Current Fluctuations in the Acoustoelectric Steady State

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A theoretical study of current fluctuations about the nonequilibrium acoustoelectric steady state of a piezoelectric semiconductor is presented. The two salient features of the observed noise are its large magnitude (typically ~60 dB) above Nyquist noise and its spectral distribution (Lorentzian, with a low-frequency cutoff ~20 Mc/sec). The former feature arises, in the present formulation, from the large fluctuations in the excess phonon concentration about its steady-state value. The low-frequency cutoff is given essentially by the temporal-gain constant for phonon growth. Physically, it represents the characteristic time for a fluctuation in drift velocity to produce a change in phonon concentration to which the drift velocity responds adiabatically. The final result for the spectral distribution bears a strong resemblance to that derived by Moore on the basis of a bunching theory.

I. INTRODUCTION

As is well known, current fluctuations in solids at thermal equilibrium are described by Nyquist's theorem.[1] Quite generally, it has been shown[2] that, in thermal equilibrium, fluctuations in dissipative systems are directly related to the kinetic coefficient connecting the linear response of the system to an arbitrarily weak external force. Hence noise measurements in thermal equilibrium provide no more information than can be obtained by conventional transport measurements. However, in considering fluctuations about a *nonequilibrium steady state*, no such general result applies; each case must be investigated separately, and new information may be obtained.

Fluctuations about a nonequilibrium steady state have been studied for many systems.[3] One particular case, studied both theoretically[4,5] and experimentally,[6] is that of semiconductors in electric fields sufficiently strong to produce, in some cases, appreciable departures from Ohm's law. These studies have shown, for example, that, in addition to the usual thermal noise associated with velocity fluctuations about the nonequilibrium electron distribution, there is an additional so-called "convective" noise associated with the energy exchange between the hot carriers and lattice.[4,5]

Moore[7] has recently measured current fluctuations about the steady state[8] saturated current in CdS. It is well established by now[9] that current saturation in this case is due to the internal generation of acoustic flux and the reaction of this flux back on the drifting carriers. Unlike the case of hot electrons, however, one must here consider the simultaneous, coupled fluctuations of both the electrons and phonons about their steady-state values.

There are two salient features of the measured noise. The first is its large magnitude. For typical semiconducting samples, the noise power per unit bandwidth is ~60 dB above thermal noise. This is shown in Fig. 1 at 10 and 100 Mc/sec. The second feature is its spectral distribution. This is approximately Lorentzian, with a rather low cutoff frequency of 20–50 Mc/sec. This is shown in Fig. 2. In both cases the data are plotted as a function of V/V_c, where V_c is the critical voltage for saturation.

Moore[7] originally explained his data in terms of a two-state, two-parameter, phenomenological model. His basic physical picture was that sound waves are coherently amplified to the level where they spatially bunch all available carriers. This requires a characteristic time τ_0. Then, after another characteristic time

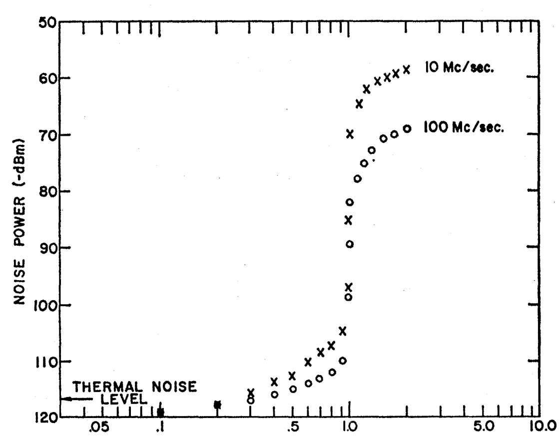

FIG. 1. Noise power at 10 and 100 Mc/sec versus applied voltage (data of A. Moore, Ref. 7).

[1] See, e.g., C. Kittel, *Elementary Statistical Physics* (John Wiley & Sons, Inc., New York, 1958), p. 141.

[2] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).

[3] See, e.g., *Fluctuation Phenomena in Solids*, edited by R. D. Burgess (Academic Press Inc., New York, 1965).

[4] P. J. Price, in Ref. 3, Chap. 8.

[5] V. L. Gurevich, Zh. Eksperim. i Teor. Fiz. **43**, 1771 (1962) [English transl.: Soviet Phys.—JETP **16**, 1252 (1963)].

[6] E. Erlbach and J. B. Gunn, Phys. Rev. Letters **8**, 280 (1962).

[7] A. R. Moore, J. Appl. Phys. **38**, 2327 (1967).

[8] It is to be emphasized that these fluctuations are *not* associated with the generation and propagation of acoustic domains observed by many investigators [e.g., A. Many and I. Balberg, Phys. Letters **20**, 463 (1966); P. O. Silva and R. Bray, Phys. Rev. Letters **14**, 372 (1965)]. Rather, these are fluctuations that occur about the steady state, reached either exponentially or via damped oscillation, which we assume is achieved for the time of application of the applied voltage pulse.

[9] R. W. Smith, Phys. Rev. Letters **8**, 87 (1962); J. H. McFee, J. Appl. Phys. **34**, 1548 (1963); R. Bray, C. S. Kumar, J. B. Ross, and P. O. Sliva, J. Phys. Soc. Japan Suppl. **21**, 483 (1963).

