Aside from the $\mathcal{G}$-dependent factors, which are of order unity, and the factor $A/\bar{G}M$ expressing the reduction in magnitude due to the many modes, note that (4.3) and (4.5) depend in the same way on the essential parameters (e.g., E, N_d, and ω). Since these dependences have been verified experimentally, the present theory may be taken as comparably satisfactory in this respect.

V. CONCLUSIONS

A theory of fluctuations in the saturation current has been presented which, it is believed, describes the essential physical features of the problem. The spectral distribution is in agreement with experiment. The cutoff frequency is found directly in terms of known properties of the system (viz., $\bar{G}$) and does not require the introduction of phenomenological parameters. The magnitude of the noise power, however, is explained less satisfactorily. It is believed that this would require a more rigorous description of the primitive statistical fluctuations about a steady state far displaced from equilibrium. As pointed out, the present theory, based on an incoherent phonon picture, agrees in its dependence on the essential parameters with the result of a bunching theory due to Moore.[7] It remains to be shown why this is so, and whether, in fact, the two theories are in some sense equivalent. In this connection, it may be of some importance to take into account space-varying effects, neglected in the present treatment. Finally, it is suggested that fluctuation analysis should serve as a sensitive test of any future theory of the acoustoelectric steady state.

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Modification of Friedel Oscillations by a Magnetic Field*

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The electron density near an impurity or "test particle" in an electron gas is investigated for the case in which a uniform magnetic field is applied to the system. The electron gas is at zero temperature and the Coulomb interaction between electrons is neglected. A δ-function potential is used for the interaction of the electrons with the test particle. The induced electron density along a line passing through the test particle *parallel* to the magnetic field is of the form $r^{-3}\cos 2k_F r$ for large r, where k_F is the Fermi wave number and r is the distance from the test particle. The induced electron density along a line passing through the test particle *perpendicular* to the magnetic field is qualitatively different. It exhibits only a finite number of oscillations in space and then falls off monotonically with increasing distance from the test particle. The number of complete oscillations corresponds to the number of occupied Landau levels in the electron gas. Similar results may be expected for the electron spin density near a magnetic impurity.

I. INTRODUCTION

A NONMAGNETIC impurity in a metal gives rise to a conduction-electron charge density[1] which varies as $r^{-3}\cos 2k_F r$ for large r, where k_F is the Fermi wave number and r is the distance from the impurity to the point in question. Similarly, a magnetic impurity in a metal gives rise to a conduction-electron spin polarization[2] of the same form. This oscillatory phenomenon is a consequence of the sharp cutoff in the momentum distribution of the conduction electrons at zero temperature.

Experiments dealing with these phenomena generally utilize an external magnetic field. In previous analyses, the effect of this field on the momentum distribution of the electrons has not been considered. Indeed, it is not apparent that one may neglect this effect. In the presence of a static homogeneous magnetic field the *angular* momentum of the electrons about the field lines is quantized and the electrons occupy angular momentum states (Landau levels) with quantum numbers ranging from zero up to some cutoff N_F. However, this does not imply a cutoff in the *linear* momentum of the electrons perpendicular to the field lines; if this cutoff is absent or if it is modified in some way, then one may expect a corresponding change in the behavior of the

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[1] J. Friedel, Nuovo Cimento Suppl. 2, 287 (1958); W. Kohn and S. H. Vosko, Phys. Rev. 119, 912 (1960); J. S. Langer and S. H. Vosko, J. Phys. Chem. Solids 12, 196 (1960).
[2] M. A. Ruderman and C. Kittel, Phys. Rev. 96, 99 (1954); T. Kasuya, Progr. Theoret. Phys. (Kyoto) 16, 45 (1956); K. Yosida, Phys. Rev. 106, 893 (1959).

spin polarization or charge density near an impurity in a metal. In this paper we calculate the conduction-electron charge density near a nonmagnetic impurity, including the effect of a static, uniform magnetic field. Similar results may be expected for the conduction-electron spin polarization near a magnetic impurity.

