITION MATRIX ELEMENT

Huby and Mines show that the "prior" form of the distorted-wave transition matrix element is equal to the "post" form (interpreted in a certain way). In contrast, we start from the exact prior form of the transition matrix element (before the application of any Gell-Mann–Goldberger relation), namely,

$$T = \langle \psi_b^- | V_a | \phi_a \rangle. \tag{21}$$

We assume that the potentials are bounded and satisfy

$$r^{3/2}V_n(r) \to 0, \quad r^{3/2}V_p(r) \to 0 \quad \text{as } r \to \infty, \tag{22}$$

which are sufficient conditions for $V_a\phi_a$ to be normalizable. These conditions are not satisfied by Coulomb forces, which need a special treatment discussed in the Appendix. If (22) is satisfied, the expression (21) for T is well defined. It has been derived for the present three-body case by Gerjuoy[5] from time-independent scattering theory. But the implied integration in the ("post") expression

$$T = \langle \phi_b | V_b | \psi_a^+ \rangle$$

[2] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).
[3] L. L. Foldy and W. Tobocman, Phys. Rev. **105**, 1099 (1957); S. T. Epstein, *ibid.* **106**, 598 (1957).
[4] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Co., New York, 1966). The symbol slim refers to a strong operator limit.
[5] E. Gerjuoy, Ann. Phys. (N. Y.) **5**, 58 (1958).

diverges by oscillation, and lacks a rigorous derivation for the three-body case.

The Gell-Mann–Goldberger relation can be derived formally by using

$$\psi_b^- = \chi_b^- + \underset{\epsilon\to 0+}{\mathrm{wlim}}\, \mathcal{G}(E-i\epsilon)(V_b-\bar{V}_b)\chi_b^-. \quad (23a)$$

The derivation of this equation is less straightforward than for the case of potential scattering. For potential scattering $(V_b-\bar{V}_b)\chi_b^-$ is normalizable, so that the resolvent acts on it to produce pure incoming waves in the limit $\epsilon\to 0+$. Accordingly the right-hand side of (23a) has the proper asymptotic behavior, and since it satisfies the appropriate Schrödinger equation, it can be identified with ψ_b^-. But in the present case the resolvent acts on a non-normalizable vector, and it is not trivial to deduce the asymptotic behavior of (23b). This remark is illustrated by the previously mentioned behavior of χ_b^-.

To avoid such ambiguities associated with the three-particle final state, we shall derive the Gell-Mann–Goldberger relation without using (23a). The derivation will in fact provide a justification of (23a), as we shall note.

Accordingly, we treat the limits in Eqs. (10) and (14) explicitly. From (10a) and (15), one obtains

$$\begin{aligned}\psi_b^- - \chi_b^- &= \underset{\epsilon\to 0+}{\mathrm{wlim}}[\mathcal{G}(E-i\epsilon)V_b\phi_b - G_b(E-i\epsilon)\bar{V}_b\chi_b^-]\\ &= \underset{\epsilon\to 0+}{\mathrm{wlim}}[\mathcal{G}(E-i\epsilon)V_b(\chi_b^- - \underset{\delta\to 0+}{\mathrm{wlim}}\, G_b(E-i\delta)\\ &\qquad \times\bar{V}_b\chi_b^-) - G_b(E-i\epsilon)\bar{V}_b\chi_b^-]\\ &= \underset{\epsilon\to 0+}{\mathrm{wlim}}\, \mathcal{G}(E-i\epsilon)(V_b-\bar{V}_b)\chi_b^-\\ &\quad + \underset{\epsilon\to 0+}{\mathrm{wlim}}\,\underset{\delta\to 0+}{\mathrm{wlim}}\, i(\epsilon-\delta)[\mathcal{G}(E-i\epsilon)-G_b(E-i\epsilon)]\\ &\qquad \times G_b(E-i\delta)\bar{V}_b\chi_b^- \quad (23b)\end{aligned}$$

by making use of such identities as

$$\mathcal{G}(E_1)-\mathcal{G}(E_2) = (E_2-E_1)\mathcal{G}(E_1)\mathcal{G}(E_2),$$
$$\mathcal{G}(E)-G_b(E) = \mathcal{G}(E)V_bG(E).$$

The second term of Eq. (23b) must be shown to make no contribution to (21) if the usual Gell-Mann–Goldberger relation is to result.