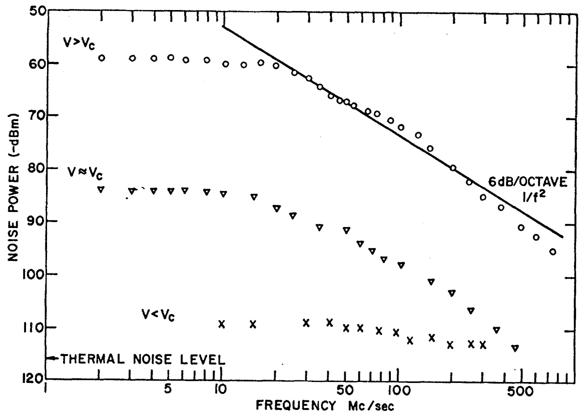

FIG. 2. Spectral distribution of the noise power (data of A. Moore, Ref. 7).

τ_s, the bunches decay, releasing the trapped charge into the Ohmic state. In this picture, the large magnitude of the noise is attributed to the complete bunching of all available charge within the individual wavelengths of the amplified sound wave. The expression for the noise spectrum so obtained provides a good fit to the data. However, aside from its phenomenological nature, the physical picture is not entirely satisfactory. In particular, it is difficult to see why the sound wave should build up in a coherent fashion only to subsequently decay, implying a statistical separation of the gain and loss mechanisms. Indeed, Moore infers that the decay is not an intrinsic process, but, for his short samples, τ_s correspond to the drift of the bunches out of the crystal. This then leaves open the physical mechanism for τ_s when the transport is not drift-limited (i.e., τ_s < drift time).

In the present paper, the noise is analyzed from an entirely different point of view, namely, by means of a fluctuation analysis of the conventional coupled Boltzmann equations of the electron-phonon system. It is to be emphasized that this is a spatially homogeneous theory[10] in which bunching effects are not taken into account. The method employed is a generalization of a method used to investigate current fluctuations in the hot-electron state,[5] mentioned previously. It will be seen that the magnitude of the noise is associated with the large relative fluctuations in the phonon number. The characteristic relaxation time in the spectral distribution is essentially given by the reciprocal gain coefficient; this is the time required for a fluctuation in drift velocity to produce a change in phonon number to which the carriers respond almost instantaneously.[11]

The Boltzmann-equation description of amplification and current saturation will first be reviewed briefly. We shall then discuss the fluctuations about the steady-state solutions provided by these equations.

II. PRELIMINARIES

Coupled Boltzmann Equations

The time-dependent Boltzmann equations describing phonon amplification and current saturation have been derived by Yamashita and Nakamura.[11] They are

$$\frac{dv_d(t)}{dt}=-\frac{v_d(t)-v_0}{\tau_0}-\frac{m}{2\pi\hbar^3 n}\sum_{\mathbf{q}} qC_q^2 f_0(\epsilon_q)\cos\theta_q \times[v_d(t)\cos\theta_q-s]\xi_q(t), \quad (2.1)$$

$$\frac{d\xi_q(t)}{dt}=\left[\frac{VC_q^2}{2\pi}\left(\frac{m}{\hbar^2}\right)^2 f_0(\epsilon_q)\right][v_d(t)\cos\theta_q-s]\xi_q(t)-\frac{\xi_q}{\tau_1}-\frac{\xi_q^2}{\tau_2}. \quad (2.2)$$

Here $v_d(t)$ is the carrier drift velocity at time t, $\xi_q(t)$ is the excess number of phonons of mode $\mathbf{q}$ (i.e., in excess of the thermal equilibrium number $N_q^{(0)}$), τ_c is the Ohmic collision time, $v_0=eE\tau_c/m$ is the Ohmic drift velocity that the electrons would acquire in the absence of amplification, C_q^2 is the electron-phonon coupling constant (later to be particularized to piezoelectric coupling), f_0 is the Maxwell distribution, $\epsilon_q=(\hbar^2/2m)\times(\frac{1}{2}q)^2$, θ_q is the angle between the phonons of mode $\mathbf{q}$ and the direction of electron drift, s is the sound velocity, and τ_1^{-1} and τ_2^{-1} are the linear and nonlinear phonon loss rates, respectively. The remaining symbols have their conventional meanings. In deriving these equations, no assumption is made about the magnitude of ξ_q; however, a displaced Maxwellian distribution is assumed for the electrons.

The physical interpretation of these equations is straightforward. The quantity multiplying ξ_q in the first term of (2.2) is the temporal-gain coefficient. Thus (2.2) gives the time rate of increase in phonon number when the phonon gain exceeds the phonon losses. As for (2.1), the first term on the right-hand side describes the Ohmic relaxation via collisions to $v_d=v_0$, and the second term gives a reduction in the rate of change of v_a due to collisions of the carriers with the excess, stimulated phonons. It is instructive to note that the coefficient of this latter term is just the gain factor when the equations are expressed as a momentum balance. Thus,

[10] We distinguish between bunching within individual wavelengths, characteristic of the classical theory [D. L. White, J. Appl. Phys. 33, 2547 (1962)], and slow spatial variations in phonon concentration and field, characteristic of the incoherent phonon theory [L. Friedman, Phys. Rev. 163, 713 (1967)]. There is always some spatial variation in practice, and this could be taken into account in the present treatment. However, we do not believe that this is an essential feature of the noise problem. This is particularly so in view of the fact that Moore (Ref. 7), in order to inhibit domain formation, used short, uniform samples over which the potential was fairly uniform. In this connection, see also the latter part of Ref. 14.

[11] J. Yamashita and K. Nakamura, Progr. Theoret. Phys. (Kyoto) 33, 1022 (1965).

multiplying (2.1) by $nm\mathcal{V}$ and (2.2) by $\hbar q$, we obtain

$$\frac{dp_d(t)}{dt}=-\frac{p_d(t)-p_0}{\tau_c}-\sum_{\mathbf{q}} G_p(q)\cos\theta_q \times[p_d(t)\cos\theta_q-p_s]p_q(t)\,, \tag{2.3}$$

$$\frac{dp_q(t)}{dt}=G_p(q)[p_d(t)\cos\theta_q-p_s]p_q(t)=\frac{p_q}{\tau_1'}-\frac{p_q^2}{\tau_2'}, \tag{2.4}$$

where

$$p_d=nm\mathcal{V}v_d,\quad p_s=nm\mathcal{V}s,\quad p_q=\hbar q\xi_q$$

and

$$G_p(q)=mC_q^2f_0(\epsilon_q)/2\pi\hbar^2n\,,$$

the gain constant for crystal momentum flux density, is independent of n.