The model for our calculations is a gas of noninteracting electrons at zero temperature in a static, uniform magnetic field **B**. A "test particle" or impurity in the system interacts with the electrons via a potential $V(\mathbf{r})$. For convenience we use a unit strength δ-function potential $V(\mathbf{r})=\delta(\mathbf{r})$. The procedure for finding the induced charge density due to the test particle is (a) find the single-particle eigenstates, $|\alpha\rangle$, i.e., calculate the wave functions, $\psi_\alpha(\mathbf{r})$, and energy eigenvalues E_α from the Schrödinger equation; (b) since the system is at zero temperature, let all eigenstates with energies less than the Fermi energy E_F be occupied; and (c) calculate the induced charge density via the relation

$$\rho(r)=\sum_{\alpha=1}^{\alpha_F}|\psi_\alpha(\mathbf{r})|^2,$$

where the upper limit α_F on the summation is used to indicate that only the *occupied* single-particle states are included in the sum. To carry out this procedure in detail we assume that it is sufficient to find the *linear* response of the system to the test particle, i.e., the wave functions $\psi_\alpha(\mathbf{r})$ are calculated only to first order in $V(\mathbf{r})$. We do *not* make an expansion in terms of the magnetic field **B**. The unperturbed system is then simply a gas of free electrons in a static, uniform magnetic field and we denote the wave functions and energy eigenvalues of this system by $\phi_\alpha(\mathbf{r})$ and ϵ_α, respectively. The wave functions of the perturbed system to first order in the test particle potential are then

$$\psi_\alpha(\mathbf{r})\approx\phi_\alpha(\mathbf{r})+\sum_{\alpha'=1,\alpha'\neq\alpha}^{\infty}\frac{\phi_{\alpha'}(\mathbf{r})V_{\alpha'\alpha}}{\epsilon_\alpha-\epsilon_{\alpha'\alpha}} \tag{1}$$

and the induced charge density is

$$\delta\rho(\mathbf{r})\approx 2\,\mathrm{Re}\sum_{\alpha=1}^{\alpha_F}\sum_{\alpha'=1,\alpha'\neq\alpha}^{\infty}\frac{\phi_{\alpha'}(\mathbf{r})V_{\alpha'\alpha}\phi_\alpha^*(\mathbf{r})}{\epsilon_\alpha-\epsilon_{\alpha'}}, \tag{2}$$

where Re denotes the real part and the matrix elements of the test particle potential are

$$V_{\alpha'\alpha}=\int d\mathbf{r}\,\phi_{\alpha'}^*(\mathbf{r})V(\mathbf{r})\phi_\alpha(\mathbf{r}). \tag{3}$$

The explicit forms of $\phi_\alpha(\mathbf{r})$ and ϵ_α are given in Sec. II. These are used in Sec. III to obtain an expression for $\delta\rho(\mathbf{r})$. In Sec. III A the induced charge density along a line passing through the test particle parallel to the magnetic field is evaluated. There is essentially no effect due to the magnetic field. In Sec. III B the induced charge density along a line passing through the test particle perpendicular to the magnetic field is evaluated. Here the induced charge density exhibits only a finite number of oscillations in space and then falls off monotonically with increasing distance from the test particle. This is in contrast to the $r^{-3}\cos 2k_F r$ form for no magnetic field.

II. UNPERTURBED ELECTRON EIGENSTATES

The Hamiltonian for an electron in a static, uniform magnetic field **B** is[3]

$$H=\frac{1}{2m}\left(\mathbf{p}+\frac{|e|}{c}\mathbf{A}\right)^2, \tag{4}$$

where **p** is the canonical momentum, m is the electron mass, $-|e|$ is the electron charge, and c is the velocity of light. The vector potential $\mathbf{A}(\mathbf{r})$ at the position **r** may be written

$$\mathbf{A}(\mathbf{r})=\tfrac{1}{2}\mathbf{B}\times\mathbf{r}. \tag{5}$$

For this problem it is most convenient to use cylindrical polar coordinates (r,θ,z) with the z axis parallel to the magnetic field **B**. In this coordinate system the components of the vector potential are

$$A_\theta=\tfrac{1}{2}Br,\quad A_r=A_z=0. \tag{6}$$

The Schrödinger equation for an electron becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi_\alpha}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi_\alpha}{\partial\theta^2}+\frac{\partial^2\phi_\alpha}{\partial z^2}+\frac{ieB}{\hbar c}\frac{\partial\phi_\alpha}{\partial\theta}-\left(\frac{eBr}{2\hbar c}\right)^2\phi_\alpha+\frac{2m\epsilon_\alpha}{\hbar^2}\phi_\alpha=0. \tag{7}$$