3. DISCUSSION OF THE REMAINDER TERM

The contribution of the second term of (23b) to (21) is

$$R = -i \lim_{\epsilon\to 0+}\lim_{\delta\to 0+}(\epsilon-\delta)\langle\chi_b^-|\bar{V}_bG_b(E+i\delta)$$
$$\times[\mathcal{G}(E+i\epsilon)-G_b(E+i\epsilon)]V_a|\phi_a\rangle, \quad (24)$$

which will be shown to vanish.

Define:

$$u_N(\epsilon) = [\mathcal{G}(E+i\epsilon)-G_b(E+i\epsilon)]V_a\phi_a. \quad (25)$$

Since $V_a\phi_a$ is normalizable, and $\mathcal{G}(E+i\epsilon)$ and $G_b(E+i\epsilon)$ are operators on Hilbert space for $\epsilon\neq 0$, $u_N(\epsilon)$ is normalizable. Therefore,

$$\begin{aligned}\lim_{\delta\to 0+}\langle\chi_b^-|\bar{V}_bG_b(E+i\delta)|u_N\rangle &= \langle\underset{\delta\to 0+}{\mathrm{wlim}}\, G_b(E-i\delta)\bar{V}_b\chi_b^-|u_N\rangle\\ &= \langle\chi_b^- - \phi_b|u_N\rangle\end{aligned}$$

by (15). Since the δ limit is the limit of a product of numbers, the limit of the factor $(\epsilon-\delta)$ can be taken independently, provided that the limit of the other factor exists. Thus

$$R = -i\lim_{\epsilon\to 0+}\epsilon\langle\chi_b^- - \phi_b|[\mathcal{G}(E+i\epsilon)$$
$$-G_b(E+i\epsilon)]V_a|\phi_a\rangle. \quad (26)$$

The ϵ limit cannot be taken inside the scalar product because $\chi_b^- - \phi_b$ is not normalizable. However, if we can show that the limit of the scalar product is *finite*, we can conclude that $R=0$, since $\epsilon\to 0$.

To test whether the scalar product tends to a finite limit, we use the lemma: *If the derivative of $f(x)$ exists for $x\geqslant a$ and tends to zero as $x\to\infty$, then the integral*

$$\int_a^\infty f(x)\exp[i(k_1+ik_2)x]dx$$

tends to a finite limit as $k_2\to 0+$, provided that the constant $k_1\neq 0$. (This result follows on integrating by parts. The integrated part is finite in the limit $k_2\to 0+$, and the remaining integral converges uniformly in k_2 for $k_2\geqslant 0$. Therefore the k_2 limit can be taken inside the integral, and gives a finite result.)

It is sufficient to examine the part of the scalar product coming from large distances. Let

$$u_N(\epsilon) = u_1 - u_2,$$
$$u_1 = \mathcal{G}(E+i\epsilon)V_a\phi_a,\quad u_2 = G_b(E+i\epsilon)V_a\phi_a.$$

The function u_1 is a normalizable solution of

$$(E+i\epsilon-H) = V_a\phi_a.$$

Now r_n and r_p may simultaneously tend to infinity in two ways:

(a) $r_n\to\infty$, $r_p\to\infty$, r_r fixed (d channel),

(b) $r_n\to\infty$, $r_p\to\infty$, $r_r\to\infty$ (np channel).

In case (a), $V_a = V_n + V_p \to 0$ but $V_{np}\nrightarrow 0$, so that $H\to K+V_{np}$. If n and p have only one bound state, then at large distances u_1 is of the form

$$u_1 = \beta(r_r)f_d(\Omega_d)\exp(ik_dr_d)r_d^{-1},$$

where $\Omega_d = (\theta_d,\phi_d)$ are the angular coordinates of $\mathbf{r}_d$, and

$$k_d = [4m\hbar^{-2}(E-Q+i\epsilon)]^{1/2},$$

which has a positive imaginary part. The normalizability of u_1 excludes a solution in $\exp(-ik_dr_d)$.