Now the excess stimulated phonons occur in a narrow Čerenkov-like cone about the drift-velocity direction and in a rather narrow range (Δq) of $\mathbf{q}$ space.[12] The upper limit of this band is determined both by the falloff of the piezoelectric interaction (as $1/q$) and by the increase of the phonon loss rate (typically, $\tau_1^{-1}\sim q^2$). The cutoff for small q is due to the screening of the piezoelectric interaction by the charge carriers when the wavelength exceeds the Debye screening length. In view of these considerations, the following approximations are made: (a) The phonon propagation for all the modes is predominantly in the forward direction,

$$\cos\theta_q=1,\quad q_0-\tfrac{1}{2}\Delta q\leq q\leq q_0+\tfrac{1}{2}\Delta q\,,$$

and (b) the q-dependent factor multiplying ξ_q in the sum appearing in (2.1) is sufficiently slowly varying over the range of interest that it can be factored out of the sum, and the sum on ξ_q taken directly:

$$\sum_{\Delta q} F(q)\xi_q\cong F(q_0)\xi_{q0}\,,$$

where

$$\xi_{q0}=\sum_{\Delta q}\xi_q.$$

Indeed, it is readily verified that the principal q-dependent factor of $F(q)$,

$$F(q)\sim\frac{1}{q}\left(\frac{q^2}{q^2+q_D^2}\right)^2,$$

with

$$q_D=(4\pi ne^2/\epsilon k_BT)^{1/2},$$

the Debye wave vector, varies by less than an order of magnitude over the range

$$5\times10^4<q<6\times10^5\ \text{cm}^{-1}$$

proposed by Yamashita and Nakamura.[12]

[12] Reference 11, pp. 1036–1037.

Finally, rewriting (2.1) and (2.2) in terms of the dimensionless wave number

$$x=\frac{q}{(2mk_BT/\hbar^2)^{1/2}},$$

one obtains

$$\frac{dv_d(t)}{dt}=A[v_0-v_d(t)]-Bf(x_0)(v_d-s)\xi(x_0,t)\,, \tag{2.5}$$

$$\frac{d\xi(x_0,t)}{dt}=Cg(x_0)(v_d-s)\xi(x_0,t)-\frac{\xi}{\tau_1}-\frac{\xi^2}{\tau_2}, \tag{2.6}$$

where

$$\xi(x_0,t)=\sum_{\Delta x}\xi(x,t)\,,$$

$$f(x)=x^2\left(\frac{x^2}{x^2+x_D^2}\right)^2e^{-x^2/4}, \tag{2.7}$$

$$g(x)=\frac{e^{-x^2/4}}{x}\left(\frac{x^2}{x^2+x_D^2}\right)^2\frac{1}{s}, \tag{2.8}$$

$$A=\tau_c^{-1}, \tag{2.9}$$

$$B=\frac{4\pi e^2\beta}{\epsilon^2}\frac{2}{\pi^{1/2}}\frac{m}{MN_ash^2}, \tag{2.10}$$

$$C=\left(\frac{4\pi e^2\beta}{\epsilon^2}\right)^2\frac{h}{2MN_a}\frac{(2\pi)^{1/2}}{(k_BT)^2}n\,, \tag{2.11}$$

and

$$x_D\approx\frac{\hbar}{(2mk_BT)^{1/2}}\left(\frac{4\pi nl^2}{\epsilon k_BT}\right)^{1/2}$$

is the dimensionless Debye screening length. Here β is the piezoelectric constant and ϵ is the static dielectric constant. The ion mass is M and N_a is the number of unit cells per unit volume.

Steady-State Solutions

The noise analysis depends critically on the form of the steady-state solution about which the fluctuations are taken. The approximation to the steady-state solution, in turn, depends upon the particular assumptions made concerning the process that limits the growth of phonon flux. Hence some attention to the steady-state solutions is warranted at this point. It should be emphasized that the purpose of the present paper is not to get a quite accurate description of the steady-state current voltage characteristic; this would require a separate study removing many of the assumptions of Ref. 11. Rather, an approximate characterization of the steady state is sought which provides an adequate basis for the fluctuation analysis.

Linear Phonon Losses Only ($\tau_2=\infty$)

For this case, the steady-state solution to (2.6) is given by

$$\bar{v}_d=s_0^*=s+[1/Cg(x_0)](1/\tau_1). \quad (2.12)$$

(Here and in what follows, steady-state values of quantities are denoted by a bar.) For this value of $\bar{v}_d$, (2.5) gives

$$\bar{\xi}_0=\bar{\xi}(x_0)=\frac{A}{Bf(x_0)}\left(\frac{v_0-\bar{v}_d}{\bar{v}_d-s}\right)=\frac{A}{Bf(x_0)}\frac{1-\mathcal{J}}{\mathcal{J}}, \quad (2.13)$$

where

$$v_0=eE\tau_c/m$$

and

$$\mathcal{J}=(\bar{v}_d-s)/(v_0-s). \quad (2.14)$$

The physical content of the solutions is that the drift velocity saturates at the fixed value (2.12) sufficient to provide the linear losses, while the steady-state phonon number increases with increasing electric field according to (2.13) in order to provide the required momentum balance.