This equation can be solved by separation of variables. The energy eigenvalues and eigenfunctions which one finds are[4]

$$\epsilon_\alpha=\hbar^2p^2/2m+\hbar(eB/mc)(n+l+\tfrac{1}{2}), \tag{8}$$

$$\phi_\alpha(\mathbf{r})=C_\alpha e^{ipz}e^{il\theta}\xi^{l/2}e^{-\xi/2}L_n{}^l(\xi), \tag{9}$$

$$\xi=(eB/2\hbar c)r^2, \tag{10}$$

where the eigenstates are labelled by the set of quantum numbers

$$\alpha=(p,l,n) \tag{11}$$

and $L_n{}^l(\xi)$ is the associated Laguerre polynomial. The quantum number p specifies the linear momentum of the electron parallel to the magnetic field and can take

[3] The electrons with spin up and spin down can be treated independently since the perturbing test particle in our problem does not interact with the spin of the electrons. Thus we neglect a constant term in the Hamiltonian representing the interaction of the intrinsic magnetic moment of the electron with the magnetic field.

[4] L. D. Landau and E. M. Lifshitz, *Non-Relativistic Quantum Mechanics* (Pergamon Press, Inc., New York, 1965), 2nd ed., p. 426.

on values ranging from $-\infty$ to $+\infty$. The quantum number l is a positive integer which specifies the angular momentum of the electron about the magnetic field lines. For positively charged particles (holes) one would use eigenstates having negative l values since these correspond to azimuthal motion about the field lines in the opposite sense. The radial quantum number n must also be a positive integer.

We normalize the wave functions in a cylindrical volume of length L and radius R, where L and R are very large and we will eventually let them become infinite. The normalization constant C_α is then given by[5]

$$|C_\alpha|^2=\frac{1}{2\pi L}\frac{eB}{\hbar c}\frac{n!}{[(n+l)!]^3}. \quad (12)$$

This completes the specification of the unperturbed electron states.

III. INDUCED CHARGE DENSITY

The matrix elements of the perturbing test-particle potential appearing in Eq. (2) for the induced charge density are easily calculated since the potential is simply a δ function. From the definition, Eq. (3), for a spherically symmetric potential, one has

$$V_{\alpha'\alpha}=2\pi\phi_{\alpha'}^*(0)\phi_\alpha(0), \quad (13)$$

and using the wave functions given in Sec. II this becomes

$$V_{\alpha'\alpha}=2\pi C_{\alpha'}^* C_\alpha \delta_{l,0}\delta_{l',0} n'!n!, \quad (14)$$

where we have used

$$L_n^0(0)=n!. \quad (15)$$

The induced charge density is obtained from Eq. (2) by substituting the expressions for the wave functions, energies, and matrix elements given in Eqs. (8), (9), and (14):

$$\delta\rho(\mathbf{r})=2\,\mathrm{Re}\sum_{\alpha=1}^{\alpha_F}\sum_{\alpha'=1}^{\infty}2\pi\delta_{l',0}\delta_{l,0}n'!n!|C_{\alpha'}|^2|C_\alpha|^2 \times\frac{e^{i(p'-p)z}e^{-\xi}L_{n'}(\xi)L_n(\xi)}{(\hbar^2/2m)(p^2-p'^2)+(\hbar eB/mc)(n-n')}. \quad (16)$$

The sum over all states α' is simply the sum over all quantum numbers:

$$\sum_{\alpha'=1}^{\infty}\equiv\sum_{n'=0}^{\infty}\sum_{l'=0}^{\infty}\frac{L}{2\pi}\int_{-\infty}^{+\infty}dp'. \quad (17)$$