In case (b), $H \to K$. Hence, at large distances u_1 is a solution of

$$(E+i\epsilon-K)u_1=0.$$

Now

$$K=\hbar^2(\nabla_n^2+\nabla_p^2)/2m=\hbar^2\nabla_6^2/2m.$$

To exploit the symmetry of the six-dimensional Laplacian, we express it in terms of new coordinates

$$r=(r_n^2+r_p^2)^{1/2},\quad \Omega_n=(\theta_n,\phi_n),\quad \Omega_p=(\theta_p,\phi_p)$$

and one of the direction cosines

$$l_n=r_n/r,\quad l_p=r_p/r=(1-l_n^2)^{1/2}.$$

The result is

$$\nabla_6^2=\partial^2/\partial r^2+5r^{-1}\partial/\partial r+r^{-2}D,$$

where D is a second-order differential operator in the angular coordinates Ω_n, Ω_l, and l_n only.

For large r the term $r^{-2}D$ is negligible, so that u_1 is a solution of

$$(\partial^2/\partial r^2+5r^{-1}\partial/\partial r+k^2)u_1=0,\quad k=[2m\hbar^{-2}(E+i\epsilon)]^{1/2}.$$

In the same way as before,[5,6] it can be shown that

$$u_1=F_1(l_n,\Omega_n,\Omega_p)r^{-5/2}\exp(ikr)$$

at large distances in the np channel. Again a solution in $\exp(-ikr)$ is excluded.

The discussion of case (b) applies also to u_2, which has no deuteron channel. Therefore at large r_n and r_p, the form of $u_N(\epsilon)$ is

$$u_N(\epsilon)=u_1-u_2=f_d(\Omega_d)\beta(r_r)r_d^{-1}\exp(ik_dr_d)+F(l_n\Omega_n\Omega_p)r^{-5/2}\exp(ikr).\quad(27)$$

We now apply the lemma to the contributions of the two terms of (27) to the scalar product part of R, namely

$$\langle\chi_b^- -\phi_b|u_N(\epsilon)\rangle$$

$$=\int\int(\chi_b^- -\phi_b)^*u_N(\epsilon)d^3\mathbf{r}_nd^3\mathbf{r}_p\quad(28a)$$

$$=\int\int(\chi_b^- -\phi_b)^*u_N(\epsilon)d^3r_dd^3\mathbf{r}_r\quad(28b)$$

$$=\int\int\int(\chi_b^- -\phi_b)^*u_N(\epsilon)r^5dr\, l_n^2l_pdl_nd\Omega_nd\Omega_p.\quad(28c)$$

Upon substituting the asymptotic forms of χ_n^- and χ_p^- into (19), the function $(\chi_b^- -\phi_b)^*$ is found to reduce at large distances to

$$(\chi_b^- -\phi_b)^*=[\exp(-i\mathbf{k}_n\cdot\mathbf{r}_n)+f_n(\Omega_n)r_n^{-1}\exp(ik_nr_n)]\times[\exp(-i\mathbf{k}_p\cdot\mathbf{r}_p+f_p(\Omega_p)r_p^{-1}\exp(ik_pr_p)]-\exp[-i(\mathbf{k}_n\cdot\mathbf{r}_n+\mathbf{k}_n\cdot\mathbf{r}_p)].\quad(29)$$

[6] M. Danos and W. Greiner, Z. Physik **202**, 125 (1967).

We assume that the amplitudes f_d, f_n, f_p, and F are bounded, so that the integrations over the finite ranges of the angular variables cannot diverge and need not be considered.

For the d-channel term of Eq. (27), we use coordinates $\mathbf{r}_d$ and $\mathbf{r}_r$. If the integral over r_d converges for fixed r_r, so will the subsequent integral over r_r, because $\beta(r_r)$ tends strongly to zero at infinity. Substituting $u_N(\epsilon)$ and $(\chi_b^- -\phi_b)^*$ in (28b) shows that the integrand contains the wave numbers (analogous to k_1 of the lemma)

$$\begin{aligned}&k_n+k_p+k_d,\\&k_p-k_n\cos\theta_{dn}+k_d,\\&k_n-k_p\cos\theta_{dp}+k_d,\end{aligned}\quad(30)$$

where

$$\cos\theta_{dn}=\mathbf{k}_n\cdot\mathbf{r}_d/k_nr_d,\quad \cos\theta_{dp}=\mathbf{k}_p\cdot\mathbf{r}_d/k_pr_d.$$

From Eqs. (9a) and (9b), one finds

$$|k_p-k_n|<k_d$$

provided that $Q<0$. Therefore, none of the wave numbers (30) can vanish, and hence the d-channel contribution to R vanishes.