Nonlinear Losses Dominant

In this case, according to (2.6), the steady-state phonon flux is limited by the nonlinear loss term and is given by

$$\bar{\xi}_0=\bar{G}_0\tau_2, \quad (2.15)$$

where

$$\bar{G}_0=Cg(x_0)(\bar{v}_d-s) \quad (2.16)$$

is the steady-state temporal-gain coefficient. From (2.5) it follows that $\bar{\xi}_0$ is also given by

$$\bar{\xi}_0=\frac{A}{Bf(x_0)}\frac{1-\mathcal{J}}{\mathcal{J}}, \quad (2.17)$$

except that $\mathcal{J}$, defined by (2.14), is no longer a constant, since $\bar{v}_d$ now depends on the field E. To find this dependence, (2.15) is substituted into (2.5). Solving, one obtains

$$\bar{v}_d-s=\tfrac{1}{2}K_0\{-1+[1+(4/K_0)(v_0-s)]^{1/2}\}, \quad (2.18)$$

where

$$K_0=\frac{A}{Bf(x_0)}\frac{1}{Cg(x_0)\tau_2}.$$

Thus in this approximation $\bar{v}_d$ increases with electric field. The increase, however, is sublinear; it does not give an Ohm's law above the knee.

III. FLUCTUATION ANALYSIS

As mentioned in Sec. II, the results of the fluctuation analysis depend upon the steady-state solutions about which fluctuations occur. Since the experimental experience in CdS indicates that the drift velocity increases with increasing field above the knee,[13] and since this is the predicted behavior for the case in which the growth of phonon flux is limited by nonlinear losses, the latter is taken as the appropriate steady-state solution. The starting equations for the nonlinear-loss case are written

$$\frac{dv_d(t)}{dt}=A[v_0-v_d(t)]-Bf_0[v_d(t)-s]\xi_0(t), \quad (3.1)$$

$$\frac{d\xi_0(t)}{dt}=Cg_0[v_d(t)-s]\xi_0(t)-\frac{\xi_0^2}{\tau_2}, \quad (3.2)$$

where the meanings of the abbreviations are obvious.

In the steady state, (3.1) and (3.2) become

$$0=A(v_0-\bar{v}_d)-Bf_0(\bar{v}_d-s)\bar{\xi}_0, \quad (3.3)$$

$$0=Cg_0(\bar{v}_d-s)\bar{\xi}_0-(\bar{\xi}_0)^2/\tau_2. \quad (3.4)$$

We now consider momentary fluctuations in both $v_d(t)$ and $\xi_0(t)$, i.e.,

$$v_d(t)=\bar{v}_d+\delta v_d(t), \quad (3.5)$$

$$\xi_0(t)=\bar{\xi}_0+\delta\xi_0(t). \quad (3.6)$$

Substituting (3.5) and (3.6) into (3.1) and (3.2), one obtains

$$\frac{d\delta v_d(t)}{dt}=A[v_0-\bar{v}_d-\delta v_d(t)]-Bf_0[(\bar{v}_d-s)+\delta v_d(t)]\times[\bar{\xi}_0+\delta\xi_0(t)], \quad (3.7)$$

$$\frac{d\delta\xi_0(t)}{dt}=Cg_0[(\bar{v}_d-s)+\delta v_d(t)][\bar{\xi}_0+\delta\xi_0(t)]-\frac{(\bar{\xi}_0)^2+2\bar{\xi}_0\delta\xi_0(t)+(\delta\xi_0)^2}{\tau_2}. \quad (3.8)$$

Neglecting $(\delta\xi_0)^2$ as higher order and subtracting (3.3) and (3.4) from (3.7) and (3.8), respectively, one gets[14]

$$\frac{d\delta v_d(t)}{dt}=-A\,\delta v_d(t)-Bf_0\{(\bar{v}_d-s)\delta\xi_0(t)+\delta v_d(t)[\bar{\xi}_0+\delta\xi_0(t)]\}, \quad (3.9)$$

[13] A. R. Moore and R. W. Smith, Phys. Rev. **138**, A1250 (1965).

[14] Note that the field term $\sim v_0$ cancels out in this subtraction recipe. The reason is the assumption of a rigidly displaced Maxwellian. Thus, in the absence of phonons, the term $-A\delta v_d$ gives the usual Nyquist noise about the steady-state distribution. In the present calculation, the field dependence later reappears via the dependence of $\bar{v}_d$ and $\bar{\xi}_0$ on v_0. In this connection, note that there are current, but no charge density fluctuations corresponding to fluctuations of the centroid of the distribution function. This assumption is self-consistent if we note that the conductivity relaxation frequency ω_c ($\sim10^{12}\sigma$ sec^{-1}) is orders of magnitude larger than the frequency of the current fluctuations ($\sim10^7$ sec^{-1}); thus any tendency toward charge bunching is immediately relaxed. Hence spatial correlations may be neglected [see Ref. 5, Eq. (2) and preceding text]; i.e., fluctuations in adjacent volume elements are mutually exclusive.

$$\frac{d\delta\xi_0(t)}{dt}=Cg_0\{(\bar{v}_d-s)\delta\xi_0(t)+\delta v_d(t)[\bar{\xi}_0+\delta\xi_0(t)]\}-\frac{2\bar{\xi}_0\delta\xi_0(t)}{\tau_2}. \quad (3.10)$$