The summation over the occupied states α requires more explanation. First, since only those states with $l=0$ contribute to $\delta\rho$ in Eq. (16), we consider the states $\alpha=(p,0,n)$. The energy of these occupied states must be less than the Fermi energy ϵ_F so the summation over n and p must be taken such that

$$\hbar^2p^2/2m+(\hbar eB/mc)(n+\tfrac{1}{2})\leq\epsilon_F. \quad (18)$$

For a given value of n, this implies that the value of p can vary between $\pm p_F(n)$, where

$$\hbar^2[p_F(n)]^2/2m+(\hbar eB/mc)(n+\tfrac{1}{2})=\epsilon_F. \quad (19)$$

Further, it is clear that n must be less than or equal to some positive integer N_F, where N_F is the largest integer satisfying the inequality

$$(\hbar eB/mc)(N_F+\tfrac{1}{2})\leq\epsilon_F. \quad (20)$$

The expression for the induced charge density can now be written

$$\delta\rho(\mathbf{r})=\left(\frac{m}{2\pi^3\hbar^2}\right)\left(\frac{eB}{\hbar c}\right)^2\mathrm{Re}\sum_{n=0}^{N_F}\sum_{n'=0}^{\infty}\frac{e^{-\xi}L_{n'}(\xi)L_n(\xi)}{n'!n!}\times\int_{-p_F(n)}^{+p_F(n)}dp\int_{-\infty}^{+\infty}dp'\times\frac{e^{i(p'-p)z}}{(p^2-p'^2)+(2eB/\hbar c)(n-n')}. \quad (21)$$

A. Parallel to the Field

We evaluate the induced charge density along a line passing through the test particle parallel to the magnetic field by setting ξ equal to zero in Eq. (21). Then, with $L_n(0)=n!$, we get

$$\delta\rho(z)=\left(\frac{m}{2\pi^3\hbar^2}\right)\left(\frac{eB}{\hbar c}\right)^2\mathrm{Re}\sum_{n=0}^{N_F}\sum_{n'=0}^{\infty}\int_{-p_F(n)}^{+p_F(n)}dp\int_{-\infty}^{+\infty}dp'\times\frac{e^{i(p'-p)z}}{(p^2-p'^2)+(2eB/\hbar c)(n-n')}. \quad (22)$$

We define a new variable of integration, $q=p'-p$, and then integrate over p:

$$\delta\rho(z)=\left(\frac{m}{2\pi^3\hbar^2}\right)\left(\frac{eB}{\hbar c}\right)^2\sum_{n=0}^{N_F}\sum_{n'=0}^{\infty}\int_{-\infty}^{+\infty}dq\,e^{iqz}\times\frac{1}{2q}\ln\left|\frac{p_F(n)+\frac{1}{2}q+(eB/q\hbar c)(n'-n)}{p_F(n)-\frac{1}{2}q-(eB/q\hbar c)(n'-n)}\right|. \quad (23)$$

For large z, one can approximately evaluate the integral in $\delta\rho$ by expanding the integrand in a power series about the singularities on the real q axis.[6] Then, to lowest

[5] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 785.

[6] M. J. Lighthill, *Fourier Analysis and Generalized Functions* (Cambridge University Press, New York, 1958), p. 46.

order in $1/z$, we have

$$\delta\rho(z)=\left(\frac{m}{2\pi^2\hbar^2}\right)\left(\frac{eB}{\hbar c}\right)^2\sum_{n=0}^{N_F}\sum_{n'=0}^{\infty}\mathrm{Re}\left[\frac{e^{iq_+(n,n')z}}{q_+(n,n')z}+\frac{e^{iq_-(n,n')z}}{q_-(n,n')z}\right],\tag{24}$$

where $q_\pm(n,n')=p_F(n)\pm p_F(n')$. According to Eqs. (19) and (20) the value of $p_F(n')$ is purely imaginary for $n'>N_F$, so that the contribution to $\delta\rho$ from those terms with $n'>N_F$ is exponentially small compared to the rest of the terms. Thus in Eq. (24) we take $n'\leq N_F$. The summations may be changed to integrations by using the differential of the defining equation for $p_F(n)$:

$$(\hbar^2/m)p_F(n)\,dp_F(n)=-(e\hbar B/mc)dn.\tag{25}$$

The induced charge density becomes

$$\delta\rho(z)=\left(\frac{m}{2\pi^2\hbar^2}\right)\mathrm{Re}\int_0^{k_F}dp\int_0^{k_F}dp'\,pp'\times\left(\frac{e^{i(p+p')z}}{(p+p')z}+\frac{e^{i(p-p')z}}{(p-p')z}\right),\tag{26}$$

where the Fermi wave number k_F is defined by

$$\hbar^2k_F^2/2m=\epsilon_F.\tag{27}$$

These integrals may be approximately evaluated for large z by using Lighthill's expansion procedure,[6] and the result is

$$\delta\rho(z)=-(mk_F/4\pi^2\hbar^2)(\cos 2k_Fz)/z^3\tag{28}$$

to lowest order in $1/z$. From this expression for $\delta\rho(z)$ it is seen that the induced charge density along the z axis has the same $r^{-3}\cos 2k_Fr$ form which one finds in the absence of a magnetic field.