For the np channel, we express the integrand in terms of r, l_n, Ω_n, Ω_p, and find that as a function of r it contains wave numbers

$$-k_nl_n\cos\theta_n+k_pl_p+k,\quad(31a)$$

$$-k_pl_p\cos\theta_p+k_nl_n+k,\quad(31b)$$

$$k_nl_n+k_pl_p+k,\quad(31c)$$

where

$$\cos\theta_n=\mathbf{k}_n\cdot\mathbf{r}_n/k_nr_n,\quad \cos\theta_p=\mathbf{k}_p\cdot\mathbf{r}_p/k_pr_p.$$

The wave number (31c) can never vanish. For (31a) to vanish, it is necessary that

$$k_n\neq0,\quad \cos\theta_n>0.$$

But

$$|k_nl_n\cos\theta_n-k_pl_p|\leqslant|k_p-k_n|\leqslant(k_p^2+k_n^2)^{1/2}=k^2$$

by (9b). The equality can apply only if

$$\theta_n=l_p=k_p=0\ \text{or}\ l_n=k_n=0.$$

Therefore (31a) does not vanish unless the exceptional condition

$$\theta_n=l_p=k_p=0$$

is satisfied. This result is independent of the sign of Q; a similar analysis applies to (31b).

Consequently, $R=0$ unless k_n or k_p vanishes.

4. FINAL FORM OF THE TRANSITION MATRIX ELEMENT

Since $R=0$, substituting (23b) into (21) yields[7]

$$T=\langle\chi_b^-|V_a|\phi_a\rangle+\lim_{\epsilon\to0+}\langle\chi_b^-|V_{np}\mathcal{G}(E+i\epsilon)V_a|\phi_a\rangle.\quad(32)$$

[7] It can now be seen that (23a) has been proved, since the only property of $V_a\phi_a$ used was its normalizability.

We want to eliminate $\mathcal{G}(E+i\epsilon)$. The result is not changed by inserting a "convergence factor"[1] $\exp(-\alpha r_n)$ in each term and taking the limit as $\alpha\to 0+$, since

$$\lim_{\alpha\to 0+}\exp(-\alpha r_n)V_{np}\mathcal{G}(E+i\epsilon)V_a\phi_a$$

is uniform in r_n and r_p. This is true as long as $\epsilon\neq 0$, for then the function that $\exp(-\alpha r_n)$ multiplies is normalizable. Hence,

$$T=\lim_{\epsilon\to 0+}\lim_{\alpha\to 0+}[\langle\chi_b^-|\exp(-\alpha r_n)V_a|\phi_a\rangle +\langle\chi_b^-|\exp(-\alpha r_n)V_{np}\mathcal{G}(E+i\epsilon)V_a|\phi_a\rangle].$$

The ϵ limit can be taken first, since the α limit is uniform in ϵ. Since $V_{np}\exp(-\alpha r_n)\chi_b^-$ is normalizable, the ϵ limit can be evaluated by taking it inside the scalar product as a weak limit, and using (11). The result is

$$T=\lim_{\alpha\to 0+}\langle\chi_b^-|\exp(-\alpha r_n)|V_a\phi_a+V_{np}(\psi_a^+-\phi_a)\rangle$$
$$=\lim_{\alpha\to 0+}[\langle\chi_b^-|\exp(-\alpha r_n)(V_a-V_{np})|\phi_a\rangle +\langle\chi_b^-|\exp(-\alpha r_n)V_{np}|\psi_a^+\rangle].$$