Following Gurevich,[5] (3.9) and (3.10) are now multiplied by $\delta v_d(0)$, and an average is taken over all possible initial times (i.e., designated $t=0$ in this case), but for a fixed time difference τ, giving,

$$\frac{d}{d\tau}\langle\delta v_d(0)\delta v_d(\tau)\rangle =-A\langle\delta v_d(0)\delta v_d(\tau)\rangle-Bf_0[(\bar{v}_d-s)\langle\delta v_d(0)\delta\xi_0(\tau)\rangle +\bar{\xi}_0\langle\delta v_d(0)\delta v_d(\tau)\rangle+\langle\delta v_d(0)\delta v_d(\tau)\delta\xi_0(\tau)\rangle], \quad (3.11)$$

$$\frac{d}{d\tau}\langle\delta v_d(0)\delta\xi_0(\tau)\rangle =Cg_0[(\bar{v}_d-s)\langle\delta v_d(0)\delta\xi_0(\tau)\rangle+\bar{\xi}_0\langle\delta v_d(0)\delta v_d(\tau)\rangle +\langle\delta v_d(0)\delta v_d(\tau)\delta\xi_0(\tau)\rangle] -(2\bar{\xi}_0/\tau_2)\langle\delta v_d(0)\delta\xi_0(\tau)\rangle. \quad (3.12)$$

Taking the Fourier transforms of (3.11) and (3.12) with respect to τ and doing an integration by parts for the time derivatives on the left-hand sides, one obtains

$$-i\omega\gamma_{vv}(\omega)-\langle[\delta v_d(0)]^2\rangle/2\pi=-A\gamma_{vv}(\omega)-Bf_0 \times[(\bar{v}_d-s)\gamma_{v\xi}(\omega)+\bar{\xi}_0\gamma_{vv}(\omega)+\gamma_{vv\xi}(\omega)], \quad (3.13)$$

$$-i\omega\gamma_{v\xi}(\omega)-\langle\delta v_d(0)\delta\xi_0(0)\rangle/2\pi =Cg_0[(\bar{v}_d-s)\gamma_{v\xi}(\omega)+\bar{\xi}_0\gamma_{vv}(\omega)+\gamma_{vv\xi}(\omega)] -2(\bar{\xi}_0/\tau_2)\gamma_{v\xi}(\omega), \quad (3.14)$$

where

$$\gamma_{vv}(\omega)\equiv\frac{1}{2\pi}\int_0^\infty d\tau\, e^{i\omega\tau}\langle\delta v_d(0)\delta v_d(\tau)\rangle, \quad (3.15)$$

$$\gamma_{v\xi}(\omega)\equiv\frac{1}{2\pi}\int_0^\infty d\tau\, e^{i\omega\tau}\langle\delta v_d(0)\delta\xi_0(\tau)\rangle, \quad (3.16)$$

and

$$\gamma_{vv\xi}(\omega)\equiv\frac{1}{2\pi}\int_0^\infty d\tau\, e^{i\omega\tau}\langle\delta v_d(0)\delta v_d(\tau)\delta\xi_0(\tau)\rangle. \quad (3.17)$$

The real part of $\gamma_{vv}(\omega)$, given by (3.15), gives the power spectrum of the drift-velocity fluctuations by the Weiner-Khintchine theorem,[1] and is the quantity for which we ultimately want to solve. Equation (3.16), on the other hand, describes the coupled correlations.

In the next subsection, dealing with the magnitude of the noise, it will be shown that

$$\langle(\delta\xi_0)^2\rangle\ll(\bar{\xi}_0)^2. \quad (3.18)$$

It follows that the triple correlator $\gamma_{vv\xi}$ is very much smaller than $\xi_0\gamma_{vv}$, and is therefore negligible. Then solving (3.14) for $\gamma_{v\xi}$, and using (2.15) and (2.16) to eliminate τ_2, one gets

$$\gamma_{v\xi}(\omega)=\frac{\bar{\xi}_0}{-i\omega/Cg_0+(\bar{v}_d-s)}\gamma_{vv}(\omega)+\frac{\langle\delta v_d\delta\xi\rangle}{2\pi} \times\frac{1/Cg_0}{-i\omega/Cg_0+(\bar{v}_d-s)}.$$

Substituting this expression into (3.13) and solving for $\gamma_{vv}(\omega)$, one obtains two contributions:

$$\gamma_{vv}(\omega)=\gamma_{vv}^{(1)}(\omega)+\gamma_{vv}^{(2)}(\omega),$$

where

$$\gamma_{vv}^{(1)}(\omega)=\frac{\langle(\delta v_d)^2\rangle}{2\pi} \times\frac{i\omega-\bar{G}}{(-i\omega+A)(i\omega-\bar{G})+Bf_0\bar{\xi}_0(i\omega-2\bar{G})} \quad (3.19)$$

and

$$\gamma_{vv}^{(2)}(\omega)=(\langle\delta v_d\delta\xi\rangle/2\pi)Bf_0(\bar{v}_d-s) \times[(-i\omega+A)(i\omega-\bar{G})+Bf_0\bar{\xi}_0(i\omega-2\bar{G})]^{-1}. \quad (3.20)$$

The order of magnitude of the noise power, as determined by $\langle(\delta v_d)^2\rangle$ and $\langle\delta v_d\delta\xi\rangle$, will be calculated first; then the spectral distribution will be investigated.

Magnitude of Noise

From (3.19) and (3.20) it is seen that the magnitude of the noise is set by the mean values $\langle\delta v_d\delta\xi\rangle$ and $\langle(\delta v_d)^2\rangle$. Now $\delta v_d=[\langle(\delta v_d)^2\rangle]^{1/2}$ is simply the rms drift-velocity fluctuation. In thermal equilibrium this is obtained[1] by noting that $\mathbf{v}_d$ is nothing more than the sum over the individual random-particle velocities $\mathbf{v}_i$, and that these are undergoing thermal fluctuations $(\delta v_i)^2\sim k_BT/m$. This leads to the well-known result for Nyquist noise.