B. Perpendicular to the Field

We evaluate the induced charge density along a line passing through the test particle perpendicular to the magnetic field by setting z equal to zero in Eq. (21):

$$\delta\rho(r)=\left(\frac{m}{2\pi^3\hbar^2}\right)\left(\frac{eB}{\hbar c}\right)^2\sum_{n=0}^{N_F}\sum_{n'=0}^{\infty}\frac{e^{-\xi}L_n(\xi)L_{n'}(\xi)}{n!n'!}\times\int_{-p_F(n)}^{+p_F(n)}dp\int_{-\infty}^{+\infty}dp'\times\frac{1}{p^2-p'^2+(2eB/\hbar c)(n-n')}.\tag{29}$$

The integrations over p' and p can be done explicitly and yield

$$\delta\rho(r)=\left(-\frac{m}{\pi^2\hbar^2}\right)\times\left(\frac{eB}{\hbar c}\right)^2\sum_{n=0}^{N_F}\sum_{n'=n+1}^{\infty}\frac{e^{-\xi}L_n(\xi)L_{n'}(\xi)g(n,n')}{n!n'!},\tag{30}$$

where

$$g(n,n')=\tfrac{1}{2}\pi,\qquad n'\leq N_F\tag{31}$$

$$g(n,n')=\sin^{-1}\left(\frac{\epsilon_F/\hbar\omega_c-(n+\frac{1}{2})}{n'-n}\right)^{1/2},\qquad n'>N_F\tag{32}$$

and $\omega_c=eB/mc$ is the electron-cyclotron frequency. The summations occurring in the expression for $\delta\rho(r)$ cannot be carried out analytically, so $\delta\rho$ has been evaluated numerically for several cases. The simplest case occurs when the magnetic field is so strong that all the electrons are in the lowest energy Landau level, i.e., the unit energy of quantized cyclotron motion $\hbar\omega_c$ is comparable to the Fermi energy ϵ_F. Then we have $N_F=0$ and the induced density is simply given by

$$\delta\rho(r)=\left(-\frac{m}{\pi^2\hbar^2}\right)\left(\frac{eB}{\hbar c}\right)^2\sum_{n'=1}^{\infty}e^{-\xi}\frac{L_{n'}(\xi)}{n'!}\times\sin^{-1}\left(\frac{\epsilon_F/\hbar\omega_c-\frac{1}{2}}{n'}\right)^{1/2}.\tag{33}$$

To evaluate $\delta\rho(r)$ numerically, we choose a magnetic field such that $\frac{1}{2}\hbar\omega_c\leq\epsilon_F<\frac{3}{2}\hbar\omega_c$; e.g.,

$$\epsilon_F/\hbar\omega_c-\tfrac{1}{2}=1\tag{34}$$

or

$$B=(\hbar c/3e)(3\pi^2\rho_0)^{2/3},\tag{35}$$

where ρ_0 is the electron number density and we have used the free-electron value for the Fermi energy, $\epsilon_F=(\hbar^2/2m)(3\pi^2\rho_0)^{2/3}$. The induced charge density then takes the form shown in curve 1 of Fig. 1. We see that the induced charge density exhibits *no oscillatory behavior* and falls off rapidly for large r. If a slightly weaker magnetic field is used, such that $N_F=1$, we get the result shown in curve 2 of Fig. 1. Here we have $\frac{3}{2}\hbar\omega_c\leq\epsilon_F<\frac{5}{2}\hbar\omega_c$; e.g.,

$$\epsilon_F/\hbar\omega_c-\tfrac{1}{2}=2\tag{36}$$

or

$$B=(\hbar c/5e)(3\pi^2\rho_0)^{2/3}.\tag{37}$$

This corresponds to the case where only *two* Landau levels are occupied and there is an additional "bump" on the induced charge density as a function of distance from the test particle. By reducing the magnetic field even further, more Landau levels become occupied and additional bumps appear in the induced charge density. As the magnetic field approaches zero, the number of occupied Landau levels increases, giving rise to more