The first term can be written as[8]

$$\lim_{\alpha\to 0+}\langle\chi_b^-|\exp(-\alpha r_n)(\bar{H}-H_a)|\phi_a\rangle$$
$$=\lim_{\alpha\to 0+}\langle\chi_b^-|\exp(-\alpha r_n)(\bar{H}-E)|\phi_a\rangle$$
$$=\lim_{\alpha\to 0+}[\langle\chi_b^-(\bar{H}-E)\exp(-\alpha r_n)|\phi_a\rangle$$
$$+\hbar\alpha(2mi)^{-1}\langle\chi_b^-|p_n\exp(-\alpha r_n)+\exp(-\alpha r_n)p_n|\phi_a\rangle].$$

The first term of this vanishes, since $\bar{H}$ can act to the left because $\exp(-\alpha r_n)\phi_a$ is normalizable, and $(\bar{H}-E)\times\chi_b^-=0$. The second term tends to zero by the type of argument used for the d channel of R. The final result is

$$T=\lim_{\alpha\to 0+}\langle\chi_b^-|\exp(-\alpha r_n)V_{np}|\psi_a^+\rangle. \qquad (33)$$

This exact expression for the transition element may be regarded as an interpretation of the nonconvergent form

$$T=\langle\chi_b^-|V_{np}|\psi_a^+\rangle,$$

which results from a purely formal derivation.

5. DISCUSSION AND CONCLUSIONS

The present result (33) reduces to the expression given by Huby and Mines if ψ_a^+ is replaced by a function χ_d^+ describing elastic scattering of d by a model potential acting only on its center of mass. This distorted-wave approximation is useful because it can be evaluated easily in zero-range approximation.

Huby and Mines were chiefly interested in sequential decay, in which p is emitted directly while n forms a resonance with A and is delayed. The new derivation shows that the accuracy of the Huby-Mines distorted-wave approximation is independent of the validity of the sequential-decay assumption, but depends on the extent to which ψ_a^+ can be represented by an elastically scattered deuteron wave. Good results can be expected if the inelastic processes (including d breakup) are weak. Furthermore, the appropriate function χ_b^- is the product of functions χ_{bn}^- and χ_{bp}^- representing scattering of n and p by potentials V_n and V_p. Even though p is emitted at a time when n still forms a resonance with A, V_p is the interaction of p with A, and *not* with $(A+n)$ in a resonant state.

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APPENDIX

If Coulomb forces U_n and U_p are added to the short-range interactions V_n and V_p, the treatment can be modified as follows. Gerjuoy[9] has shown that the transition-matrix element can be written in the form

$$T=\langle\psi_b^-|V_n+V_p+U_n+U_p-U_d|\chi_{ac}^+\rangle, \qquad (34)$$

where the Coulomb forces are

$$U_n=Z_AZ_n/r_n,\quad U_p=Z_AZ_p/r_p,\quad U_d=Z_A(Z_n+Z_p)/r_d,$$

and χ_{ac}^+ is the solution of

$$(K+U_d+V_{np}-E)\chi_{ac}^+=0,$$

which asymptotically contains a Coulomb-distorted incident wave with d in its ground state, plus outgoing waves. Sinde U_d acts only on the center of mass of d, this potential cannot by itself induce d breakup—unlike the true Coulomb potential U_n+U_p.

Now the function $(V_n+V_p+U_n+U_p-U_d)\chi_{ac}^+$ is normalizable if V_n and V_p satisfy (22), since in the Coulomb terms the monopole part (which falls off as r^{-1}) cancels, leaving only dipole and higher terms, which fall off at least as fast as r^{-2}. Therefore Eq. (34) can be treated in the same way as (21). The logarithmic phases appearing in ψ_b^- and χ_{ac}^+ do not affect the application of the lemma, because the factor

$$\exp[-i\eta\ln(2kr)]$$

can be absorbed in f of the lemma, since its derivative tends to zero at infinity.

The final result is just Eq. (33), where ψ_a^+ and χ_b^- are appropriate eigenfunctions of Hamiltonians including the true Coulomb interaction U_n+U_p.

[8] We have omitted a term arising from the discontinuity of the derivative of $\exp(-\alpha r_n)$ at the origin. This term tends to zero as α^2, and could be eliminated by choosing a convergence factor with continuous derivative at the origin.

[9] E. Gerjuoy, Ref. 5, p. 79.