However, it would be incorrect to use such a result in the present case. The reason is that, by definition, acoustoelectric current saturation occurs when the scattering of electrons by the stimulated phonons [i.e., the second term on the right side of (3.1)] dominates the usual Ohmic collisions, else there would be no current saturation in the first place. However, the phonons themselves are undergoing fluctuations. The relative fluctuations of Bose systems in thermal equilibrium is, in fact, known to be quite large.[1] In the present case, fluctuations are occurring, of course, about the steady state ξ_0, and not about the thermal-equilibrium Planck distribution. However, *assuming* that the relative fluctuations *per mode* are the same in the two cases, it is possible to calculate the magnitude of the measured noise power.

Equation (3.9) is considered at $t=0$. One then notes that the electrons follow the phonons essentially instantaneously.[15] Thus the time derivative on the left-

[15] See Ref. 11, Eq. (51) and preceding text. The same assumption could also have been made in the derivation of the correlation function itself (that is, the neglect of the term $i\omega$ with respect to A) with but minor modification in the final results.

hand side can be neglected. Solving for $\delta v_d(0)$, one gets

$$\delta v_d(0^+) = -\frac{Bf_0(\bar{v}_d - s)}{A + Bf_0\bar{\xi}_0(0)}\delta\xi_0(0)\,, \qquad (3.21)$$

where $\delta\xi$ has been dropped with respect to $\bar{\xi}_0$ in view of the inequality (3.18). The proof for this will be given later.

From (3.21) it follows directly that

$$-Bf_0(\bar{v}_d - s)\langle\delta v_d\delta\xi\rangle = (A + Bf_0\bar{\xi}_0)\langle(\delta v_d)^2\rangle = (A/\mathcal{g})\langle(\delta v_d)^2\rangle\,.$$

Using this result, it is seen that (3.20) is larger than (3.19) by the factor

$$|A/(i\omega - G)| \sim 10^5\,.$$

Hence $\gamma_{vv}^{(1)}$ is neglected with respect to $\gamma_{vv}^{(2)}$; henceforth, the superscript (2) will be dropped. One has

$$\gamma_{vv}(\omega) = (\langle\delta v_d\delta\xi\rangle/2\pi)Bf_0(\bar{v}_d - s)f(\omega)\,, \qquad (3.22)$$

where

$$f(\omega) = [(-i\omega + A)(i\omega - \bar{G}) + Bf_0\bar{\xi}_0(i\omega - 2\bar{G})]^{-1}\,. \qquad (3.23)$$

Using (3.21) and (2.17), one has

$$-(1/2\pi)Bf_0(\bar{v}_d - s)\langle\delta v_a\delta\xi\rangle = \frac{1}{2\pi}\frac{(Bf_0)^2(\bar{v}_d - s)^2}{A + Bf_0\bar{\xi}_0}\langle(\delta\xi_0)^2\rangle = \frac{1}{2\pi}A\frac{(1-\mathcal{g})^2}{\mathcal{g}}(v_0 - s)^2\frac{\langle(\delta\xi_0)^2\rangle}{(\bar{\xi}_0)^2}\,. \qquad (3.24)$$

The problem then reduces to a calculation of the primitive fluctuation in phonon number $\delta\xi_0$ about its steady value $\bar{\xi}_0$.

Recall that ξ_0 refers to the sum of the excess phonon numbers in the hot-phonon band [cf. text preceding (2.5)]. For convenience, reverting back to $\mathbf{q}$ space,

$$\xi_0 = \xi_{q0} = \sum_{\Delta\mathbf{q}} \xi_q\,.$$

Then

$$(\delta\xi_0)^2 = \sum_{q,q' \text{ in } \Delta q} \delta\xi_q\delta\xi_{q'}\,,$$

$$\langle(\delta\xi_0)^2\rangle = \sum_{q,q' \text{ in } \Delta q} \delta_{qq'}\bar{\xi}_q(\bar{\xi}_q + 1) = \sum_{\Delta q} \bar{\xi}_q(\bar{\xi}_q + 1)\,,$$

where, as stated just prior to (3.21), it is *assumed* that equilibrium phonon statistics for the individual modes can be applied to fluctuations about the steady state. It then follows that

$$\frac{\langle(\delta\xi_0)^2\rangle}{(\bar{\xi}_0)^2} = \frac{\sum_{\Delta q}\xi_q(\xi_q + 1)}{(\sum_{\Delta q}\xi_q)^2} \cong \frac{M\xi_{q0}^2}{M^2\xi_{q0}^2} = \frac{1}{M}\,, \qquad (3.25)$$

where M is the number of modes in the phonon band and, certainly, $\xi_q \gg 1$.

From (3.22), (3.24), and (3.25) one obtains

$$-\mathrm{Re}\gamma_{vv}(\omega) = \frac{1}{2\pi}\frac{A}{M}(v_0 - s)^2\frac{(1-\mathcal{g})^2}{\mathcal{g}}\mathrm{Re}f(\omega)\,, \qquad (3.26)$$

where $f(\omega)$ is given by (3.23).

The numerical magnitude of the noise will be calculated in Sec. IV.