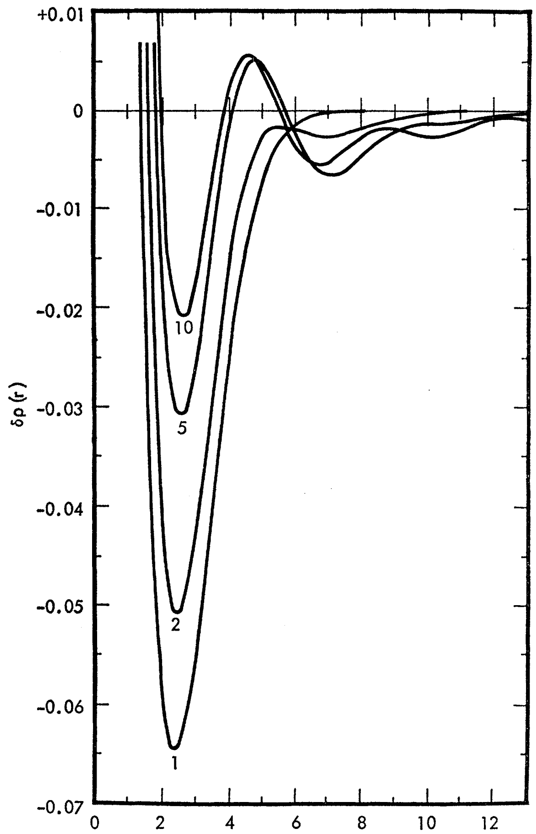

FIG. 1. The induced electron number density (in units of $-3mk_F/\hbar^2$) versus distance (in units of $1/k_F$) along a line passing through the test particle perpendicular to the magnetic field. The number of occupied Landau levels labels the curves.

and more oscillations until the form $r^{-3}\cos 2k_F r$ is obtained. It should also be noted from Fig. 1 that the magnitudes of the maxima and minima in the charge density decrease with the magnetic field.

This modification of the induced charge density by a strong magnetic field can be attributed to a change in the momentum distribution of the electrons. When there is no magnetic field the unperturbed electrons must all have momenta (or wavelengths) less than the Fermi value. Thus only a limited number of Fourier components are available for constructing the induced charge density, and this necessarily leads to an oscillatory behavior for large distances. The momentum distribution for the electrons in a strong magnetic field can be obtained from the Fourier transform of the coordinate-space eigenfunctions given by Eq. (9). For the case where only the lowest energy Landau level is occupied, one finds that the momentum distribution perpendicular to the direction of the magnetic field is of the form $\exp(-k_\perp^2 r_c^2)$, where $\hbar k_\perp$ is the perpendicular component of the momentum and $r_c=(2\hbar c/eB)^{1/2}$ is the electron-cyclotron radius. All possible momenta or wavelengths are present to some extent, so the electron density does not have the oscillatory behavior at large distances from the test particle.

These results indicate that in the presence of a strong magnetic field the conduction-electron *spin polarization* near a magnetic impurity may also differ qualitatively from the zero-magnetic-field case, since the sharp cutoff in the momentum distribution is similarly modified in this case. However, the simplicity of the model used in these calculations precludes any quantitative estimate of this effect. The possibility of observing this behavior depends on the density of conduction electrons, the applicability of the free-electron model, and the interaction between the impurity and the conduction electrons. The magnetic field B_0 needed to reach the extreme quantum limit $N_F=0$ is related to the electron density ρ_0 by $B_0=(\hbar c/3e)(3\pi^2\rho_0)^{2/3}$. In a semiconductor such as InSb the conduction-electron density can be on the order of 10^{15} cm^{-3} and the required magnetic field B_0 is about 2 kG. However, in a metal such as copper, the density ρ_0 is on the order of 10^{21} cm^{-3} and the magnetic field needed to reach the extreme quantum limit is beyond present capabilities. The problem which one encounters is that while the extreme quantum limit can be reached for low-electron densities, the number of electrons may be too small to produce a measureable effect. The strength of the interaction between the impurity or test particle and the conduction electrons also has a direct influence on the perturbed charge or spin density.

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