Spectral Distribution

The frequency dependence of the noise power is given by $\mathrm{Re}f(\omega)$, with $f(\omega)$ given by (3.23). Taking the real part and using (2.17) to eliminate $\bar{\xi}_0$, one gets

$$\mathrm{Re}f(\omega) = \frac{\omega^2 - \bar{G}A(2-\mathcal{g})/\mathcal{g}}{[\omega^2 - \bar{G}A(2-\mathcal{g})/\mathcal{g}]^2 + \omega^2(\bar{G} + A/\mathcal{g})^2}\,.$$

In the frequency range of interest,

$$\omega \sim 3\times 10^7\ \mathrm{sec}^{-1}\,,$$

and taking typical values[7,16]

$$A = \tau_c^{-1} \sim 10^{13}\ \mathrm{sec}^{-1}, \quad \bar{G} \sim 10^8\ \mathrm{sec}^{-1}\,,$$

it is seen that

$$\bar{G} \ll A$$

and

$$\omega^2 \ll A\bar{G}\,.$$

Under these conditions, $\mathrm{Re}f(\omega)$ simplifies to

$$-\mathrm{Re}f(\omega) \cong \frac{1}{\bar{G}A}\frac{\mathcal{g}}{2-\mathcal{g}}\frac{1}{1+\omega^2\tau^2}\,, \qquad (3.27)$$

where

$$1/\tau = \omega_{c0} = \bar{G}(2-\mathcal{g})\,. \qquad (3.28)$$

Hence, in the frequency range of interest, the spectral distribution is very nearly Lorentzian. Moreover, the cutoff frequency is essentially equal to the steady-state temporal-gain coefficient, to within a numerical factor of order unity. As pointed out in the Introduction, this represents the characteristic time for an initial fluctuation in drift velocity to produce a change in phonon number to which the drift velocity responds adiabatically. The cutoff frequency will be estimated numerically in Sec. IV.

IV. FINAL RESULTS AND DISCUSSION

From (3.26) and (3.27) one obtains

$$\mathrm{Re}\gamma_{vv}^{(\xi)}(\omega) \cong \frac{1}{2\pi}\frac{1}{A}\frac{A}{\bar{G}M}(v_0 - s)^2\frac{(1-\mathcal{g})^2}{2-\mathcal{g}}\frac{1}{1+\omega^2\tau^2}\,, \qquad (4.1)$$

with τ given by (3.28).

[16] A. Moore (private communication) quotes a spatial-gain constant $\alpha_s \sim 10^2\ \mathrm{cm}^{-1}$ corresponding to $n \sim 10^{15}$–$10^{16}\ \mathrm{cm}^{-3}$ and nearly complete current saturation for a typical semiconducting sample.

This is to be compared with the Nyquist result for thermal noise,

$$\mathrm{Re}\gamma_{vv}{}^{(\mathrm{th})}(\omega)=\frac{1}{2\pi}\frac{1}{A}\frac{4k_BT}{m}\frac{1}{n\mathcal{V}}. \tag{4.2}$$

It proves to be more convenient for later discussion to express the final results in terms of the total current fluctuations rather than drift-velocity fluctuations. The total current and drift velocity are related by

$$I=A'nev_d=(N_d/L)ev_d,$$

where A' is the sample cross-sectional area, L is its length, n is the electron concentration, and N_d is the total electron number. Noting that

$$v_0=\mu E$$

and that there are no fluctuations in carrier concentration in the present theory,[14] one finally obtains

$$\mathrm{Re}\gamma_{II}{}^{(\xi)}(\omega)=(N_d{}^2/L^2)e^2\,\mathrm{Re}\gamma_{vv}{}^{(\xi)}(\omega)$$
$$=\frac{1}{2\pi}\frac{N_d{}^2}{L^2}e^2\left[\left(E-\frac{s}{\mu}\right)^2\mu^2\right]\frac{1}{A}\frac{A}{\bar{G}M}\times\frac{(1-\mathcal{J})^2}{2-\mathcal{J}}\frac{1}{1+\omega^2\tau^2}. \tag{4.3}$$

The corresponding result for Nyquist noise is

$$\mathrm{Re}\gamma_{II}{}^{(\mathrm{th})}(\omega)=(1/2\pi)4k_BT/R, \tag{4.4}$$

where R is the total Ohmic resistance.

Frequency Cutoff

The cutoff frequency of the Lorentzian, given by (3.28), is essentially equal to $\bar{G}$. With a spatial gain of typically 10^2 cm^{-1},[16] and with $s\cong 2\times10^5$ cm sec^{-1}, one has

$$(2\pi)^{-1}\omega_{c0}\approx\bar{G}\approx 2\times10^7\ \mathrm{sec}^{-1},$$

in order-of-magnitude agreement with the data shown in Fig. 2.

Magnitude of Noise

To establish the magnitude of the acoustoelectric noise, one takes the ratio of (4.1) to (4.2) [or, equivalently, (4.3) to (4.4)] in the low-frequency limit $\omega\tau\ll1$. This gives

$$\lim_{\omega\tau\to0}\frac{\mathrm{Re}\gamma_{vv}{}^{(\xi)}(\omega)}{\mathrm{Re}\gamma_{vv}{}^{(\mathrm{th})}(\omega)}=\frac{m}{4k_BT}n\mathcal{V}(v_0-s)^2\frac{(1-\mathcal{J})^2}{2-\mathcal{J}}\frac{A}{\bar{G}M}.$$

Taking[7]

$$n=10^{15}\ \mathrm{cm}^{-3},\quad \mathcal{V}=10^{-3}\ \mathrm{cm}^3,$$
$$v_0-s\sim10^5\ \mathrm{cm\ sec}^{-1},$$
$$T=300^\circ\mathrm{K},\quad \mathcal{J}=\tfrac{1}{2},$$

it is found that

$$\frac{\mathrm{Re}\gamma_{vv}{}^{(\xi)}}{\mathrm{Re}\gamma_{vv}{}^{(\mathrm{th})}}\approx10^7\frac{A/\bar{G}}{M}.$$

Now

$$A/\bar{G}\sim10^5.$$

Recall that M is the number of modes in the phonon band. For a Debye phonon spectrum, which certainly applies for the low wave vectors of interest ($q_0\sim10^4$ cm^{-1}),[11] the number of modes with q between q_0 and $q_0+\Delta q_0$ is given by

$$\omega(q_0)\Delta q=(q_0{}^2/2\pi)\Delta q\sim(10^8/2\pi)\times10^5\approx10^{12}$$

The number of modes M is smaller than this by the ratio of the area of the Čerenkov-like cone to 4π, denoted by Θ. For the particular case for which the previously quoted parameters apply (i.e., $n=10^{15}$ cm^{-3}, $\bar{G}=10^8$ sec^{-1}), the current-voltage curve is highly saturated[16] ($v_d\approx s$) and one would expect a very small Čerenkov-like angle, perhaps $\Theta\sim10^{-3}$. Then

$$M=\omega(q_0)\Delta q\Theta\approx10^9$$

and

$$\mathrm{Re}\gamma_{vv}{}^{(\xi)}/\mathrm{Re}\gamma_{vv}{}^{(\mathrm{th})}\sim10^3,$$

which is smaller than the experimental ratio of $\sim10^6$ shown in Fig. 1. In essence, the relative fluctuation in total phonon number is reduced (with respect to the fluctuations of a *single* mode) by the number of modes within the phonon band, and the noise level is reduced proportionally. The width of this band, as estimated in Ref. 11, is quite uncertain, principally because of uncertainties in the correct description of the phonon losses, and this appears as an uncertainty in the present calculation. In addition, it must be emphasized that the use of equilibrium boson statistics for each phonon mode should be regarded only as a physically plausible assumption.

Aside from this factor, it is of interest to note that (4.3) is of the same form as the result derived by Moore.[7] The physical basis of his theory was discussed in the Introduction. It reads

$$\mathrm{Re}\gamma_{II}{}^{(\mathrm{Moore})}(\omega)=\frac{1}{2\pi}\frac{N_d{}^2}{L^2}e^2\left[\left(E-\frac{s}{\mu}\right)^2\mu^2\right]\tau_s\times[\mathcal{J}^3(1-\mathcal{J})]\frac{4\lambda^3}{\mathcal{V}}\frac{1}{1+\omega^2\tau^2}, \tag{4.5}$$

where

$$1/\tau=1/\tau_0+1/\tau_s;$$

here τ_0 is the lifetime in the Ohmic state (the time required for charge bunching) and τ_s is the lifetime in the perfectly saturated state (the time required for decay of the sound wave and charge debunching). Also, λ is the wavelength of the amplified sound (a single frequency is assumed rather than a band of frequencies as in the present case) and $\mathcal{V}$ is the sample volume.

Aside from the $\mathcal{G}$-dependent factors, which are of order unity, and the factor $A/\bar{G}M$ expressing the reduction in magnitude due to the many modes, note that (4.3) and (4.5) depend in the same way on the essential parameters (e.g., E, N_d, and ω). Since these dependences have been verified experimentally, the present theory may be taken as comparably satisfactory in this respect.

V. CONCLUSIONS

A theory of fluctuations in the saturation current has been presented which, it is believed, describes the essential physical features of the problem. The spectral distribution is in agreement with experiment. The cutoff frequency is found directly in terms of known properties of the system (viz., $\bar{G}$) and does not require the introduction of phenomenological parameters. The magnitude of the noise power, however, is explained less satisfactorily. It is believed that this would require a more rigorous description of the primitive statistical fluctuations about a steady state far displaced from equilibrium. As pointed out, the present theory, based on an incoherent phonon picture, agrees in its dependence on the essential parameters with the result of a bunching theory due to Moore.[7] It remains to be shown why this is so, and whether, in fact, the two theories are in some sense equivalent. In this connection, it may be of some importance to take into account space-varying effects, neglected in the present treatment. Finally, it is suggested that fluctuation analysis should serve as a sensitive test of any future theory of the acoustoelectric steady state.

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Modification of Friedel Oscillations by a Magnetic Field*

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The electron density near an impurity or "test particle" in an electron gas is investigated for the case in which a uniform magnetic field is applied to the system. The electron gas is at zero temperature and the Coulomb interaction between electrons is neglected. A δ-function potential is used for the interaction of the electrons with the test particle. The induced electron density along a line passing through the test particle *parallel* to the magnetic field is of the form $r^{-3}\cos 2k_F r$ for large r, where k_F is the Fermi wave number and r is the distance from the test particle. The induced electron density along a line passing through the test particle *perpendicular* to the magnetic field is qualitatively different. It exhibits only a finite number of oscillations in space and then falls off monotonically with increasing distance from the test particle. The number of complete oscillations corresponds to the number of occupied Landau levels in the electron gas. Similar results may be expected for the electron spin density near a magnetic impurity.

I. INTRODUCTION

A NONMAGNETIC impurity in a metal gives rise to a conduction-electron charge density[1] which varies as $r^{-3}\cos 2k_F r$ for large r, where k_F is the Fermi wave number and r is the distance from the impurity to the point in question. Similarly, a magnetic impurity in a metal gives rise to a conduction-electron spin polarization[2] of the same form. This oscillatory phenomenon is a consequence of the sharp cutoff in the momentum distribution of the conduction electrons at zero temperature.

Experiments dealing with these phenomena generally utilize an external magnetic field. In previous analyses, the effect of this field on the momentum distribution of the electrons has not been considered. Indeed, it is not apparent that one may neglect this effect. In the presence of a static homogeneous magnetic field the *angular* momentum of the electrons about the field lines is quantized and the electrons occupy angular momentum states (Landau levels) with quantum numbers ranging from zero up to some cutoff N_F. However, this does not imply a cutoff in the *linear* momentum of the electrons perpendicular to the field lines; if this cutoff is absent or if it is modified in some way, then one may expect a corresponding change in the behavior of the

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